Quiz 2

VECTOR CALCULUS
ELECTROSTATICS
CURRENT & RESISTANCE

Name ___Solution_______________

Section ________________________

Multiple Choice

1. (5 Pts) ____________

2. (6 Pts) ____________

3. (16 Pts)
or ____________
3. (16 Pts)

4. (6 Pts) ____________

5. (8 Pts) ____________

Regular Questions

6. (12 Pts) ____________

7. (12 Pts) ____________

8. (20 Pts) ____________

9. (15 Pts) ____________

10. (Ex Cred) ____________

Total (100 Pts)

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Notes:
1. In the multiple choice questions, each question may have more than one correct answer; circle all of them.
2. For multiple choice questions, you may add some comments to justify your answer.
3. Make sure your calculator is set to perform trigonometric functions in radians & not degrees.

Some Comments and Helpful Info:
In this test, we use two types of notation for unit vectors. Keep in mind that

$\hat{a}_x = \hat{x}$ $\hat{a}_y = \hat{y}$ $\hat{a}_z = \hat{z}$ $\hat{a}_r = \hat{r}$ $\hat{a}_\phi = \hat{\phi}$ $\hat{a}_\theta = \hat{\theta}$

Be sure to show your work for the multiple choice questions.
Draw pictures for each problem to be sure that you understand the problem statement.
Please read the note explaining how the extra credit problem will be graded, since there is a penalty for an incorrect answer.

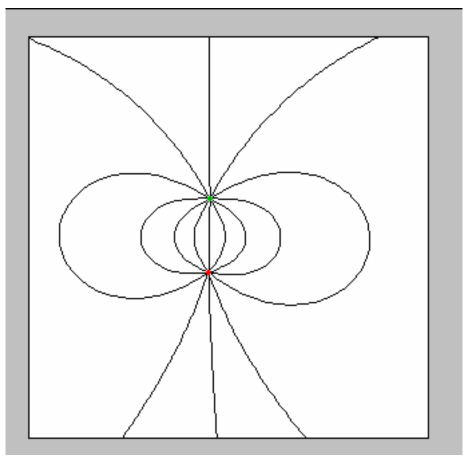
MULTIPLE CHOICE AND SHORT ANSWER QUESTIONS

1. Miscellaneous Questions (5 points)

a) Unlike charges attract, like charges repel.
   i) True
   ii) False

b) The force between two charged particles decays as the square of the distance between the particles.
   i) True
   ii) False

c) Volts are the units of the electric field.
   i) True
   ii) False



d) The electric field lines shown above are for two identical positive charges.
   i) True
   ii) False

e) There are free positive and negative charges in the air we breathe.
   i) True
   ii) False

2. Gauss' Law (6 points)

Gauss' Law is given in integral form as $\oint \vec{D}\cdot d\vec{S} = \int \rho_v dv = Q_{encl}$.

a) Assuming that $\rho_v = \rho_v(r)$, where *r* is cylindrical radius, simplify the electric flux density vector $\vec{D} = D_r(r,\phi,z)\hat{r} + D_\phi(r,\phi,z)\hat{\phi} + D_z(r,\phi,z)\hat{z} = ?$
   i) $\vec{D} = D_\phi(r,\phi,z)\hat{\phi}$
   ii) $\vec{D} = D_\phi(r)\hat{\phi}$
   iii) $\vec{D} = D_r(r,\phi,z)\hat{r}$
   iv) $\vec{D} = D_r(r)\hat{r}$
   v) $\vec{D} = D_z(r,\phi,z)\hat{z}$
   vi) $\vec{D} = D_z(r)\hat{z}$

b) Again, assuming that $\rho_v = \rho_v(r)$, what Gaussian surface will work well if we are to use Gauss' Law to find $\vec{D}$? Also, what $d\vec{S}$ goes with the surface?

i) An open cylinder at radius $r = const$ with $d\vec{S} = \hat{r}rd\phi dz$

ii) A closed cylinder consisting of a surface at radius $r = const$ with $d\vec{S} = \hat{r}rd\phi dz$, length l and end caps at $z = 0$ and $z = l$ with $d\vec{S} = \pm\hat{z}drrd\phi$

iii) A closed cylinder consisting of a surface at radius $r = const$ with $d\vec{S} = \hat{\phi}rd\phi dz$, length l and end caps at $z = 0$ and $z = l$ with $d\vec{S} = \pm\hat{z}drrd\phi$

iv) A closed cylinder consisting of a surface at radius $r = const$ with $d\vec{S} = \hat{r}rd\phi dz$, length l and end caps at $z = 0$ and $z = l$ with $d\vec{S} = \pm\hat{\phi}drrd\phi$

c) Given your solution to questions a) and b), evaluate the left hand side of Gauss' Law $\oint \vec{D} \cdot d\vec{S}$

i) $\oint \vec{D} \cdot d\vec{S} = D_r(r)2\pi rl$

ii) $\oint \vec{D} \cdot d\vec{S} = D_r(r)\pi r^2 l$

iii) $\oint \vec{D} \cdot d\vec{S} = D_r(r)\pi r^2$

iv) $\oint \vec{D} \cdot d\vec{S} = 0$

3. Capacitance, Voltage, and Electric Field (16 points) – Option 1

A spherical capacitor consists of an inner conductor of radius a and an outer conductor of inner radius b and outer radius c. The total charge on the inner conductor is Q_o and the total charge on the outer conductor is $-Q_o$. The space between the conductors is empty (air).

