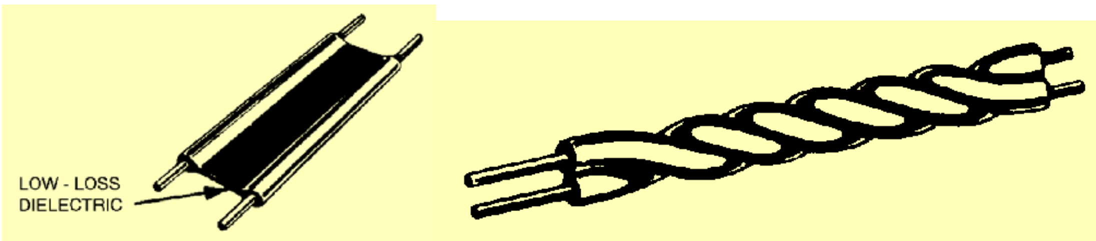
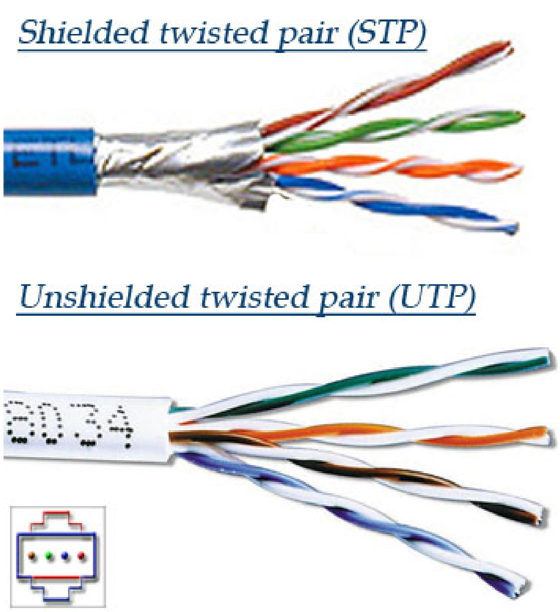
Homework 3

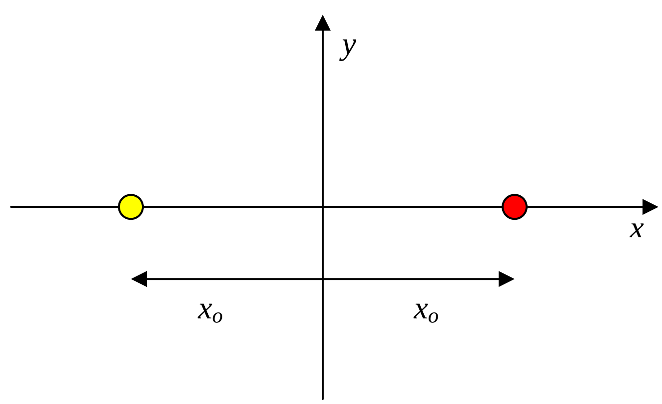
Due 8 October at 4:00 pm

1. Superposition, Electric Flux Integral, and Two-Wire Lines





Two wire transmission lines come in many configurations. When they are carrying a voltage signal of some kind, one wire will be positively charged and one will be negatively charged. A highly simplified, but useful model of such a configuration is two infinitely long parallel line charges, as shown below.



a. The electric field due to a line charge, located at the origin, is given by $\vec{E} = \frac{\rho_L}{2\pi\varepsilon_o r}\hat{a}_r$. Assume that the line charge at $x=-x_o$ has a density of ρ_{Lo} and the line charge at $x=+x_o$ is $-\rho_{Lo}$. Rewrite the two expressions for the electric field in rectangular coordinates and then modify the expressions so that they represent the line charges at the two different locations (coordinate transformation).
b. Evaluate the electric flux passing through the plane $x=0$ for one of the two line charges. $\int \vec{E} \cdot d\vec{S} = ?$ Why does your answer make sense?
c. The flux through the $x=0$ plane produced by the other will be the same. Can you explain why?

