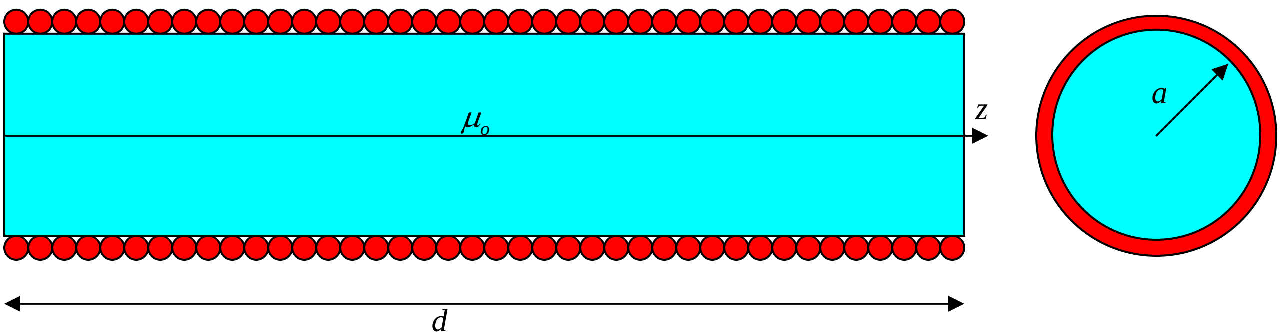
Homework 7

Due 18 November at 6:00 pm

1. **Maxwell's Equations – Quasi-statics**



An air core, N turn, cylindrical solenoid of length d and radius a, carries a current $I = I_o \cos\omega t$.

a. Using Ampere's Law, determine the magnetic flux density $\vec{B}$ and magnetic field intensity $\vec{H}$ for this solenoid, assuming that the solenoid is very long and there is no fringing. Note that, while you may know the answer to this question by inspection, show all steps. That is, set up and evaluate the line integral, find the current enclosed by the line, etc.

b. Evaluate the time derivative of the magnetic flux density $\frac{\partial \vec{B}}{\partial t}$.

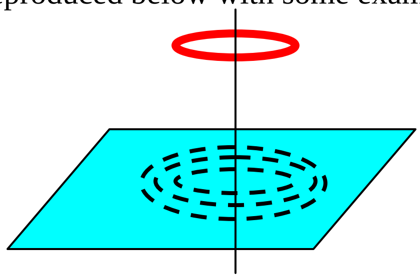
c. From the point form of Faraday's Law, we know that $\nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t}$ or

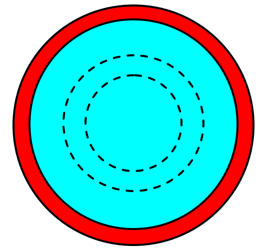
$$\hat{r}\left(\frac{1}{r}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right) + \hat{z}\frac{1}{r}\left(\frac{\partial (rE_\phi)}{\partial r} - \frac{\partial E_r}{\partial \phi}\right) = \hat{z}\frac{\partial B_z}{\partial t}$$ where we have

used the fact that the magnetic flux density $\vec{B} = \hat{z}B_z(r)$ to drop the x and y directed terms from the right hand side. We can use this information to also drop the same terms from the left hand side. $\hat{z}\frac{1}{r}\left(\frac{\partial (rE_\phi)}{\partial r} - \frac{\partial E_r}{\partial \phi}\right) = \hat{z}\frac{\partial B_z}{\partial t}$. So far, we have shown that the electric field $\vec{E}$ can have, at most a ϕ or r directed component. In fact, $\vec{E} = \hat{\phi}E_\phi(r)$ only because the electric field in this case has to be in the same direction as the currents that produce the solenoid $\vec{B}$ field. Why this must be is a bit subtle, but maybe best seen by looking at what would happen if the core material was iron instead of air. If that was the case, then we would observe eddy currents being created in the iron that mimic the current in the solenoid winding in an attempt to reduce the applied field as much as possible.