a) The electric field inside the inner conductor ($r<a$) is

i) $\vec{E} = \dfrac{Q_o}{2\pi\varepsilon_o r}\hat{r}$

ii) $\vec{E} = \dfrac{Q_o}{2\pi\varepsilon_o r^2}\hat{r}$

iii) $\vec{E} = 0$

iv) $\vec{E} = \dfrac{Q_o}{4\pi\varepsilon_o r^2}\hat{r}$

v) $\vec{E} = \dfrac{Q_o}{4\pi\varepsilon_o r}\hat{r}$

b) The field in the region between the conductors ($a<r<b$) is

i) $\vec{E} = \dfrac{Q_o}{2\pi\varepsilon_o r}\hat{r}$

ii) $\vec{E} = \dfrac{Q_o}{2\pi\varepsilon_o r^2}\hat{r}$

iii) $\vec{E} = 0$

iv) $\vec{E} = \dfrac{Q_o}{4\pi\varepsilon_o r^2}\hat{r}$

v) $\vec{E} = \dfrac{Q_o}{4\pi\varepsilon_o r}\hat{r}$

c) The field in the outer conductor ($b<r<c$) is

i) $\vec{E} = \dfrac{Q_o}{2\pi\varepsilon_o r}\hat{r}$

ii) $\vec{E} = \dfrac{Q_o}{2\pi\varepsilon_o r^2}\hat{r}$

iii) $\vec{E} = 0$

iv) $\vec{E} = \dfrac{Q_o}{4\pi\varepsilon_o r^2}\hat{r}$

v) $\vec{E} = \dfrac{Q_o}{4\pi\varepsilon_o r}\hat{r}$

d) The field in the region outside the outer conductor ($c<r$) is

i) $\vec{E} = \frac{Q_o}{2\pi\varepsilon_o r}\hat{r}$

ii) $\vec{E} = \frac{Q_o}{2\pi\varepsilon_o r^2}\hat{r}$

iii) $\vec{E} = 0$

iv) $\vec{E} = \frac{Q_o}{4\pi\varepsilon_o r^2}\hat{r}$

v) $\vec{E} = \frac{Q_o}{4\pi\varepsilon_o r}\hat{r}$

e) The voltage between the inner and outer conductors $V(a) - V(b) = -\int_b^a \vec{E}\cdot d\vec{l}$ is

i) $V(a) - V(b) = \frac{Q_o}{2\pi\varepsilon_o}\ln\frac{b}{a}$

ii) $V(a) - V(b) = \frac{Q_o}{2\pi\varepsilon_o}\left(\frac{1}{a} - \frac{1}{b}\right)$

iii) $V(a) - V(b) = 0$

iv) $V(a) - V(b) = \frac{Q_o}{4\pi\varepsilon_o}\left(\frac{1}{a} - \frac{1}{b}\right)$

v) $V(a) - V(b) = \frac{Q_o}{4\pi\varepsilon_o}\ln\frac{b}{a}$

f) The capacitance of the spherical capacitor is given by

i) $\dfrac{2\pi\varepsilon_o}{\ln\frac{b}{a}}$

ii) $\dfrac{2\pi\varepsilon_o}{\left(\frac{1}{a} - \frac{1}{b}\right)}$

iii) ∞

iv) $\dfrac{4\pi\varepsilon_o}{\left(\frac{1}{a} - \frac{1}{b}\right)}$

v) $\dfrac{4\pi\varepsilon_o}{\ln\frac{b}{a}}$

Assume that the inner conductor is replaced by a uniform volume charge distribution but the total charge is still equal to Q_o.

g) What is the volume charge density in this case?

i) $\rho_V = \dfrac{Q_o}{4\pi a^2}$

ii) $\rho_V = \dfrac{Q_o}{\frac{4}{3}\pi a^3}$

iii) $\rho_V = Q_o 4\pi a^2$

iv) $\rho_V = Q_o \dfrac{4}{3}\pi a^3$

h) For the region outside the inner conductor ($a < r$), the electric field will remain the same as for the solid inner conductor, since the charge enclosed remains the same. However inside the charge, the answer will be different. What is the electric field inside the charge ($0 < r < a$)?

i) $\vec{E} = \dfrac{Q_o}{4\pi\varepsilon_o a^2}\hat{r}$

ii) $\vec{E} = \dfrac{Q_o r}{4\pi\varepsilon_o a^3}\hat{r}$

iii) $\vec{E} = \dfrac{Q_o r^2}{4\pi\varepsilon_o a^4}\hat{r}$

iv) $\vec{E} = \dfrac{Q_o}{4\pi\varepsilon_o r^2}\hat{r}$

3. Charge, Voltage and Electric Field (16 points) – Option 2
A coaxial cable consists of an inner conductor of radius *a* and an outer conductor of inner radius *b* and outer radius *c*. The total charge per unit length on the inner conductor is Q_o and the total charge per unit length on the outer conductor is $-Q_o$. The space between the conductors is empty (air). For the following questions, keep in mind that the units of Q_o are Coulombs per meter, not just Coulombs when you check your units.

a. The electric field inside the inner conductor ($r<a$) is

   i. $\vec{E} = \frac{Q_o}{2\pi\varepsilon_o r}\hat{r}$

   ii. $\vec{E} = \frac{Q_o}{2\pi\varepsilon_o r^2}\hat{r}$

   iii. $\vec{E} = 0$

   iv. $\vec{E} = \frac{Q_o}{4\pi\varepsilon_o r^2}\hat{r}$

   v. $\vec{E} = \frac{Q_o}{4\pi\varepsilon_o r}\hat{r}$

b. The field in the region between the conductors ($a<r<b$) is

   i. $\vec{E} = \frac{Q_o}{2\pi\varepsilon_o r}\hat{r}$

   ii. $\vec{E} = \frac{Q_o}{2\pi\varepsilon_o r^2}\hat{r}$

   iii. $\vec{E} = 0$

   iv. $\vec{E} = \frac{Q_o}{4\pi\varepsilon_o r^2}\hat{r}$

   v. $\vec{E} = \frac{Q_o}{4\pi\varepsilon_o r}\hat{r}$

c. The field in the outer conductor ($b<r<c$) is

   i. $\vec{E} = \frac{Q_o}{2\pi\varepsilon_o r}\hat{r}$

   ii. $\vec{E} = \frac{Q_o}{2\pi\varepsilon_o r^2}\hat{r}$

   iii. $\vec{E} = 0$

   iv. $\vec{E} = \frac{Q_o}{4\pi\varepsilon_o r^2}\hat{r}$

   v. $\vec{E} = \frac{Q_o}{4\pi\varepsilon_o r}\hat{r}$

d. The field in the region outside the outer conductor ($c<r$) is

i. $\vec{E} = \frac{Q_o}{2\pi\varepsilon_o r}\hat{r}$

ii. $\vec{E} = \frac{Q_o}{2\pi\varepsilon_o r^2}\hat{r}$

iii. $\vec{E} = 0$

iv. $\vec{E} = \frac{Q_o}{4\pi\varepsilon_o r^2}\hat{r}$

v. $\vec{E} = \frac{Q_o}{4\pi\varepsilon_o r}\hat{r}$

e. The voltage between the inner and outer conductors $V(a) - V(b) = -\int_b^a \vec{E}\cdot d\vec{l}$ is

