Fields and Waves I

Lecture 3
Input Impedance on Transmission Lines

K. A. Connor
Electrical, Computer, and Systems Engineering Department
Rensselaer Polytechnic Institute, Troy, NY
These Slides Were Prepared by Prof. Kenneth A. Connor Using Original Materials Written Mostly by the Following:

- Kenneth A. Connor – ECSE Department, Rensselaer Polytechnic Institute, Troy, NY
- J. Darryl Michael – GE Global Research Center, Niskayuna, NY
- Thomas P. Crowley – National Institute of Standards and Technology, Boulder, CO
- Sheppard J. Salon – ECSE Department, Rensselaer Polytechnic Institute, Troy, NY
- Lale Ergene – ITU Informatics Institute, Istanbul, Turkey
- Jeffrey Braunstein – Chung-Ang University, Seoul, Korea

Materials from other sources are referenced where they are used. Those listed as Ulaby are figures from Ulaby’s textbook.
Overview

- Review
- Voltages and Currents on Transmission Lines
- Standing Waves
- Input Impedance
- Lossy Transmission Lines
- Low Loss Transmission Lines

Henry Farny *Song of the Talking Wire*
What do we know so far?

- Solutions look like \( A \cos(\omega t + \beta z) \)

\[
\begin{align*}
\beta &= \frac{\omega}{u} = \omega \sqrt{lc} = \omega \sqrt{\mu \varepsilon} = \frac{2\pi}{\lambda} \\
\omega &= 2\pi f = \frac{2\pi}{T} \\
\lambda &= \frac{2\pi}{\beta} = \frac{u}{f} \\
u &= \frac{1}{\sqrt{lc}} = \frac{1}{\sqrt{\mu \varepsilon}} \\
\varepsilon &= \varepsilon_r \varepsilon_o \\
\mu &= \mu_r \mu_o
\end{align*}
\]

Figure from http://www.emc.maricopa.edu/
Phase Velocity

- A simple way to find the phase velocity
  - Identify some feature of the sine wave. For this we choose constant phase
    \[ \omega t + \beta z = \text{const} \]
  - Determine its velocity. Since the phase is a constant, we know that
    \[ \frac{\partial}{\partial t} (\omega t + \beta z) = 0 \]
    \[ \omega + \beta \frac{\partial z}{\partial t} = 0 \]
    \[ \frac{\partial z}{\partial t} = u = \frac{\omega}{\beta} \]
Phasor Notation

- For ease of analysis (changes second order partial differential equation into a second order ordinary differential equation), we use phasor notation.

\[ f(z, t) = A \cos(\omega t \mp \beta z) = \text{Re}\left(\left\{ Ae^{\mp j\beta z}\right\} e^{j\omega t}\right) \]

- The term in the brackets is the phasor.

\[ f(z) = Ae^{\mp j\beta z} \]
Phasor Notation

- To convert to space-time form from the phasor form, multiply by $e^{j\omega t}$ and take the real part.

$$f(z, t) = \text{Re}(Ae^{\mp j\beta z}e^{j\omega t}) = A\cos(\omega t \mp \beta z)$$

- If $A$ is complex

$$A = |A|e^{j\theta_A}$$

$$f(z, t) = \text{Re}(|A|e^{j\theta_A}e^{\mp j\beta z}e^{j\omega t}) = |A|\cos(\omega t \mp \beta z + \theta_A)$$
What is the phasor of the time derivative?