Recall that currents induced in the copper pipe or copper plate by the strong permanent magnet (demo done in class) we in a direction to produce a field in the opposite direction. To qualitative show this for the solenoid, the end view of the solenoid has been reproduced below with some example eddy currents shown as dashed lines.

Also shown is a generic circular current inducing eddies in a flat plate.





Now, since there is only one component for $\vec{E} = \hat{\phi} E_\phi(r)$, Faraday's Law simplifies to $\frac{1}{r}\left(\frac{\partial (rE_\phi)}{\partial r}\right) = \frac{\partial B_z}{\partial t}$. Solve this equation for $\vec{E} = \hat{\phi} E_\phi(r)$. Note that your solution must be a function of radius *r* and cannot be a constant like $\vec{B}$ since $\vec{E}$ must have a curl.

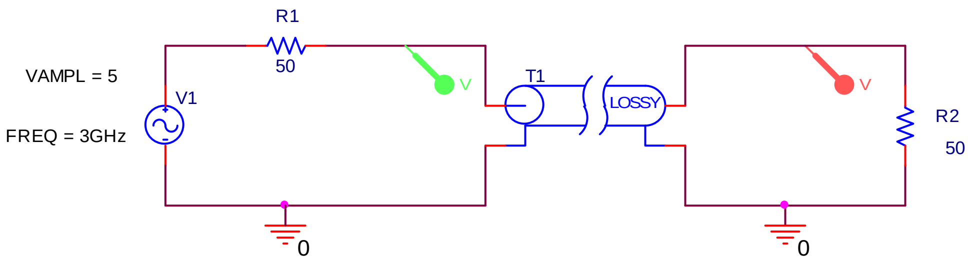
d. The solutions to parts a and c are the quasi-static field. However, to determine the accuracy of this solution, we need to use the $\vec{E}$ just found to find a correction to $\vec{B}$. This correction to the original field must also be a function of *r*. To start this process, determine the electric flux density $\vec{D}$ from $\vec{E}$ and then find its time derivative $\frac{\partial \vec{D}}{\partial t}$. This is the displacement current density.

e. Use Ampere's Law, following the same steps as in part a, to find the new contribution to the magnetic field intensity $\vec{H}$. Note that, since the displacement current density is distributed throughout the region $0 \le r \le a$, evaluating the right hand side of Ampere's Law is a bit more complicated than it was for part a. Once you have the new $\vec{H}$, also find the new $\vec{B}$.

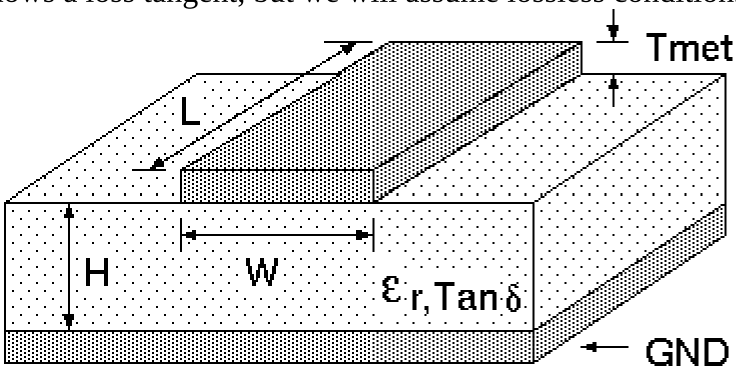
f. At this point, we can represent our field solutions as $\vec{B} = \vec{B}_0 + \vec{B}_2$ and $\vec{E} = \vec{E}_1$ where the subscripts indicate the order of the corrections. For quasi-statics, we should have that $\frac{B_2}{B_0} << 1$. Evaluate this ratio and determine the range of frequencies for which the quasi-static solution (i.e. $\vec{B} = \vec{B}_0$ and $\vec{E} = \vec{E}_1$) is reasonably accurate.

Hint: There is an extensive discussion of quasi-statics for two other geometries in the class notes, Unit 9, pages 17-24.

