1. a. $R = \int \frac{dz}{\sigma A(z)}$ $\qquad A(z) = \text{area} = \pi r^2$

$\qquad r(z) = a\left(1 + (z/h)^2\right)$

$$= \frac{1}{\pi a^2 \sigma} \int_0^h \frac{dz}{\left(1 + (z/h)^2\right)^2}$$

From Maple: (for $h = 10$ & $a = 0.1$)

$$\int_0^h \frac{dz}{\left(1+(z/h)^2\right)^2} = \frac{5}{2} + \frac{5\pi}{4} = \frac{h}{4}\left[1 + \frac{\pi}{2}\right]$$

$$R = \frac{1}{\pi a^2 \sigma} \frac{h}{4}\left[1+\frac{\pi}{2}\right] = \frac{h}{4 a^2 \sigma \pi}\left[1 + \frac{\pi}{2}\right]$$

Can also be done using integral tables

$$\int_0^h \frac{dz}{\left(1 + \frac{z^2}{h^2}\right)^2} = h \int_0^h \frac{d\, z/h}{\left(1 + (z/h)^2\right)^2} = h \int_0^1 \frac{dx}{(1+x^2)^2}$$

$$= h\left[\frac{1}{4} + \frac{1}{2}\frac{\pi}{4}\right] = \frac{h}{4}\left(1 + \frac{\pi}{2}\right)$$

To see if the answer is reasonable, compare with a cylinder of area πa^2 (which should have a larger resistance) Then $R_{cyl} = \frac{h}{\sigma \pi a^2}$

$$\frac{R_{cyl}}{R_{tree}} = \frac{\cancel{h}}{\cancel{\sigma}\cancel{\pi}\cancel{a^2}} \frac{4\cancel{\pi}\cancel{a^2}\cancel{\sigma}}{\cancel{h}\left[1+\frac{\pi}{2}\right]} = \frac{4}{1+\frac{\pi}{2}} \simeq 1.5$$ which is larger as it should be

For the numbers given

$$R = \frac{10}{4(.1)^2\, 2\pi}\left[1 + \frac{\pi}{2}\right] \cong 102$$

Answer should be near 100.

b. $i = 200{,}000$ A $\qquad V = iR \cong 2 \times 10^7$ Volts

power $= (i^2 R) = (2\times10^5)^2 \, 10\Omega \cong 4 \times 10^{12}$ W

energy $= pt = 4\times10^{12} \times 2\times10^{-4} = 8\times10^8$
$= 0.8\times10^9$ J
$= 0.8$ GJ

c. Volume of tree = h x average area
average area is determined from the radius that gives the same resistance.

$$\pi r_{ave}^2 = 4(.1)^2$$
$$r_{ave}^2 = \frac{4}{\pi}(.1)^2 \qquad r_{ave} = \sqrt{4/\pi}\,.1 = 11.3 \text{ cm}$$
$$= .113 \text{ m}$$

half the volume is $\dfrac{10 \times 4(.1)^2}{2} = .2 \text{ m}^3$

Energy to boil water.
Assume room temp 30 °C
Boiling is 100 °C
Need to raise the temp of water 70° & then change its phase.

1 calorie = energy to raise 1 g (1cc also) of water 1°C
= 4.18 J. per calorie

$.2 \text{ m}^3$ of $H_2O \Rightarrow .2(10^6 \text{ cm}^3) = 2\times10^5 \text{ cm}^3$
$= 2\times10^5$ gm

$2\times10^5 \times 4.18 \times 70 = 5.85\times10^7$ Joules

To boil water requires 540 cal/g

$\underset{cal/g}{540} \times \underset{g}{2\times10^5} \times \underset{J/cal}{4.18} = \underline{4.5\times10^8 \text{ J}}$ So there is enough energy.

Some references for problem 1
Latent heat of vaporization for water

http://www.school-for-champions.com/science/heat_ice_steam.htm

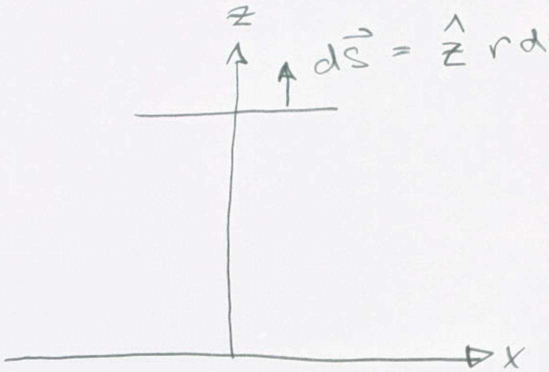
Includes audio

2. Faraday's law

$$\vec{B} \cong \frac{\mu_0 a^2 I}{4 r^3}\left[\hat{r}\, 2\cos\theta + \hat{\theta}\sin\theta\right]$$

