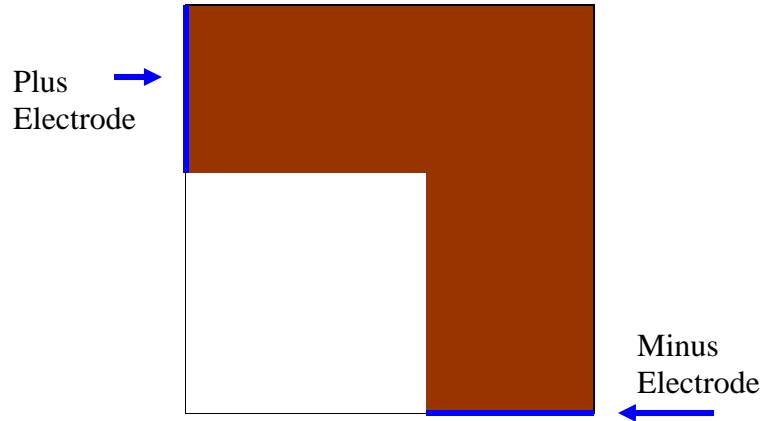


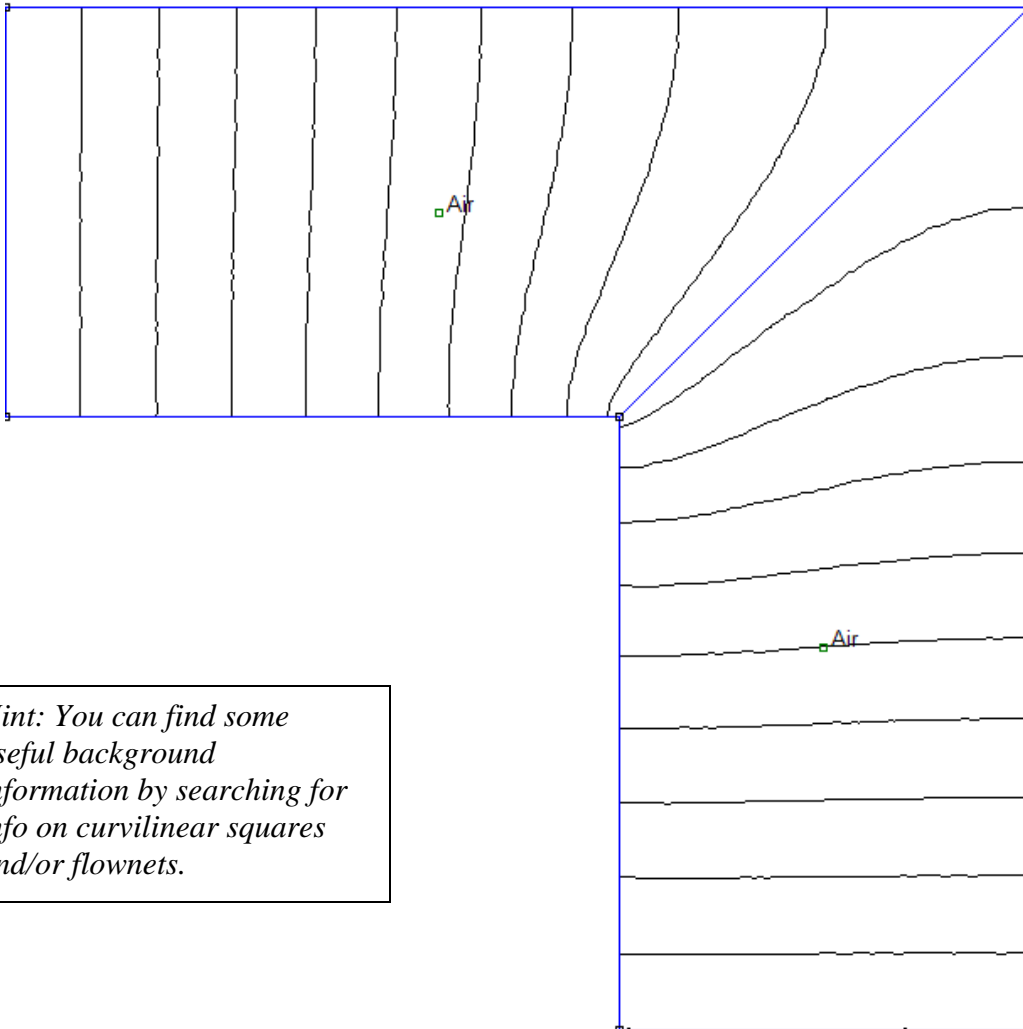


1. Resistance Calculations



Assume that we have a thin conductor deposited onto the surface of a plastic board (as on a printed circuit board). The conductor is L-shaped, so any current passing through it must turn a corner. The square region in which the conductor is found is 10cm by 10cm. The conductor is 4cm wide and 1mm thick. To keep things simple, we will assume a conductivity of $\sigma = 10$ as might be obtained using some kind of salty water solution.

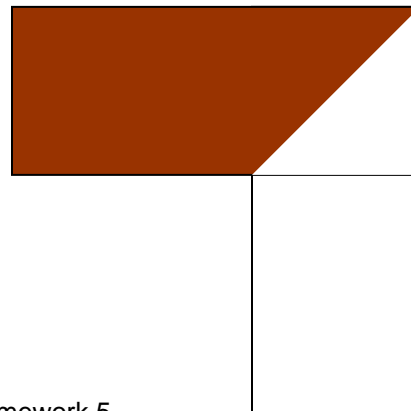
- Using the simplest possible model, find the resistance of this conductor. That is, assume it has the given width and assume a length that includes all of the conducting material. Then, being careful about units, find the resistance.
- Next, use the plot below to find a more accurate value for the resistance. This plot, generated using FEMM, shows the equipotentials. To use this plot, ***carefully*** draw electric field lines in sufficient number to form curvilinear squares. That is, each cell formed by two equipotentials and two field lines should be approximately square. Once you have this diagram, then find the resistance but first finding the resistance of a single cell and then combine the cells in series and parallel to get the final answer. The plot labels two regions as air, but that is just an artifact of the method used. Assume the region shown is a conductor.
- Now you should solve this problem using a spreadsheet. To do this, you assume that the region of interest is made of a single material. It really does not matter whether the material is a dielectric or conductor, since the potentials will be the same, because you only need to use Laplace's equation. Solve for the voltages in the region of interest. Once you have obtained your solution, save your results and then load them into Matlab using a modification of the file provided on the handout page. This file applies to an excel solution to a square blob in a parallel plate region, also provided. (continued below)



Hint: You can find some useful background information by searching for info on curvilinear squares and/or flownets.

