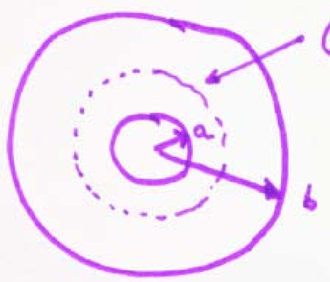
Homework 4

Due 8 March 2005

1. **Increasing the Breakdown Voltage:** This first question is a mini design project. Your first step is to find a commercial cable (coaxial or two wire line) for which you have the following information: the capacitance per meter, the type of material used for the insulator (to make it simpler to grade this problem, choose a cable that uses solid polyethylene (PE) for the insulator), the dimensions of the inner and outer conductors, and the maximum electric field or voltage that the cable can sustain. The latter parameter may not be given, but since you will know the insulating material, you can look up the value of its breakdown field.

   a. Using the parameters you have for your selected cable, calculate the capacitance from first principles. That is, find $\vec{D}$, $\vec{E}$, and V as functions of radius in the region between the two conductors and then use this information to determine the capacitance, either using the charge or the energy method. If you use the energy method it is not necessary to find the voltage.

This calculation was done in the lectures. Thus, it is only necessary to reproduce it here and put in the correct variables.



Gaussian Surface

$$\vec{D} = D_r(r)\,\hat{a}_r$$

$$\oint \vec{D}\cdot d\vec{S} = Q_{encl}$$

$$D_r(r)\,2\pi r\ell \qquad \rho_{sa}\,2\pi a\ell$$

$$\Rightarrow \hat{a}_r D_r(r) = \rho_{sa}\frac{a}{r}\,\hat{a}_r = \vec{D}$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{\rho_{sa}}{\varepsilon}\frac{a}{r}\,\hat{a}_r \qquad \varepsilon = 2.6\,\varepsilon_0$$

$$V(a) - \underbrace{V(b)}_{0} = -\int_b^a E_r\,dr = -\frac{\rho_{sa}}{\varepsilon}\,a\int_b^a \frac{dr}{r}$$

$$= \frac{\rho_{sa}\,a}{\varepsilon}\ln\frac{b}{a} = V_{ab}$$

$$C = \frac{Q}{V} = \frac{2\pi a \ell \rho_{sa}}{V} = \frac{2\pi a \ell}{V_0}\,\frac{\varepsilon V_0}{a \ln\frac{b}{a}} = \boxed{\frac{2\pi\varepsilon\ell}{\ln b/a}}$$

or

$$W_e = \int \frac{1}{2}\varepsilon E^2 dv = \int_0^{\ell}\int_0^{2\pi}\int_a^b \frac{1}{2}\varepsilon \frac{V_0^2}{r^2(\ln\frac{b}{a})^2}\, r\,dr\,d\phi\,dz$$

$$= \frac{1}{2}\varepsilon\frac{V_0^2}{(\ln b/a)^2}\int_0^{\ell}\int_0^{2\pi}\int_a^b \frac{dr}{r}\,d\phi\,dz = \frac{1}{2}\varepsilon\frac{V_0^2}{(\ln b/a)^2}\cdot \ln\frac{b}{a}\,2\pi\ell$$

$$= \boxed{\frac{\pi\varepsilon\ell V_0^2}{\ln b/a}}$$

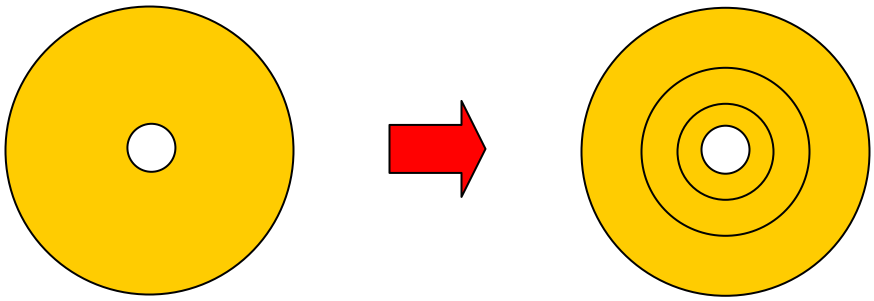
$$\frac{1}{2}CV^2 = W_e \Rightarrow \quad C = \frac{2W_e}{V^2} = \frac{2}{V_0^2}\,\frac{\pi\varepsilon\ell V_0^2}{\ln b/a} = \boxed{\frac{2\pi\varepsilon\ell}{\ln b/a}}$$

Fictitious Belden 75 Ohm Cable RG59

Inner wire diameter 0.032 inches
Outer wire diameter 0.144 inches
Insulator is Foam Polyethylene with an ε_r that produces a velocity of propagation of 83% the speed of light.
Breakdown field up to 150kV/mm

For this cable with PE insulation. $a = 0.81mm$, $b = 3.66mm$, $\varepsilon_r = \frac{1}{(.64)^2} = 2.44$ *and the capacitance is* $c = 90\,{pF}/{m} = 27.5\,{pF}/{ft}$

We wish to improve on this cable in some way. Our specific goal will be to change the insulator so that the breakdown voltage increases while maintaining the same capacitance per unit length. We will assume that the inner and outer conductors remain the same. We will only change the insulator. The approach we will take will be to replace the PE with two or three other insulators as shown.



