# Fields and Waves I

## Lecture 2

Sine Waves on Transmission Lines

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Materials from other sources are referenced where they are used.
Those listed as Ulaby are figures from Ulaby's textbook.

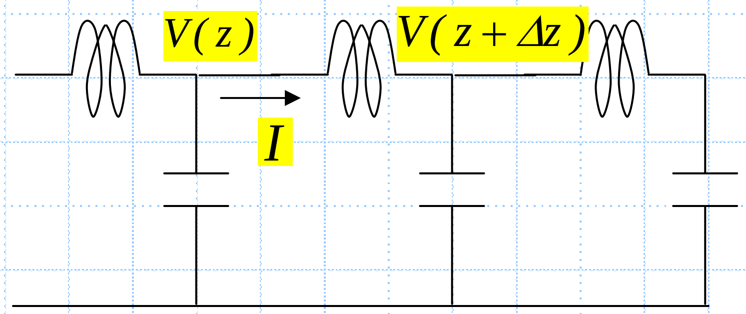
# Overview

Henry Farny *Song of the Talking Wire*

- Review
- Voltages and Currents on Transmission Lines
- Standing Waves
- Input Impedance
- Lossy Transmission Lines
- Low Loss Transmission Lines

# Transmission Line Representation

As $\Delta z \Rightarrow 0$ limit




$$V(z+\Delta z)-V(z)=-L\frac{\partial I}{\partial t}$$

$l\cdot\Delta z$

$$-l\cdot\frac{\partial I}{\partial t}=\frac{\Delta V}{\Delta z}=\frac{\partial V}{\partial z}$$

# Transmission Line Representation

Similarly, $\frac{\partial I}{\partial z} = -c \cdot \frac{\partial V}{\partial t}$ looks like $\frac{\partial V}{\partial z} = -l \cdot \frac{\partial I}{\partial t}$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial z}(-l \cdot \frac{\partial I}{\partial t}) = -l \cdot \frac{\partial}{\partial t}(\frac{\partial I}{\partial z}) = lc\frac{\partial^2 V}{\partial t^2}$$

Obtain the following PDE: $\frac{\partial^2 V}{\partial z^2} = lc\frac{\partial^2 V}{\partial t^2}$

Solutions are: $f(t \pm \frac{z}{u})$

These are functions that move with velocity ***u***

# Transmission Line Representation

$\beta = \frac{\omega}{u}$

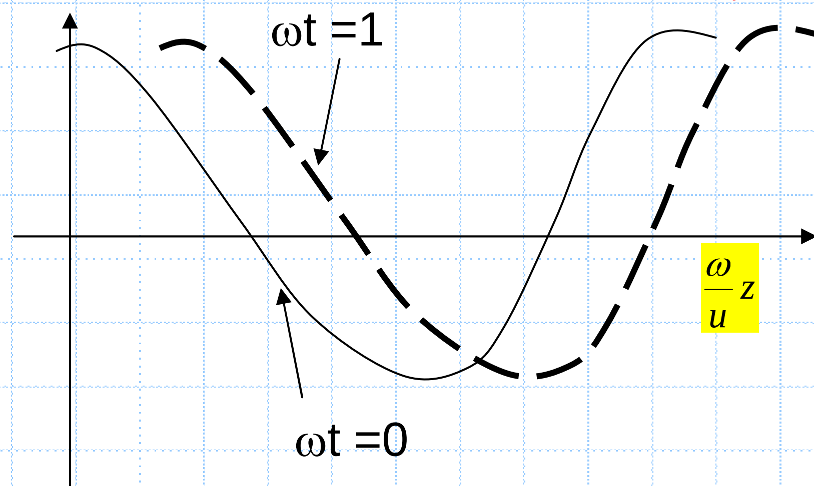
Functions that move with velocity ***u***

$V = V^{+}\cos(\omega t - \beta z) + V^{-}\cos(\omega t + \beta z)$

Example: $cos(\omega t \pm \frac{\omega}{u} z)$

At t=0, $cos(-\frac{\omega}{u} z)$ Wave moving to the right

At $\omega$t =1 $cos(1 - \frac{\omega}{u} z)$



## Workspace – look at the general form of the solution

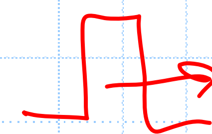
$$\frac{\partial^2 f}{\partial z^2} = lc\,\frac{\partial^2 f}{\partial t^2}$$

$$f\left(t \pm \frac{z}{u}\right)$$

$$f\left(t - \frac{z}{u}\right)$$

$$\frac{\partial f}{\partial z} = f'\left(-\frac{1}{u}\right)$$

$$\frac{\partial f}{\partial t} = f'$$



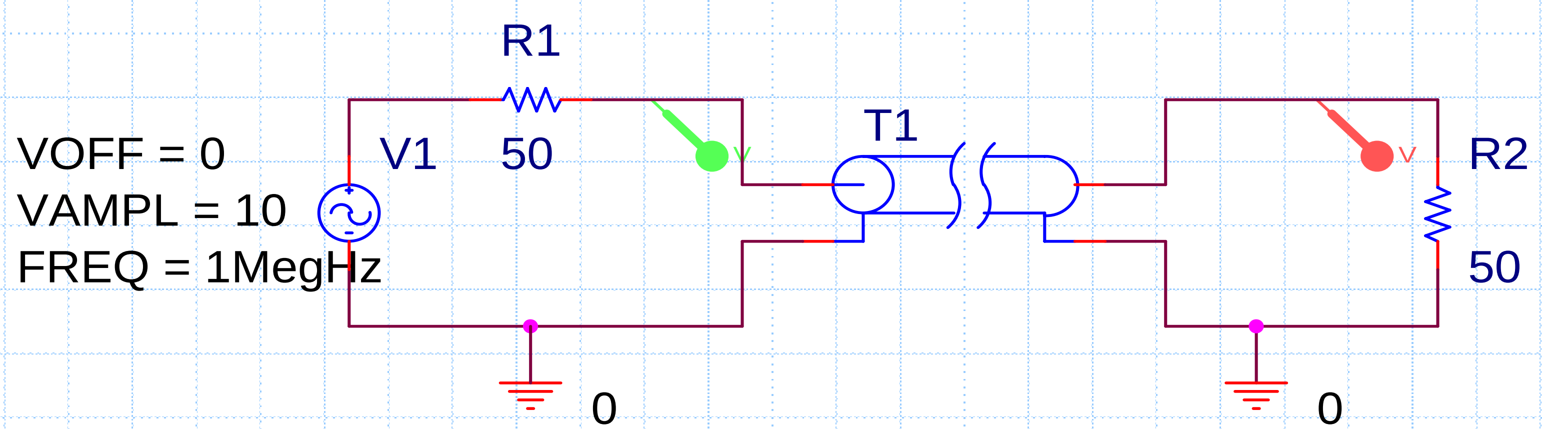

$$\frac{\partial^2 f}{\partial z^2} = f''\left(\frac{1}{u^2}\right)$$

$$\frac{\partial^2 f}{\partial t^2} = f''$$

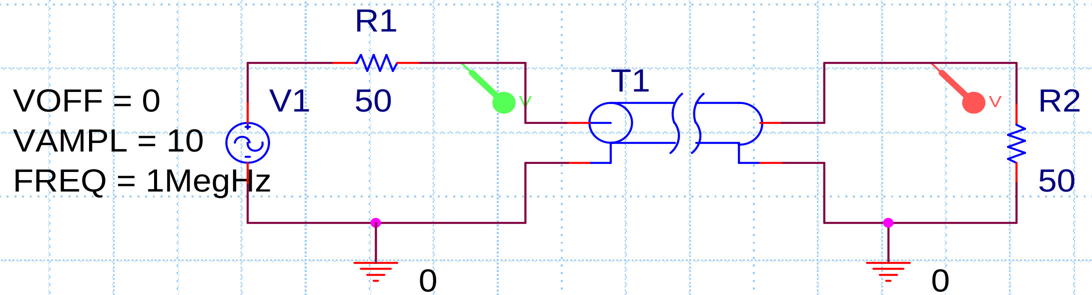
$$\frac{1}{u^2} = lc \quad \text{or} \quad u = \frac{1}{\sqrt{lc}}$$

# Some Numerical Experiments

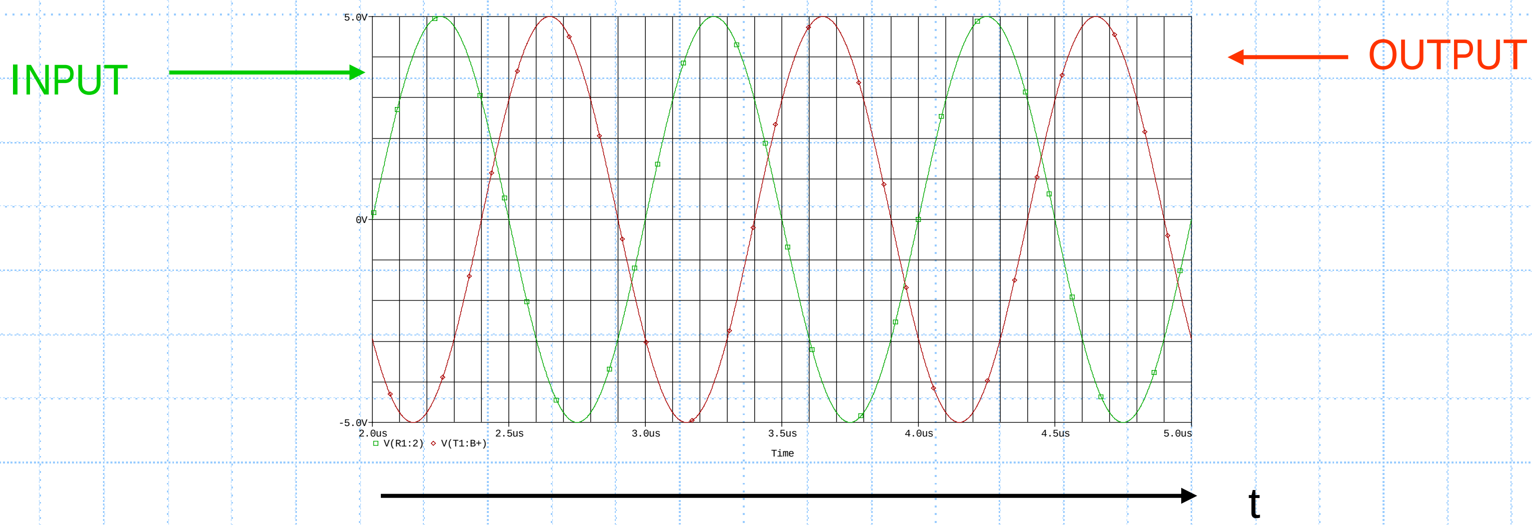
- PSpice can be used to do simple numerical experiments that demonstrate how transmission lines work



# PSpice

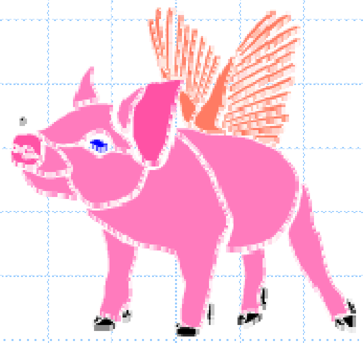


R1
50
V1
VOFF = 0
VAMPL = 10
FREQ = 1MegHz
T1
R2
50
0
0

INPUT
OUTPUT
5.0V
0V
-5.0V
2.0us
2.5us
3.0us
3.5us
4.0us
4.5us
5.0us
V(R1:2)
V(T1:B+)
Time
t

# Computer-Based Tools



- When you use a program like PSpice,  applets, or any handy tools available online … remain skeptical.
- Do not assume that the answers are correct.
- Apply crude plausibility checks.
- Know the assumptions and limitations of the tools you are using.
- Test all tools on problems you can solve other ways or with tools you have already tested.
- Use even sometimes incorrect tools as long as errors are recognized.

