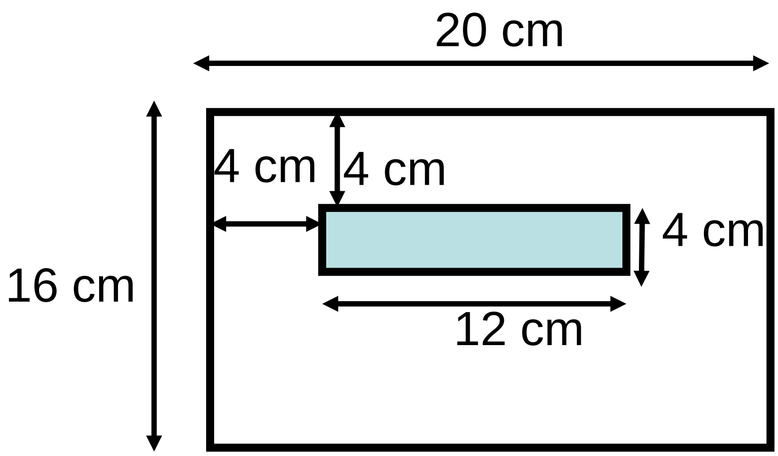
Homework 4
Fields and Waves I
Fall 2007

1. In the configuration below, we have a rectangular conductor (inner rectangle) charged to 100 Volts in a grounded metal box (outer rectangle). Find the capacitance per meter using the finite difference iteration method on a spreadsheet. Check for the reasonableness of the solution. There is air in between the two conductors.

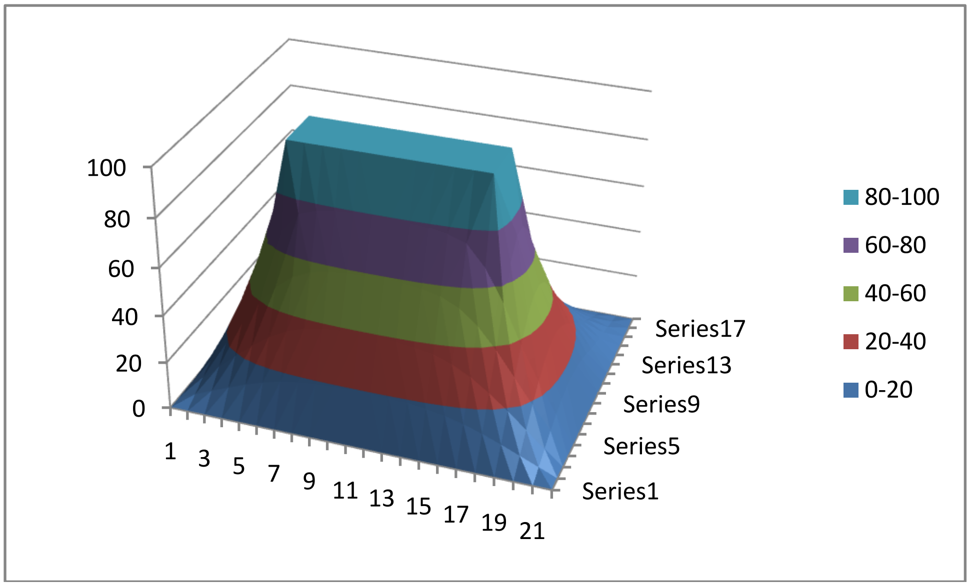


Solution :

Excel Grid :