The center insulator will be chosen to have the largest breakdown voltage, the other one or two insulators will be chosen to obtain the same capacitance. Recall that each will have a different dielectric constant. To find material properties, you can go to MatWeb http://matweb.com/ which has almost any property one can think of for commercially available materials.

b. Begin by solving for the capacitance of a coaxial cable with 2 insulating regions. Assume that the first insulator is in the region $a \le r \le b$ and the second insulator is in the region $b \le r \le c$ where a is the radius of the inner conductor and c is the radius of the outer conductor. The dielectric constant of region *1* is ε_1 and the dielectric constant of region *2* is ε_2. Again, do this from first principles and find $\vec{D}, \vec{E}$, and V as functions of radius in the region between the two conductors and then use this information to determine the capacitance, either using the charge or the energy method. If you use the energy method, it is not necessary to find the voltage.

The good news is that $\vec{D}$ *is the same because the dielectric properties do not show up in Gauss' Law. Thus* $\vec{D} = \rho_s \frac{a}{r}\hat{a}_r$ *so that* $\vec{E} = \frac{\rho_s}{\varepsilon}\frac{a}{r}\hat{a}_r$ *with a different epsilon in each reation. To find the voltage we need to integrate the electric field. For the outer insulator*

$$V(r) = -\int_c^r \vec{E}\cdot d\vec{l} = -\int_c^r \frac{\rho_s}{\varepsilon_2}\frac{a}{r}\hat{a}_r \cdot \left(\hat{a}_r dr + \hat{a}_\phi r d\phi + \hat{a}_z dz\right)$$ *while for the inner insulator*

$$V(r) - V(b) = -\int_b^r \vec{E}\cdot d\vec{l} = -\int_b^r \frac{\rho_s}{\varepsilon_1}\frac{a}{r}\hat{a}_r \cdot \left(\hat{a}_r dr + \hat{a}_\phi r d\phi + \hat{a}_z dz\right)$$ *thus we have that*

$$V(r) = -\int_c^r \frac{\rho_s}{\varepsilon_2}\frac{a}{r}dr = \frac{\rho_s a}{\varepsilon_2}\ln\frac{c}{r}$$ *for the outer insulator and*

$$V(r) - \frac{\rho_s a}{\varepsilon_2}\ln\frac{c}{b} = -\int_b^r \frac{\rho_s}{\varepsilon_1}\frac{a}{r}dr = \frac{\rho_s a}{\varepsilon_1}\ln\frac{b}{r}$$. *The total voltage difference between the outer*

and inner conductor is then $V(a) = \frac{\rho_s a}{\varepsilon_2}\ln\frac{c}{b} + \frac{\rho_s a}{\varepsilon_1}\ln\frac{b}{a} = \rho_s a\left(\frac{1}{\varepsilon_2}\ln\frac{c}{b} + \frac{1}{\varepsilon_1}\ln\frac{b}{a}\right)$. *The capacitance per unit length is then*

$$C = \frac{Q}{\rho_s a\left(\frac{1}{\varepsilon_2}\ln\frac{c}{b} + \frac{1}{\varepsilon_1}\ln\frac{b}{a}\right)} = \frac{\rho_s 2\pi a}{\rho_s a\left(\frac{1}{\varepsilon_2}\ln\frac{c}{b} + \frac{1}{\varepsilon_1}\ln\frac{b}{a}\right)} = \frac{2\pi\varepsilon_1\varepsilon_2}{\varepsilon_1\ln\frac{c}{b} + \varepsilon_2\ln\frac{b}{a}}$$

. Note that this could also be determined by taking the parallel combination of the two capacitors formed by the two insulators. $C_2 = \frac{2\pi\varepsilon_2}{\ln\frac{c}{b}}$ *and* $C_1 = \frac{2\pi\varepsilon_1}{\ln\frac{b}{a}}$*. Since the problem stated that first principles must be used, this latter approach cannot be used to solve the problem. However, it can be used to check the answer.*