\[ v(z, t) = \text{Re} \left( V_o e^{-j\beta z} e^{j\omega t} \right) \]

\[ \frac{\partial}{\partial t} v(z, t) = \frac{\partial}{\partial t} \text{Re} \left( V_o e^{-j\beta z} e^{j\omega t} \right) \]

\[ = \text{Re} \left( V_o e^{-j\beta z} \frac{d}{dt} e^{j\omega t} \right) \]

\[ = \text{Re} \left( V_o e^{-j\beta z} (j\omega) e^{j\omega t} \right) \]

\[ \text{Phasors:} \ \beta \rightarrow j\omega \]
Transmission Lines

Incident Wave

Mismatched load

Reflected Wave

Standing wave due to interference
Transmission Lines – Phasor Voltage Solution

Phasor Form of the Wave Equation:

\[ \frac{\partial^2 V}{\partial z^2} = l \cdot c \cdot \frac{\partial^2 V}{\partial t^2} \]

where:

\[ V = V^\mp \cdot e^{\pm j \cdot \beta \cdot z} \]

General Solution:

\[ V = V^+ e^{-j \cdot \beta \cdot z} + V^- e^{+j \cdot \beta \cdot z} \]
Transmission Lines - Standing Wave Derivation

\[ V = V^+ e^{-j \beta z} + V^- e^{+j \beta z} \]

Forward Wave: \[ \cos(\omega t - \beta z) \]
Backward Wave: \[ \cos(\omega t + \beta z) \]

\( V_{\text{max}} \) occurs when Forward and Backward Waves are in Phase
\( V_{\text{min}} \) occurs when Forward and Backward Waves are out of Phase

CONSTRUCTIVE INTERFERENCE
DESTRUCTIVE INTERFERENCE
Transmission Line Voltages and Currents

- General Solution
  \[ v(z) = V^+ e^{-j\beta z} + V^- e^{+j\beta z} \]

- The latter expression is derived from
  \[ i(z) = \frac{V^+ e^{-j\beta z} - V^- e^{+j\beta z}}{Z_o} = \frac{V^+}{Z_o} e^{-j\beta z} - \frac{V^-}{Z_o} e^{+j\beta z} \]

Can we easily explain the minus sign?
Workspace

\[ v(t) = V^+ e^{-j\beta z} + V^- e^{+j\beta z} \]

\[ i(t) = \frac{d}{dz} \frac{d}{dt} (\text{Phasors}) \]

\[ i(t) = V^+ \beta e^{-j\beta z} - \frac{V^+}{\omega L} e^{-j\beta z} + \frac{V^-}{\omega L} e^{+j\beta z} \]

\[ Z_0 = \frac{\omega L}{\beta} = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{jwL}{jwC}} \]
Define the Reflection Coefficient:

\[ |V_m^-| = |\Gamma_L| \cdot |V_m^+| \]

Maximum Amplitude when in Phase:

\[ V_{\text{max}} = |V_m^+| + |V_m^-| \]
\[ \therefore V_{\text{max}} = |V_m^+| \cdot (1 + |\Gamma_L|) \]

Similarly:

\[ V_{\text{min}} = |V_m^+| \cdot (1 - |\Gamma_L|) \]

Standing Wave Ratio (SWR) = \[ \frac{V_{\text{max}}}{V_{\text{min}}} = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} \]
Another General Form for the Solution

- Using the reflection coefficient

\[ v(z) = V^+ e^{-j\beta z} + \Gamma_L V^+ e^{+j\beta z} \]

\[ i(z) = \frac{V^+}{Z_o} e^{-j\beta z} - \frac{\Gamma_L V^+}{Z_o} e^{+j\beta z} \]

Note that we will be rewriting the solution in different forms.
Transmission Lines

Both from Ulaby
Reflection Coefficient

Let $z=0$ at the LOAD

\[ \Rightarrow V_{load} = V^+ \cdot e^{-j\beta L} + V^- \cdot e^{+j\beta L} \]

\[ = V^+ + V^- \quad \text{Voltage at load} \]

\[ = V^+ \cdot (1 + \Gamma_L) \]

\[ I_{load} = \frac{V^+}{Z_0} - \frac{\Gamma_L V^+}{Z_0} = \frac{V^+}{Z_0} (1 - \Gamma_L) \]
Reflection Coefficient