2. Coaxial Cable Electric Field

a. The coaxial cable most of you used for your channel blocker design is the F6TSV from Commscope, one of the many RG-6/U cables available commercially. From the given data you have for this cable, you should be able to determine the capacitance and inductance per unit length as well as the radii of the inner and outer conductors. You will need this information to do this problem.
b. The data provided for this cable does not include a maximum voltage rating, but similar cables are designed to meet an 800V spec. Assume that the voltage on the line is +800V (on the center conductor) and the outer conductor is grounded. From the capacitance per unit length, determine the charge per unit length from the usual relationship between the voltage, capacitance and charge for a capacitor.
c. We know that charges on a capacitor are found on the surfaces of the conducting electrodes. From your knowledge of the dimensions of the cable and the charge per unit length from b, determine the surface charge density found on the inner and outer conductors of the cable.
d. Your answer to part c specifies the charge distribution that produces the electric field of the coaxial cable, so you now have sufficient information to find $\vec{E}$, except for one parameter, the dielectric constant ε. Again, while this is not given in the spec sheet, there is sufficient information available to determine ε, as discussed in lecture. Once you have a value for ε, use Gauss' Law now to find the electric field in the region between the inner and outer conductors. Plot the magnitude of $\vec{E}$ as a function of radius.
e. What is the maximum value for $\vec{E}$ and where is it located? Compare the value you obtained for $|\vec{E}|_{\max}$ with a published value for the dielectric strength of gas expanded polyethylene (provide the reference).

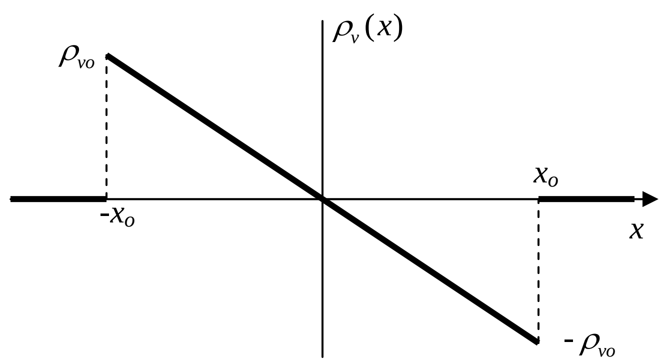
3. The Earth-Ionosphere Capacitor

a. The typical value for the electric field magnitude near the surface of the earth is $\left|\vec{E}\right| = 100\frac{V}{m}$. What is the surface charge density associated with this field magnitude? *Hint: this can be answered either with boundary conditions or using Gauss' Law.* Look up the radius of the earth and then determine the total charge on the earth.

b. Using your value for the surface charge density, find the expression for the electric field vector $\vec{E}$ in the region between the surface of the earth and the ionosphere (assume the altitude of the ionosphere is about *100km*). Plot your result as a function of altitude and determine the value of the electric field at the altitude of the ionosphere. Does it seem reasonable to use a constant electric field throughout this region of space?

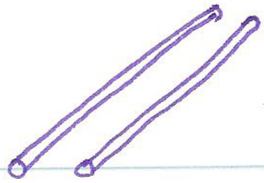
4. Using Gauss' Law to Find the Electric Field of a Charge Distribution

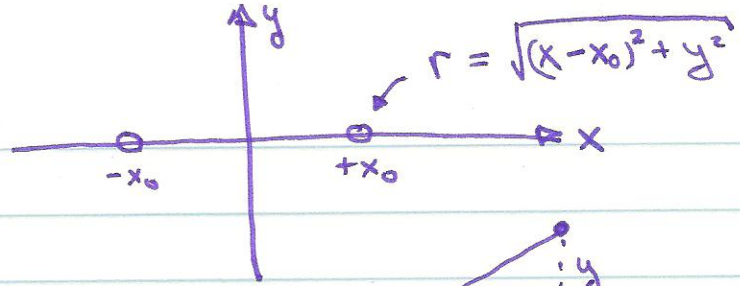
a. The volume charge distribution in a planar structure is described by the plot shown below. Note that the charge density is zero for $|x| > x_o$. Using this information, write the mathematical expression for the charge distribution $\rho_v(x)$. Using your expression, show that the total charge is equal to zero.