i. $V(a) - V(b) = \frac{Q_o}{2\pi\varepsilon_o}\ln\frac{b}{a}$

ii. $V(a) - V(b) = \frac{Q_o}{2\pi\varepsilon_o}\left(\frac{1}{a} - \frac{1}{b}\right)$

iii. $V(a) - V(b) = 0$

iv. $V(a) - V(b) = \frac{Q_o}{4\pi\varepsilon_o}\left(\frac{1}{a} - \frac{1}{b}\right)$

v. $V(a) - V(b) = \frac{Q_o}{4\pi\varepsilon_o}\ln\frac{b}{a}$

f. The capacitance per unit length of the coaxial cable is given by

i. $\frac{2\pi\varepsilon_o}{\ln\frac{b}{a}}$

ii. $\frac{2\pi\varepsilon_o}{\left(\frac{1}{a} - \frac{1}{b}\right)}$

iii. ∞

iv. $\frac{4\pi\varepsilon_o}{\left(\frac{1}{a} - \frac{1}{b}\right)}$

v. $\frac{4\pi\varepsilon_o}{\ln\frac{b}{a}}$

Assume that the inner conductor is replaced by a uniform volume charge distribution but the total charge per unit length is still equal to Q_o.

g. What is the volume charge density in this case?

i. $\rho_V = \dfrac{Q_o}{2\pi a}$

ii. $\rho_V = \dfrac{Q_o}{\pi a^2}$

iii. $\rho_V = Q_o 2\pi a$

iv. $\rho_V = Q_o \pi a^2$

h. For the region outside the inner conductor ($a < r$), the electric field will remain the same as for the solid inner conductor, since the charge enclosed remains the same. However inside the charge, the answer will be different. What is the electric field inside the charge ($0 < r < a$)?

i. $\vec{E} = \dfrac{Q_o}{2\pi\varepsilon_o a}\hat{r}$

ii. $\vec{E} = \dfrac{Q_o r}{2\pi\varepsilon_o a^2}\hat{r}$

iii. $\vec{E} = \dfrac{Q_o r^2}{2\pi\varepsilon_o a}\hat{r}$

iv. $\vec{E} = \dfrac{Q_o}{2\pi\varepsilon_o r^2}\hat{r}$

4. Resistance and Conductivity (6 points)

a) A typical value for the conductivity of a good conductor (e.g. copper, silver, gold, aluminum, platinum, etc.) that we might make wire out of is:

i) $\sigma = 200 \,{}^{S}\!/_{m}$

ii) $\sigma = 0.003 \,{}^{S}\!/_{m}$

iii) $\sigma = 2x10^4 \,{}^{S}\!/_{m}$

iv) $\sigma = 6x10^7 \,{}^{S}\!/_{m}$

v) $\sigma = 10^{-6} \,{}^{S}\!/_{m}$

b) The resistance of a cylindrical conductor with a uniform circular cross section of radius a, a uniform conductivity $\sigma_o \,{}^{S}\!/_{m}$, and a length d, is given by:

i) $R = \dfrac{\sigma_o d}{\pi a}$

ii) $R = \dfrac{\sigma_o d}{\pi a^2}$

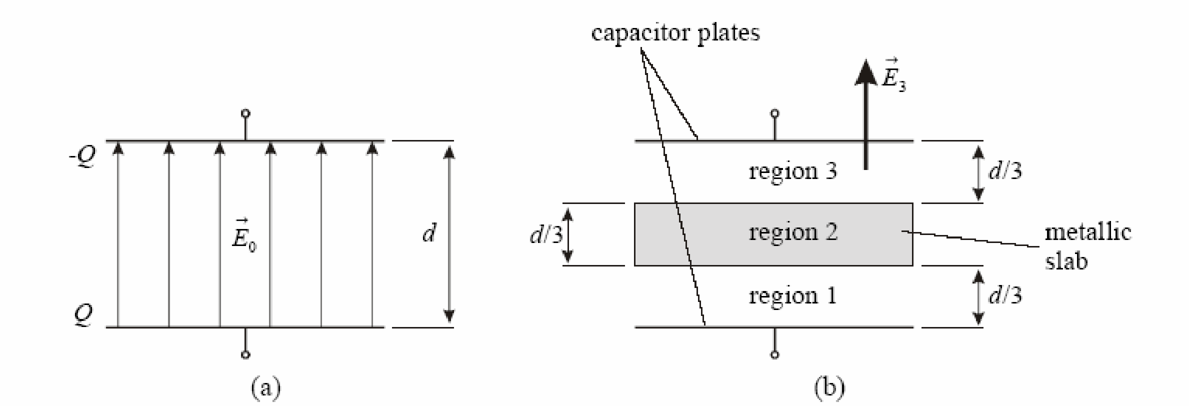
iii) $R = \dfrac{d}{\sigma_o \pi a^2}$

iv) $R = \dfrac{d}{\sigma_o 2\pi a}$

v) $R = \dfrac{\sigma_o \pi a^2}{d}$

5. Capacitance (8 points)

An air-filled parallel-plate capacitor is charged and its terminals are left open. The charges of the lower and upper plates are Q and $-Q$, respectively, the fringing effects can be neglected, and the electric field intensity vector between the plates is $\mathbf{E}_0$ [Figure (a)]. An uncharged metallic slab is then inserted between the plates, without touching the plates by hands or any other conducting body [Figure (b)].



