## Homework  3

### 3Due 1 March 2005

Some general advice on doing basic vector mathematics in Fields and Waves I.

- Always draw as many views of the problem you are considering as you find necessary to fully understand the configuration.
- Always write out the full expression for the line, surface or volume element before attempting any integrals. Then, for line or surface integrals, take any dot products before doing anything else. This will usually reduce the problem to a more manageable scalar integral.
- Simplify the mathematical expressions before you try to solve them. Usually the math, once simplified, will be relatively simple.
- When doing surface integrals, it is usually possible to check one's answer against Maxwell's equations or, if the integrals are used to find a field expression, the differential forms of Maxwell's equations can be used to check answers.

## 1. Flux Integrals

a.  The electric field due to a point charge is given by $\vec{E} = \dfrac{q}{4\pi\varepsilon_o R^2}\hat{a}_R$ where $\vec{R}$ is the

vector from the charge to the observation point. Assume that the charge is located at the origin of a cylindrical coordinate system $(r,\phi,z)=(0,0,0)$. Determine the total

electric flux $\int \vec{E}\cdot d\vec{S}$ passing through the surface $z=d$. Begin by drawing a diagram

showing the point charge and the surface in the *r-z* plane below. Also indicate the value of $d\vec{S}$. Recall that the $z=d$ surface goes from *r = 0 to infinity*.
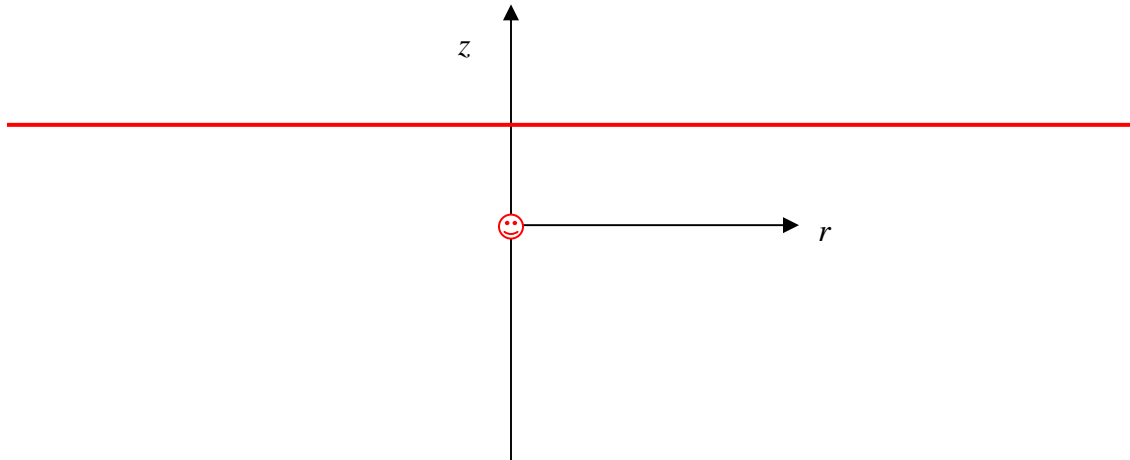
*The happy face is the point charge, while the horizontal line is the z=d plane. For this plane the surface element is $d\vec{S} = \hat{a}_z r dr d\phi$.*

*The flux through the z=d surface is given by* $\int \vec{E}\cdot d\vec{S} = \dfrac{q}{4\pi\varepsilon_o}\int\limits_0^{\infty} dr \int\limits_0^{2\pi} d\phi \dfrac{r}{R^2}\hat{a}_R \cdot \hat{a}_z$

$R^2 = r^2 + d^2$ ; $\hat{a}_R \cdot \hat{a}_z = \cos\theta = \dfrac{d}{\sqrt{r^2+d^2}}$ *so that* $\int \vec{E}\cdot d\vec{S} = \dfrac{q2\pi d}{4\pi\varepsilon_o}\int\limits_0^{\infty} dr \dfrac{r}{\left(r^2+d^2\right)^{3/2}} = \dfrac{q}{2\varepsilon_o}$

*where we have used* $\int\limits_0^{\infty} dr \dfrac{r}{\left(r^2+d^2\right)^{3/2}} = \dfrac{1}{d}$ *which you can check with Maple if you wish.*