- At the load

\[
\frac{V_{\text{load}}}{I_{\text{load}}} = Z_L
\]

\[
\frac{V^+ (1 + \Gamma_L)}{Z_L (1 - \Gamma_L)} = Z_L = \varepsilon_0 \text{ at match}
\]

\[
\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = 0 \text{ matched}
\]
$Z_{in} = \frac{V_1}{I_1}$

$Z_{out} = \frac{V_2}{I_2}$
Workspace – Short Circuit Load

\[ P_L = \frac{Z_L - Z_0}{Z_L + Z_0} \]

\[ Z_L = 0 \]

\[ \frac{0 - Z_0}{0 + Z_0} = -1 \]
Workspace - Open Circuit Load

\[ V = \frac{t_0 - t_c}{t_c + t_{\infty}} \]

\[ t_{\infty} \rightarrow \infty \]

\[ 2t_{\infty} = 1 \]
Short Circuit Load

For $Z_L = 0$, we have

\[ \Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{0 - Z_o}{0 + Z_o} = -1 \]

\[ v(z) = V^+ e^{-j\beta z} + \Gamma_L V^+ e^{+j\beta z} = V^+ \left( e^{-j\beta z} - e^{+j\beta z} \right) \]

\[ \begin{cases} 
  e^{+j\beta z} = \cos \beta z + j \sin \beta z \\
  e^{-j\beta z} = \cos \beta z - j \sin \beta z 
\end{cases} \]

\[ v(z) = -V^+ (j2 \sin \beta z) \]
Short Circuit Load

- Convert to space-time form

\[ v(z, t) = \text{Re}\left(v(z)e^{j\omega t}\right) = \text{Re}\left(V^+ (-j2 \sin \beta z)e^{j\omega t}\right) \]

\[ \text{Re}\left((-j2 \sin \beta z)e^{j\omega t}\right) = \text{Re}\left(-2 \sin \beta z (j \cos \beta z - \sin \beta z)\right) \]

\[ v(z, t) = 2V^+ \sin \beta z \sin \omega t \]

- This is a standing wave
Short Circuit Load

- What are the voltage maxima and minima?

\[ v(z, t) = 2V^+ \sin \beta z \sin \omega t \]

- Where are they?

\[ \beta z = 0, \pi, 2\pi \]

\[ \beta = \frac{2\pi}{\lambda}, \quad \frac{2\pi z}{\lambda} = \pi \]

\[ \lambda = \frac{\lambda}{2} \]

- The standing wave pattern is the envelope of this function.
Reflection

Note that we are free to choose either the load end or the source end as $z=0$. 

2 Vpp
1.5 MHz

$50 \, \Omega$

80 m

RG-58

$93 \, \Omega$

$z=0$

$z'=80$

$z=80$

$z'=0$
Reflection Coefficient

- Determine the reflection coefficient at the load, $\Gamma_L$, and the standing wave ratio, VSWR. Start with a short circuit load and then consider a 25 Ohm load. Then do an open circuit and 93 Ohm load.
- Assume that the forward traveling wave has an amplitude of 10 Volts. Sketch the standing wave pattern for voltage and current for the short circuit load. Include numbers for amplitudes and distances.
- Under what conditions do you get a voltage maximum at the load? a minimum? Can you answer this in general?
- If the load is a 3.3 nF capacitor, what is the reflection coefficient at the load? Where is the location of the first minimum? To answer this, we need a bit more development.
First Sketch the Standing Wave Patterns by Hand

- The reflection coefficient

\[
\Gamma_L = \frac{0 - 50}{0 + 50} = -1 \\
\Gamma_L = \frac{25 - 50}{25 + 50} = -\frac{1}{3} = -0.333 \\
\Gamma_L = \frac{\infty - 50}{\infty + 50} = 1 \\
\Gamma_L = \frac{93 - 50}{93 + 50} = \frac{43}{143} = +0.3
\]
Using Matlab for the Voltage Standing Wave Patterns
Current Standing Wave Patterns

- Can we use what we just displayed to find the current standing wave patterns?