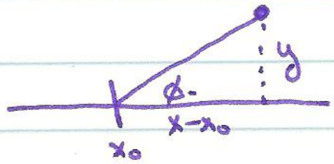
b. Use your charge distribution and Gauss' Law to determine the electric field everywhere in space $\vec{E}$. Be sure that you simplify everything in your expressions as much as possible.

1.





$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\,\hat{a}_r$$



Brute Force Method:

Negative line charge $\quad \vec{E} = \dfrac{-\rho_{L0}}{2\pi\varepsilon_0\sqrt{(x-x_0)^2+y^2}}\,\hat{a}_{r-}$

$$\hat{a}_{r-} = \hat{a}_x \frac{(x-x_0)}{\sqrt{(x-x_0)^2+y^2}} + \hat{a}_y \frac{y}{\sqrt{(x-x_0)^2+y^2}}$$

$$\Rightarrow \vec{E} = \frac{-\rho_{L0}}{2\pi\varepsilon_0}\,\frac{1}{(x-x_0)^2+y^2}\left(\hat{a}_x(x-x_0) + \hat{a}_y\, y\right)$$

Then take dot product with $d\vec{S} = \hat{a}_x\, dy\, dz$

slightly easier method

Start with $d\vec{S} = \hat{a}_x\, dy\, dz$ so we know that we will only need the x component of $\vec{E}$

$$E_x = \frac{-\rho_{L0}}{2\pi\varepsilon_0}\,\frac{1}{r}\,\frac{(x-x_0)}{r} \qquad \text{where } r^2 = (x-x_0)^2+y^2$$

$$\vec{E}_x\,\hat{a}_x\cdot d\vec{S} = -\frac{\rho_{L0}}{2\pi\varepsilon_0}\,\frac{x-x_0}{(x-x_0)^2+y^2}\,dy\,dz$$

the integration in z will be per unit length

$$\int_{-\infty}^{\infty} dy \int_0^1 dz \left(\frac{-\rho_{L0}}{2\pi\varepsilon_0}\,\frac{x-x_0}{(x-x_0)^2+y^2}\right)_{x=0}$$

$$= 1\int_{-\infty}^{\infty}\left(\frac{+\rho_{L0}x_0}{2\pi\varepsilon_0}\right)\frac{1}{x_0^2+y^2}\,dy$$

$$= \frac{\rho_{L0}x_0}{2\pi\varepsilon_0}\,\frac{1}{x_0}\tan^{-1}\frac{y}{x_0}\Big|_{-\infty}^{\infty} = \frac{\rho_{L0}}{2\pi\varepsilon_0}\pi = \frac{\rho_{L0}}{2\varepsilon_0}$$