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	3.16	6.32	9.43	12.3	14.8	16.8	18.1	18.9	19.4	19.6	19.6	19.4	18.9	18.1	16.8	14.8	12.3	9.43	6.32	3.16	0
0	6.29	12.7	19	25.1	30.3	34.2	36.7	38.2	39	39.3	39.3	39	38.2	36.7	34.2	30.3	25.1	19	12.7	6.29	0
0	9.3	18.9	28.9	38.7	47.1	53.1	56.5	58.2	59	59.3	59.3	59	58.2	56.5	53.1	47.1	38.7	28.9	18.9	9.3	0
0	12	24.6	38.3	54	66.8	75.1	77.7	78.8	79.3	79.5	79.5	79.3	78.8	77.7	75.1	66.8	54	38.3	24.6	12	0
0	14.1	29.1	46	66.1	100	100	100	100	100	100	100	100	100	100	100	100	66.1	46	29.1	14.1	0
0	15.4	31.8	50.5	73.5	100	100	100	100	100	100	100	100	100	100	100	100	73.5	50.5	31.8	15.4	0
0	15.7	32.4	51.3	74	100	100	100	100	100	100	100	100	100	100	100	100	74	51.3	32.4	15.7	0
0	15.1	30.9	48.4	68.5	100	100	100	100	100	100	100	100	100	100	100	100	68.5	48.4	30.9	15.1	0
0	13.7	27.9	42.9	59.3	72.5	81.1	84.2	85.7	86.4	86.8	86.8	86.4	85.7	84.2	81.1	72.5	59.3	42.9	27.9	13.7	0
0	12	24	36.3	47.9	57.9	65	69.5	72	73.3	73.9	73.9	73.3	72	69.5	65	57.9	47.9	36.3	24	12	0
0	9.99	19.9	29.6	38.6	46.3	52.3	56.5	59.2	60.8	61.5	61.5	60.8	59.2	56.5	52.3	46.3	38.6	29.6	19.9	9.99	0
0	8.06	16	23.6	30.5	36.5	41.4	45.1	47.6	49.2	49.9	49.9	49.2	47.6	45.1	41.4	36.5	30.5	23.6	16	8.06	0
0	6.24	12.3	18.1	23.4	28	31.9	34.8	36.9	38.3	38.9	38.9	38.3	36.9	34.8	31.9	28	23.4	18.1	12.3	6.24	0
0	4.55	8.98	13.2	17	20.4	23.2	25.4	27	28.1	28.6	28.6	28.1	27	25.4	23.2	20.4	17	13.2	8.98	4.55	0
0	2.97	5.85	8.59	11.1	13.3	15.1	16.6	17.7	18.4	18.8	18.8	18.4	17.7	16.6	15.1	13.3	11.1	8.59	5.85	2.97	0
0	1.46	2.89	4.24	5.47	6.56	7.48	8.21	8.76	9.12	9.3	9.3	9.12	8.76	8.21	7.48	6.56	5.47	4.24	2.89	1.46	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Surface plot of the potential distribution :



The total charge induced in the inner conductor can be figured out using Gauss's law. For this we could use any closed surface around the inner metal conductor and integrate ∫D.ds. We use a rectangular surface just inside the outer box.

The electric field at each point is the given by the change in the potential difference between consecutive grid points divided by distance between grid points (1cm).

$$Q = \oint_s D.ds \cong \sum D.dA = dA \times \sum D$$

where dA is the area of each small square box on our surface, which is given by 1cm times the depth of the box. The depth is taken to be 1m for convenience.
From the excel sheet we determine the electric fields perpendicular to our gaussian surface at each grid point, and then sum it up.

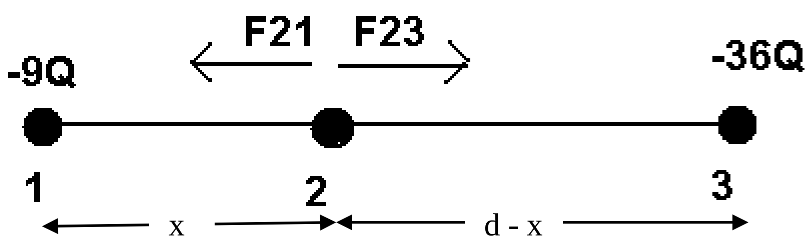
$$\Sigma \mathrm{D} = \varepsilon(\Sigma \mathrm{E}) = 8.85 \times 10^{-12} \times 70441.40 \frac{\mathrm{Coulombs}}{\mathrm{m}^2} = 6.23 \text{ x } 10^{-7} \frac{\mathrm{C}}{\mathrm{m}^2}$$

$$Q = dA\,\Sigma D = 6.23 \times 10^{-9} C/m$$

$$C = \frac{Q}{V} = \frac{6.23 \times 10^{-9}}{100} = 6.23 \times 10^{-11} F/m$$

2. A charge of -9Q is placed on the origin and one of -36Q on the x axis at x = d. We want to place another charge on the x axis so that the system is in equilibrium. Find the charge and location.

If the forces on the new charge have to be cancelled out, it has to be placed at a suitable point on the line joining the two existing charges.
The force on Q_1due to Q_3 is in the negative x direction. To counter this force, the force on Q_1 due to Q_2 has to be in the positive x direction. Therefore Q_2 must be a positive charge.



$$F_{12} = F_{23}$$

Therefore,

$$\frac{-9Q.Q_2}{4\pi\epsilon_0 x^2} = \frac{-36Q.Q_2}{4\,\pi\epsilon_0 (d-x)^2}$$