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b. The magnetic field outside of a long straight wire of radius $r = a$, carrying a current $I$

   is given by $\vec{B} = \dfrac{\mu_o I}{2\pi r}\hat{a}_\phi$. Determine the total magnetic flux passing through the

   surface defined by $a \le r \le b$ and $0 \le z \le d$. Begin by drawing a picture of the surface
   in the *r-z* plane. Also indicate the value of $d\vec{S}$. What is the solution for the specific
   case where $b \to \infty$?
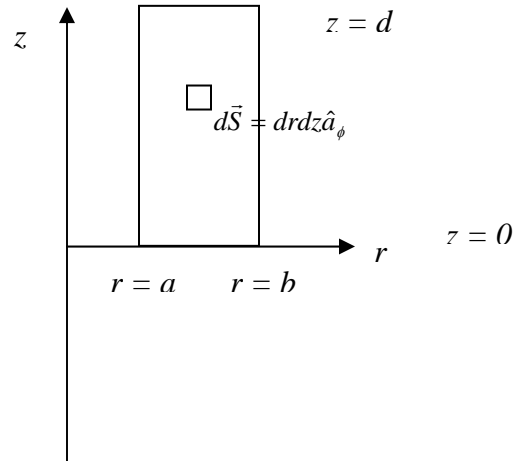


$$\int \vec{B}\cdot d\vec{S} = \frac{\mu_o I}{2\pi}\int_a^b dr \int_0^d dz\, \frac{1}{r} = \frac{\mu_o I}{2\pi} d\int_a^b dr\, \frac{1}{r} = \frac{\mu_o I d}{2\pi}\left(\ln b - \ln a\right) = \frac{\mu_o I d}{2\pi}\ln\frac{b}{a}$$

*as $b \to \infty$ $\int \vec{B}\cdot d\vec{S} \to \infty$ since $\ln\infty \to \infty$. The reason this goes to infinity is that the field*

*expression we are using is only correct for $b \to \infty$ if the line is infinitely long. Such lines*
*will produce infinite flux.*

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**2.   The Electric Field due to a Volume Charge Distribution**

a. Assume that there is a volume charge distribution in the spherical region

$0 \leq r \leq a$ given by $\rho = \rho_o \left( 1 - \left( \dfrac{r}{a} \right)^{33} \right)$. First, plot this expression as a function of $r$.
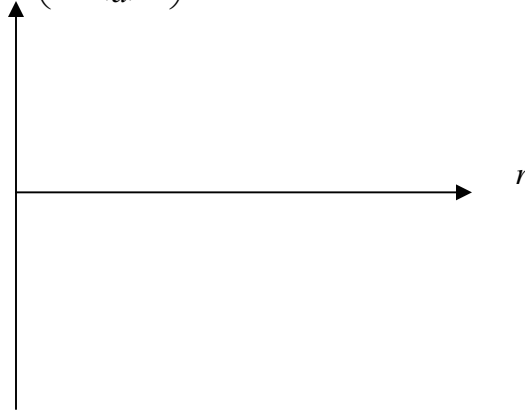
*Matlab was used to plot the charge distribution as a function of r, which is shown below. Note that it looks very much like a uniform charge distribution, except that it is a little more rounded at r = a. We had to choose a = 30 for this plot, but that does not matter.*
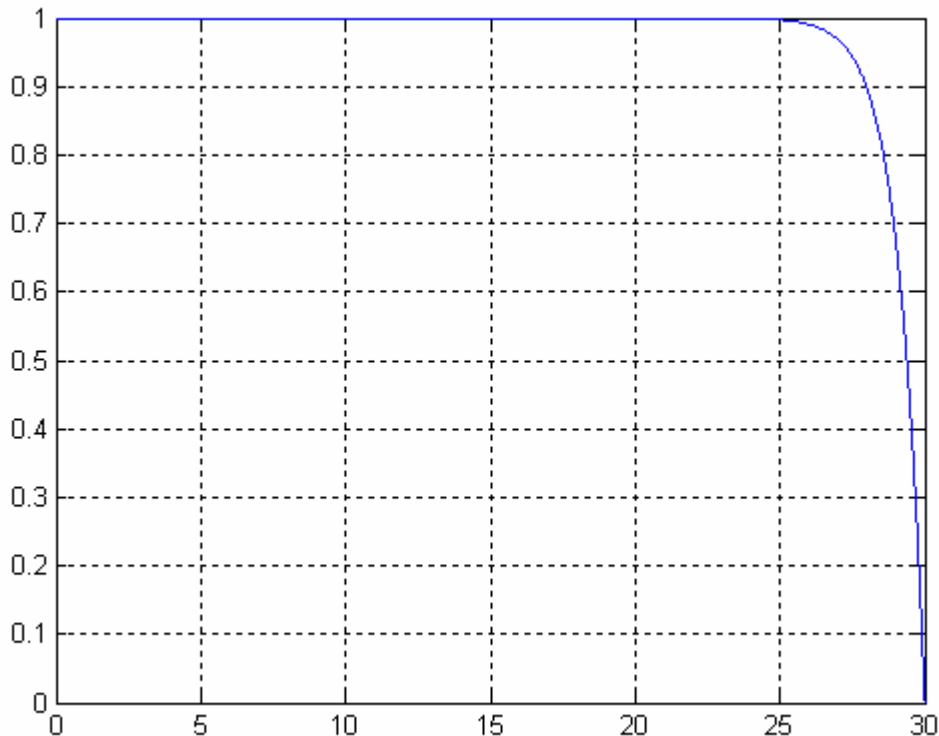
Next, determine the total amount of charge in this distribution. Repeat for a uniform distribution in the same region, that is, for $\rho = \rho_o$. Compare your results for the two cases.