Pig from http://www.cincinnatiskeptics.org

# PSpice Example

- Let us return to the configuration shown above and simulate it using PSpice
- List some conclusions from this exercise.
  - ?
  - ?
  - ?

# Sine Waves

- The form of the wave solution

$$A\cos\left(\omega t \mp \frac{\omega}{u} z\right) = A\cos(\omega t \mp \beta z)$$

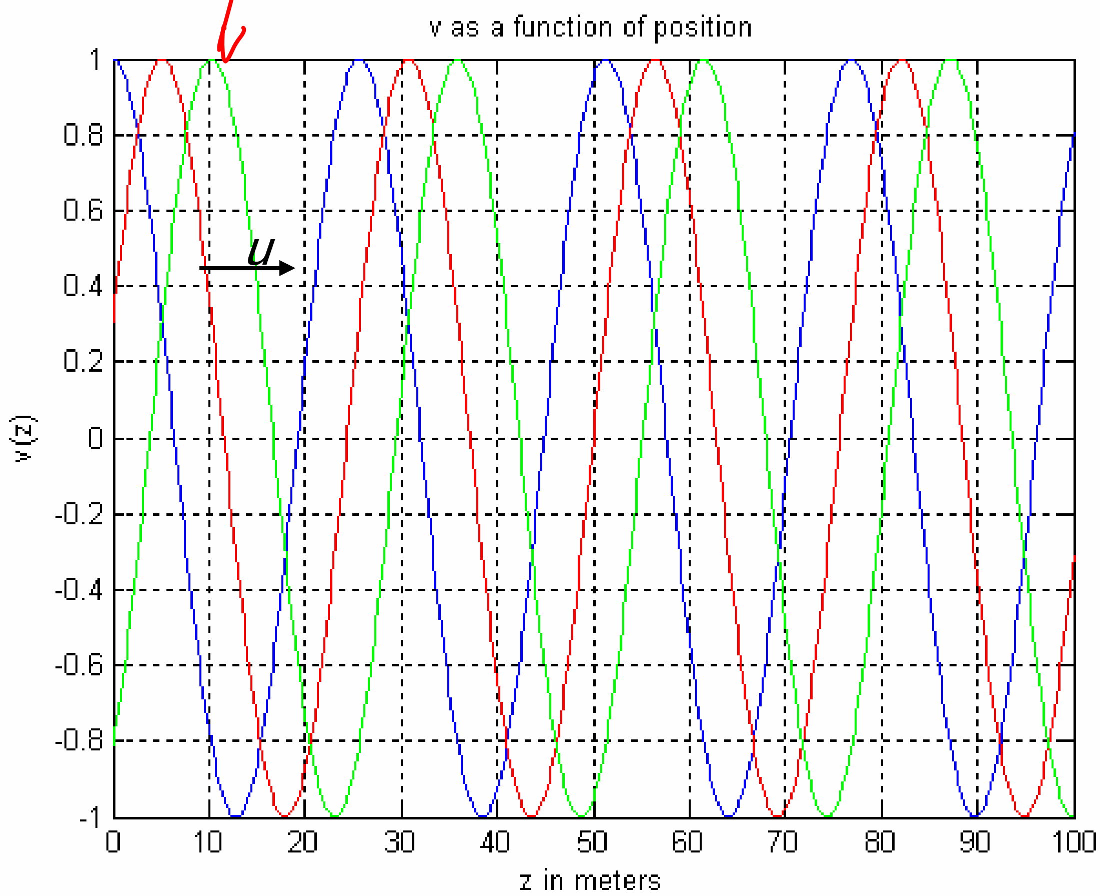
- First check to see that these solutions have the properties we expect by plotting them using a tool like Matlab

# Sine Waves

Apply frequency and wavelength analogy argument to show this is reasonable

$$\cos\left(\omega t - \frac{\omega}{u} z\right) = \cos\left(2\pi f t - \frac{2\pi f}{u} z\right)$$

- The positive wave



# Solutions to the Wave Equation

- Thus, our sine wave is a solution to the voltage or current equation

$$\frac{\partial^2 V}{\partial z^2} = lc \frac{\partial^2 V}{\partial t^2}$$

- if $\beta = \frac{\omega}{u} = \omega\sqrt{lc}$ or $u = \frac{1}{\sqrt{lc}}$ $u = \frac{\omega}{\beta}$

- *u = the speed of wave propagation = the speed of light*

## Sine Waves

$$\cos\left(\omega t - \frac{\omega}{u}z\right) = \cos\left(2\pi f t - \frac{2\pi f}{u}z\right)$$

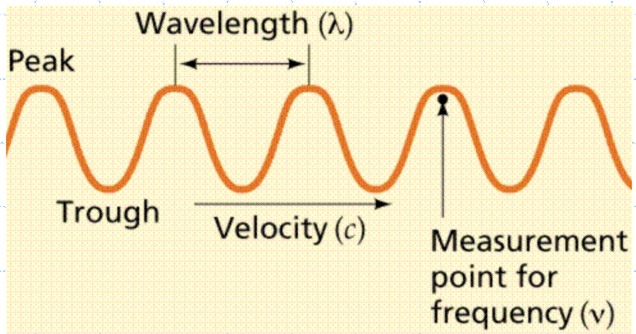
- Consider one other property. What is the distance required to change the phase of this expression by $2\pi$ ? We just did this qualitatively.
- This distance is called the wavelength or

$$\beta z = \frac{\omega}{u}z = \frac{2\pi f}{u}z = 2\pi$$

$$\beta\lambda = \frac{\omega}{u}\lambda = \frac{2\pi f}{u}\lambda = 2\pi$$

$$\lambda = \frac{2\pi}{\beta} = \frac{u}{f}$$

# Sine Waves -- Summary




- Solutions look like $A\cos(\omega t \mp \beta z)$

$$\beta = \frac{\omega}{u} = \omega\sqrt{lc} = \omega\sqrt{\mu\varepsilon} = \frac{2\pi}{\lambda}$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

$$u = \frac{1}{\sqrt{lc}} = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{u}{f}$$

$$\varepsilon = \varepsilon_r \varepsilon_o \qquad \mu = \mu_r \mu_o$$

Figure from http://www.emc.maricopa.edu/

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

# Phasor Notation

- For ease of analysis (changes second order partial differential equation into a second order ordinary differential equation), we use phasor notation.

$$f(z,t) = A\cos(\omega t \mp \beta z) = \mathrm{Re}\left(\left\{Ae^{\mp j\beta z}\right\}e^{j\omega t}\right)$$

- The term in the brackets is the phasor.

$$f(z) = Ae^{\mp j\beta z}$$

$$V = V^{+}e^{-j\beta z} + V^{-}e^{+j\beta z}$$

# Phasor Notation

- To convert to space-time form from the phasor form, multiply by $e^{j\omega t}$ and take the real part.

$$f(z,t) = \mathrm{Re}(Ae^{\mp j\beta z}e^{j\omega t}) = A\cos(\omega t \mp \beta z)$$

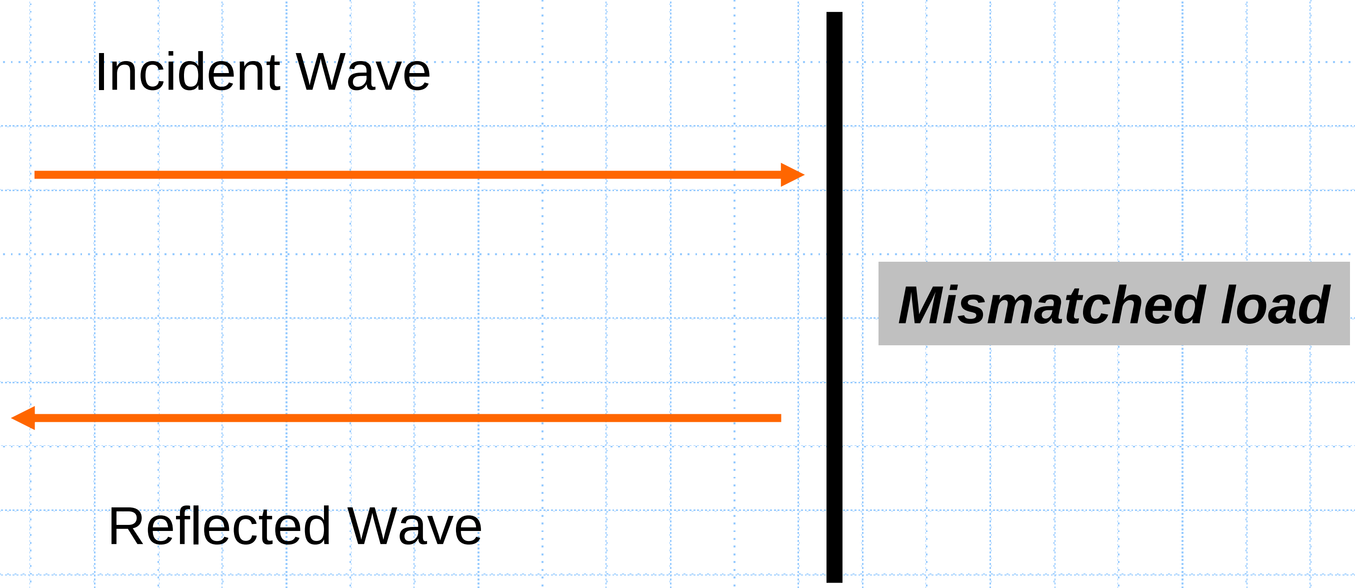
- If A is complex

$$A = |A|e^{j\theta_A}$$

$$f(z,t) = \mathrm{Re}(|A|e^{j\theta_A}e^{\mp j\beta z}e^{j\omega t}) = |A|\cos(\omega t \mp \beta z + \theta_A)$$

