Homework 2
Fields and Waves
Fall 2007

1.) Show that any function of *(x + ct)* satisfies the wave equation.

Solution :

Let

$$z = x + ct$$
$$V = f(x+ct) = f(z)$$

The wave equation can be written as:

$$\frac{1}{c^2}\frac{d^2 V}{d t^2} = \frac{d^2 V}{d x^2} \qquad \text{....................} \quad (i)$$

$$\frac{d V}{dt} = \frac{d V}{d z}\frac{d z}{d t}$$

$\frac{d z}{d t} = c$, therefore

$$\frac{d V}{dt} = c\frac{d V}{d z}$$

$$\frac{d^2 V}{d t^2} = \frac{d}{d t}\left(c\frac{d V}{d z}\right) = c^2\frac{d^2 V}{d z^2} \qquad \text{.....................} \quad (ii)$$

$$\frac{d V}{d x} = \frac{d V}{d z}\frac{d z}{d x}$$

$$\frac{d z}{d x} = 1$$

$$\frac{dV}{dx} = \frac{dV}{dz}$$

$$\frac{d^2 V}{d x^2} = \frac{d}{d x}\left(\frac{d V}{d z}\right) = \frac{d^2 V}{d z^2} \qquad \text{.....................} \quad (iii)$$

From (ii) and (iii) ,

$\frac{d^2 V}{dt^2} = c^2\frac{d^2 V}{d x^2}$, which proves the wave equation.

2.a) Calculation: The inductance per unit length of a coaxial cable is $l = \left(\frac{\mu_0}{2\pi}\right)\left\{\ln\frac{b}{a} + 0.25\right\}$. For this problem you can ignore the internal effect (the 0.25) and use $l = \left(\frac{\mu_0}{2\pi}\right)\left\{\ln\frac{b}{a}\right\}$. The capacitance per unit length is $c = \frac{2\pi\varepsilon}{\ln\frac{b}{a}}$. See Appendix B for the values of permeability and permittivity of free space. For the coax cable in the lab (RG 58A/U) there is polyethylene dielectric insulation with $\varepsilon_r = 2.3$ and copper conductors with inner radius $a = 0.4$ mm and outer radius $b = 1.4$ mm. To model a 4 meter section of the cable what values of L and C should we use? (As in the text, lower case *l* and *c* are per unit length and upper case L and C are total values.) Compare these values to the lumped circuit transmission lines in the lab;
$l = 2.5\times 10^{-7} H/m,\ c = 10^{-10} F/m$. This should represent 80 meters of RG 58 by 20 sections.
We have seen that $V = V_0\cos(\omega s)$ where $s = t \pm z/u$ is a solution to
$\frac{\partial^2 V(z,t)}{\partial z^2} - lc\frac{\partial^2 V(z,t)}{\partial t^2} = 0$ Find the velocity, *u*, for the RG 58 A/U cable. What is the time delay for the 4 meter section? What is the characteristic impedance of the cable?

Solution:

$$\mu_0 = 4\pi\times 10^{-7} H/m \text{ and } \epsilon_0 = 8.854\times 10^{-12} F/m$$

$$l = \left(\frac{\mu_0}{2\pi}\right)\left(\ln\frac{b}{a}\right) = 2.5\times 10^{-7} H/m$$

$$c = \frac{2\pi\epsilon_r\epsilon_0}{\ln\frac{b}{a}} = 1.02\times 10^{-10} F/m$$

Each section would represent a $\frac{80\text{m}}{20} = 4\text{m}$ long line.

The total values of L and C are given by :

$$L = l \times \textit{length of the section} = 2.5\times 10^{-7} H/m \times 4m = 1\,mH$$

$$C = c \times \textit{length of the section} = 1.02\times 10^{-10} F/m \times 4m = 4.08\times 10^{-10} F$$

If $$V = V_0\cos(\omega s) = V_0\cos(\omega t \pm \omega\frac{z}{u}),$$

$$\frac{\partial^2 V(z,t)}{\partial z^2} = -\frac{\omega^2}{u^2} V(z,t) \qquad \text{..................................} (iv)$$

And,
$$\frac{\partial^2 V(z,t)}{\partial t^2} = -\omega^2 V(z,t) \qquad \text{.....................................} (v)$$

For the voltage wave,

$$\frac{\partial^2 V(z,t)}{\partial z^2} - lc\frac{\partial^2 V(z,t)}{\partial t^2} = 0$$

Using (iv) and (v) we have,

$$\frac{-\omega^2}{u^2} V(z,t) + lc\omega^2 V(z,t) = 0$$

Therefore,

$$\frac{1}{u^2} = lc \ , \ u = \frac{1}{\sqrt{(lc)}}$$

For an ideal co-ax,

$$l = \left(\frac{\mu_0}{2\pi}\right)\left(\ln\frac{b}{a}\right) \text{ and } c = \frac{2\pi\epsilon_r\epsilon_0}{\ln\frac{b}{a}} \ .$$

It follows that

$$u = \frac{1}{\sqrt{(lc)}} = \frac{1}{\sqrt{\mu_0\epsilon_0\epsilon_r}}$$

Thus u depends only on the material properties if the line is ideal. Plugging in the numbers,

$$u = 1.98\times10^8 \, m/s$$

Time delay for the 4m section would be

$$t_d = \frac{d}{u} = \frac{4\text{m}}{1.98\times10^8 \, m/s} = 2.02\times10^{-8} s$$

The characteristic impedance

$$Z_0 = \sqrt{\frac{L}{C}} = 49.5\,\Omega$$