- Yes, because the reflection coefficient for current is always just the negative of the voltage reflection coefficient.
Standing Wave Pattern

We have just seen that:

Minimum occurs at LOAD for \( Z_L \rightarrow 0 \)

Is it also true that:

Maximum occurs at LOAD for \( Z_L \rightarrow \infty \)

Or, in general, that:

\[
\Gamma_L > 0 \quad \Rightarrow \quad Z_L > Z_0 \quad \text{Max at LOAD}
\]

\[
\Gamma_L < 0 \quad \Rightarrow \quad Z_L < Z_0 \quad \text{Min at LOAD}
\]

IF \( Z_L \) is REAL
Standing Wave Pattern

If \( z=L \) at LOAD and \( z=0 \) at SOURCE,

\[
\Gamma(z) = \Gamma_L \cdot e^{-j \cdot 2 \cdot \beta \cdot (L-z)}
\]

\[
= |\Gamma_L| \cdot e^{j \cdot \theta_\Gamma} \cdot e^{-j \cdot 2 \cdot \beta \cdot (L-z)}
\]

When Phase = \( \pi \), the FIRST MINIMUM occurs

\[
\theta_\Gamma - 2 \cdot \beta \cdot (L-z) = \pi
\]

\[
\Rightarrow (L-z) = \frac{\lambda}{4} + \frac{\theta_\Gamma}{4 \cdot \pi} \cdot \lambda
\]

Other MINs are displaced by \( \lambda/2 \)
A Repeat of HW2 Experiment (600kHz)

- What did you see at the 20 nodes?
  - Time Delay
  - Amplitude
- Did any of you try an open or short circuit?
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Change the Frequency to 1.5MHz

\[ \lambda = \frac{2\pi}{B} = 132 \text{ m} \]

Distance between max and min = \( \frac{32}{13} \lambda = 0.24 \lambda \)

Distance between 2 maxima = \( \frac{\lambda}{2} \)

\[ \frac{\lambda}{2} = 66 \text{ m} \]

\( \beta = \frac{2\pi}{\lambda} \)

\[ \beta = \frac{2\pi \times 1.5 \times 10^6}{132} = 0.0477 \text{ m}^{-1} \]
Capacitive Load?

![Graph showing 3.3 nF load with Volts P-P on the y-axis and node # on the x-axis. The graph peaks at node 10 and shows a symmetric waveform.](image)
Capacitive Load

Now we can answer the question about where the first minimum is located for a capacitive load.

\[ \Gamma_L = \frac{1}{j\omega C} - \frac{Z_0}{1} + Z_0 \]

\[ = \frac{1 - j\omega CZ_0}{1 + j\omega CZ_0} = 1 \exp \left( \tan^{-1} \left( \frac{-\omega CZ_0}{1 - \left(\omega CZ_0\right)^2} \right) \right) \]

\[ \frac{a - jb}{a + jb} = \frac{a - jb}{a + jb} = \left( \frac{a^2 + b^2}{a^2 + b^2} \right) \exp \left( \tan^{-1} \left( \frac{-2ab}{a^2 - b^2} \right) \right) = \exp \left( \tan^{-1} \left( \frac{-2ab}{a^2 - b^2} \right) \right) \]
Capacitive Load

- For the specific case here

\[
\Gamma_L = \frac{1}{j\omega C} - \frac{Z_o}{1 + \frac{1}{j\omega C} + Z_o} = \frac{-j32.2 - 50}{-j32.2 + 50} = -0.4 - j0.91 = 1e^{-j0.6362\pi}
\]

\[
L - z = \frac{\lambda}{4} + \frac{\theta_\Gamma}{2\pi} \lambda = \frac{\lambda}{4} + \frac{-0.6362\pi}{4\pi} \lambda = \frac{\lambda}{4} - 0.1591\lambda = 0.09\lambda
\]
Standing Wave Pattern for Capacitive Load