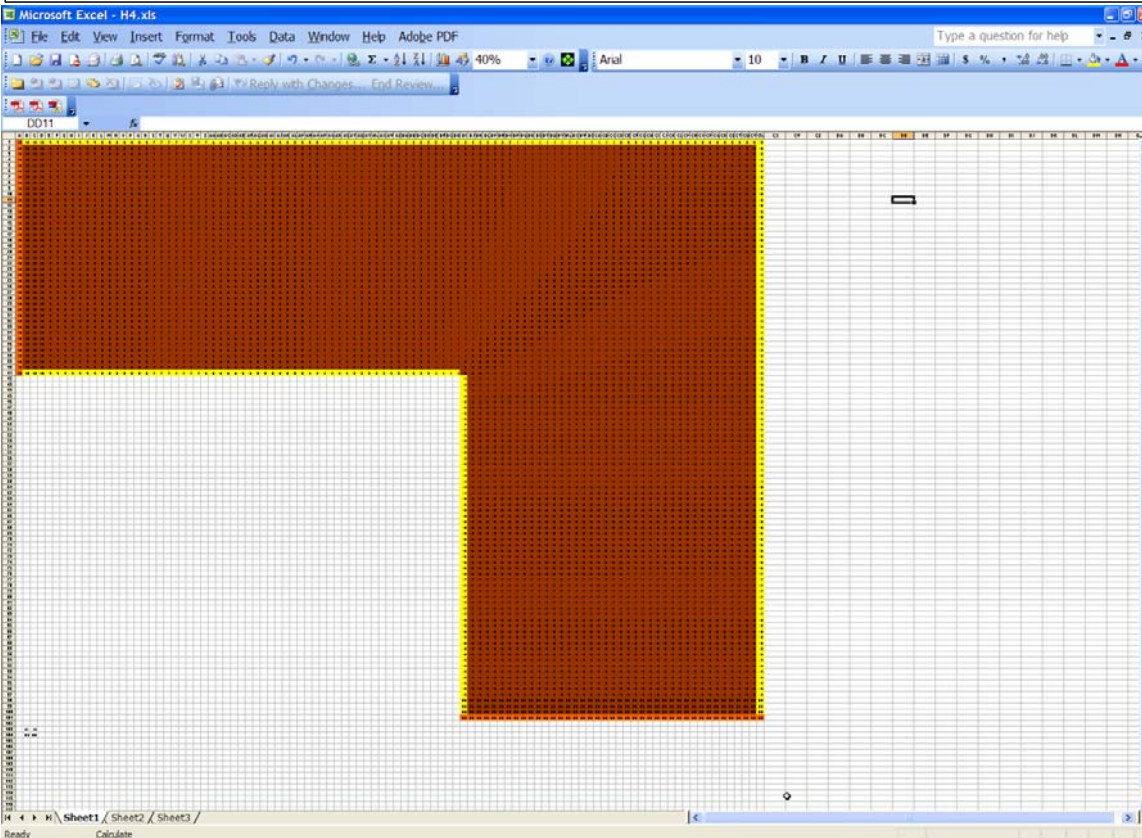
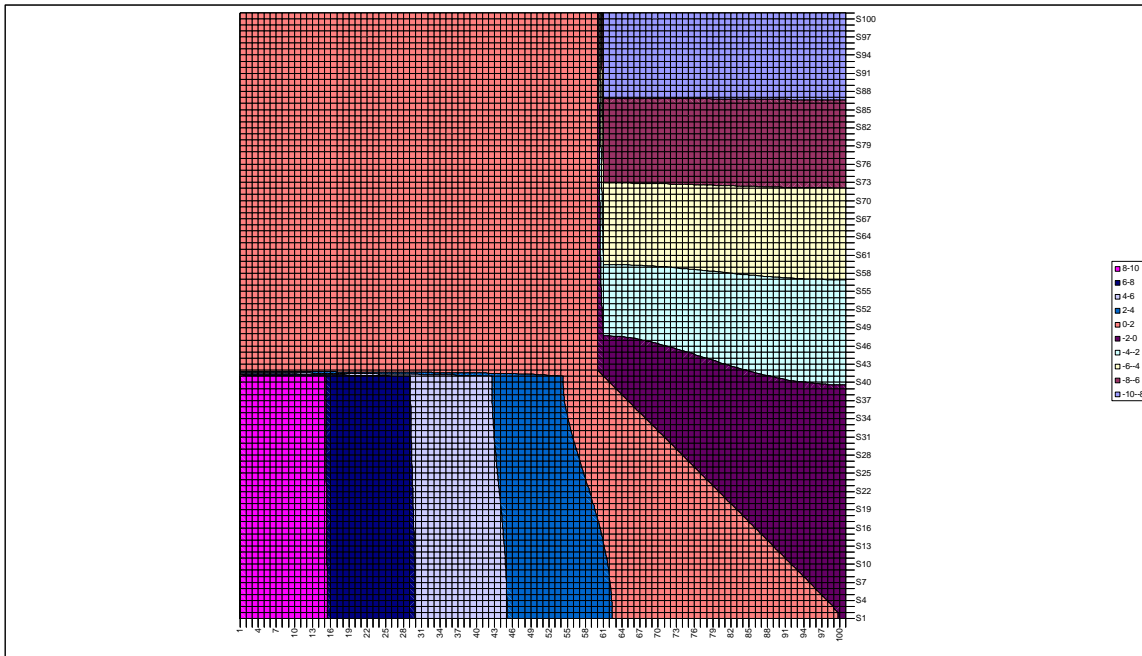
The Matlab file allows us to produce streamlines from equipotentials. Our streamlines are the electric field lines. The method is not perfect, so that it is really only accurate for about half of the structure. This is shown in the example provided. To achieve the best accuracy using this method, only use the field lines from the upper half of the configuration, as shown, and then double your answer.

The advantage of using this method is that you do not have to sketch the field lines by hand. However, we are stuck with its inaccuracies, such as they are. Be sure to point out the inaccuracies in your results and fully label your plots.





For the spreadsheet solution, the equipotentials look like.



The yellow cells are set equal to their interior neighbor cell, the orange cells are at fixed voltages and the brown cells are set equal to the average of their four neighbors.



The modified m-file for this case:

```
% Plotting Data from Excel  
% K. A. Connor
```

```
% Input Data from Excel for a Laplace's Equation Solution  
% First, you need to know what part of the excel file holds your data  
% Second, copy the file into the Matlab Workspace so you have access to it  
% V=xlsread('file.xls','upper left cell: lower right cell');
```

```
% Name the array (usually some kind of voltage)  
v=xlsread('H4.xls','a1:cw101');
```

```
% Position array for plotting  
d=size(v);  
[x,y]=meshgrid(0:(d(1)-1)/100:(d(1)-1),0:(d(2)-1)/100:(d(2)-1));
```

```
% Calculate the E field using the gradient function  
% [Ex,Ey]=gradient(-V);%Don't forget the negative sign  
[ex,ey]=gradient(-v);
```

```
% For streamlines one needs to specify where they should start  
% It is a good idea to start them uniformly distributed around the largest  
% electrode. [sx,sy]=meshgrid(0:100,1); %Note that the range of points is  
% shown in one or the other of the parameters.;  
[sx,sy]=meshgrid(1,4:4:36);  
[tx,ty]=meshgrid(64:4:96,100);
```