2. **Plane Waves and Transmission Lines**



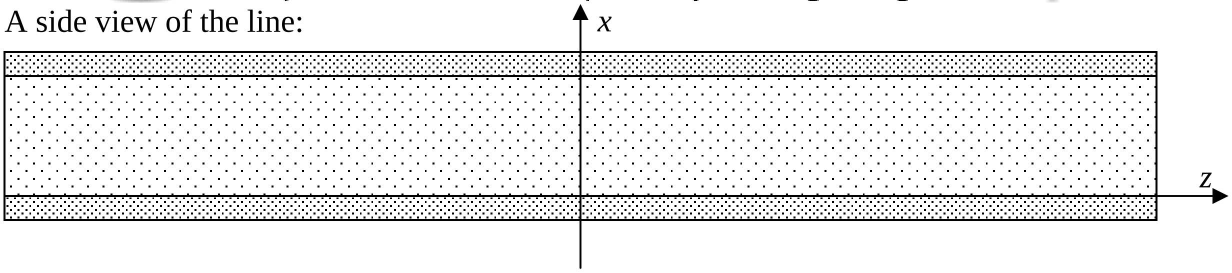
A transmission line is driven by a 5V, 3GHZ voltage source. Both the source and load resistances are 50Ohms. The length of the line is 10cm. Now that we have addressed finding capacitance and inductance, we should begin such a problem by first describing the configuration of the transmission line. Assume that we have a microstrip line for which there are many, many tools available to determine the line characteristics. The parameters of the line are as follows: $\varepsilon_r = 4.2$, dielectric thickness = *0.79375mm*, the trace width is *1.5mm*, the trace thickness is *0.035mm*. A view of the line is shown below. Note that this shows a loss tangent, but we will assume lossless conditions.



a. Using this information and one of the impedance calculators found on the Resources webpage, determine the characteristic impedance Z_o, the delay time t_d, the capacitance and inductance per unit length for this line. You should find that the line is matched to the source and load.
b. Use your answers to part a to set up the PSpice simulation shown above. Do time-domain analysis and show roughly 3-5 periods of the source and load voltages.
c. Write the phasor expressions for the voltage and current on the line and verify that your expression is consistent with your PSpice solution.

Now, will use the information have on voltage and current to find the electric and magnetic fields that exist in the region between the plates. For this analysis, we will assume that the plates are infinite so that there is no fringing and the field structure is that of an ideal parallel plate. This approach will give us a good answer for the middle of the field region, but probably not near the edges of the plates. We will first begin by finding the electric field. We will not find the magnetic field directly. Rather, we will find it from the electric field.

A side view of the line:

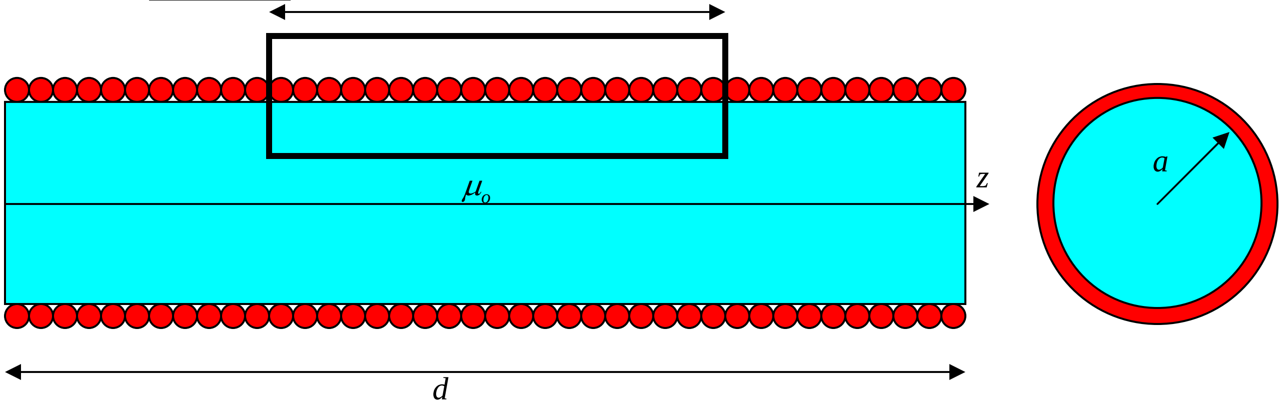


d. Assuming that the phasor voltage on the line creates an electric field in the insulator region between the plates, find the electric field $\vec{E}$. Note the coordinate system shown above and that this field is a uniform plane wave.
e. From $\vec{E}$ find $\vec{D}$ and then determine the surface charge density ρ_S on each conductor using the boundary condition on the normal component of $\vec{D}$.
f. Determine the intrinsic impedance of the dielectric medium η.
g. From your answers for $\vec{E}$ and η, determine the magnetic field intensity $\vec{H}$
h. Show that your answer for $\vec{H}$ is consistent with your expression for the current on the line (from part c) by evaluating the boundary condition for the tangential component of $\vec{H}$.
i. Using your expressions for $\vec{E}$ and $\vec{H}$, determine the average power density (Poynting vector) and then find the total power flowing in the line (see below).
j. Show that the total power flowing in the line is equal to the power delivered to the load resistor from your transmission line analysis.

Note: Because the configuration addressed here is not an ideal parallel plate structure, we would have to solve for the fields using a numerical method to answer this question directly. However, we can address an equivalent ideal structure to see that the voltages and fields are properly connected. To answer parts i and j, use the expressions in table 2.1 of Ulaby for inductance and capacitance per unit length for a parallel plate structure. Adjust the values for ε and *w* (keeping the value of *d* the same) to obtain the identical values for Z_o, *l*, and *c* for the transmission line. Then answer the questions.

Solution:

Problem 1



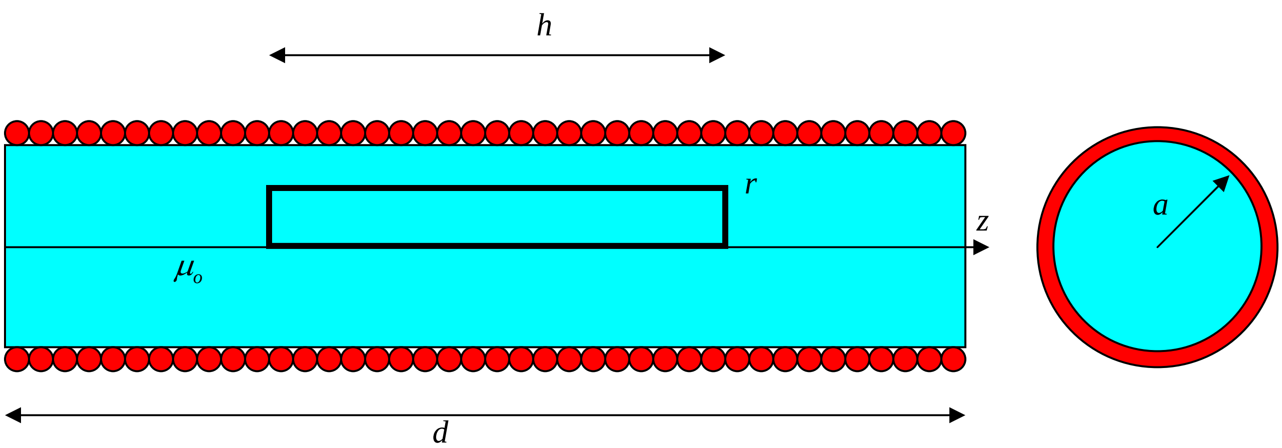
a. The integration path for Ampere's Law is shown above. The field outside the solenoid is assumed to be zero because the solenoid is infinitely long. The magnetic field is only in the *z-direction* so that: $\oint \vec{H} \cdot d\vec{l} = H_z h = I_{enclosed} = \frac{NI}{d} h$ or $H_z = \frac{NI}{d}$ and $B_z = \mu_o H_z = \frac{\mu_o NI}{d}$

b. The time derivative is $\frac{\partial B_z}{\partial t} = \frac{\mu_o N}{d}\frac{\partial I}{\partial t} = -\frac{\mu_o N}{d}\omega I_o \sin \omega t$