a) The electric field intensity vector in region 3 between the slab and the upper plate in the new stage is

i) $\vec{E}_3 = 0$

ii) $\vec{E}_3 = -\vec{E}_o$

iii) $\vec{E}_3 = \vec{E}_o/3$

iv) $\vec{E}_3 = 3\vec{E}_o/2$

v) $\vec{E}_3 = \vec{E}_o$ *The charge on the plates remains the same so the magnetic flux density and the electric field must remain the same.*

b) If the capacitance in figure (a) is C_a, the capacitance of figure (b) is

i) $C_b = 0$

ii) $C_b = C_a$

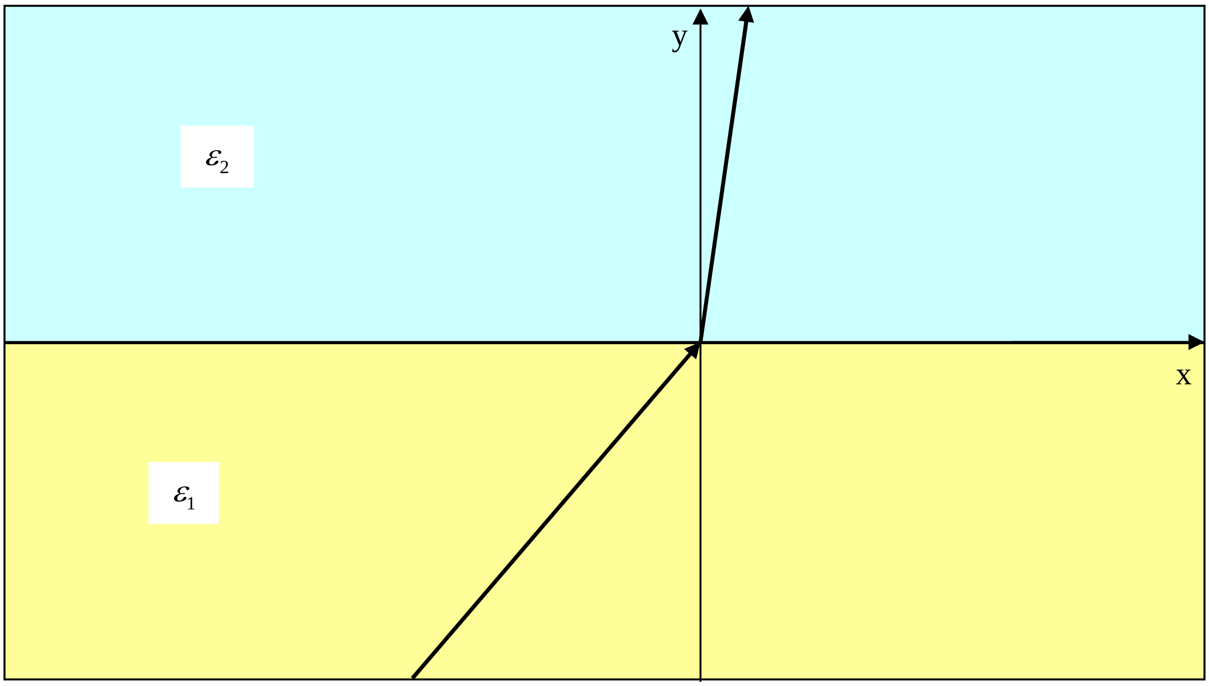
iii) $C_b = 3C_a$

iv) $C_b = (2/3)C_a$

v) $C_b = (3/2)C_a$ *The effective separation reduces to 2/3 of its previous value so the voltage drops while the charge stays the same.*

REGULAR QUESTIONS

6. Boundary Conditions (12 points)



The electric field in region 1 is $\vec{E}_1 = E_o\left(\hat{a}_x + \hat{a}_y\right)$. The electric field in region 2 is $\vec{E}_2 = E_o\left(\hat{a}_x + 5\hat{a}_y\right)$. Assuming that one of these regions is free space, what is the dielectric constant ε of the other region? Identify which region is free space (air), region 1 or region 2.

The upper region is free space while the lower region is the dielectric. The normal component for each region can be read from the given expressions.

$E_{n1} = 1$

$E_{n2} = 5$

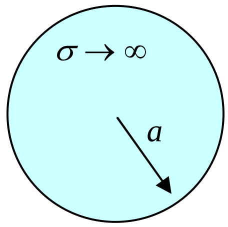
$D_{n1} = D_{n2}$

$\varepsilon_1 E_{n1} = \varepsilon_2 E_{n2}$

$\varepsilon_1 = 5\varepsilon_o$ *dielectric*

$\varepsilon_2 = \varepsilon_o$ *free space (air)*

7. Simple Problem (12 points)



A solid cylindrical perfect conductor with a radius of *a* is shown above. The surrounding medium is free space. There is a total charge Q_o per meter distributed on this conductor.

What is the charge density? (6)

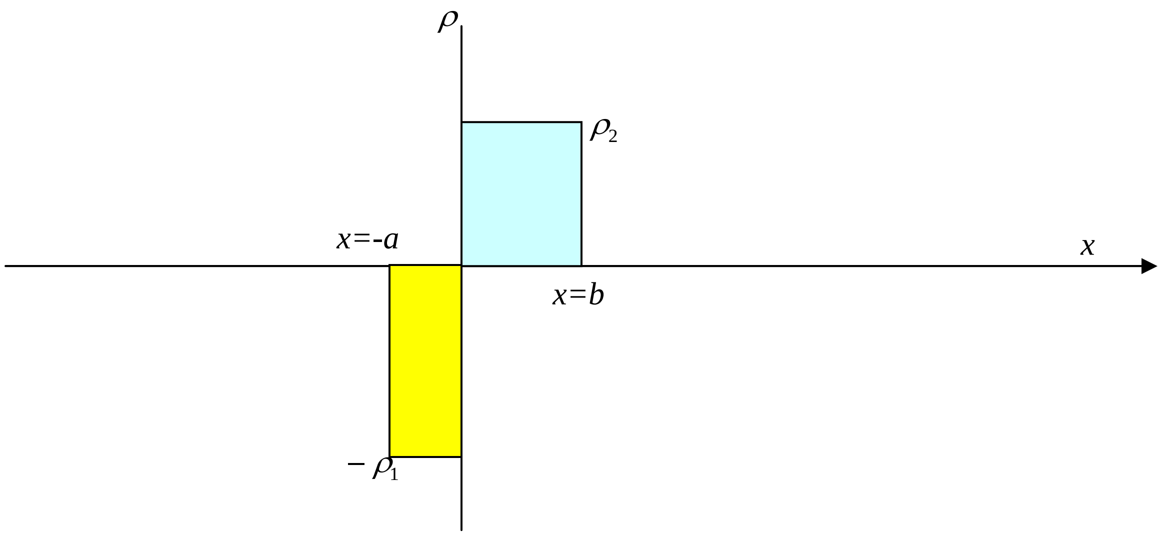
The total charge is distributed over the surface of the cylinder of unit length, thus the surface charge density must be $\frac{Q_o}{2\pi a}$