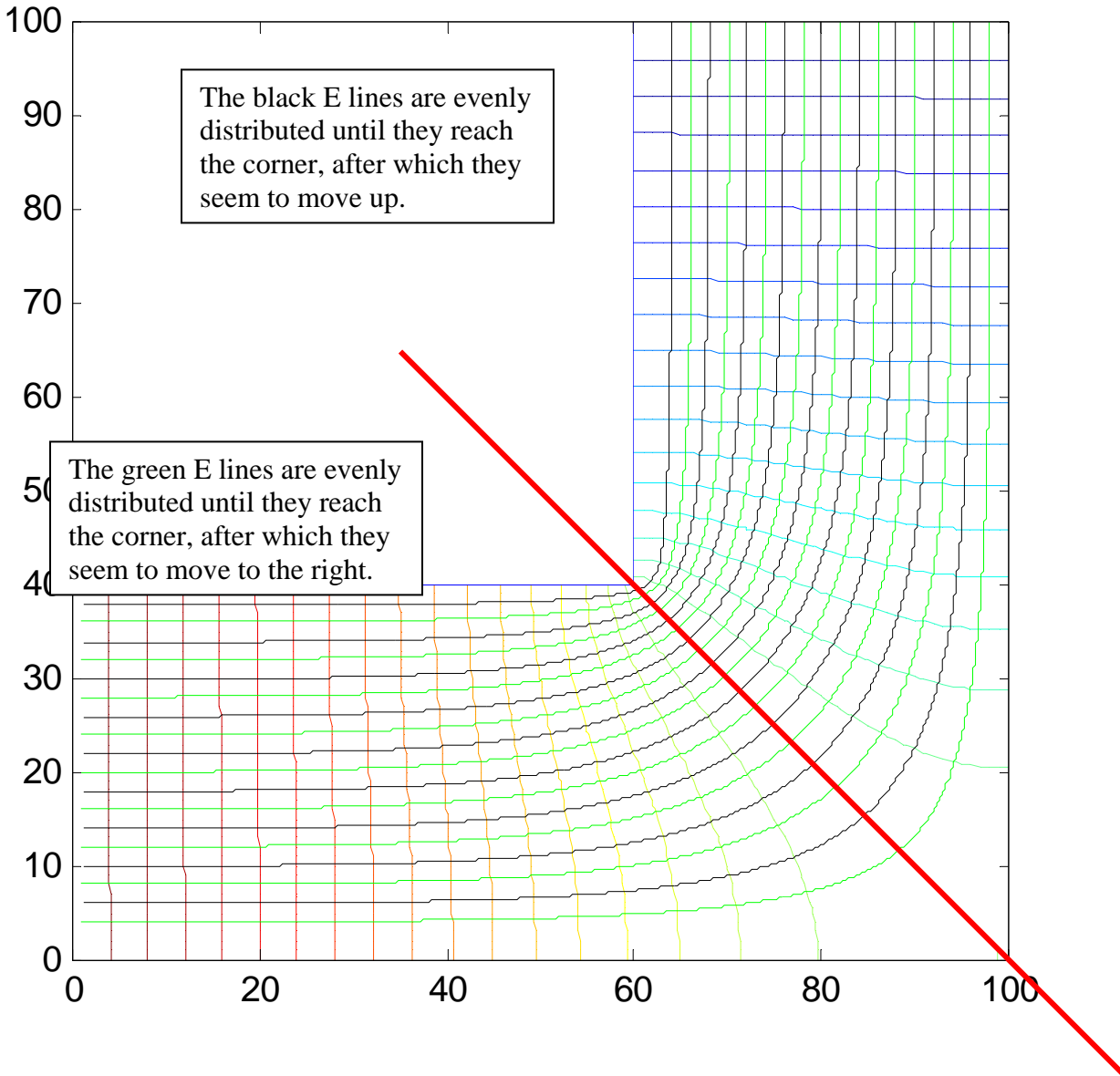
```
% Plotting  
contour(x,y,v,35);hold on;axis square;  
h=streamline(ex,ey,sx,sy);set(h, 'Color', 'green');  
g=streamline(-ex,-ey,tx,ty);set(g, 'Color', 'black');
```

```
x1=[0 60];y1=[40 40];  
line(x1,y1)  
x2=[60 60];y2=[40 100];  
line(x2,y2)
```

Note: The last two line functions show the edges of the region.



The plot from Mablab shows streamlines (E field lines) plotted from each electrode. Note that they become less accurate after they turn the corner. Thus, we only use half of the figure and then multiply the result by 2 to obtain the full resistance. Note that the E lines would be the same drawn from both electrodes if they had no errors. The solution is probably not even totally accurate as far as the corner, but using only half of the field structure seems to be a reasonable approximation.



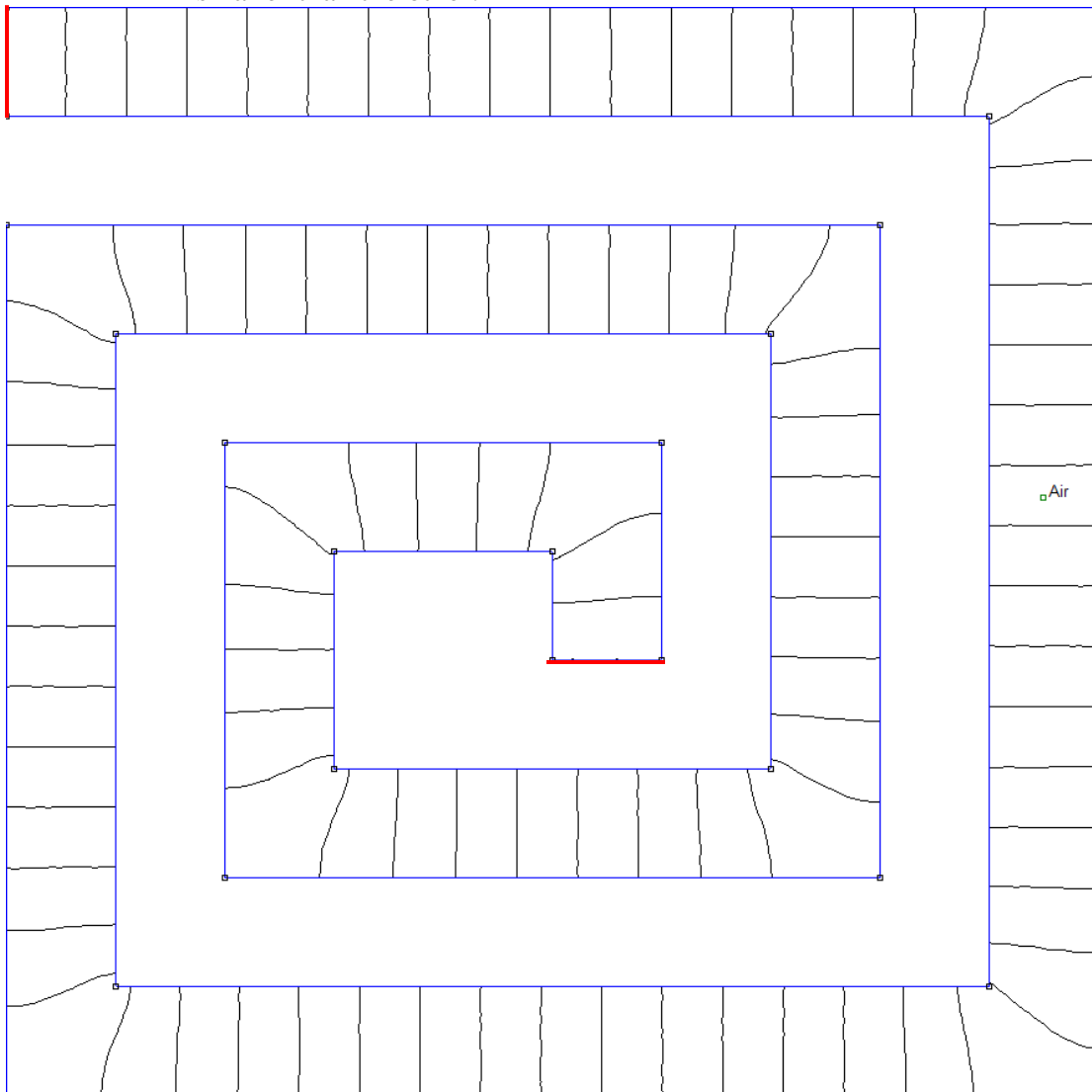
We use a red line to mark the useful half of the structure and then focus on the black cells in the upper leg. There are 10 cells across and 18 down. Each cell has a resistance of 100 Ohms as before so the total is $\frac{18}{10} 100$ times 2 or 360 Ohms. Both of the numerical solutions are more accurate than the simple formula (see explanation in problem 2).