Solving

$$\frac{(d-x)^2}{x^2} = 4$$

Or

$$\frac{d-x}{x} = 2$$

Therefore,

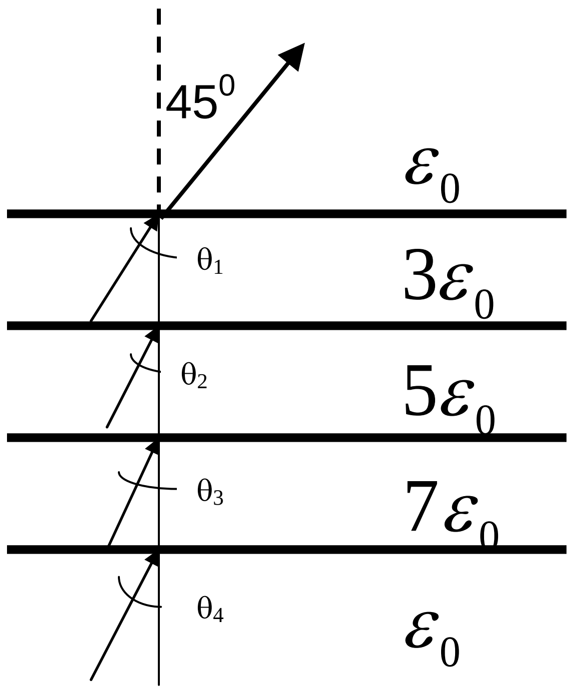
$$x = d/3$$

To solve for charge use the equilibrium condition on Q_1:

$$F_{13} + F_{12} = 0$$

$$-\frac{9Q.36Q}{4\pi\epsilon_0 d^2} + \frac{9Q.Q_2}{4\pi\epsilon_0 x^2} = 0$$

$$Q_2 = \frac{36(x^2)Q}{d^2} = 4Q$$



3. In the above figure we see that an electric field line makes a 45 degree angle to the surface of a layered dielectric. Find the angles of the electric field in the other regions.

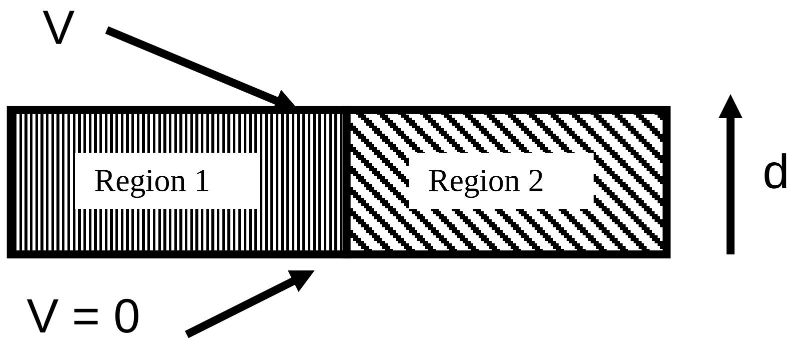
$$\theta_1 = \tan^{-1}\left(\frac{\epsilon_1}{\epsilon_0}\tan\theta_0\right) = \tan^{-1}(3\tan 45) = 71.6°$$

$$\theta_2 = \tan^{-1}\left(\frac{\epsilon_2}{\epsilon_1}\tan\theta_1\right) = \tan^{-1}((5/3)\tan 71.6) = 78.7°$$

$$\theta_3 = \tan^{-1}\left(\frac{\epsilon_3}{\epsilon_2}\tan\theta_2\right) = \tan^{-1}(7/5\tan 78.7) = 81.9°$$

$$\theta_4 = \tan^{-1}\left(\frac{\epsilon_0}{\epsilon_3}\tan\theta_3\right) = \tan^{-1}(1/7\tan 81.9) = 45°$$

4. In the figure below we have a capacitor which half of the capacitor is filled with air and half with a dielectric of relative permittivity equal to 3. The total area of the plate is A and the spacing is d. We apply a potential difference of V between the plates. Find the energy stored and relate this to the capacitance.



The electric field $$E = V/d$$

In Region 1,

$$D_1 = \frac{\epsilon_0 V}{d}$$

In Region 2,

$$D_2 = \frac{3\ \epsilon_0 V}{d}$$

The energy stored,

$$W_1 = \frac{1}{2} D\ E\ (Volume) = \frac{V^2 \epsilon_0}{2\ d^2}\ \frac{A}{2}\ d = \frac{V^2 \epsilon_0 A}{4d}$$

$$W_2 = \frac{3\ V^2 \epsilon_0 A}{4\ d}$$

Total Energy,

$$W_{TOT} = W_1 + W_2 = \frac{V^2 \epsilon_0 A}{d}$$

We can confirm this using the capacitances.

$$C_1 = \frac{\epsilon_0}{d}\frac{A}{2} \qquad C_2 = \frac{3\epsilon_0}{d}\frac{A}{2}$$

$$W_1 = \frac{1}{2} C_1 V^2 = \frac{1}{2}\frac{\epsilon_0 A}{2d} V^2$$

$$W_2 = \frac{1}{2} C_2 V^2 = \frac{1}{2}\frac{3\epsilon_0 A}{2d} V^2$$

$$W_{TOT} = W_1 + W_2 = \frac{V^2 \epsilon_0 A}{d}$$