What is the electric field $\vec{E}$ everywhere (for all values of radius)? (6)

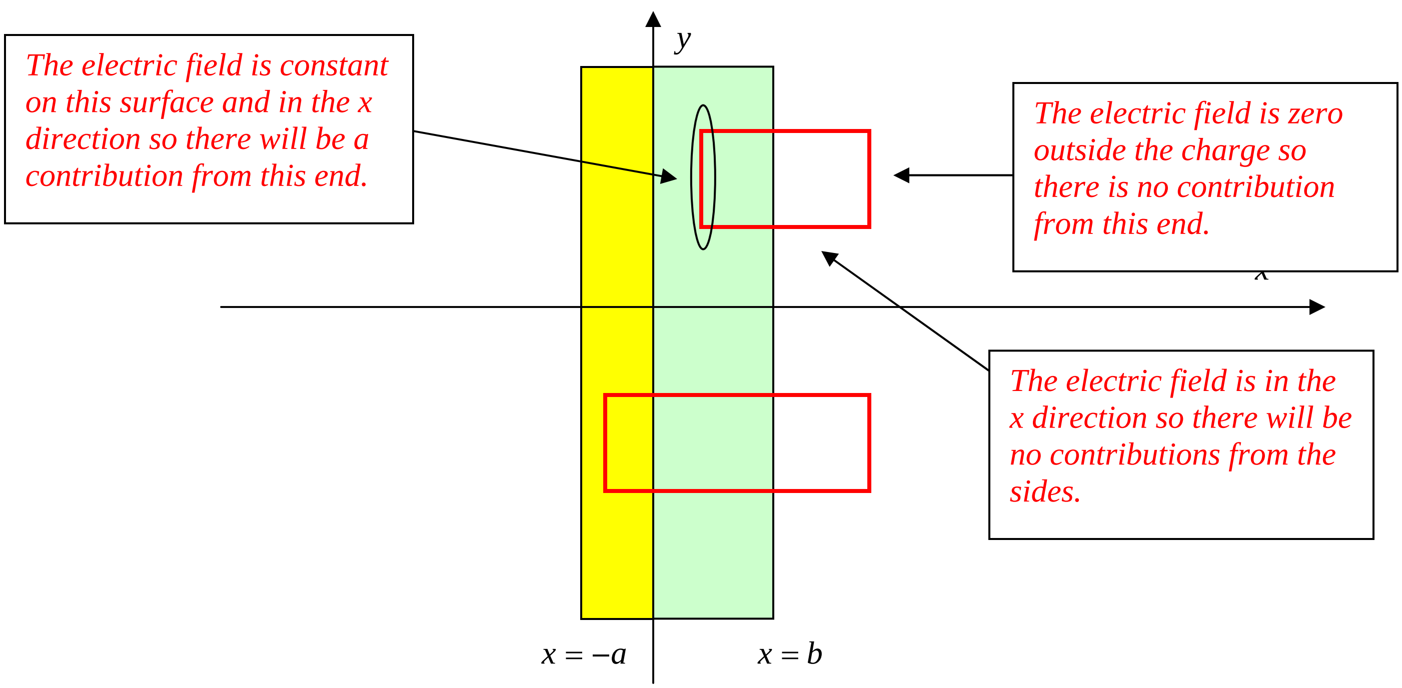
The electric field outside of the uniform cylindrical surface charge will look exactly like the field of a line charge. Thus, $\vec{E} = \frac{Q_o}{2\pi\varepsilon_o r}\hat{r}$ *which should be recognized as being identical to one of the answers for problem 3. Inside the cylinder the electric field is zero, which is also consistent with the potential being a constant in the conductor.* $\vec{E} = 0$

8. Gauss' Law (20 points)

In a planar region, there is a uniform negative volume charge density $\rho_v = -\rho_1$ for $-a \le x \le 0$ and a uniform positive volume charge density $\rho_v = \rho_2$ for $0 \le x \le b$



The total charge is zero. This configuration looks like the following in space.



Draw the figure you would use to apply Gauss' Law. Position your shape exactly where you intend to apply the integral equation. Describe why you chose that shape and location (i.e. how the field behaves on each surface, etc.). (5) *The field components will be in the x-direction so the surfaces should align partially with x. The sides not pointed in the x-direction will give no contribution. The field outside the charge will be zero since the total charge is zero. Thus, these surfaces have a contribution only on the left side.*

Determine the static electric field $\vec{E}$ everywhere. (10)

The left hand side of Gauss' Law will just be equal to the product of E_x and S, the area of the end. The sign should be minus since the electric field is in the negative x direction. The right hand side is equal to the charge enclosed. For x>0, the charge enclosed is just equal to (b-x)S times the charge density or $\rho_2(b-x)S$ *. For negative x, the charge enclosed will be* $\rho_1 xS + \rho_2 bS$ *where it is important to reiterate that x is negative. The electric field then is found by setting the two sides equal to one another. For x>0* $E_x = \frac{\rho_2(b-x)}{\varepsilon_o}$ *while for x<0* $E_x = \frac{\rho_1 x + \rho_2 b}{\varepsilon_o}$*. Note that for x = -a, the electric field will be zero as it should be outside of the charge distribution.*