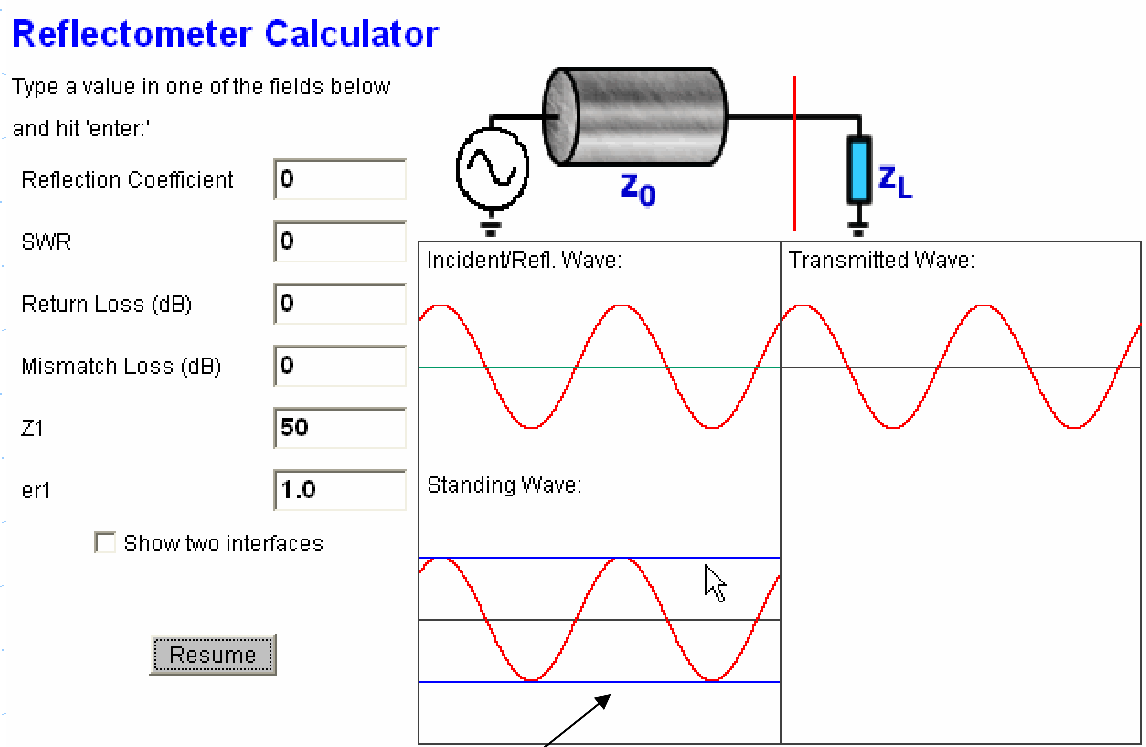
$$\mathrm{Re}\, e^{j\theta} = \cos\theta$$

# Transmission Lines




Standing wave due to interference

# Standing Waves



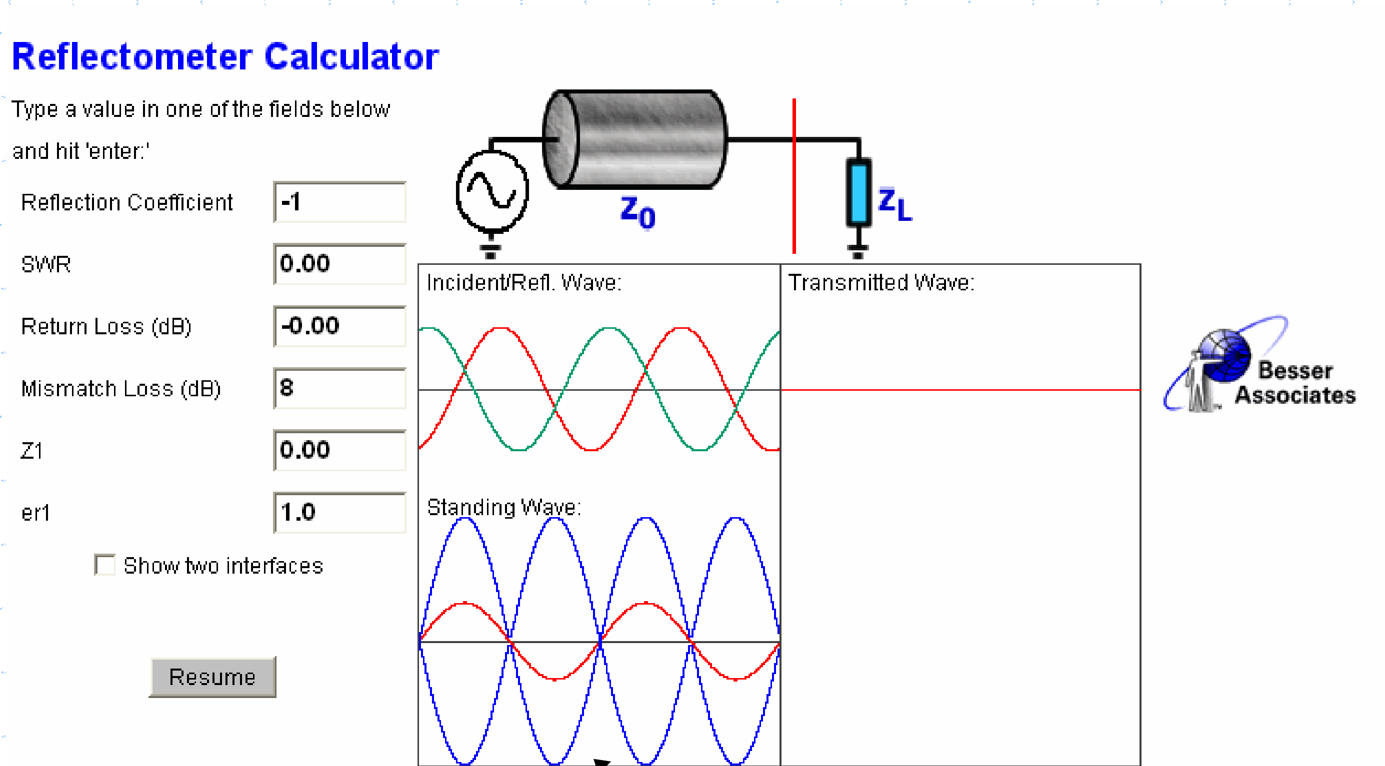

Besser Associates

http://www.bessernet.com

No Standing Wave

[Besser Associates](http://www.bessernet.com)