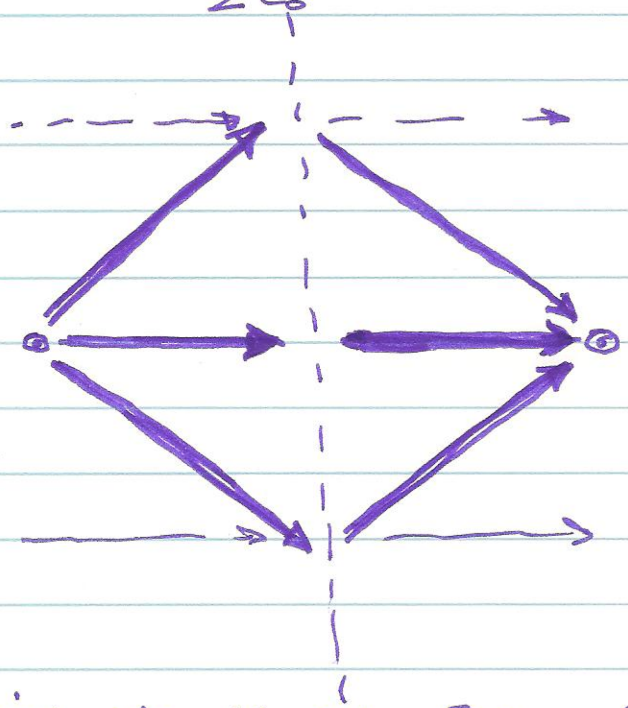
For the positive line charge

$$E_x = \frac{+\rho_{l0}}{2\pi\varepsilon_0}\,\frac{1}{r}\,\frac{x+x_0}{r}$$

$$E_x\,\hat{a}_x\cdot d\vec{S} = \frac{\rho_{l0}}{2\pi\varepsilon_0}\,\frac{x+x_0}{(x+x_0)^2+y^2}\,dy\,dz$$

$$\int_{\text{at } x=0} \bar{E}\cdot d\bar{S} = \frac{\rho_{l0}\,x_0}{2\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{1}{x_0^2+y^2}\,dy$$

$$= \frac{\rho_{l0}}{2\varepsilon_0}$$



The line charge are symmetrically located so the field are mirror images & each contributes identically to the total flux.

Also notice that the ∞ plane at $x=0$ will have half of the flux from each line pass through it so the flux is just half of $\frac{\rho_l}{\varepsilon_0}$ for each one.

2.a. $Z_0 = 75\,\Omega$ $\quad u$ = vel of prop $= .85\,(3\times10^8\,m/s)$

$= \sqrt{\frac{l}{c}}$ $\quad = \frac{1}{\sqrt{lc}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0 \varepsilon_r}} \Rightarrow \varepsilon_r = \left(\frac{1}{.85}\right)^2$

$= 1.38$

$$u Z_0 = \frac{1}{c} \Rightarrow c = \frac{1}{u Z_0} = 52.3\ pF/m$$

$$l = Z_0^2 c = 2.94\times10^{-7} = .294\ \mu H/m$$

$$c = \frac{2\pi\varepsilon}{\ln b/a} = 51.3\ pF/m \qquad \text{using} \quad a = .51\,mm \quad b = 2.285\,mm$$

$$l = \frac{\mu_0}{2\pi}\ln\frac{b}{a} = .3\ \mu H/m$$

* These are ideal formulas which give close answers as a check. Actually none of the numbers from the spec sheet are accurate to the significant digits shown here. Z_0 is, for example, given within $3\,\Omega$

$\Rightarrow$ Reduce Digits $\qquad c = 52\ pF/m$

$\qquad l = .3\ \mu H/m$

Note answers are OK within 10%

b. $\rho_{L0} = 800 \cdot 52\ pF/m$
$= 42\ nC/m$

$Q = CV$

$\Rightarrow \rho_L = cV$ ← per unit length

c. $\rho_{sa} = \frac{\rho_L}{2\pi a} = 13\ \mu C/m^2$

$\rho_{sb} = \frac{-\rho_L}{2\pi b} = -3\ \mu C/m^3$

$\rho_L\, dl = \rho_s\, dS$

$= \rho_s\, 2\pi r\, dl$

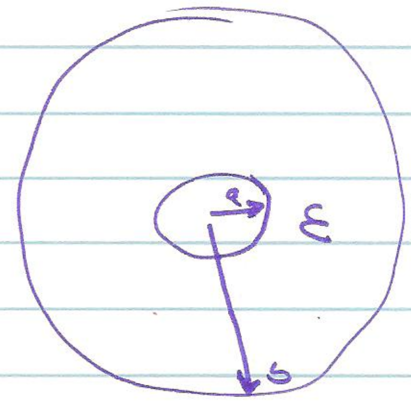
$\rho_{sa} = \frac{\rho_L}{2\pi a}$

d. $\varepsilon = \varepsilon_r \varepsilon_0$ $\varepsilon_r = \left(\frac{3\times 10^8}{u}\right)^2 = \frac{1}{(.85)^2} = 1.38$