Determine the voltage difference between $x = -a$ and $x = b$. (5)

Assuming that the voltage is zero on the left side (actually anywhere to the left of x = -a), then $V(x) - V(-a) = V(x) = -\int_{-a}^{x} \frac{\rho_1 x + \rho_2 b}{\varepsilon_o} dx = -\left(\frac{\rho_1 \left(x^2 - a^2\right)}{2\varepsilon_o} + \frac{\rho_2}{\varepsilon_o} b(x+a) \right)$ *for x<0*

and for x>0 $V(x) - V(0) = V(x) + \frac{\rho_1}{2\varepsilon_o} a^2 - \frac{\rho_2}{\varepsilon_o} ab = -\int_{0}^{x} \frac{\rho_2 (b-x)}{\varepsilon_o} dx = -\frac{\rho_2}{\varepsilon_o}\left(bx - \frac{x^2}{2} \right)$

The total voltage difference between -a and b is

$$V(b) - V(-a) = -\frac{\rho_1 a^2}{2\varepsilon_o} + \frac{\rho_2}{\varepsilon_o} ab - \frac{\rho_2}{\varepsilon_o}\left(b^2 - \frac{b^2}{2} \right) = -\frac{\rho_1 a^2}{2\varepsilon_o} + \frac{\rho_2}{\varepsilon_o} ab - \frac{\rho_2}{\varepsilon_o} \frac{b^2}{2}$$

9. Finite Difference – Capacitance and Resistance (15 points)

A set of electrodes consisting of two parallel plates with one or more very thin fins attached was analyzed using a spreadsheet resulting in the numbers given below. Remember that this configuration is two-dimensional. That is, it extends indefinitely into the page. Thus, for capacitance or resistance calculations, we are interested in per unit length values or the values for a 1 meter depth.

AG27

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA	AB	AC	AD	AE
1	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
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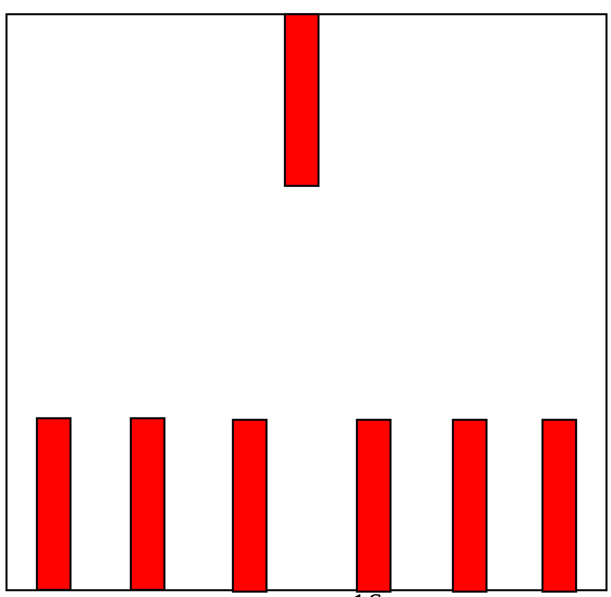
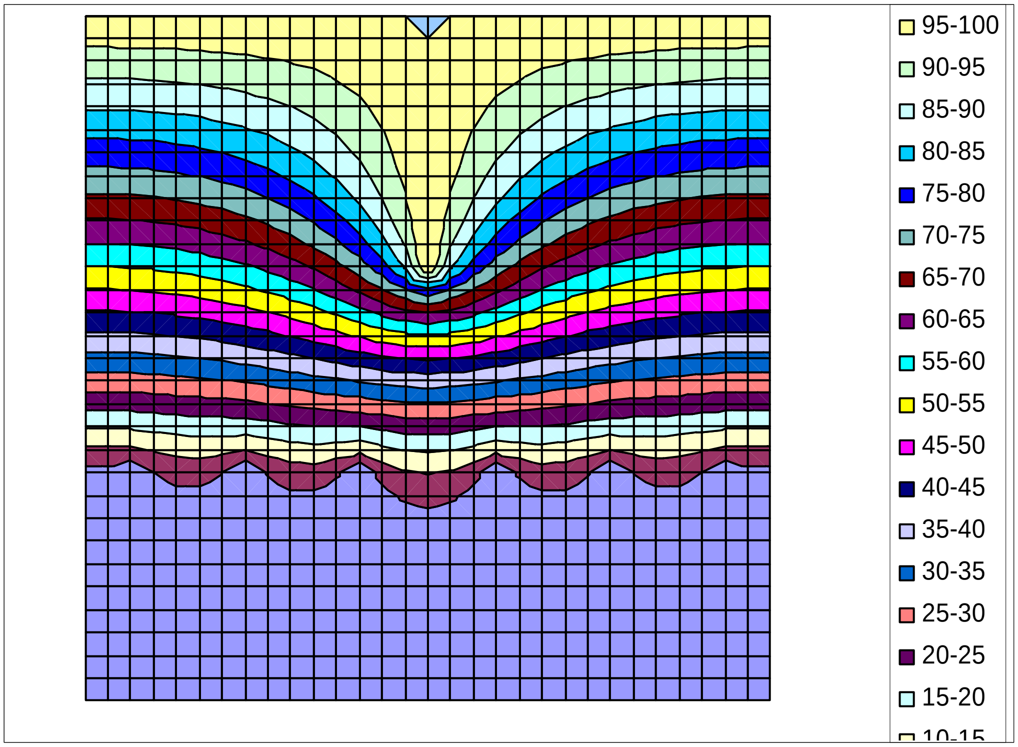
18	22	22	22	22	23	23	24	24	25	26	26	27	28	29	29	30	29	29	28	27	26	26	25	24	24	23	23	22	22	22	22
19	16	16	16	16	17	17	17	17	18	19	20	20	20	21	22	22	22	21	20	20	20	19	18	17	17	17	17	16	16	16	16
20	9.3	9.3	8.8	10	11	12	11	10	12	13	13	13	12	14	15	16	15	14	12	13	13	13	12	10	11	12	11	10	8.8	9.3	9.3
21	3.6	3.6	0	4.9	6.7	6.8	5.1	0	5.5	7.6	7.8	5.8	0	6.7	9.5	10	9.5	6.7	0	5.8	7.8	7.6	5.5	0	5.1	6.8	6.7	4.9	0	3.6	3.6
22	1.4	1.4	0	2.5	3.8	3.8	2.6	0	2.8	4.3	4.3	2.9	0	3.6	5.7	6.4	5.7	3.6	0	2.9	4.3	4.3	2.8	0	2.6	3.8	3.8	2.5	0	1.4	1.4
23	0.5	0.5	0	1.3	2.1	2.1	1.3	0	1.5	2.4	2.4	1.5	0	2.1	3.5	3.9	3.5	2.1	0	1.5	2.4	2.4	1.5	0	1.3	2.1	2.1	1.3	0	0.5	0.5
24	0.2	0.2	0	0.7	1.1	1.1	0.7	0	0.8	1.3	1.3	0.8	0	1.2	2.1	2.4	2.1	1.2	0	0.8	1.3	1.3	0.8	0	0.7	1.1	1.1	0.7	0	0.2	0.2
25	0.1	0.1	0	0.4	0.6	0.6	0.4	0	0.4	0.7	0.7	0.4	0	0.7	1.2	1.4	1.2	0.7	0	0.4	0.7	0.7	0.4	0	0.4	0.6	0.6	0.4	0	0.1	0.1
26	0	0	0	0.2	0.3	0.3	0.2	0	0.2	0.4	0.4	0.2	0	0.4	0.7	0.9	0.7	0.4	0	0.2	0.4	0.4	0.2	0	0.2	0.3	0.3	0.2	0	0	0
27	0	0	0	0.1	0.2	0.2	0.1	0	0.1	0.2	0.2	0.1	0	0.3	0.4	0.5	0.4	0.3	0	0.1	0.2	0.2	0.1	0	0.1	0.2	0.2	0.1	0	0	0
28	0	0	0	0.1	0.1	0.1	0.1	0	0.1	0.1	0.1	0.1	0	0.1	0.3	0.3	0.3	0.1	0	0.1	0.1	0.1	0.1	0	0.1	0.1	0.1	0.1	0	0	0
29	0	0	0	0	0.1	0.1	0	0	0	0.1	0.1	0	0	0.1	0.1	0.2	0.1	0.1	0	0	0.1	0.1	0	0	0	0.1	0.1	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.1	0.1	0.1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Since these numbers are hard to read, they are repeated on the next page for the left half of the configuration. Symmetry tells us that the numbers are a mirror image on the right half. Assume that the cell size in this case is quite small. Each cell is $1\text{x}10^{-6}$ meters in width and height. The insulator between the plates has $\varepsilon = 2\varepsilon_o$.