# Standing Waves



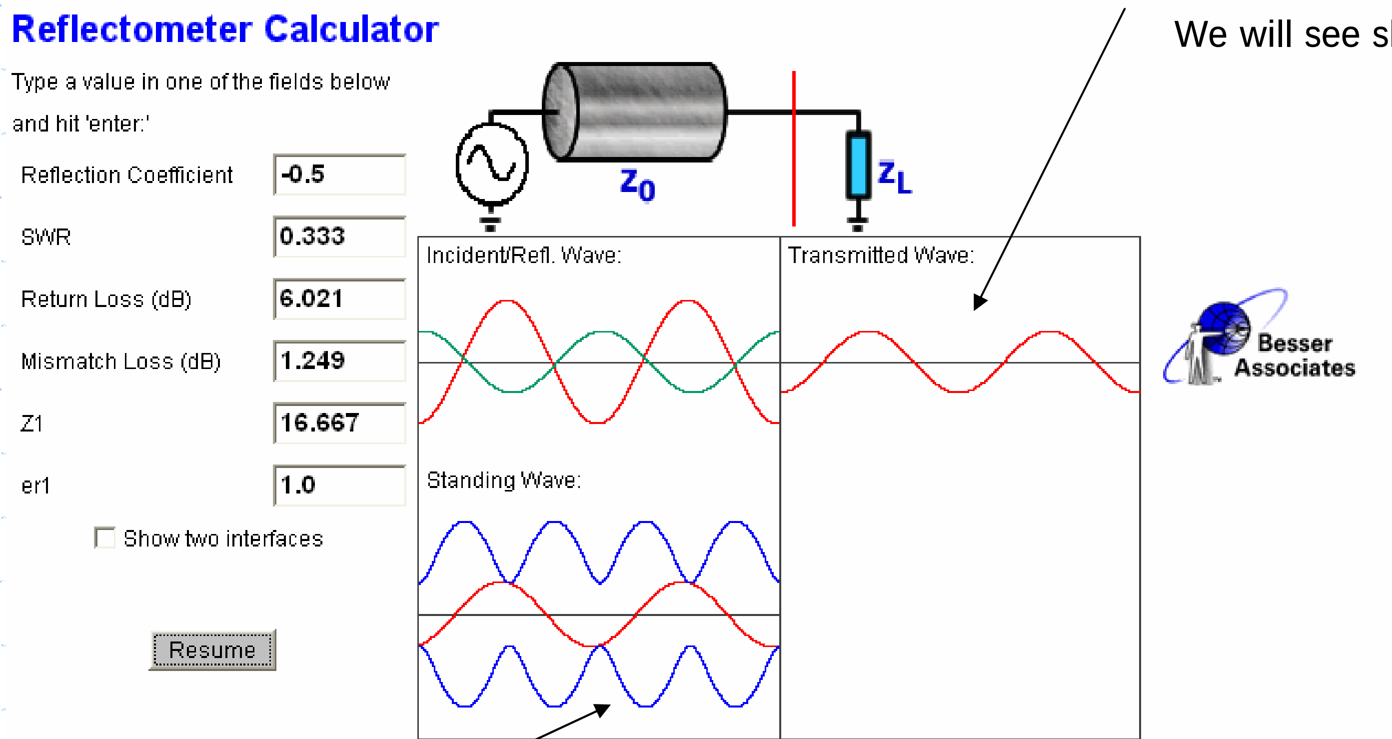

Standing Wave

Besser Associates

# Standing Waves

This may be wrong

We will see shortly




Standing Wave

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# Standing Waves

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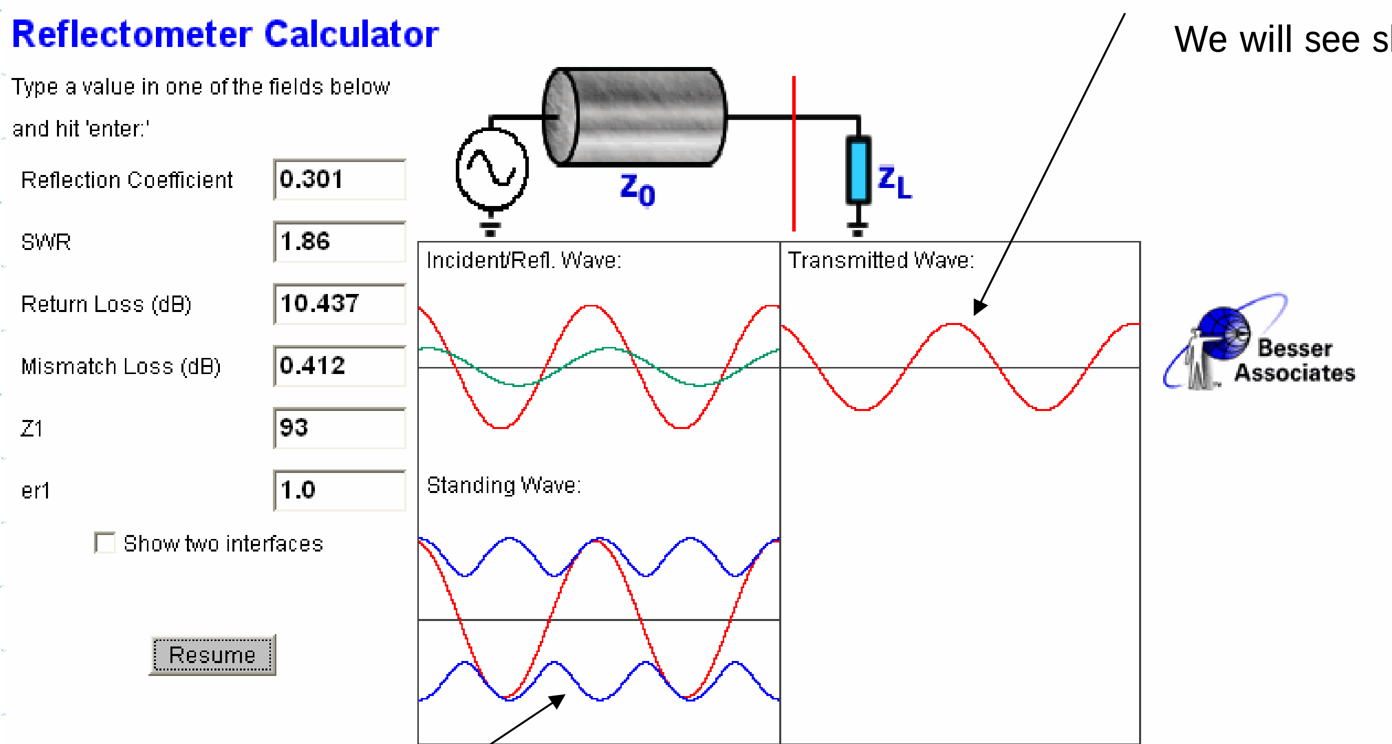
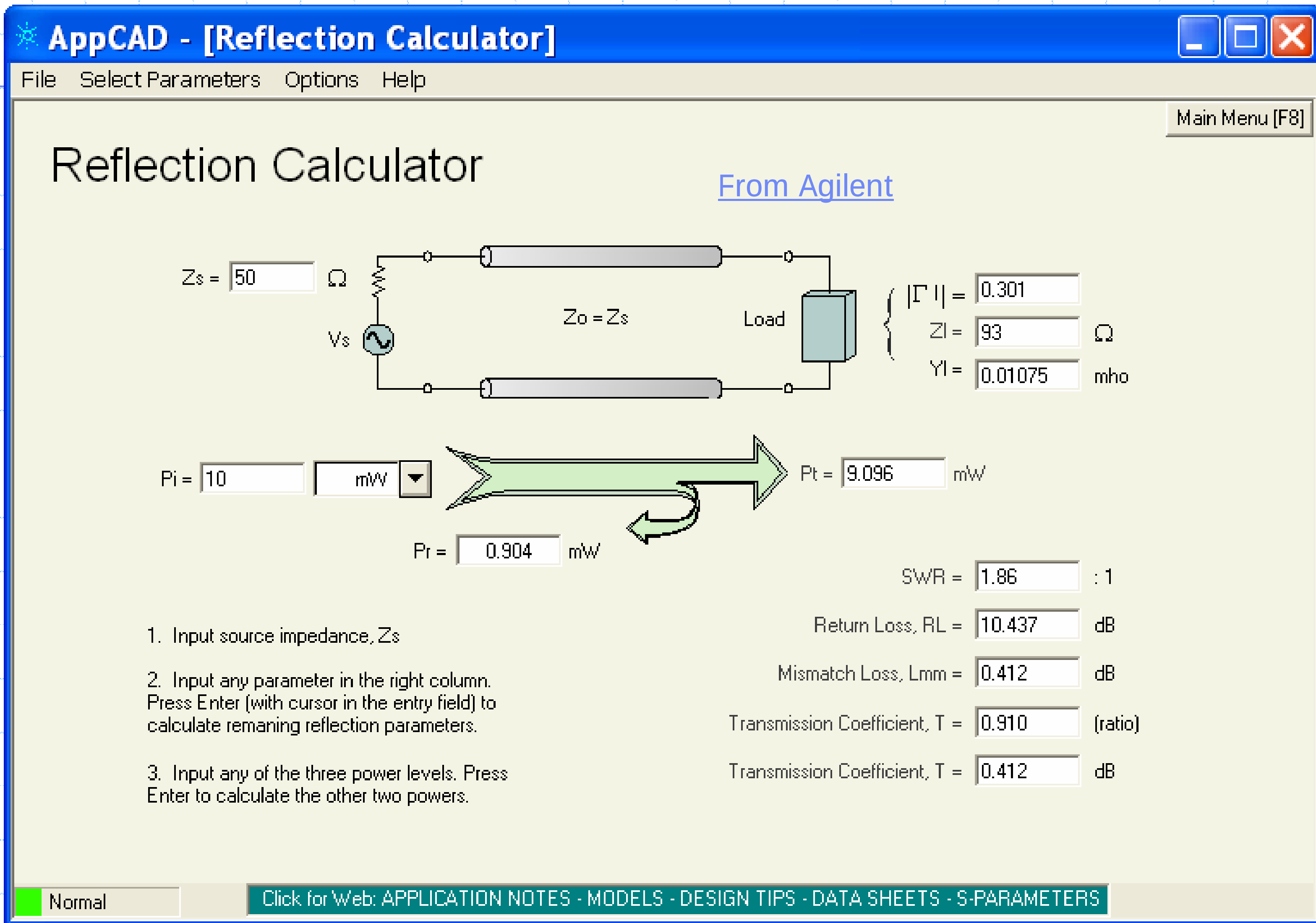



Standing Wave

[Besser Associates]

# Standing Waves



$\beta = \omega\sqrt{lc}$

# Transmission Lines - Standing Wave Derivation

$(\omega t - \beta z)$

$e^{j\omega t}$

Phasor Form of the Wave Equation:

$$\frac{\partial^2 V}{\partial z^2} = l \cdot c \cdot \frac{\partial^2 V}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 V}{\partial z^2} = -\omega^2 \cdot l \cdot c \cdot V$$

where:

$$V = V^{\mp} \cdot e^{\pm j \cdot \beta \cdot z}$$

lossless

General Solution:

$$V = V^{+} e^{-j \cdot \beta \cdot z} + V^{-} e^{+j \cdot \beta \cdot z}$$

Starting Pt

# Workspace

$$\beta^2 = \omega^2 lc$$

$$\frac{\partial^2 \tilde{V}}{\partial t^2} = -\omega^2 lc\,\tilde{V} \qquad \frac{\partial}{\partial t} \rightarrow j\omega$$

$$\tilde{V} = \tilde{V}^+ e^{-j\beta z} + \tilde{V}^- e^{+j\beta z}$$

$$\frac{\partial \tilde{V}}{\partial z} = \tilde{V}^+(-j\beta)e^{-j\beta z}$$

$$\frac{\partial^2 \tilde{V}}{\partial z^2} = \tilde{V}^+(-j\beta)^2 e^{-j\beta z} = -\tilde{V}^+\beta^2 e^{-j\beta z}$$