(repeated from above)

$$\oint \vec{E} \cdot d\bar{S} = \frac{1}{\varepsilon} Q_{encl}$$

$$\text{or } \oint \bar{D} \cdot d\bar{S} = Q_{encl}$$



For a Gaussian surface
$r = \text{const} < a$
$Q_{encl} = 0$

For a Gaussian surface
$r = \text{const} > b$
$Q_{encl} = 0$ because the pos & neg charges are equal.

For a surface $a \le r \le b$ Q_{encl} per unit length is ρ_L or $\rho_{sa} 2\pi a$

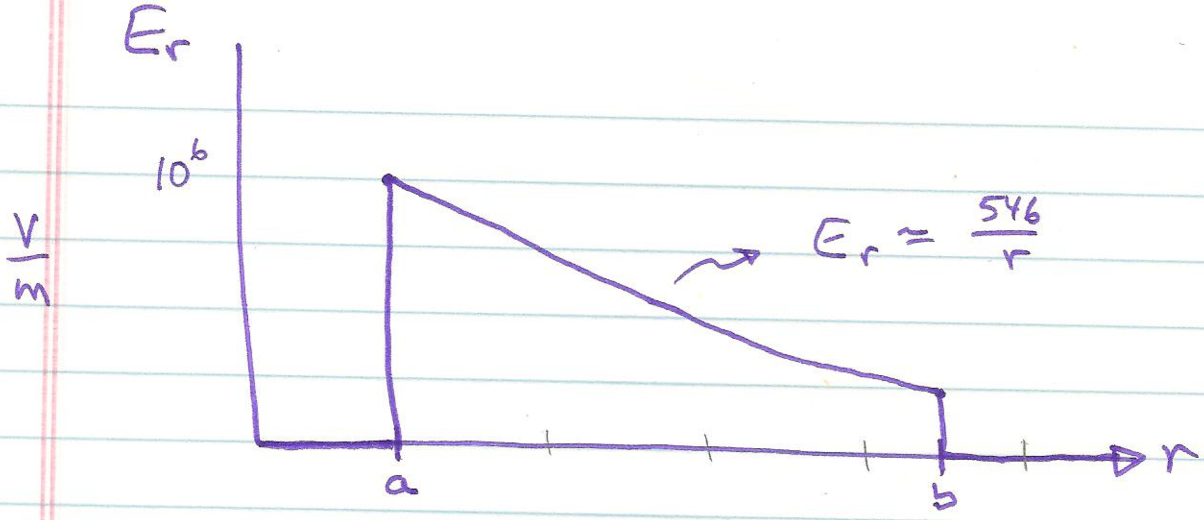
$\rho_L = 42 nC/m$ $\rho_{sa} 2\pi a = 42 nC/m$

$$\int_0^1 \int_0^{2\pi} D_r \hat{a}_r \cdot \hat{a}_r \, r \, d\phi \, dz = D_r(r) \, r \, 2\pi$$

$$D_r(r) = \frac{\rho_L}{2\pi r} = \frac{42 nC/m}{2\pi r}$$

$$E_r(r) = \frac{D_r}{\varepsilon} = \frac{42 \, nC/m}{2\pi \varepsilon_r \varepsilon_0 r}$$

$$= \frac{546}{r}$$



e. $E_{r\,max} \approx 1.1 \times 10^6$ V/m
Air (dry) can support 3×10^6 V/m
Polyethylene typical values 50×10^6 V/m
Gas expanded will be about half that
because it includes air. $\Rightarrow$ $\sim 20\text{-}30 \times 10^6$ V/m
which is way over the max E_r calculated
Ref www.matweb.com.