2. Resistance of Spiral

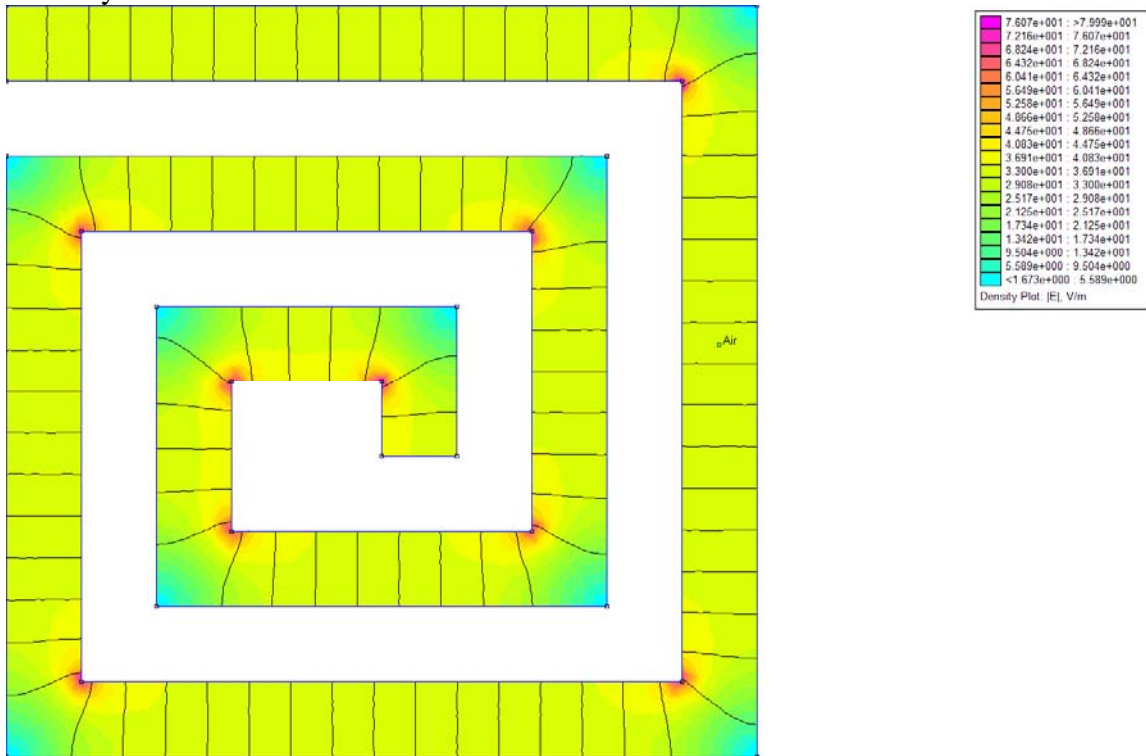
The configuration below shows a spiral conductor formed on the surface of a plastic board. (From FEMM) The overall region is again 10cm by 10cm with the conducting stripe exactly 1cm wide and 1mm thick. The two connections for the conductor are shown in red. That is the current flows from one to the other

- Again, begin by determine the simplest possible solution for the resistance of this configuration.
- Next, use the information from FEMM to determine a more accurate solution. Again, you want to form the individual square cells of the curvilinear squares method.
- Discuss why your answers differ and why it is reasonable that one is smaller than the other.





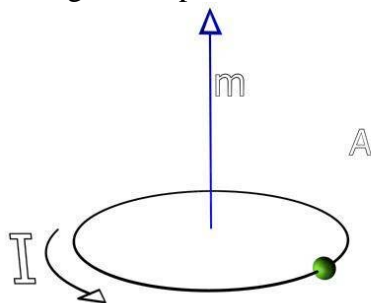
The same information is shown below, along with a false color plot of the electric field intensity.



Label the regions in this plot where the electric field is the largest and where it is the smallest. Also note the regions in which the electric field intensity is reasonably uniform. Provide an explanation to accompany your labels.

3. Magnetic Vector Potential & Magnetic Flux

A magnetic dipole is one of the most fundamental and useful magnetic configurations.



The orientation of the dipole is along the z-axis. The magnitude of the dipole is