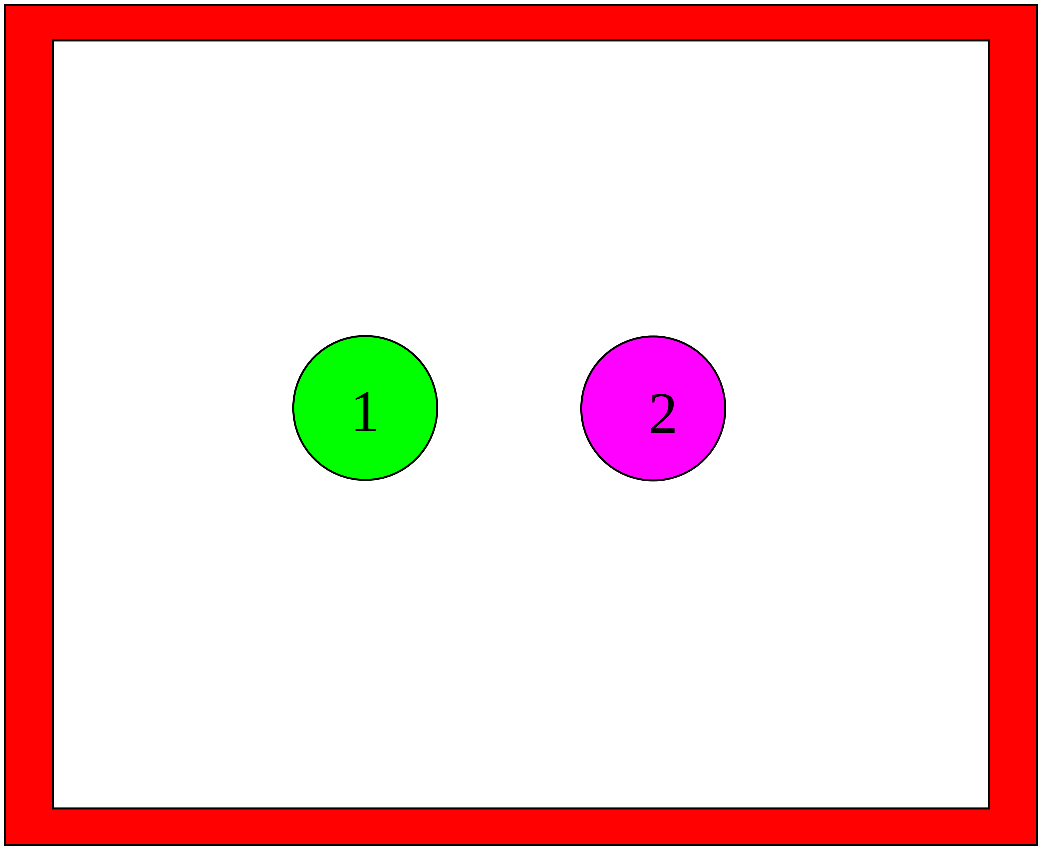
To get a higher value for the operating voltage, we can choose an insulator from the MatWed site. There is a property search page that makes this very simple. http://matweb.com/search/SearchProperty.asp *On this page, select a value for the Dielectric Strength of at least 200 and maybe a dielectric constant of 1.45 (about) to see if anything fits the bill. Unfortunately, nothing shows up. Thus, just pick a dielectric constant that is close. The lowest dielectric constant will be the easiest to match. Thus, choose S*aint-Gobain Norton® FEP Fluoropolymer Film *which has a dielectric constant of 2.1 and a dielectric strength of 236. We also need something with a higher dielectric constant to make up for the lower value with this material. We can choose many different materials. Try D*uPont Kapton® 120FN616 Polyimide/FEP Composite Film with *3.1 and 272. Then use the lower value dielectric constant on the inner region and the higher in the outer region. Then if we choose the thickness of the inner region to be 0.075inches or 1.9mm, we get the same value of capacitance using two materials with larger breakdown fields.*

To complete your design, you need to identify two new dielectric materials, one of which has a higher breakdown voltage than PE and the other has a dielectric constant that compensates for the first dielectric in the capacitance. You can choose the distance *b* to get the capacitance you need. If you cannot make two insulators work, you will have to try three.

c. What are the final parameters of your design? Demonstrate that the capacitance is the same and the breakdown voltage is larger.

The final parameters are therefore the Saint-Gobain dielectric in the inner region and the Dupont in the outer region. The maximum field occurs in the inner region so the dielectric strength is limited by the properties of the Saint-Gobain material. Note that using two different dielectrics actually does not give the full improvement in this case because the field in the lower dielectric region is increased relative to the uniform case. Here, for example, the field at r=a is given by the usual expression for $\vec{E} = \frac{\rho_s}{\varepsilon}\frac{a}{r}\hat{a}_r$ *. Since the charge density must be the same if the capacitance is the same, the field is larger here by the ratio* $\frac{\varepsilon}{\varepsilon_1} = \frac{2.44}{2.1} = 1.16$*. Fortunately the dielectric strength increases by more than this value* $\frac{236}{150} = 1.57$ *so we really have made things better. Any student who recognizes this gets extra credit.*

2. Using a Spreadsheet to Find Capacitance: Assume that you have the following two dimensional configuration of conductors (all dielectrics are air).