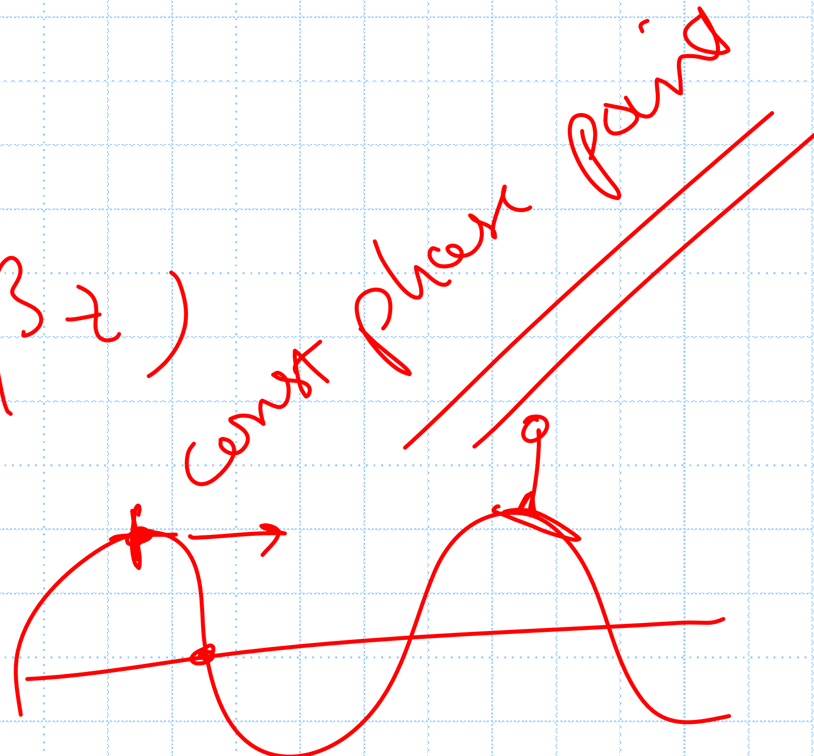
$$= -\beta^2 \tilde{V}$$

# Workspace

$$\cos(\omega t - \beta z)$$

$$\frac{d}{dt}(\omega t - \beta z) = \omega - \beta \frac{dz}{dt} = 0$$

$$\frac{dz}{dt} = \frac{\omega}{\beta}$$



## Transmission Lines - Standing Wave Derivation

$$V = V^{+} e^{-j\cdot\beta\cdot z} + V^{-} e^{+j\cdot\beta\cdot z}$$

Forward Wave

$$cos(\omega \cdot t - \beta \cdot z)$$

Backward Wave

$$cos(\omega \cdot t + \beta \cdot z)$$

*TIME DOMAIN*

$V_{max}$ occurs when Forward and Backward Waves are in Phase

→ *CONSTRUCTIVE INTERFERENCE*

$V_{min}$ occurs when Forward and Backward Waves are out of Phase

→ *DESTRUCTIVE INTERFERENCE*

# Transmission Lines Formulas

matched $V^- = 0$

$$\frac{V(z)}{i(z)} = Z_o$$

- Fields and Waves I Quiz Formula Sheet
- In the class notes

$$v(z) = V_+ e^{-j\beta z} + V_- e^{+j\beta z}$$

$$i(z) = \frac{V_+ e^{-j\beta z} - V_- e^{+j\beta z}}{Z_o} = \frac{V_+}{Z_o} e^{-j\beta z} - \frac{V_-}{Z_o} e^{+j\beta z}$$

- Note: $V_+ = V^+ = V_m^+ \qquad V_- = V^- = V_m^-$

All are used in various handouts, texts, etc. There is no standard notation.

RG58U $Z_o = 50\Omega$

# RG58/U Cable

$$\frac{2\pi(1.5\times10^{6})}{2\times10^{8}}$$

- Assume 2 $V_{P\text{-}P}$ 1.5MHz sine wave is launched on such a line. Find $\beta = \dfrac{\omega}{u} = \omega\sqrt{lc} = \omega\sqrt{\mu\varepsilon} = \dfrac{2\pi}{\lambda}$ and $\lambda$

$$u = \frac{2}{3}c \triangleq 2\times10^{8}\ \text{m/s}$$

- Answers?

$$l = .25\times10^{-6}\ \text{H/m}$$

$$\frac{3\pi}{200}$$

$$c = 100\times10^{-12}\ \text{F/m}$$

$$\cos(\omega t - \beta z)$$

$$\beta z = 2\pi$$

$$z = \lambda$$

# Reflection Coefficient Derivation

$$V^- = \Gamma_L V^+$$

Define the Reflection Coefficient:

$$\left|V_m^-\right| = \left|\Gamma_L\right| \cdot \left|V_m^+\right|$$

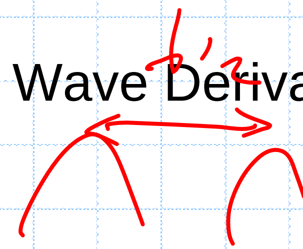
Maximum Amplitude when in Phase: $V_{max} = \left|V_m^+\right| + \left|V_m^-\right|$

$$\therefore V_{max} = \left|V_m^+\right| \cdot \left(1 + \left|\Gamma_L\right|\right)$$

Similarly: $V_{min} = \left|V_m^+\right| \cdot \left(1 - \left|\Gamma_L\right|\right)$

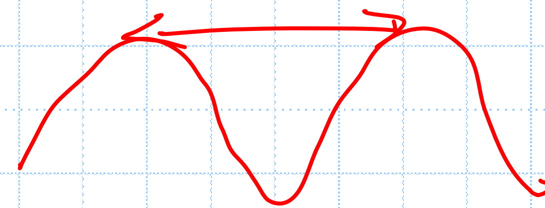
Standing Wave Ratio (SWR) = $\dfrac{V_{max}}{V_{min}} = \dfrac{1+\left|\Gamma_L\right|}{1-\left|\Gamma_L\right|}$ $= \dfrac{V^+(1+|\Gamma_L|)}{V^+(1-|\Gamma_L|)}$

# Transmission Lines - Standing Wave Derivation



Distance between Max and Min is $\lambda/2$ ⇦ *Show this*

Assume $V^{\pm}$ are real (will be complex if the load is complex)



Forward Phase is = $-j \cdot \beta \cdot z$

Backward Phase is = $+j \cdot \beta \cdot z$

Difference in Phase is = $-2 \cdot j \cdot \beta \cdot z$

Varies by $2\pi$ (distance between maxima)

$$\therefore 2 \cdot \beta \cdot \Delta z = 2 \cdot \pi \quad \Rightarrow \quad \Delta z = \frac{\pi}{\beta} = \frac{\pi}{2 \cdot \pi / \lambda} = \frac{\lambda}{2}$$

# Reflection Coefficient Derivation

Let z=0 at the LOAD

$$\Rightarrow V_{load} = V^{+} \cdot e^{-j\cdot\beta\cdot 0} + V^{-} \cdot e^{+j\cdot\beta\cdot 0}$$

$$= V^{+} + V^{-}$$

$$= V^{+} \cdot (1 + \Gamma_L)$$

Need a relationship between current and voltage:

$$\frac{\partial V}{\partial z} = -l \cdot \frac{\partial I}{\partial t} \qquad \Rightarrow \frac{\partial V}{\partial z} = -j \cdot l \cdot \omega \cdot I$$

# Reflection Coefficient Derivation

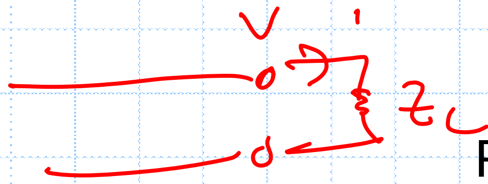
$$I = -\frac{1}{j\cdot\omega\cdot l}\cdot\frac{\partial V}{\partial z}$$

$$= -\frac{1}{j\cdot\omega\cdot l}\cdot(-j\cdot\beta\cdot V^{+}\cdot e^{-j\cdot\beta\cdot z} + j\cdot\beta\cdot V^{-}\cdot e^{+j\cdot\beta\cdot z})$$

$$= \frac{\beta}{\omega\cdot l}\cdot(V^{+}\cdot e^{-j\cdot\beta\cdot z} - V^{-}\cdot e^{+j\cdot\beta\cdot z})$$

$$\therefore I = \frac{\omega\cdot\sqrt{l\cdot c}}{\omega\cdot l}\cdot(V^{+}\cdot e^{-j\cdot\beta\cdot z} - V^{-}\cdot e^{+j\cdot\beta\cdot z})$$

$$= \frac{V^{+}}{\sqrt{\frac{l}{c}}}\cdot e^{-j\cdot\beta\cdot z} - \frac{V^{-}}{\sqrt{\frac{l}{c}}}\cdot e^{+j\cdot\beta\cdot z} \qquad = \frac{V^{+}}{\sqrt{\frac{l}{c}}}\cdot e^{-j\cdot\beta\cdot z} - \frac{V^{+}\cdot\Gamma_L}{\sqrt{\frac{l}{c}}}\cdot e^{+j\cdot\beta\cdot z}$$




# Reflection Coefficient Derivation

$V(z)$
$i(z)$

At LOAD: $$\frac{V}{I} = Z_L$$

$$\frac{V(\text{load})}{i(\text{load})} = Z_L$$

*Use derived terms of V and I at z=0 (position of the LOAD)*

$e^{-j\beta z}$

$e^{+j\beta z}$

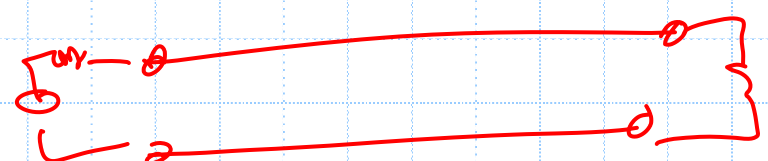
$$\left( \frac{V^+}{\sqrt{l/c}} - \frac{V^+ \cdot \Gamma_L}{\sqrt{l/c}} \right)^{-1} \cdot \left( V^+ + \Gamma_L \cdot V^+ \right) = Z_L$$

$$Z_L = \sqrt{\frac{l}{c}} \cdot \left( \frac{1 + \Gamma_L}{1 - \Gamma_L} \right)$$

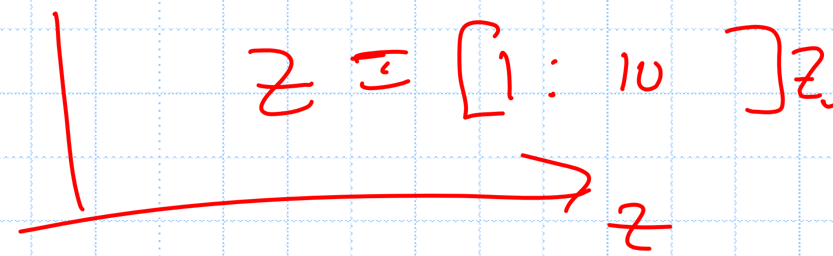
*Note that* $Z_0 = \sqrt{\dfrac{l}{c}}$

OR

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$



# Short Circuit Load




- For $Z_L=0$, we have $\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{0 - Z_o}{0 + Z_o} = -1$

$$v(z) = V^+ e^{-j\beta z} + \Gamma_L V^+ e^{+j\beta z} = V^+\left(e^{-j\beta z} - e^{+j\beta z}\right)$$

$$e^{+j\beta z} = \cos\beta z + j\sin\beta z$$

$$e^{-j\beta z} = \cos\beta z - j\sin\beta z$$

$$\Rightarrow \quad v(z) = -V^+\left(j2\sin\beta z\right)$$

Standing Wave

# Short Circuit Load

- Convert to space-time form

$$v(z,t) = \mathrm{Re}\left(v(z)e^{j\omega t}\right) = \mathrm{Re}\left(V^{+}\left(-j2\sin\beta z\right)e^{j\omega t}\right)$$

$$\mathrm{Re}\left(\left(-j2\sin\beta z\right)e^{j\omega t}\right) = \mathrm{Re}\left(-2\sin\beta z\left(j\cos\beta z - \sin\beta z\right)\right)$$