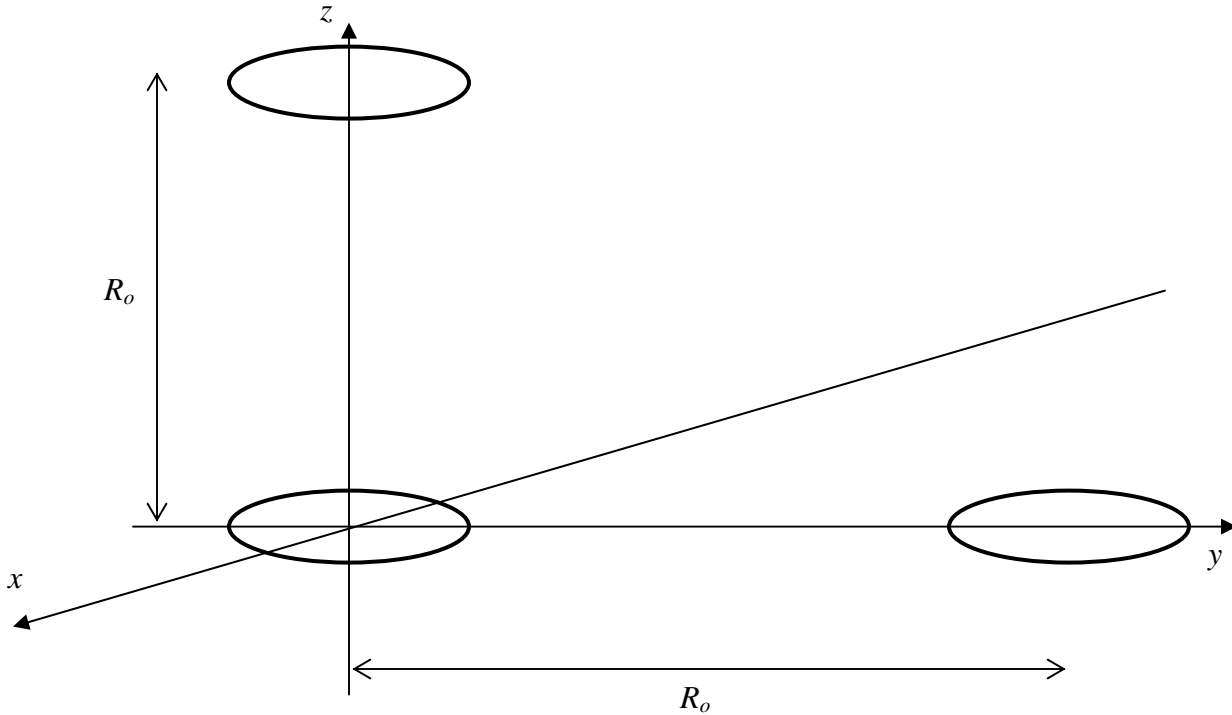
$$m = I \cdot \text{Area} = I\pi a^2$$

One of the most useful characteristics of the magnetic vector potential is that it has vector components only in the direction of the current that generates it. In this case, that means that the vector potential must only have a ϕ component. For distances that are not too

close to the loop, the vector potential is given by $A_\phi = \frac{\mu_0 I \pi a^2}{4\pi r^2} \sin \theta$.



- a. Determine the magnetic field \vec{B} in vector form from the expression above for the vector potential.
- b. Determine the magnetic flux passing through the two loops shown below for the given magnetic vector potential field. Determine the flux using both the expressions for \vec{A} and \vec{B} . The two answers should be the same. For all four integrals, you can assume that the distances between the loops is large compared to their diameters (all three $=2a$)



4. Ampere's Law

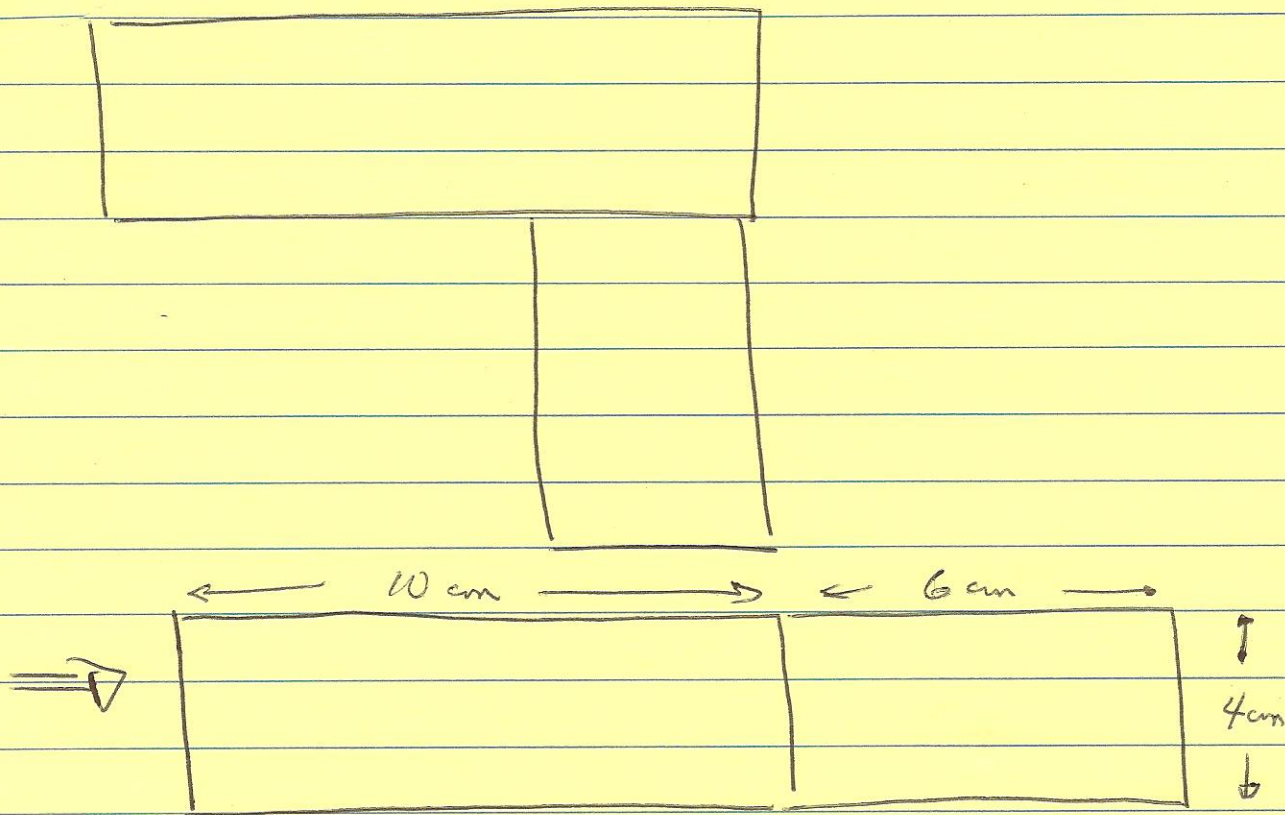
- a. A beam of charged particles is directed axially down the center of a conducting pipe of radius b . The current density in the beam is given by

$$\vec{J} = J_o \left(1 - \frac{r^4}{a^4} \right) \hat{a}_z \text{ in the region } a \leq r \leq b \text{ and zero elsewhere. Determine}$$

the total current carried by this beam. Assume that this current must be returned as a surface current at $r=b$. Determine \vec{J}_s .

- b. From the source distribution you have given in part a, determine the magnetic field \vec{B} everywhere in space using Ampere's Law in integral form.
- c. Confirm that your answer to part b is correct by showing that it satisfies the differential forms of Maxwell's equations for magnetostatics (both the curl and divergence equations).