Earth Radius: 6378km

3. a. $\rho_{so} = \varepsilon_0 E = 100\,\varepsilon_0 = 8.85 \times 10^{-10}\ C/m^2$

$Q_{TOTAL} = 4\pi r^2 \rho_{so} = 4\pi (6.38 \times 10^6)^2\, 8.85 \times 10^{10} = 4.5 \times 10^5\ C$

b. Spherical problem.

$Q_{encl} = \text{Area} \cdot \rho_{so} = 4\pi (r)^2 \rho_{so}$

$= 4.5 \times 10^5\ C$

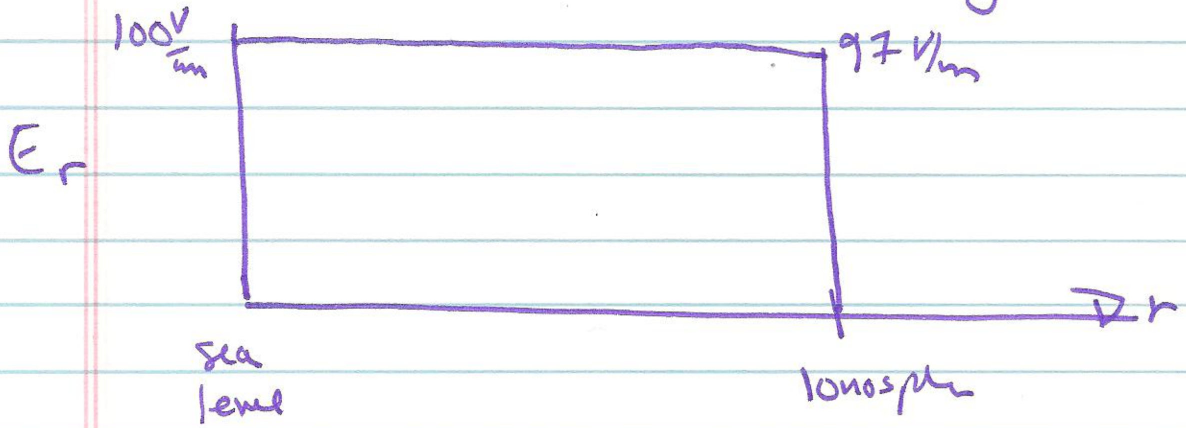
$$E_r(r) = \frac{Q_{encl}}{4\pi\varepsilon_0 r^2} = \frac{4.5 \times 10^5}{4\pi\varepsilon_0}\frac{1}{r^2} = \frac{\cancel{1.62 \times 10^{20}}\ 4.07 \times 10^{15}}{r^2}$$

$E_r(a) = 100$ V/m

(Typo correction)

$$E_r(b) = \frac{\cancel{1.62 \times 10^{20}}\ 4.07 \times 10^{15}}{(6.378 \times 10^6 + 10^5)^2} = 97\ V/m$$

The const E model is a good one.



4. Gauss' Law for E of charge dist.

$$\rho_v(x) = 0 \qquad x > x_0 \,,\; x < -x_0$$

$$\rho_v(x) = a_1 + a_2 x$$
$$\rho_v(0) = 0 = a_1$$
$$\rho_v(x_0) = -\rho_{v0} = a_2 x_0$$
$$a_2 = -\frac{\rho_{v0}}{x_0}$$

$$\rho_v(-x_0) = \rho_{v0} = -a x_0 \Rightarrow a_2 = -\frac{\rho_{v0}}{x_0}$$

$$\Rightarrow \rho_v(x) = -\frac{\rho_{v0}}{x_0} x \qquad -x_0 \le x \le x_0$$

Q_{TOTAL} per unit area $$= 1 \int_{-\infty}^{\infty} \rho_v(x)\, dx$$

$$= -\int_{-x_0}^{x_0} \frac{\rho_{v0}}{x_0} x \, dx$$

$$= -\frac{\rho_{v0}}{x_0} \left(\frac{x^2}{2} \right)_{-x_0}^{x_0} = -\frac{\rho_{v0}}{x_0} \left(\frac{x_0^2}{2} - \frac{x_0^2}{2} \right) = 0$$

b. Since there is no net charge, the E field outside the charge dist will be zero.