$$v(z,t) = 2V^{+}\sin\beta z\sin\omega t$$

- This is a standing wave

# Short Circuit Load

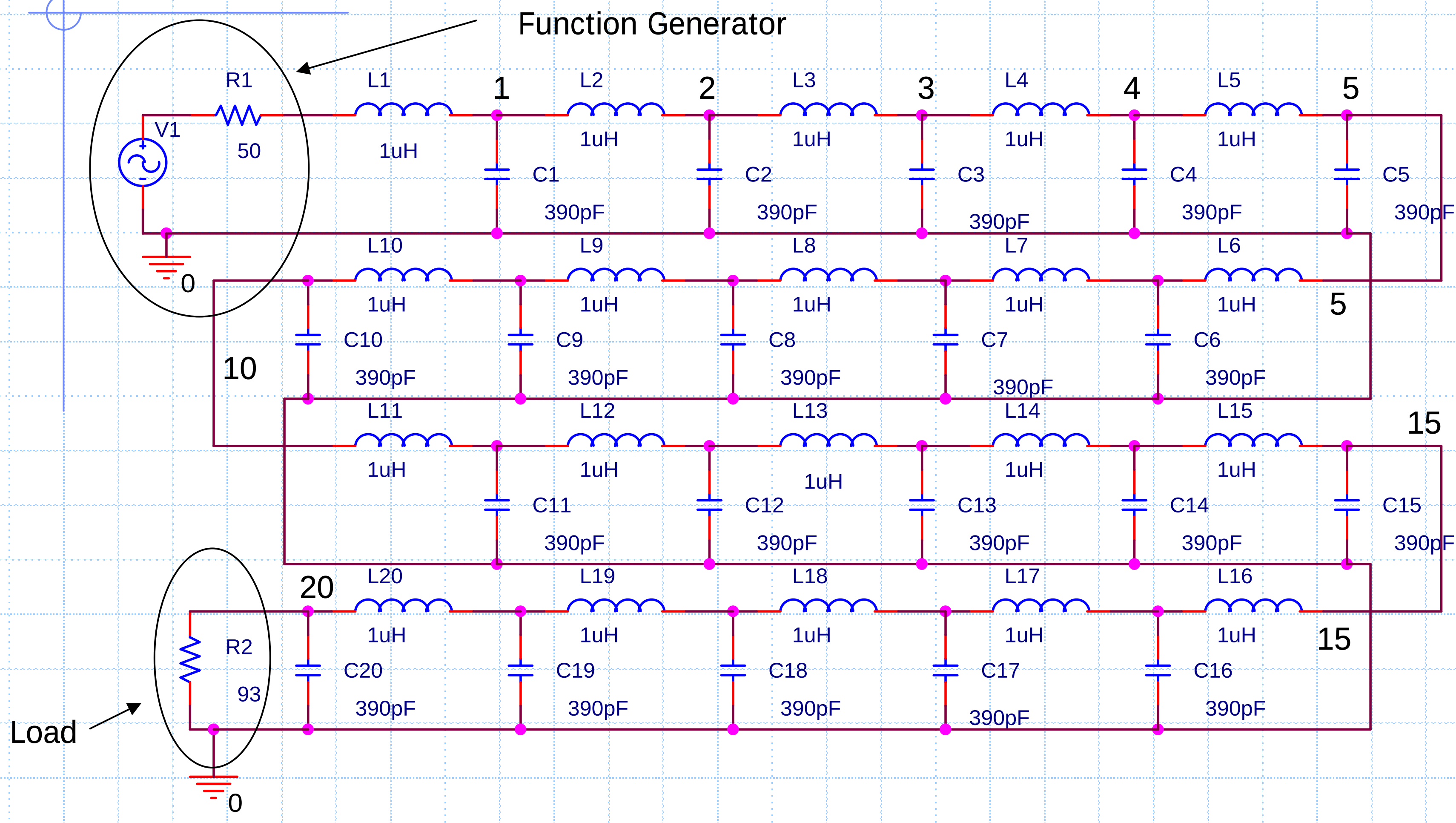
- What are the voltage maxima and minima?

$$v(z,t) = 2V^{+}\sin\beta z \sin\omega t$$

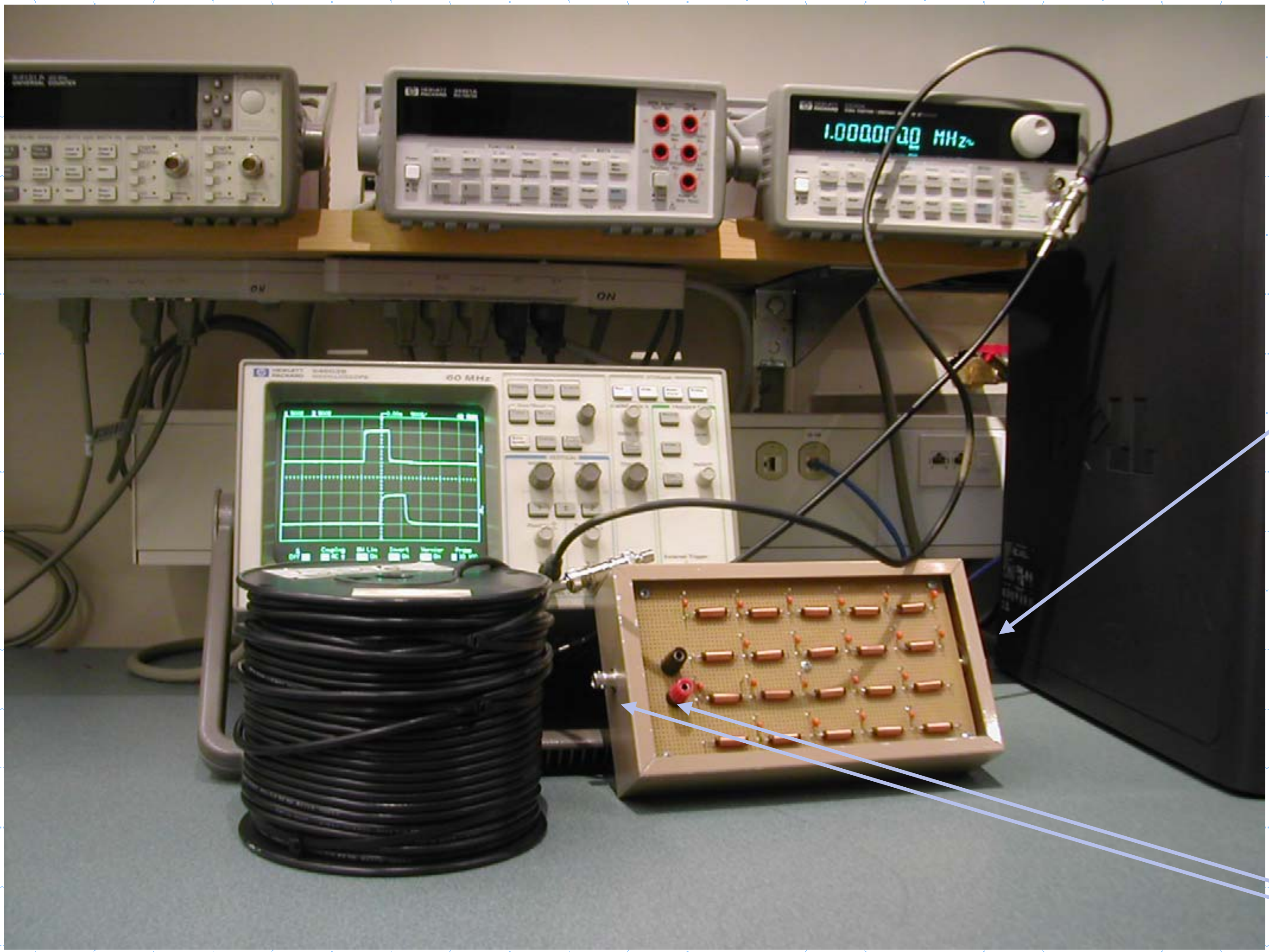
- Where are they?

- The standing wave pattern is the envelope of this function.

# Lumped Transmission Line



# Lumped Transmission Line

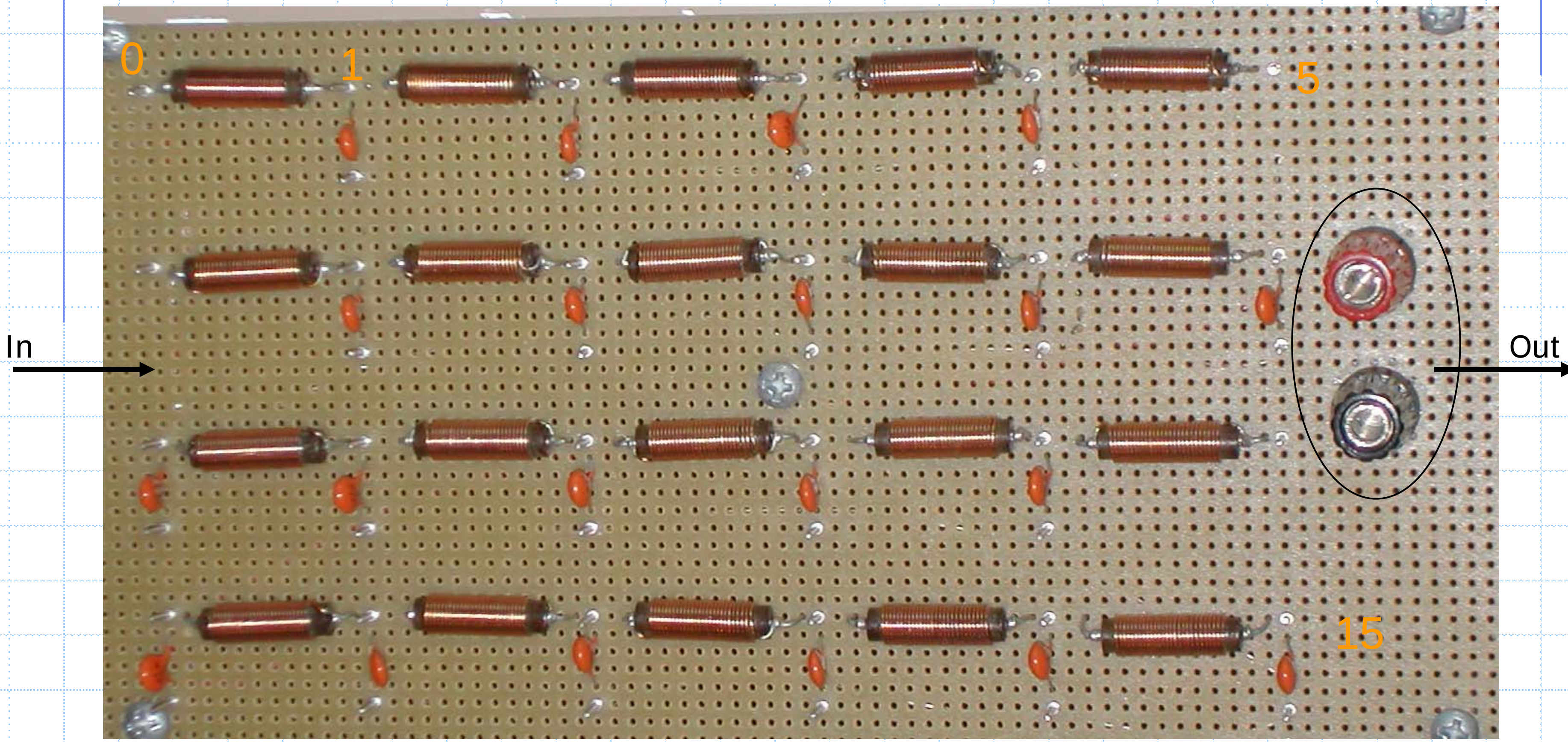



Input Not Shown

Both Outputs Shown

# Lumped Transmission Line

Input is BNC    Output is both BNC and Banana Plugs (for some loads)