a. Carefully sketch 5 E field lines and 5 equipotentials on the diagram below. (5)



Using the spreadsheet, we can easily plot the equipotentials.



It is not easy to draw in the E field lines electronically, so you just have to sketch them in perpendicular to the equipotentials.

100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
96	96	96	97	97	97	97	97	97	98	98	98	99	99	100	100
93	93	93	93	93	93	94	94	95	95	96	97	97	98	99	100
89	89	89	89	90	90	91	91	92	93	94	95	96	97	99	100
85	85	86	86	86	87	87	88	89	90	91	93	94	96	98	100
82	82	82	82	82	83	84	85	86	87	89	91	93	95	97	100
78	78	78	78	79	79	80	81	82	84	86	88	91	94	97	100
73	73	74	74	75	75	76	77	79	81	83	85	88	92	96	100
69	69	69	70	70	71	72	73	75	77	79	82	86	90	95	100
65	65	65	65	66	67	68	69	71	73	75	79	82	87	93	100
60	60	60	60	61	62	63	64	66	68	71	74	78	83	91	100
55	55	55	56	56	57	58	59	61	63	66	69	73	78	86	100
50	50	50	50	51	52	53	54	56	58	60	63	66	70	74	78
45	45	45	45	46	46	47	49	50	52	53	56	58	61	64	65
39	39	39	40	40	41	42	43	44	45	47	49	51	52	54	55
33	33	34	34	34	35	36	37	38	39	40	42	43	44	45	46
28	28	28	28	29	29	30	30	31	32	33	34	35	36	37	37
22	22	22	22	23	23	24	24	25	26	26	27	28	29	29	30
16	16	16	16	17	17	17	17	18	19	20	20	20	21	22	22
9.3	9.3	8.8	10	11	12	11	10	12	13	13	13	12	14	15	16
3.6	3.6	0	4.9	6.7	6.8	5.1	0	5.5	7.6	7.8	5.8	0	6.7	9.5	10
1.4	1.4	0	2.5	3.8	3.8	2.6	0	2.8	4.3	4.3	2.9	0	3.6	5.7	6.4
0.5	0.5	0	1.3	2.1	2.1	1.3	0	1.5	2.4	2.4	1.5	0	2.1	3.5	3.9
0.2	0.2	0	0.7	1.1	1.1	0.7	0	0.8	1.3	1.3	0.8	0	1.2	2.1	2.4
0.1	0.1	0	0.4	0.6	0.6	0.4	0	0.4	0.7	0.7	0.4	0	0.7	1.2	1.4
0	0	0	0.2	0.3	0.3	0.2	0	0.2	0.4	0.4	0.2	0	0.4	0.7	0.9
0	0	0	0.1	0.2	0.2	0.1	0	0.1	0.2	0.2	0.1	0	0.3	0.4	0.5
0	0	0	0.1	0.1	0.1	0.1	0	0.1	0.1	0.1	0.1	0	0.1	0.3	0.3
0	0	0	0	0.1	0.1	0	0	0	0.1	0.1	0	0	0.1	0.1	0.2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0.1	0.1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

b. Using the information given, find the total charge per unit length on each plate. (5)

Ave Volts Near Top	95.28624855	
Average E	4713751.45	
Average D	9427502.901	
Ave Charge Density	9427502.901	
Area per Unit Length	0.000051	
Charge per Length	480.8026479	
Cap per Unit Length	4.808026479	in epso
	4.25123E-11	

The charge per unit length is 480 times epsilon. Actually any value between 450 and 550 times epsilon is fine.

c. Find the capacitance per unit length. (3)

The capacitance per unit length is 41nF/m. Again, a range is OK.

d. Determine the total electric flux $\psi_e = \int \vec{D} \cdot d\vec{S}$ leaving the 100V electrode. (2)

The flux leaving either electrode is equal to the charge on the electrode, from Gauss' Law. This is a correct response, but if one goes ahead and evaluates it from the information given, one finds a disagreement due to the fact that we are dealing with an approximate approach.