$d\vec{S} = \hat{z}\, r\, dr\, d\phi$ where r is cyl. radius

$r = \sqrt{x^2+y^2}$



We need the z component of $\vec{B}$ or $\hat{z}\cdot\vec{B}$

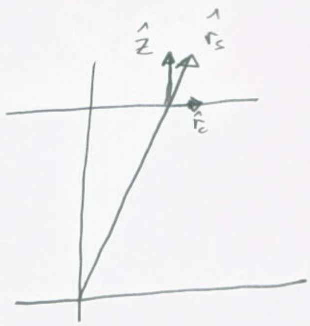
$$\hat{z}\cdot\vec{B} \cong \frac{\mu_0 a^2 I}{4 r^3}\left[\hat{z}\cdot\hat{r}\, 2\cos\theta + \hat{z}\cdot\hat{\theta}\sin\theta\right]$$

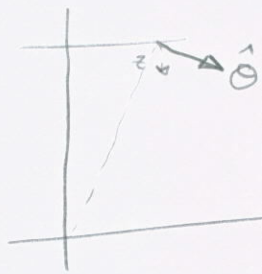
$\hat{z}\cdot\hat{r} = \cos\theta$

$\hat{z}\cdot\hat{\theta} = -\sin\theta$

From these drawings

or look them up in a table





Useful references

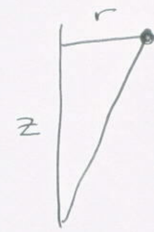
http://en.wikipedia.org/wiki/Unit_vector

http://www.csupomona.edu/~ajm/materials/de

Also appendix from Orfanidis' book (on FW

website)

$$\hat{z}\cdot\vec{B} = \frac{\mu_0 a^2 I}{4}\frac{1}{r^3}\left[2\cos^2\theta - \sin^2\theta\right]$$



$$\frac{1}{r^3} = \frac{1}{(z^2+r^2)^{3/2}} \qquad \cos\theta = \frac{z}{(z^2+r^2)^{1/2}}$$

$$\sin\theta = \frac{r}{(z^2+r^2)^{1/2}}$$

$$\hat{z}\cdot\vec{B} = \frac{\mu_0 a^2 I}{4}\left[\frac{2z^2 - r^2}{(z^2+r^2)^{5/2}}\right]$$

$$ds = r\,dr\,d\phi$$

$$\int_0^{2\pi} d\phi$$

$$\int \bar{B}\cdot d\bar{S} = \frac{\mu_0 a^2 I\, 2\pi}{4}\int_0^b \frac{2z^2 - r^2}{(z^2+r^2)^{5/2}}\, r\,dr$$

let $b = 2a$ $\quad z_0 = 4a$ $\quad$ & $\quad a = 1\text{ cm}$

$$\int \bar{B}\cdot d\bar{S} = \frac{\mu_0 (.01)^2 I 2\pi}{4}\, 4.5$$

$$\simeq 7\times10^{-4}\mu_0 I$$

Using the approximation that the B field is a constant over the area of the loop, where we only use the first term in the $\hat{r}$ direction.

$$\int \bar{B}\cdot d\bar{S} \simeq B\cdot\pi b^2 \simeq \frac{\mu_0 a^2 I}{4}\frac{2}{z_0^3}\pi(.02)^2 \simeq \frac{\mu_0 I\,(.01)^2}{2\,(.04)^3}$$

$$\simeq 9.8\times10^{-4}\mu_0 I$$

Which is reasonably close

We should obtain almost perfect agreement for $Z_0 >>> a$. Choose $Z_0 = 9 >> .01$

Then from the exact integral

$$\int \bar{B} \cdot d\bar{S} = \frac{\mu_0 (.01)^2 I \, 2\pi}{4} (5.5 \times 10^{-7}) = 8.6 \times 10^{-11} \mu_0 I$$

ξ $\quad B \cdot \text{Area} = B \pi b^2 = \dfrac{\mu_0 a^2 I}{4} \dfrac{2\pi b^2}{Z_0^3}$

$$= \frac{\mu_0 (.01)^2 I \; 2\pi (.02)^2}{4 \quad 9^3}$$

$$= 8.6 \times 10^{-11} \mu_0 I$$

So for very far distances, the two approaches give the same result, as they should