# Lumped Transmission Line Experiment

- Treat the lumped version just like the reel of cable. (Connectors are opposite so you will need connector cables.)
- Monitor the output of the function generator on one channel
- Monitor the voltages on each node (one at a time) on the other channel. You can use just the signal (red) lead, since the ground (black) lead is connected through the other cables. Use the voltage cursors to obtain $V_{P-P}$ *for each node. Record your values and plot with Excel, Matlab, etc.*

# Lumped Transmission Line Numerical Experiment (Not required)

- Use PSpice to set up the standard transmission line, matched and not
- Look at the output for a variety of frequencies
- Set up the lumped line in PSpice (more work) and repeat
- Use the lumped line model to show the standing wave pattern
- Will there be any obvious differences between the physical and numerical experiments?

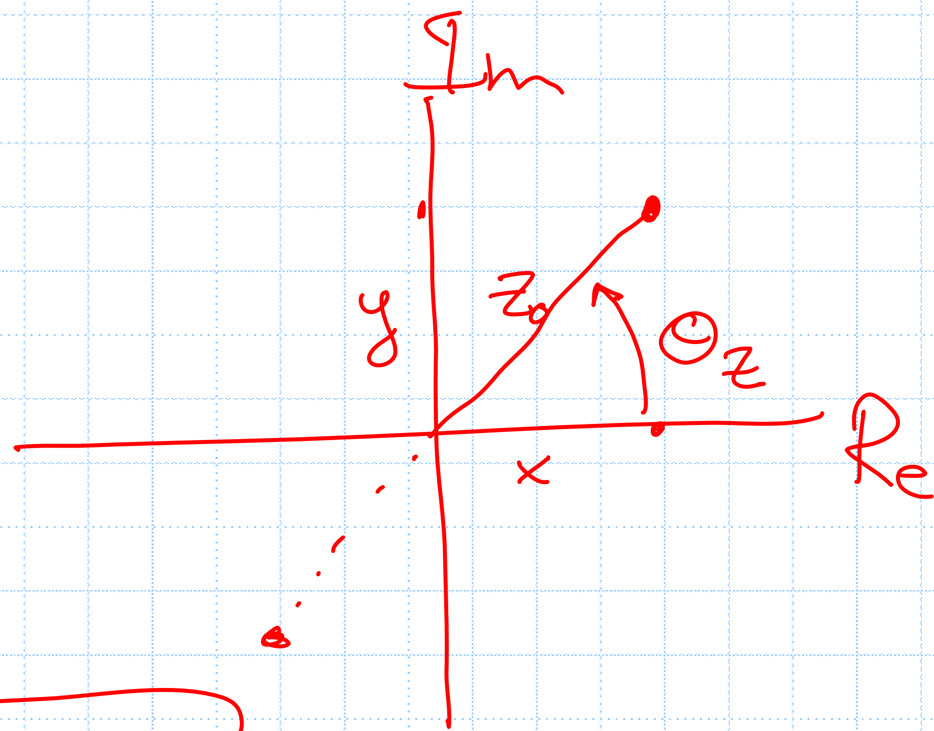
# Workspace

$$z = Z_0 e^{j\theta_z}$$

$$= x + jy$$

$$Z_0 = \sqrt{x^2 + y^2}$$

$$\theta_z = \operatorname{atan}\frac{y}{x}$$



# Workspace

$$\frac{\partial}{\partial t} e^{j\omega t} = j\omega e^{j\omega t}$$

$$V(z) = V^+ e^{-j\beta z} + V^- e^{+j\beta z}$$

$$\frac{\partial i}{\partial z} = -c \frac{\partial v}{\partial t} \qquad \frac{\partial v}{\partial z} = -\ell \frac{\partial i}{\partial t}$$

$c$ cap/length $\quad$ $\ell$ ind/length

$$\frac{\partial}{\partial t} \rightarrow j\omega$$

$$\frac{\partial V}{\partial z} = -j\omega \ell i$$

# Workspace

$$\beta = \omega\sqrt{lc} \qquad Z_0 = \frac{\omega l}{\beta} = \frac{\omega l}{\omega\sqrt{lc}}$$

$$i = \frac{-1}{j\omega l}\frac{\partial V}{\partial z}$$

$$V = V^+ e^{-j\beta z} + V^- e^{+j\beta z}$$

$$\frac{\partial V}{\partial z} = V^+(-j\beta)e^{-j\beta z} + V^-(+j\beta)e^{+j\beta z}$$

$$i(z) = V^+ \frac{j\beta}{j\omega l} e^{-j\beta z} - \frac{j\beta}{j\omega l} e^{+j\beta z} V^-$$

$$= \frac{V^+}{Z_0} e^{-j\beta z} - \frac{V^-}{Z_0} e^{+j\beta z}$$

# Workspace

$$V(z) = V^+ e^{-j\beta z} + V^- e^{+j\beta z}$$

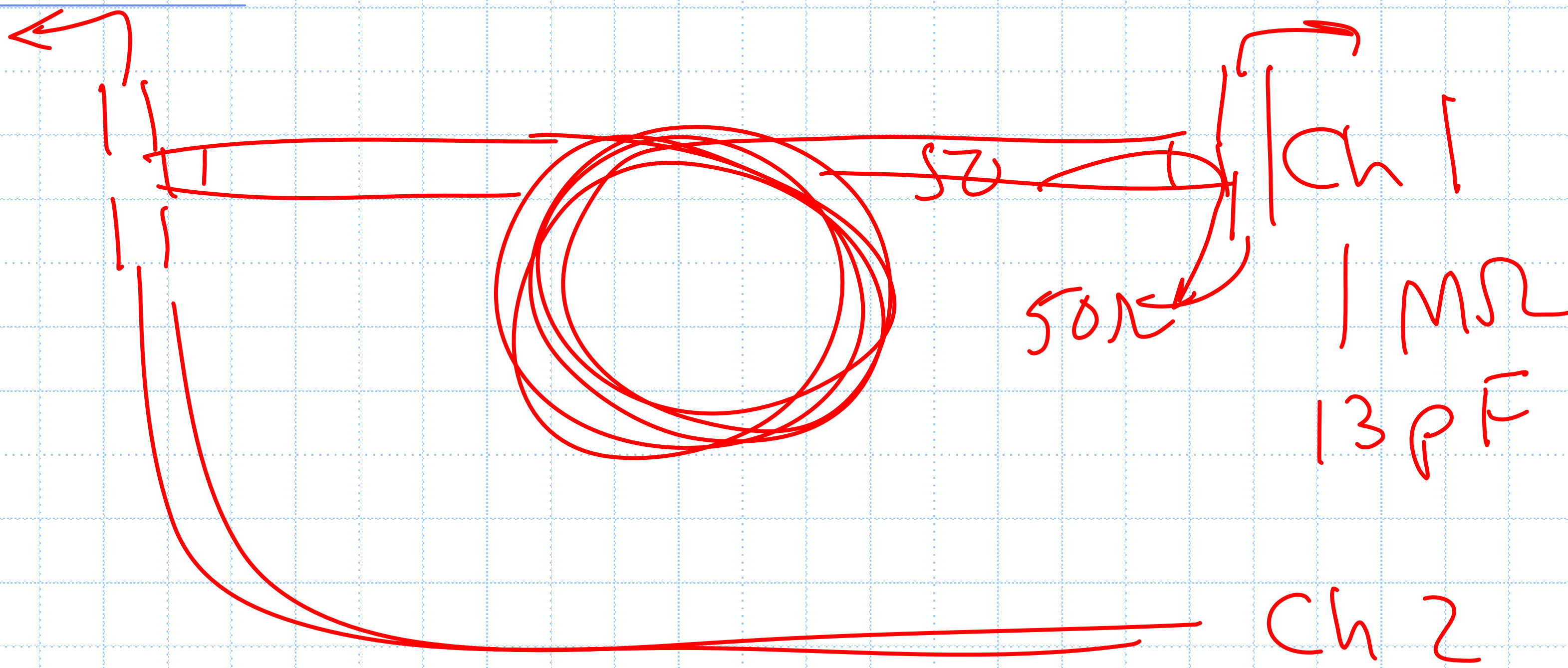
$$i(z) = \frac{V^+}{Z_0} e^{-j\beta z} - \frac{V^-}{Z_0} e^{+j\beta z}$$