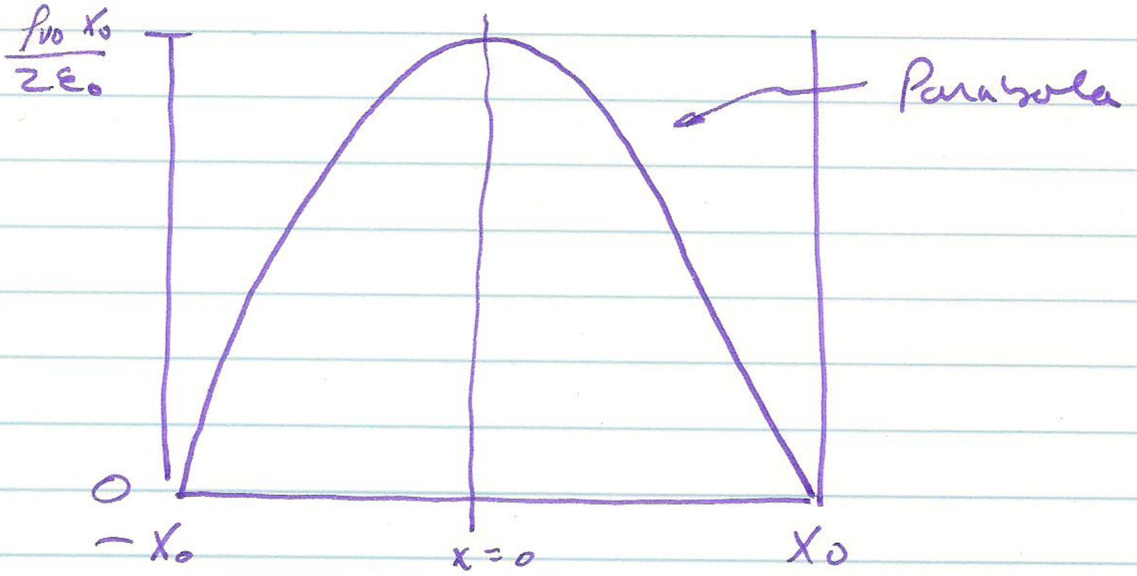
$$\oint \vec{D} \cdot d\vec{S} = Q_{enc}$$

$$\varepsilon_0 E_x \cancel{S} \Big|_{-x_0}^{x} = \cancel{S} \int_{-x_0}^{x} \left(-\frac{\rho_{v0}}{x_0} x\right) dx$$

$$\varepsilon_0 \Big(E_x(x) - \underbrace{E_x(x_0)}_{= 0} \Big) = -\frac{\rho_{v0}}{x_0} \frac{x^2 - x_0^2}{2}$$

$$E_x(x) = \frac{\rho_{v0}}{\varepsilon_0 x_0} \frac{x_0^2 - x^2}{2} \qquad -x_0 \le x \le x_0$$

$$\text{Peak } E_x(x) = E_x(0) = \frac{\rho_{v0}}{\varepsilon_0} \frac{x_0}{2}$$



Checking the result: $\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon_0}$ ✓

$$\frac{\partial E_x}{\partial x} = -\frac{\rho_{v0}}{\varepsilon_0 x_0}\left(\frac{\cancel{2}x}{\cancel{2}}\right) = -\frac{\rho_{v0}}{\varepsilon_0}\frac{x}{x_0} = \frac{\rho_v(x)}{\varepsilon_0}$$

7/7