Back to the original flux $\Psi_m = 7 \times 10^{-4} \mu_0 I$

b. $\dfrac{d\Psi_m}{dt} = 7 \times 10^{-4} \mu_0 \dfrac{dI}{dt} = 7 \times 10^{-4} \mu_0 I_0 \omega \cos \omega t$

let $f = 1000$, then $\omega = 2000\pi$

$$\Rightarrow \frac{d\Psi_m}{dt} = 7 \times 10^{-4} \mu_0 I_0 \, 2000\pi \cos \omega t$$

$$= 4.4 \, \mu_0 I_0 \cos \omega t$$

which is also the voltage since $-\dfrac{d\Psi}{dt} = \oint \bar{E} \cdot d\bar{l}$

$$\vec{v}\times\vec{B} = v_0\,\hat{z}\times\frac{\mu_0 a^2 I}{4r^3}\left(\hat{r}_s\, 2\cos\theta + \hat{\theta}\sin\theta\right)$$

now the $\hat{z}$ component in $\vec{B}$ does not contribute

$\Rightarrow$ we need the $\hat{r}_c$ component.
"cylindrical"

$$\hat{r}_c\cdot\vec{B} = \frac{\mu_0 a^2 I}{4 r_s^3}\left(\underbrace{2\cos\theta\sin\theta + \sin\theta\cos\theta}_{3\cos\theta\sin\theta}\right.$$

$$\hat{r}_c\cdot\hat{r}_s = \sin\theta$$

$$\hat{r}_c\cdot\hat{\theta} = \cos\theta$$

$$= \frac{\mu_0 a^2 I}{4}\,\frac{3zr}{(z^2+r^2)^{5/2}}$$

$$\int \vec{v}\times\vec{B}\cdot d\vec{l} = 2\pi\,\frac{v_0\mu_0 a^2 I}{4}\,3\,\frac{zb}{(z^2+b^2)^{5/2}}$$

cut around circle of radius b

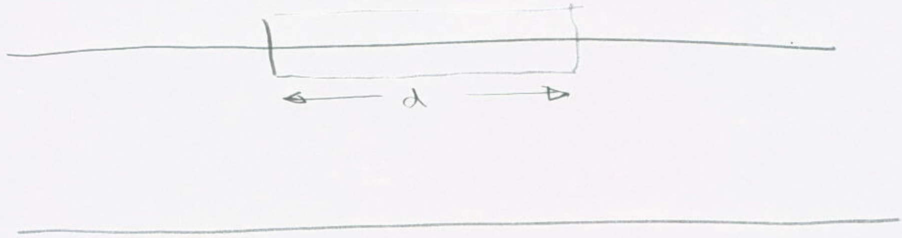
$$= \frac{2\pi v_0 \mu_0 a^2 I\, 3z}{4\,(z^2+b^2)^{5}}$$

$$= \frac{2\pi v_0\mu_0 I\,(.01)^2\, 3\,(.04)(.02)}{4\,(.04^2+.02^2)^{5/2}}$$

$$= v_0\mu_0 I\; 2.1$$

$B = 0$ outside

3. a.



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{encl} = \mu_0 \frac{N}{H_0} d$$

call the height H_0 to avoid confusion

$$B_z d = \mu_0 \frac{N}{H_0} d$$

$$B_z = \mu_0 \frac{N}{H_0}$$

Can also determine this from B. C.

$$\underbrace{H_{tan\,1} - H_{tan\,2}}_{= 0} = J_s = \frac{N}{H_0} I$$

$$H = \frac{B}{\mu_0} = \frac{N}{H_0} I$$

b. $$\Psi_m = \int \vec{B}\cdot d\vec{S} = \underbrace{\mu_0 \frac{N}{H_0} I}_{B} \underbrace{\pi a^2}_{\text{area}}$$

$$\frac{d\Psi_m}{dt} = \mu_0 \frac{N}{H_0} \pi a^2 \frac{dI}{dt}$$

$$= \mu_0 \frac{N}{H_0} \pi a^2 \omega I_0 \cos\omega t$$

c. $$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Psi_m}{dt} = -\mu_0 \frac{N}{H_0} \pi a^2 \omega I_0 \cos\omega t$$

Voltage

$$R = \frac{l}{\sigma A} = \frac{2\pi a}{\sigma \Delta H_0}$$

Magnitude: $$I = \frac{V}{R} = \frac{\mu_0 \frac{N}{\cancel{H_0}} \cancel{\pi} a^{\cancel{2}} \omega I_0 \cos\omega t \; \sigma \Delta \cancel{H_0}}{2\cancel{\pi}\cancel{a}}$$

$$= \mu_0 N a \omega I_0 \cos\omega t \frac{\sigma \Delta}{2}$$

only one turn

d. $$\psi_{ma} = \mu_0 \frac{1}{H_0} I \pi a^2 = L I$$

$$L = \frac{\mu_0 \pi a^2}{H_0}$$

Using numbers $$R = \frac{2\pi(.03)}{(4\times 10^7)(.12)(.0002)} = 2\times 10^{-4}\,\Omega$$

depends on the Source

$$L = \frac{(4\pi \times 10^{-7})(\pi)(.03)^2}{.12} = 3\times 10^{-8} \text{ Henries}$$

$$\tau = \frac{L}{R} = \frac{3\times 10^{-8}}{2\times 10^{-4}} = 1.5\times 10^{-4} = 150\ \mu s$$

The exact answer will depend on the conductivity of aluminum, usually $3.5 \to 4\times 10^7$