$$Z_0 = \sqrt{\frac{\ell}{c}} \qquad \beta = \omega\sqrt{\ell c}$$

$$V^- = \Gamma_L V^+$$

# Workspace

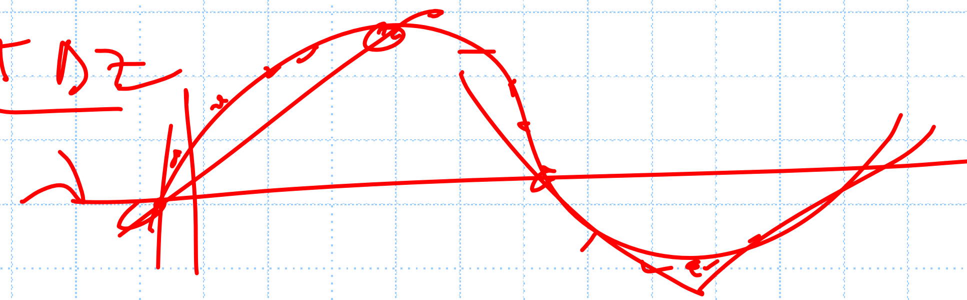
$u = 2\times10^8$ m/s

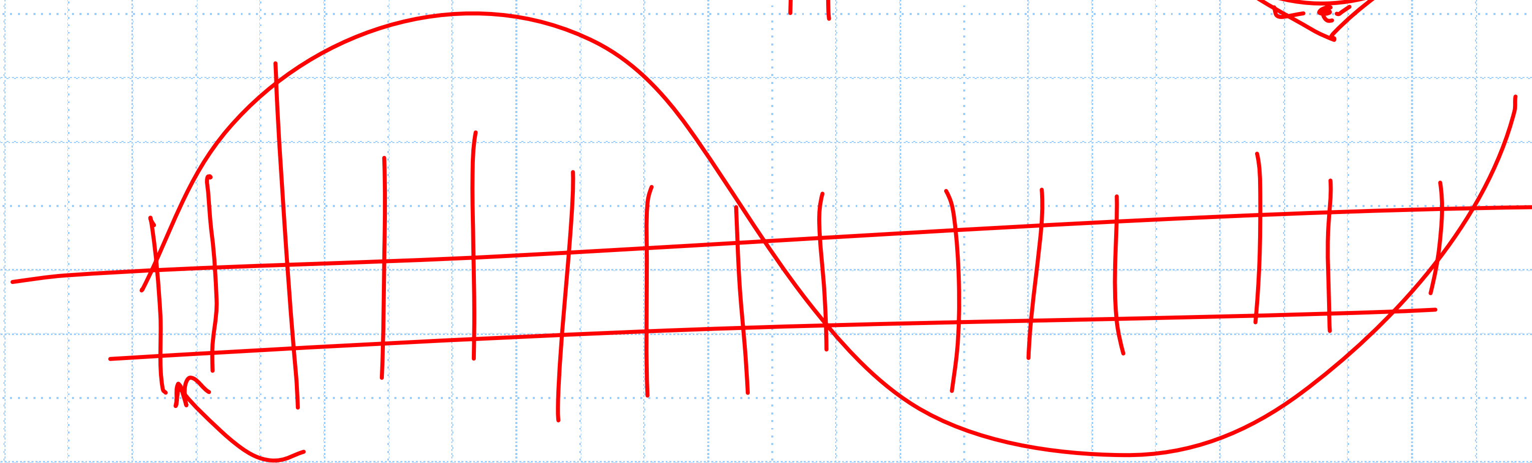


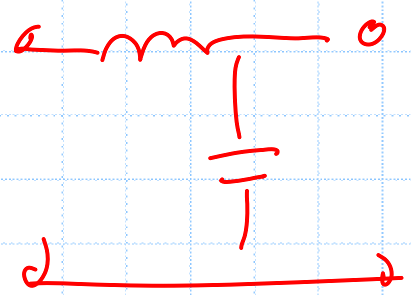
# Workspace

$\beta = \frac{2\pi}{\lambda}$

$\frac{2\pi \Delta z}{\lambda}$





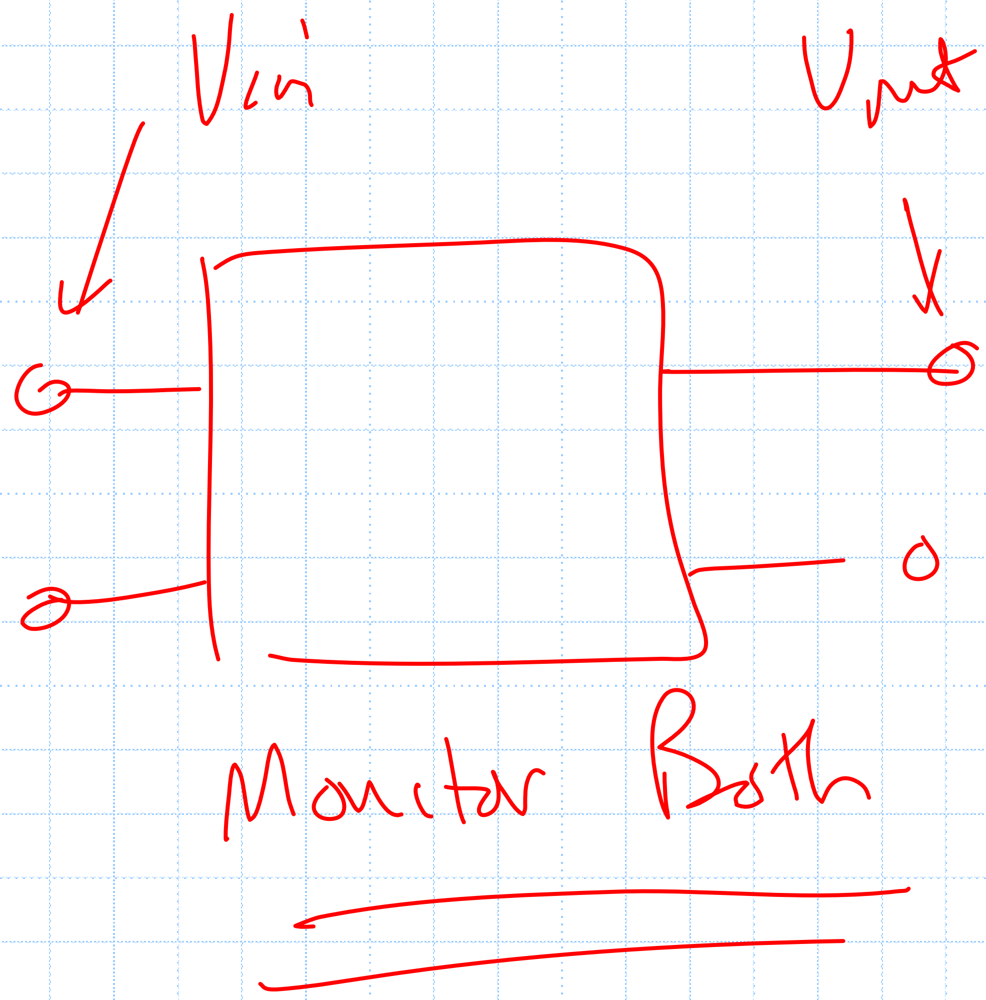


$\cos(\beta z)$

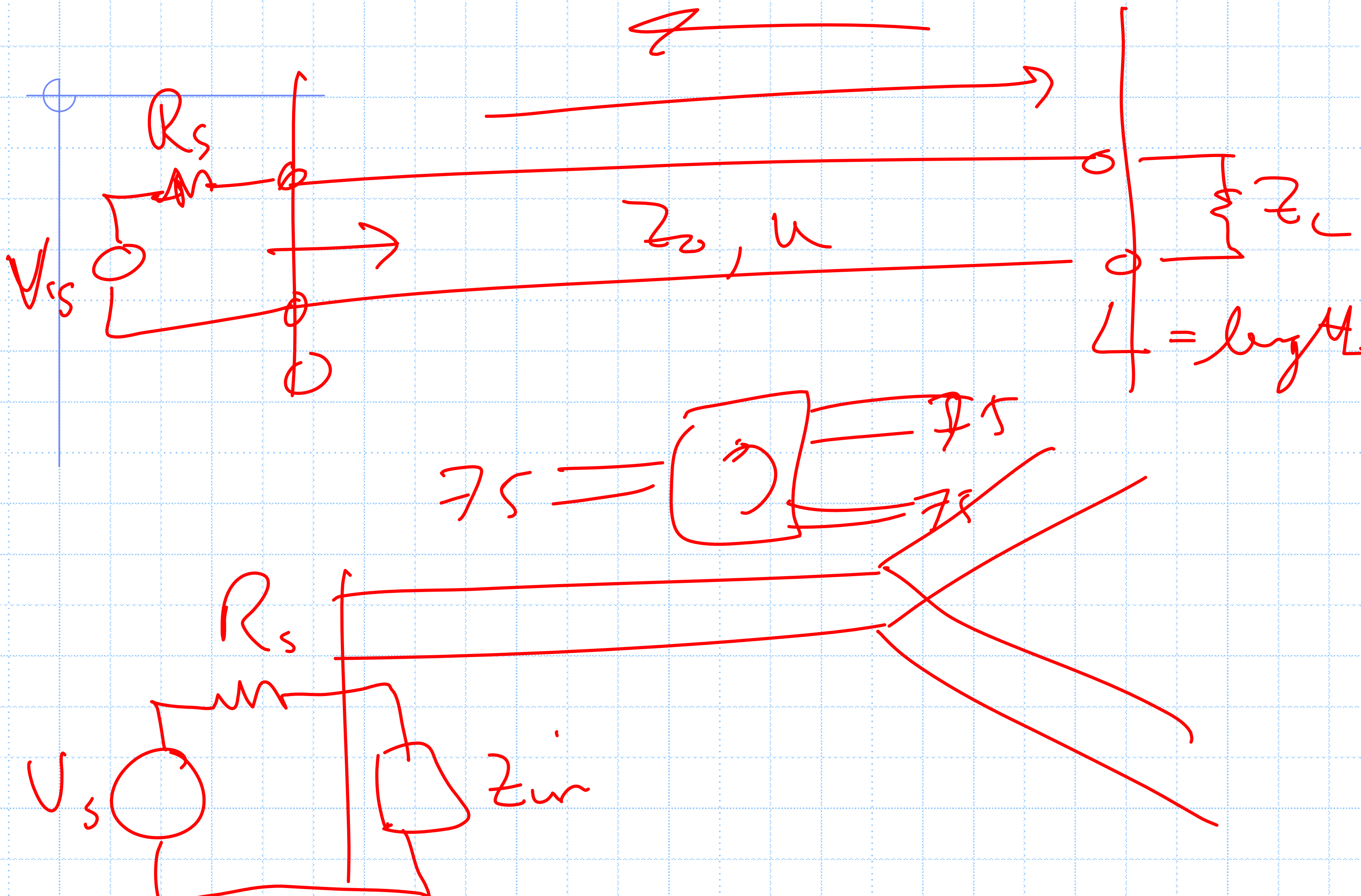
$\frac{\sin(\beta z)}{\sin \beta \Delta z} \simeq \beta \Delta z$

$\Delta z << \lambda$

# Workspace



# Workspace



# Workspace

# Workspace

# Workspace