Reflection Coefficient

$$\Gamma_L = 1e^{-j0.6362\pi}$$
PSpice – Input Impedance

For the same source and line, but different load:

- **T1**
  - VOFF = 0
  - VAMPL = 10
  - FREQ = 1meg

- **T2**
  - VOFF = 0
  - VAMPL = 10
  - FREQ = 1meg

**RL1**
- Value: 300

**RL2**
- Value: 50
Changing the Load

- The voltages and currents at the input change → the input impedance changes.
Changing the Length and Line Properties

- From the standing wave patterns or the expressions for the voltages and the currents on the line, we can see that the ratio of the voltage to the current will depend on the length of the line and the line properties.
\[ V(t) = V^+ \left( e^{-i\beta t} + \Gamma e^{i\beta t} \right) \]
\[ = V^+ e^{-i\beta t} \left( 1 + \Gamma e^{i2\beta t} \right) \]
Input Impedance

What does $Z_{in} = \frac{V_{in}}{I_{in}}$, look like?

When $Z_L$ is complex, so is $\Gamma_L$. To address the input impedance, we need to generalize the reflection coefficient.

Define:

$$\Gamma(z) = \frac{V^- \cdot e^{+j \beta z}}{V^+ \cdot e^{-j \beta z}} = \frac{V^-}{V^+} \cdot e^{2j \beta z}$$

$$= \Gamma_L \cdot e^{j \cdot 2 \beta z} \quad \text{if } z = 0 \text{ at LOAD}$$
Another Form for the General Solution

- Using the Generalized Reflection Coefficient

\[
v(z) = V^+ e^{-j\beta z} (1 + \Gamma(z))
\]

\[
i(z) = \frac{V^+}{Z_o} e^{-j\beta z} (1 - \Gamma(z))
\]
Previously, we have seen:

$$\hat{V}(z) = V^+(z) + V^-(z) = V^+ \cdot e^{-j\beta z} \cdot (1 + \Gamma(z))$$

What about I?

$$\hat{I}(z) = \frac{V^+(z)}{Z_o} - \frac{V^-(z)}{Z_o} = \frac{V^+}{Z_o} \cdot e^{-j\beta z} \cdot (1 - \Gamma(z))$$

Also,

$$\Gamma(z) = \Gamma_L \cdot e^{-2j\beta(L-z)}$$
Input Impedance

Form the Ratio (the generalized impedance):

\[
\frac{\hat{V}(z)}{\hat{I}(z)} = Z_o \cdot \frac{1 + \Gamma(z)}{1 - \Gamma(z)} = Z(z)
\]

We are primarily interested in \( z = 0 \) value

- treat connection to rest of circuit as 2 port with,

\[
Z_{in}(z = 0) = Z_o \cdot \frac{1 + \Gamma(z = 0)}{1 - \Gamma(z = 0)}
\]
Input Impedance

After lots of algebra, one can show:

\[
Z_{in}(z = 0) = Z_o \cdot \frac{Z_L + j \cdot Z_o \cdot \tan(\beta \cdot L)}{Z_o + j \cdot Z_L \cdot \tan(\beta \cdot L)}
\]

Special Case example: \(Z_L=0\) (short circuit)

\[
Z_{in}(z = 0) = Z_o \cdot \frac{0 + j \cdot Z_o \cdot \tan(\beta \cdot L)}{Z_o + j \cdot 0 \cdot \tan(\beta \cdot L)} = j \cdot Z_o \cdot \tan(\beta \cdot L)
\]
Input Impedance - SHORT CIRCUIT

\[ Z_{in}(z = 0) = j \cdot Z_o \cdot \tan(\beta \cdot L) \]

\[ \beta = \omega \sqrt{\varepsilon \mu} \]

Can change \( Z_{in} \) by changing these two parameters

- Fix \( \beta \), vary \( L \) - different effects
- Vary \( \beta \), fix \( L \) - get same effects