c. The electric field is found from $\frac{1}{r}\left(\frac{\partial\left(rE_\phi\right)}{\partial r}\right) = \frac{\partial B_z}{\partial t}$. Multiply through by *r* and integrate $\left(\frac{\partial\left(rE_\phi\right)}{\partial r}\right) = \frac{\partial B_z}{\partial t} r = -r\frac{\mu_o NI\omega}{d}\sin \omega t$ and $\left(rE_\phi\right) = -\frac{r^2}{2}\frac{\mu_o NI\omega}{d}\sin \omega t$ or finally $E_\phi = -\frac{r}{2}\frac{\mu_o NI\omega}{d}\sin \omega t$. Note that this field is not a constant, but increases linearly with radius. Look at the discussions mentioned from the class notes to see similar behavior.

d. The displacement current density is $\frac{\partial D_\phi}{\partial t} = \varepsilon_o \frac{\partial E_\phi}{\partial t} = -\frac{r}{2}\frac{\varepsilon_o \mu_o NI\omega^2}{d}\cos \omega t$

e. We now have the first order term $\vec{E} = \vec{E}_1$ and would like to find the second order correction to the original magnetic field $\vec{B} = \vec{B}_0 + \vec{B}_2$ due to the displacement current density. This current is different from the original current in the solenoid windings because it is distributed throughout the core region $0 \le r \le a$. Thus, the integral will be a little more involved, although it is fundamentally the same. First we draw a slightly different integration path.



The integration path now goes from the *z-axis* to the radius *r*.

$$\oint \vec{H}\cdot d\vec{l} = H_z(0)h - H_z(r)h = I_{enclosed} = -h\int \frac{r}{2}\frac{\varepsilon_o\mu_o NI\omega^2}{d}\cos\omega t\,dr \quad \text{or}$$

$$H_z(r) = \frac{r^2}{2}\frac{\varepsilon_o\mu_o NI\omega^2}{d}\cos\omega t$$ so we have that $$B_z(r) = \frac{r^2}{2}\frac{\varepsilon_o\mu_o{}^2 NI\omega^2}{d}\cos\omega t$$

f. The ratio $\frac{B_2}{B_0} = \frac{r^2}{2}\omega^2\mu_o\varepsilon_o << 1$ gives us $\omega^2 << \frac{2c^2}{a^2}$ where *c* is the speed of light. We also note that for the quasi-static solution to work, we see that the dimensions must be much smaller than a wavelength. $a^2 << \frac{2c^2}{\omega^2} = \frac{\lambda^2}{2\pi^2}$

Problem 2

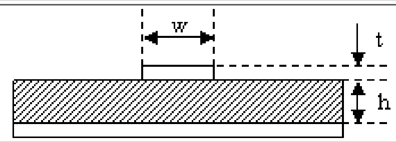
a. Microstripline Parameters from http://www.mantaro.com/resources/impedance_calculator.htm#microstrip_impedance

Microstrip Impedance Calculator:

$$Z_0 = \frac{87}{\sqrt{\varepsilon_r + 1.41}} \ln\left(\frac{5.98h}{(0.8w + t)}\right) \qquad T_{pd} = 3.333\sqrt{0.475 \cdot \varepsilon_r + 0.67}$$