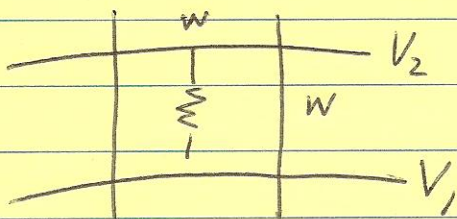
1. a. Simplest possible model



$$R = \frac{16 \text{ cm}}{\sqrt{(4 \text{ cm})(1 \text{ mm})}} = \frac{.16}{10 (.04)(.001)}$$

$$= \frac{4}{.01} = 400 \Omega$$

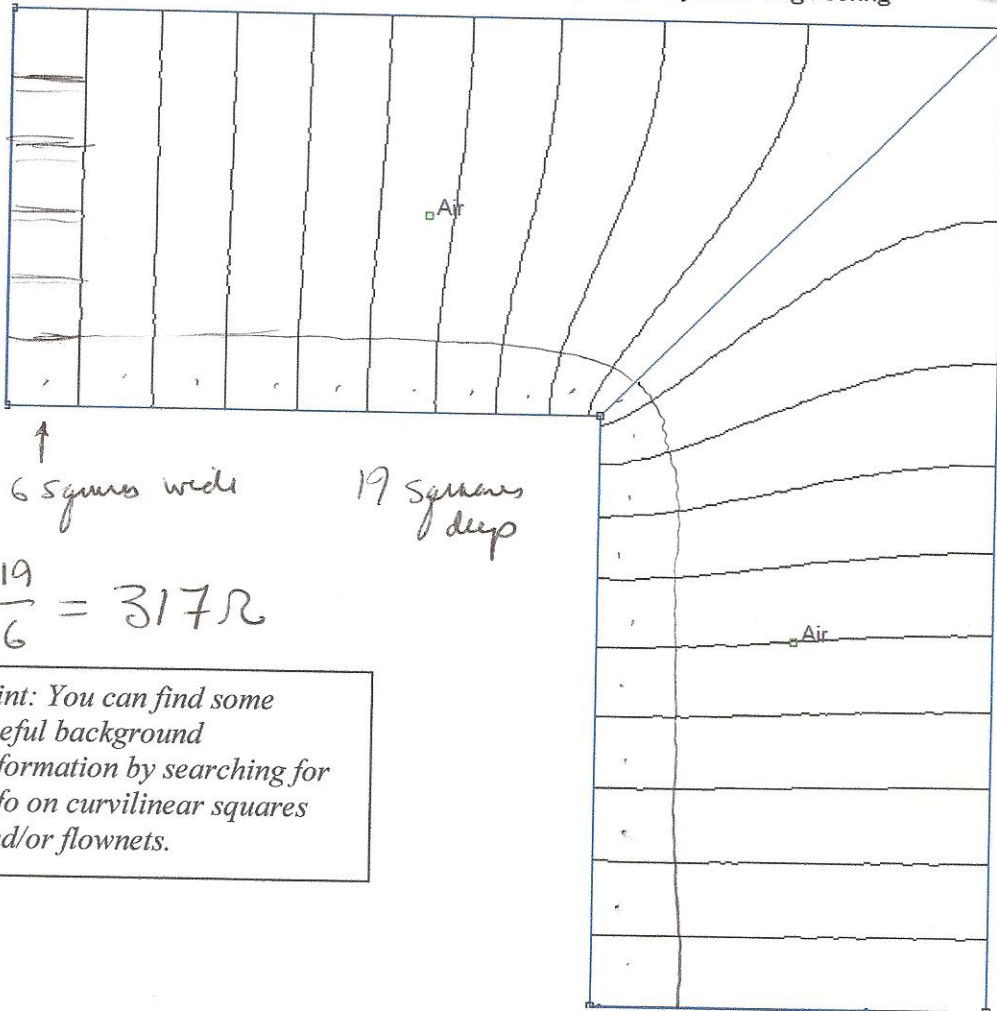
5. Use curvilinear squares method



The square resistor, 1 meter deep

$$R = \frac{W}{\sqrt{(.001)W}} \quad \underbrace{\text{deep}}_{1 \text{ mm}}$$

$$= \frac{1}{10(.001)} = 100 \text{ } \mu\text{square}$$

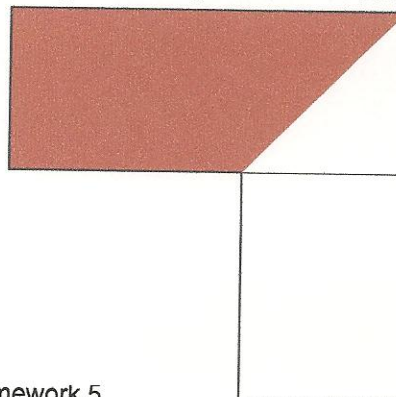


$$\Rightarrow 100 \frac{19}{6} = 317 \Omega$$

Hint: You can find some useful background information by searching for info on curvilinear squares and/or flownets.

The Matlab file allows us to produce streamlines from equipotentials. Our streamlines are the electric field lines. The method is not perfect, so that it is really only accurate for about half of the structure. This is shown in the example provided. To achieve the best accuracy using this method, only use the field lines from the upper half of the configuration, as shown, and then double your answer.

The advantage of using this method is that you do not have to sketch the field lines by hand. However, we are stuck with its inaccuracies, such as they are. Be sure to point out the inaccuracies in your results and fully label your plots.





2. Resistance of Spiral

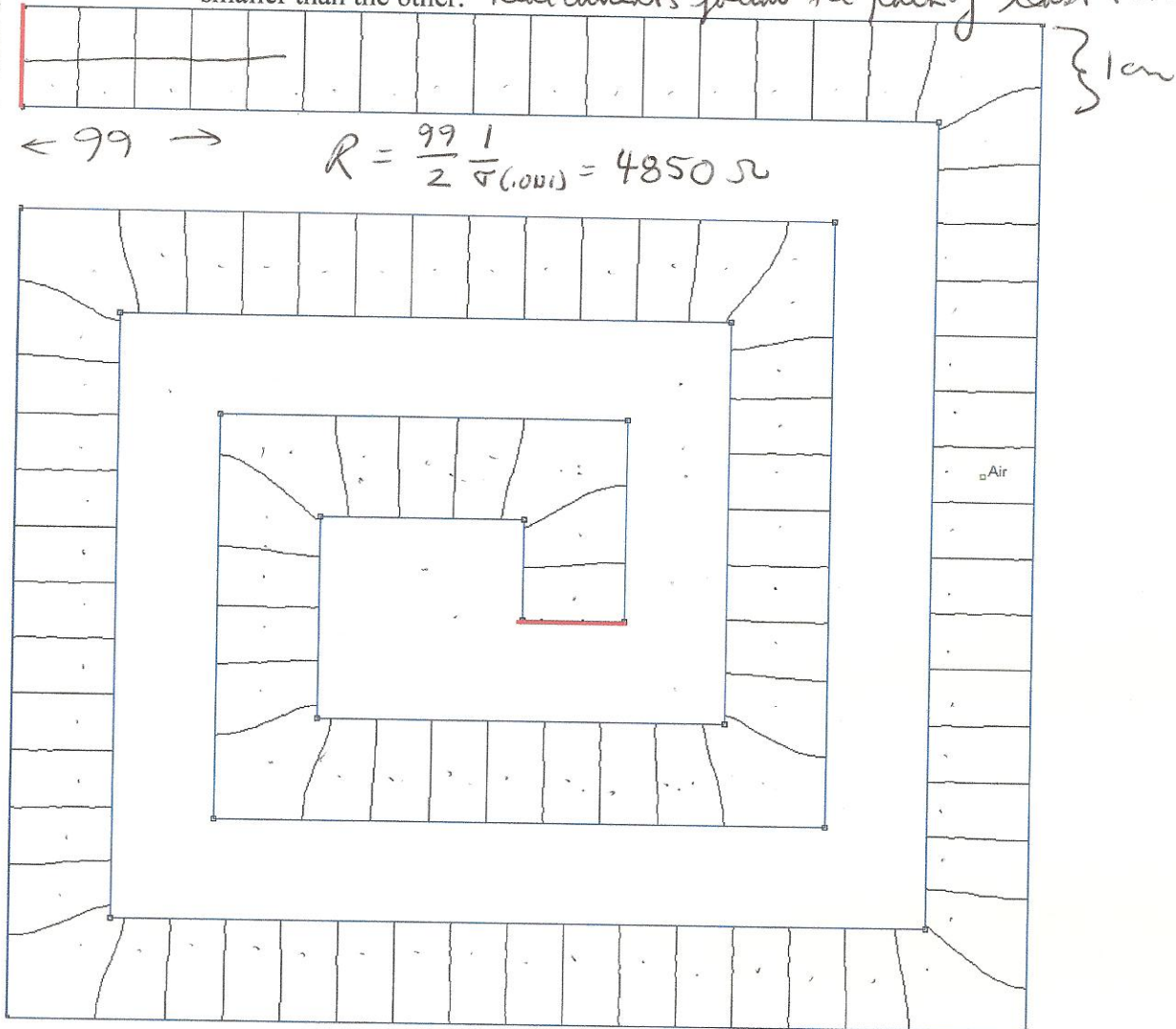
Length = 10 + 9 + 9 + 7 + 7 + 5 + 5 + 3 + 3 + 1