Note that \( L \) is the length of the Transmission Line
For varying frequency, the input impedance is imaginary and can achieve any value.
Consider some other cases
Open Circuit Case

\[ Z_L = \infty \]

\[ Z_L = Z_o \frac{Z_L + j \cdot Z_o \cdot \tan(\beta \cdot L)}{Z_o + j \cdot Z_L \cdot \tan(\beta \cdot L)} \]

\[ Z_L = 93 \Omega - \text{lots of complex algebra, but straightforward} \]
Using the Input Impedance

We know $Z_{in} (z=0)$ - treat as 2-PORT

\[
\begin{align*}
\text{Power} &= \frac{1}{2} \Re \{ V_{in} \times I_{in}^* \} \\
&= \frac{1}{2} \Re \left\{ \frac{V_{in} \times V_{in}^*}{Z_{in}^*} \right\} \\
&= \frac{1}{2} \Re \left\{ \frac{|V_{in}|^2}{Z_{in}^*} \right\}
\end{align*}
\]
Using the Input Impedance

In a Lossless Transmission Line, $P_{\text{in}}$ flows into the Transmission Line and it is dissipated at the LOAD

\[ P_{\text{in}} = \frac{1}{2} \cdot \frac{|V_L|^2}{Z_L} \]

**What is the voltage at the load?**

\[ \hat{V}(z) = V^+ \cdot e^{-j \cdot \beta \cdot z} \cdot (1 + \Gamma(z)) \]

\[ \hat{V}(z = 0) = V_{\text{in}} = V^+ \cdot e^{-j \cdot \beta \cdot z} \cdot (1 + \Gamma(z = 0)) \]

\[ \Rightarrow V^+ = \frac{V_{\text{in}}}{1 + \Gamma(0)} \]

Can then plug back and get the full phasor expression
Using the Input Impedance

- The full form of the voltage

\[ V(z) = V^+ e^{-j\beta z} + V^- e^{+j\beta z} \]

- All information is now available to determine the voltage and current everywhere on the line. You will be doing this on the project.
Special Cases

- Recall that the standing wave pattern repeated every half wavelength. Thus, we expect that this will also happen for $Z_{in}$. First, consider the trivial case of $L=0$.

$$Z_{in} = Z_o \frac{Z_L + jZ_o \tan \beta L}{Z_o + jZ_L \tan \beta L} = Z_L$$

- Now let the line be a half wavelength long

$$\tan \beta L = \tan \left( \frac{2\pi \frac{\lambda}{2}}{\frac{\lambda}{2}} \right) = \tan(\pi) = 0 \quad Z_{in} = Z_o \frac{Z_L + 0}{Z_o + 0} = Z_L$$
Special Cases

- Thus, for a line that is exactly an integer number of half wavelengths long

\[ Z_{in} = Z_L \]

- Thus, if you have a transmission line with the wrong characteristic impedance, you can match the load to the source by selecting a length equal to a half wavelength.
Special Cases

- If the line is an odd multiple of a quarter wavelength, we also get an interesting result.

\[ Z_{in} = Z_o \frac{Z_L + jZ_o \tan \beta L}{Z_o + jZ_L \tan \beta L} = Z_o \frac{jZ_o \tan \beta L}{jZ_L \tan \beta L} = \frac{Z_o^2}{Z_L} \]

\[ \tan \beta L = \tan \frac{2\pi \lambda}{\lambda 4} = \tan \frac{\pi}{2} \rightarrow \infty \]

- Thus, such a transmission line works like an impedance transformer and has a real input impedance.
Today’s Major Result

- **Input Impedance**

\[
Z_{in}(z = 0) = Z_o \cdot \frac{Z_L + j \cdot Z_o \cdot \tan(\beta \cdot L)}{Z_o + j \cdot Z_L \cdot \tan(\beta \cdot L)}
\]