Flux in Middle	
Average E	6571710.924
Area	0.00003
Flux in Eps	394.3026554
Cap per Unit Length	3.943026554

Note that the answer is a bit smaller. This is probably more correct than the first solution since we are not using any numbers near sharp objects. Thus, in general, it is more accurate to find the flux well away from any physical objects. However, if one increases the resolution of the grid, then the results will come very close.

e. Now assume that the insulator is quite lossy, having a conductivity of $\sigma = 1x10^{-3}\ {}^{S}\!/_{m}$. Determine the resistance or conductance per unit length. (Extra Credit 4 Pts)

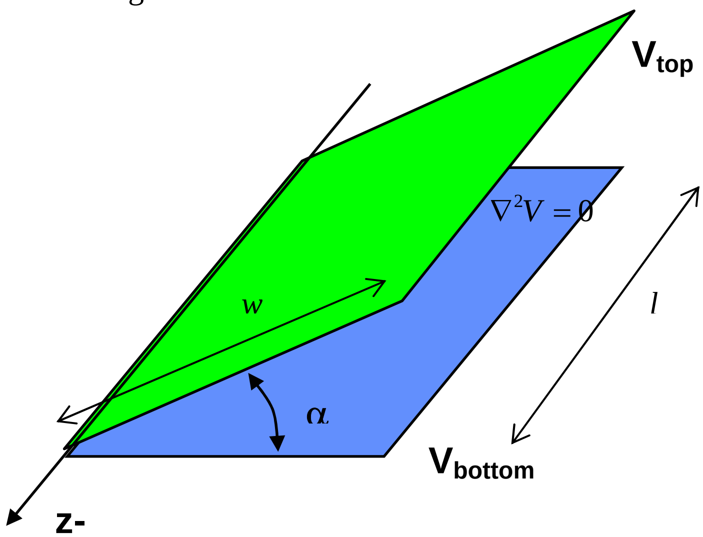
From the conductance analog of capacitance, one only needs to multiply the capacitance per unit length by $\frac{\sigma}{\varepsilon}$ to get the conductance per unit length.

Note: There are many ways to do this problem. We learned one way to find the capacitance and applied it in a homework assignment. Since there are many choices, you may want to consider using other methods to check your results. Also, for the resistance or conductance calculation, recall that you should be able to use the result you obtained for capacitance to make this a much easier question to answer.

10. Laplace's Equation, Capacitance (Extra Credit)

Note- For this problem, guessing is not permitted. If your answer to part a is incorrect, you will lose 5 pts for attempting this problem. Thus, you should be sure that you understand it.

a. In this problem, we will address the electric potential structure between two electrodes that form a wedge with the upper electrode at a voltage V_o and lower electrode grounded. The plates are *l* meters long and *w* meters wide.



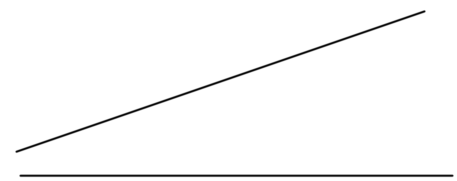
We assume that the angle between the two plates is small and the plates are very large. Thus, we can analyze this by assuming that $\frac{\partial}{\partial r} = \frac{\partial}{\partial z} = 0$. Using the given information, Laplace's Equation was solved for the voltage as a function of position. Unfortunately, the person who solved the problem mixed up the solution to three similar problems and now cannot recall which is the correct solution for this problem. The three solutions are as follows: $V(r,\phi,z) = V_o \frac{r\phi}{w\alpha}$ $\quad V(r,\phi,z) = V_o \frac{\phi}{\alpha}$ $\quad V(r,\phi,z) = V_o \frac{\phi^2}{\alpha}$

Circle the correct solution and then show that it does indeed satisfy Laplace's equation and the given boundary conditions. (5 Pts)

The Laplacian for cylindrical coordinates has only one term that depends on the angle ϕ. Applying this expression, we see that the Laplacian is equal to zero as it must be.

$$\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}V(r,\phi,z) = \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}V_o\frac{\phi}{\alpha} = 0$$

b. From your solution for $V(\vec{r})$, determine the electric field in the region between the plates $\vec{E}(\vec{r})$. Note that the electric field will become infinite if the plates almost touch. Thus, we assume that the plates do not go all the way to $r = 0$, but rather start at a small value of $r = a$. (5 Pts)



$$\vec{E} = -\nabla V(r,\phi,z) = -\frac{1}{r}\frac{\partial}{\partial\phi}V_o\frac{\phi}{\alpha}\hat{\phi} = -\frac{1}{r}V_o\frac{1}{\alpha}\hat{\phi}$$

The electric field would blow up at r = 0 if our solution was to hold at that location.

c. From your solution for $\vec{E}(\vec{r})$ find the charge on one of the plates and the capacitance of this structure. (5 Pts)

To get the charge on one of the plates, convert E to D. Since D is only normal it is equal to the charge density and one only needs to integrate it over the surface to get the charge.

$$Q = \int \rho_s dS = \int D_\phi dS = \int \varepsilon_o E_\phi dS = -l\int_a^w \frac{1}{r}V_o\frac{\varepsilon_o}{\alpha}dr = lV_o\frac{\varepsilon_o}{\alpha}\ln\frac{w}{a}$$

$$C = l\frac{\varepsilon_o}{\alpha}\ln\frac{w}{a}$$