The configuration below shows a spiral conductor formed on the surface of a plastic board. (From FEMM) The overall region is again 10cm by 10cm with the conducting stripe exactly 1cm wide and 1mm thick. The two connections for the conductor are shown in red. That is the current flows from one to the other

- a. Again, begin by determine the simplest possible solution for the resistance of this configuration. $R = \frac{59cm}{10(1cm)(1mm)} = 5900 \Omega$
- b. Next, use the information from FEMM to determine a more accurate solution. Again, you want to form the individual square cells of the curvilinear squares method.

- c. Discuss why your answers differ and why it is reasonable that one is smaller than the other. *Real currents follow the path of least resistance*

2 cells wide

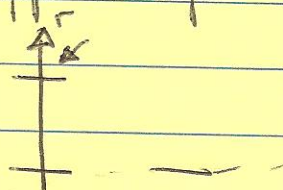


$$3. \vec{A} = A_\phi(r) \hat{\phi} = \frac{\mu_0 I \pi a^2}{4\pi r^2} \sin\theta \hat{\phi}$$

From vector formula sheet

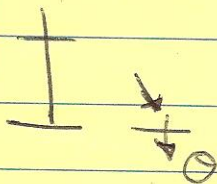
$$\begin{aligned} a. \vec{B} &= \nabla \times \vec{A} = \hat{r} \frac{1}{r \sin\theta} \frac{d}{d\theta} (A_\phi \sin\theta) + \hat{\theta} \frac{1}{r} \left(-\frac{d}{dr} (r A_\phi) \right) \\ &= \hat{r} \frac{\mu_0 I \pi a^2}{4\pi r^3 \sin\theta} \frac{d}{d\theta} (\sin^2\theta) - \hat{\theta} \left(\frac{1}{r} \frac{d}{dr} \frac{1}{r} \right) \left(\frac{\mu_0 I \pi a^2}{4\pi} \sin\theta \right) \\ &= \hat{r} \frac{\mu_0 I \pi a^2}{4\pi r^3} \frac{2 \sin\theta \cos\theta}{\sin\theta} + \hat{\theta} \frac{\mu_0 I \pi a^2 \sin\theta}{4\pi} \frac{1}{r^3} \\ &= \frac{\mu_0 I a^2}{4r^3} \left[\hat{r} 2 \cos\theta + \hat{\theta} \sin\theta \right] \end{aligned}$$

$$\begin{aligned} b. \Psi_m &= \int \vec{B} \cdot d\vec{S} = B_r \int dS_r = B_r S_r \quad \text{upper loop} \\ &= \frac{\mu_0 I a^2}{4R_0^3} \underbrace{2 \cos\theta}_{=1} \pi a^2 \end{aligned}$$



$$= \frac{\mu_0 I a^4 \pi}{2R_0^3} = \frac{\mu_0 I a^2}{4R_0^3} \underbrace{2 \pi a^2}_{\text{area of loop}}$$

$$\begin{aligned} \text{Lower loop} \quad \Psi_m &= \frac{\mu_0 I a^2}{4R_0^3} \underbrace{\sin\theta}_{=1} \pi a^2 \\ &= \frac{\mu_0 I a^2}{4R_0^3} \underbrace{\pi a^2}_{\text{area of loop}} \end{aligned}$$



Using \vec{A} to find the flux

$$a. \quad \Psi_m = \oint \vec{A} \cdot d\vec{l} = A \phi \underbrace{2\pi a}_{\substack{\text{circumference} \\ \text{of loop}}} \quad \text{Upper Loop}$$

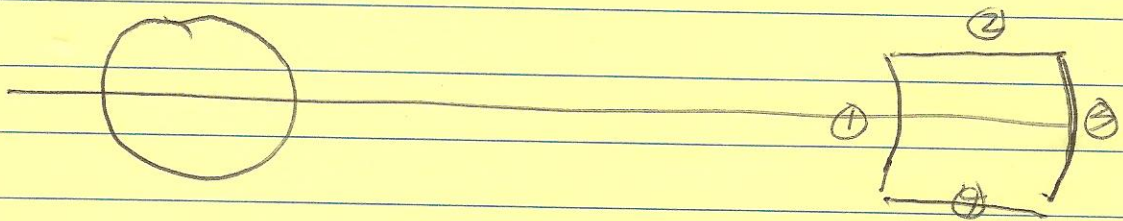
$$= \frac{\mu_0 a^2 I \sin\theta}{4r^2} 2\pi r$$

$$= \frac{\mu_0 a^2 I \pi}{2r^2} \sin\theta a = \frac{\mu_0 a^3 I \pi}{2R_0^2} \frac{a}{R_0}$$

$$= \frac{\mu_0 a^4 I \pi}{2R_0^3} = \frac{\mu_0 I a^2 \pi a^2}{4R_0^3} \underbrace{\sin\theta}_{\text{area of loop}}$$

which agrees with the previous result.

b. $\Psi_m = \oint \vec{A} \cdot d\vec{l}$ This one is more complex. However, since we can use the approx of $\Psi_m = \vec{B} \cdot \text{Area}$ regardless of the shape of the area, we can choose the loop shape to be something we can easily evaluate.