Note: valid for (w/h) from 0.1 to 3.0

Dimensional units: (•) mm ( ) mils

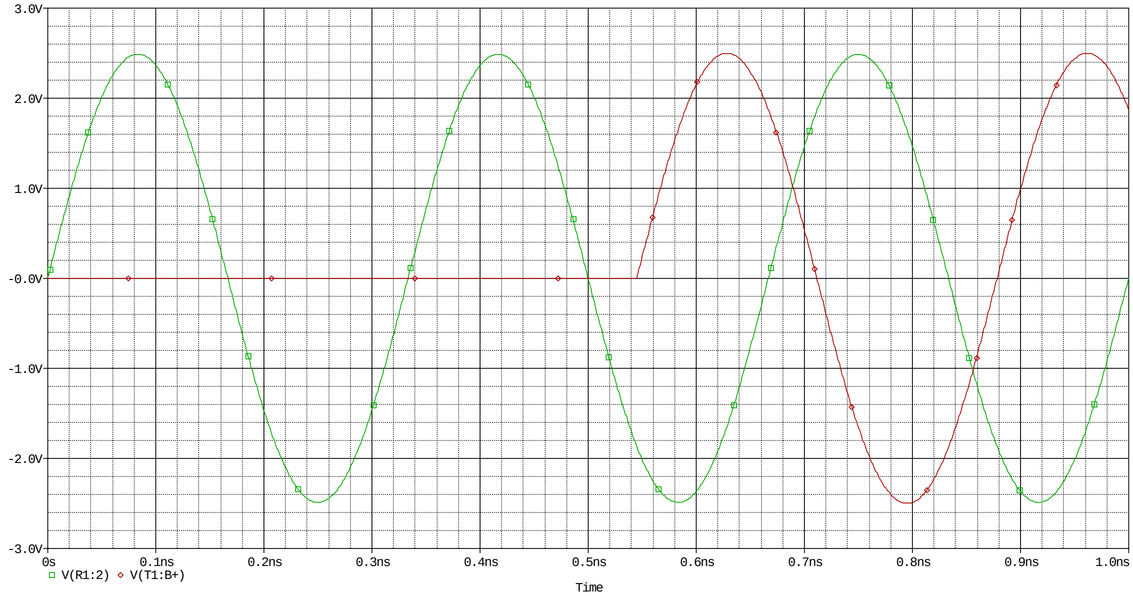
w (trace width) =	1.5
t (trace thickness) =	.035
h (dielectric thickness) =	.79375
er (relative dielectric constant) =	4.2
	Calculate
Zo (Single Ended Impedance, Ohms) =	49.454
Propagation Dealy, Tpd (ps/cm) =	54.411
Inductance, L (nH/cm) =	2.691
Capacitance, C (pF/cm) =	1.10023

Note: 1oz = 1.4mils = 0.03556mm

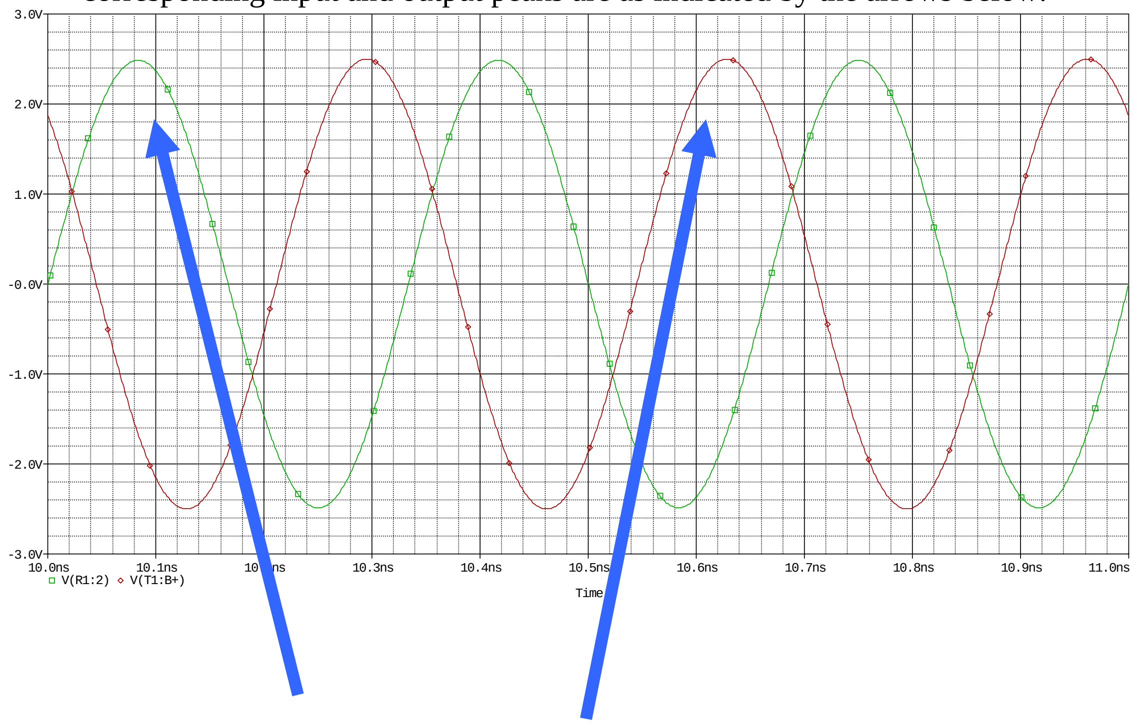
Note that the impedance of the line is roughly 50 Ohms, so everything is matched.