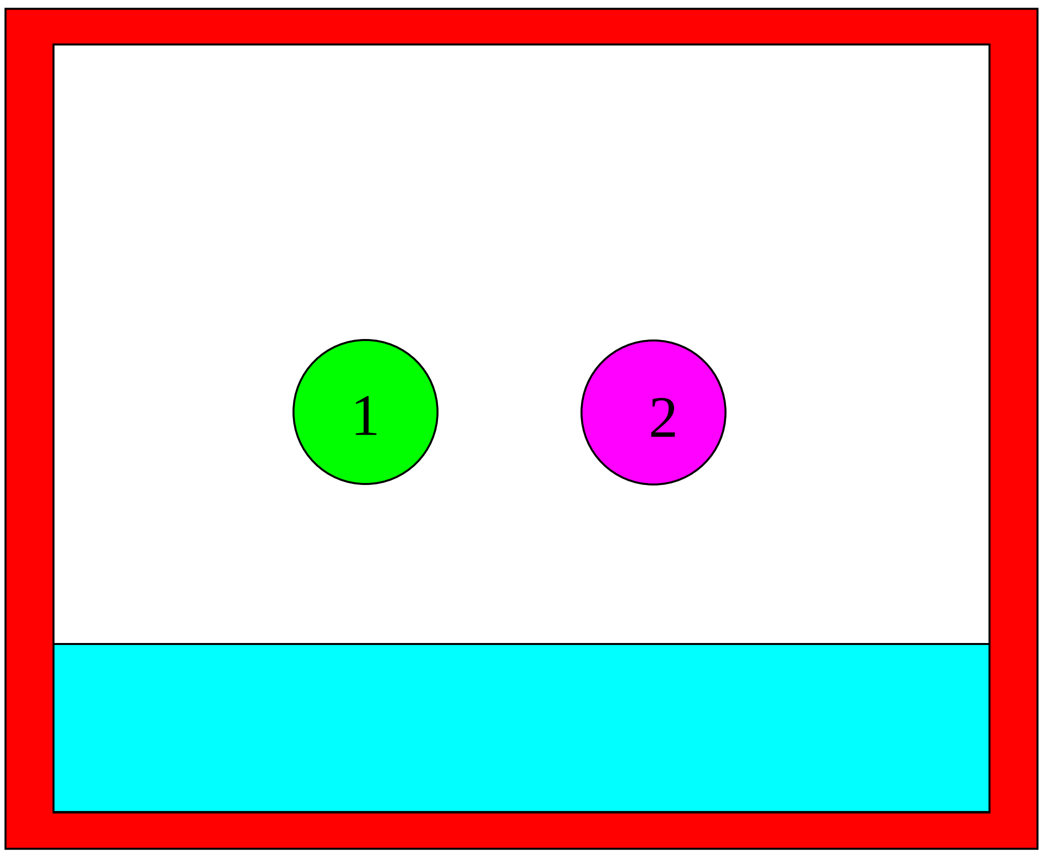


For this problem, see the Excel spreadsheets.

Homework 4

Assume that the voltage on conductor 1 is *-100V* and on conductor 2 is *+100V*. The diameter of each is *10mm*, the outer box is *50mm* by *50mm*, and the distance between the conductors is also *10mm*.

a. Use the spreadsheet method to find the capacitance per unit length of this shielded, two wire transmission line.
b. Produce a plot showing at least 8 equipotentials. Sketch a representative set of electric field lines on this plot.
c. Find the charge per unit length.
d. Using a standard analytic formula, find the capacitance per unit length of the two wire line without the shield.
e. Assume that about half of the lower half of the insulating region is filled with a dielectric with $\varepsilon = 81\varepsilon_o$. Repeat all steps and find the capacitance per unit length.



3. Poisson's Equation: The electric scalar potential in a spherical region $0 \le r \le a$ is given by $V(r) = V_o$. This region is known to be filled with charge with an unknown density distribution $\rho = \rho(r)$. Determine the charge distribution responsible for this potential.

This is simply found by applying the Laplacian to this voltage.

$$-\frac{\rho}{\varepsilon} = \nabla^2 V(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V(r)}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V_o}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 0\right) = 0$$

Thus, the charge density is zero or there is no charge. Any time the voltage is a constant, the charge must be zero, regardless of the coordinate system.