Note two sides in the ϕ direction & 2 in the r direction

$$\oint \vec{B} \cdot d\vec{l} = (1) + (2) + (3) + (4)$$

$$= \frac{\mu_0 I a^2}{4R_0^2} W - \frac{\mu_0 I a^2}{4R_0^2} W + \frac{\mu_0 I a^2}{4R_0^3} W^2$$

$$= \frac{\mu_0 I a^2}{4R_0^3} W^2 \quad \checkmark$$

Which is the same as for the approach using
 $\Psi_m = \int \vec{B} \cdot d\vec{S}$ (For More detail see last page)

4. a. $I = \int \vec{J} \cdot d\vec{S} = \int_0^a \int_0^{2\pi} J_0 \left(1 - \frac{r^4}{a^4}\right) r dr d\phi$

$$= 2\pi J_0 \int_0^a \left(r dr - \frac{r^5}{a^4} dr \right)$$

$$= 2\pi J_0 \left(\frac{a^2}{2} - \frac{a^6}{6a^4} \right)$$

$$= 2\pi J_0 \left(\frac{a^2}{2} - \frac{a^2}{6} \right)$$

$$= J_0 \pi a^2 \left(1 - \frac{1}{3} \right)$$

$$= \frac{2}{3} J_0 \pi a^2$$

note that if the current was uniform, the

the total current would be $J_0 \pi a^2$

so this makes sense.

To have the same current return at $r=b$

$$I = - J_s 2\pi b$$

$$\begin{aligned} J_s &= - \frac{2}{3} J_0 \pi a^2 / 2\pi b \\ &= - \frac{1}{3} \frac{J_0 a^2}{b} \end{aligned}$$

b. For the region $a \leq r \leq b$, the \vec{B} field is identical to that of a long straight wire carrying the same total current.

$$\begin{aligned} \vec{B} &= \hat{\phi} \frac{\mu_0 I}{2\pi r} = \hat{\phi} \frac{\mu_0 2}{2\pi r 3} J_0 \pi a^2 \\ &= \hat{\phi} \frac{\mu_0 J_0 a^2}{3 r} \end{aligned}$$

For the region $0 \leq r \leq a$, $\oint \vec{B} \cdot d\vec{l} = 2\pi r B_\phi(r)$

$$\begin{aligned} I_{\text{enc}} &= \int_0^r \int_0^{2\pi} J_0 \left(1 - \frac{r^4}{a^4}\right) r dr d\phi \\ &= 2\pi J_0 \left(\frac{r^2}{2} - \frac{r^6}{6a^4}\right) \\ &= \pi J_0 r^2 \left(1 - \frac{r^4}{3a^4}\right) \end{aligned}$$

$$\Rightarrow B_\phi(r) = \frac{\mu_0 \pi J_0 r^2}{2\pi r} \left(1 - \frac{r^4}{3a^4}\right) = \frac{\mu_0 J_0 r}{2} \left(1 - \frac{r^4}{3a^4}\right)$$

b. continued for $r > b$, $I_{enc} = 0$

$$\Rightarrow B_{\phi} = 0$$

c. $\nabla \cdot \vec{B} = 0$ \vec{B} is only in $\hat{\phi}$ direction
 $\hat{\phi}$ depends on $r \Rightarrow \nabla \cdot \vec{B} = 0$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\nabla \times \vec{B} = \hat{z} \frac{1}{r} \frac{d}{dr} r B_{\phi}$$

$$= \hat{z} \frac{1}{r} \frac{d}{dr} \left(\frac{\mu_0 J_0 r^2}{2} \left(1 - \frac{r^4}{3a^4} \right) \right)$$

$$= \hat{z} \frac{1}{r} \frac{d}{dr} \left(r^2 - \frac{r^6}{3a^4} \right) \left(\frac{\mu_0 J_0}{2} \right)$$

$$= \hat{z} \frac{1}{r} \left(2r - 2 \frac{r^5}{3a^4} \right) \left(\frac{\mu_0 J_0}{2} \right)$$

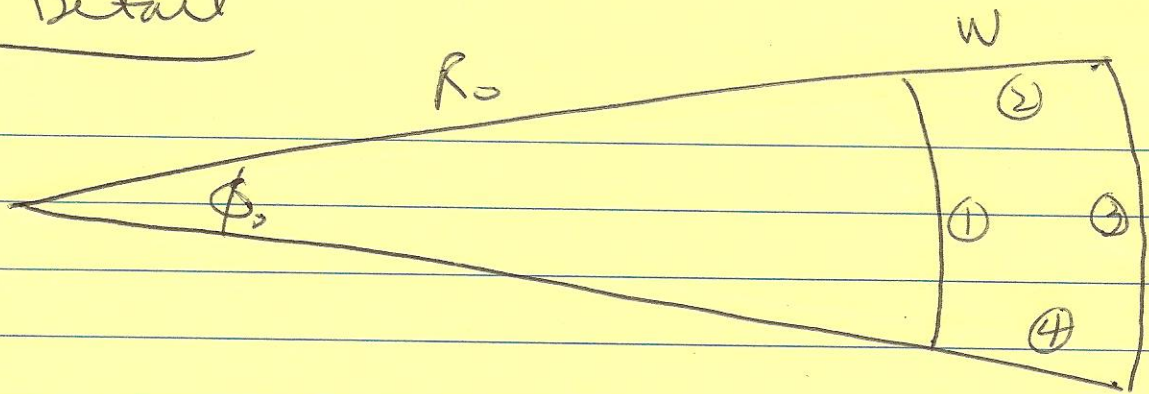
$$= \hat{z} \frac{1}{r} \left(\frac{\mu_0 J_0}{2} \right) \left(2r - 2 \frac{r^5}{3a^4} \right)$$

$$= \hat{z} \frac{\mu_0 J_0}{2} \left(2 - 2 \frac{r^4}{3a^4} \right)$$

$$= \hat{z} \mu_0 J_0 \left(1 - \frac{r^4}{3a^4} \right)$$

\vec{J} ✓

More Detail



$$\oint \vec{A} \cdot d\vec{l} = \frac{\mu_0 I a^2}{4 R_0^2} R_0 \phi_0 + 0 + 0$$

$$\underbrace{\hspace{10em}}_{A_\phi} \quad \underbrace{\hspace{10em}}_{d_l}$$

(1)

(2)

(4)

$$+ \frac{\mu_0 I a^2}{4 (R_0 + w)^2} (R_0 + w) \phi_0$$

(4)

$$= \frac{\mu_0 I a^2}{4 R_0} \phi_0 - \frac{\mu_0 I a^2}{4 (R_0 + w)} \phi_0$$

$$= \frac{\mu_0 I a^2}{4 R_0} \phi_0 - \frac{\mu_0 I a^2 \phi_0}{4 R_0 \left(1 + \frac{w}{R_0}\right) \left(1 - \frac{w}{R_0}\right)}$$

$$= \frac{\mu_0 I a^2}{4 R_0} \phi_0 - \frac{\mu_0 I a^2 \phi_0}{4 R_0} + \frac{\mu_0 I a^2 \phi_0 w}{4 R_0^2}$$

$$= \frac{\mu_0 I a^2 \phi_0 R_0 w}{4 R_0^3} = \frac{\mu_0 I a^2}{4 R_0^3} \underbrace{(w^2)}_{\text{area of } w \times w} \quad w = R_0 \phi_0$$

✓