b. PSpice Simulation

The transmission line parameters can be found in the diagram above. The lossless assumption is used so that resistance and conductance are both zero. The line length is 0.1m.



The output (red) shows the delay from the line. This can be eliminated from the plot by showing the same amount of time but later. The first plot shows us that the corresponding input and output peaks are as indicated by the arrows below.



c. The phasor form of the voltage and current on the line. $V(z) = V^{+}e^{-j\beta z} = 2.5e^{-j\beta z}$

$$I(z) = \frac{V^{+}}{Z_o}e^{-j\beta z} = \frac{2.5}{50}e^{-j\beta z} = .05e^{-j\beta z}.\ Z_o = 50$$

$$\beta = \omega\sqrt{lc} = 2\pi(3x10^{9})\sqrt{(.27x10^{-6})(110x10^{-12})} = 102.7$$

Note that both V_{in} and V_{out} for the line in PSpice have amplitudes equal to *2.5V* which checks with this result.

d. Since the plate separation is *0.79375mm*, the electric field in the center of the structure is roughly equal to $E_x(z) = \frac{2.5}{d}e^{-j\beta z} = 3.15e^{-j\beta z}$

e. The electric flux density in the dielectric is $D_x(z) = \varepsilon 3.15e^{-j\beta z} = 13.2\varepsilon_o e^{-j\beta z}$ so that $\rho_S = D_x(z) = 13.2\varepsilon_o e^{-j\beta z}$ with the positive charge on the top plate and negative on the bottom.

f. $\eta = \sqrt{\frac{\mu_o}{\varepsilon}} = \frac{377}{\sqrt{4.2}} = 184$

g. $H_y(z) = \frac{3.15}{\eta}e^{-j\beta z} = 0.0135e^{-j\beta z}$

h. The boundary condition is for the tangential component of the magnetic field $J_S(z) = 0.0135e^{-j\beta z}$ To check against the total current, we need the equivalent parallel plate structure, developed in the following.

i. For a parallel plate transmission line, the parameters are given in Table 2-1 of Ulaby, where $c = \frac{\varepsilon w}{d}$, $l = \frac{\mu_o d}{w}$ and $Z_o = \sqrt{\frac{l}{c}} = \sqrt{\frac{\mu_o}{\varepsilon}}\frac{d}{w}$. We can find the equivalent parallel plate structure to the microstripline we have been analyzing, by fixing *d* at its present value and then solving for *w* and ε. The results are *w=3.7mm* and $\varepsilon = 2.6\varepsilon_o$. Both parameters changed in reasonable directions from the values we used above. Charges cover more area than just the original width of *1.5mm* and there is field in regions with no dielectric so that the average value of ε should be smaller. These changes also produce a new value for $\eta = \sqrt{\frac{\mu_o}{\varepsilon}} = \frac{377}{\sqrt{2.6}} = 233$. Using these numbers, we can multiply the current density in part h by the new value of *w* to obtain $I(z) = wJ_S(z) = (3.7)0.0135e^{-j\beta z} = 0.05e^{-j\beta z}$ as above. For the average power density $P_{ave} = \frac{1}{2}\frac{E^2}{\eta} = \frac{3.15^2}{2(233)} = .02$. The area between the plates is *(3.7)(.8)=2.96* so that the power is *P = 0.063 Watts.*

j. From the original transmission line, the power is $P = \frac{1}{2}\frac{(2.5)^2}{50} = 0.063$ which agrees with the previous result.