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*First, for a uniform distribution, the total charge is given by*

$$Q_{total} = \int \rho_o r^2 dr 4\pi = \frac{4}{3}\pi a^3 \rho_o \text{ while for the given charge distribution}$$

$$\int \rho dv = \int \rho_o\left(1-\left(\frac{r}{a}\right)^{33}\right)dv = \rho_o 4\pi \int dr r^2\left(1-\left(\frac{r}{a}\right)^{33}\right)$$

$$\int dr r^2\left(1-\left(\frac{r}{a}\right)^{33}\right) = \frac{a^3}{3}-\frac{a^{36}}{36a^{33}} = \frac{a^3}{3}-\frac{a^3}{36} = \frac{a^3}{3}\frac{11}{12}$$

$$\int \rho dv = 4\pi\rho_o\frac{a^3}{3}\frac{11}{12}\text{ which is }\frac{11}{12}\text{ as much as the uniform distribution. This also shows that}$$

*the given distribution is indeed pretty close to uniform.*

b. Using Gauss' Law in integral form, determine the electric field $\vec{E}$ for all values of radius for both charge distributions. Plot the magnitude of the electric field as a function of radius $\left|\vec{E}\right|$.
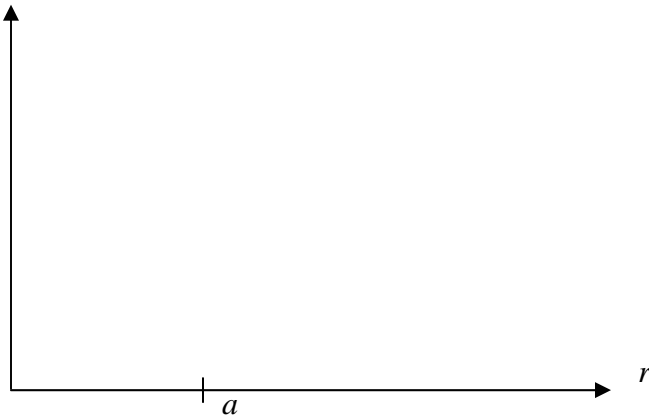


*To solve for the electric field using Gauss' Law, we need the total charge and the charge enclosed by a Gaussian surface with a radius smaller than the charge distribution (0<r<a). We can use the integrals above and just replace the upper limit a by r. Then*

$$\int \rho dv = \int \rho_o\left(1-\left(\frac{r}{a}\right)^{33}\right)dv = \rho_o\left(\frac{4}{3}\pi r^3 - 4\pi\frac{r^{36}}{36a^{33}}\right) = \frac{4}{3}\pi r^3\rho_o\left(1-\frac{r^{33}}{12a^{33}}\right)\text{ and}$$

$$\int \rho dv = \int \rho_o dv = \frac{4}{3}\pi r^3\rho_o\text{ for the two charge distributions. The left hand side of Gauss'}$$

*Law is* $\oint \vec{D}\cdot d\vec{S} = D_r 4\pi r^2 = \varepsilon_o E_r 4\pi r^2$ *for all radii. Thus, for the uniform charge*

*distribution* $E_r(r) = \frac{\rho_o r}{3\varepsilon_o}$ *inside the charge and* $E_r(r) = \frac{\rho_o a^3}{3\varepsilon_o r^2}$ *outside the charge. For the*

*given distribution* $E_r(r) = \frac{\rho_o r}{3\varepsilon_o}\left(1-\frac{1}{12}\frac{r^{33}}{a^{33}}\right)$ *inside the charge and* $E_r(r) = \frac{\rho_o a^3}{3\varepsilon_o r^2}\left(\frac{11}{12}\right)$.

*The two field expressions in each region are very similar to one another. Plotting them using Matlab, we see the following:*

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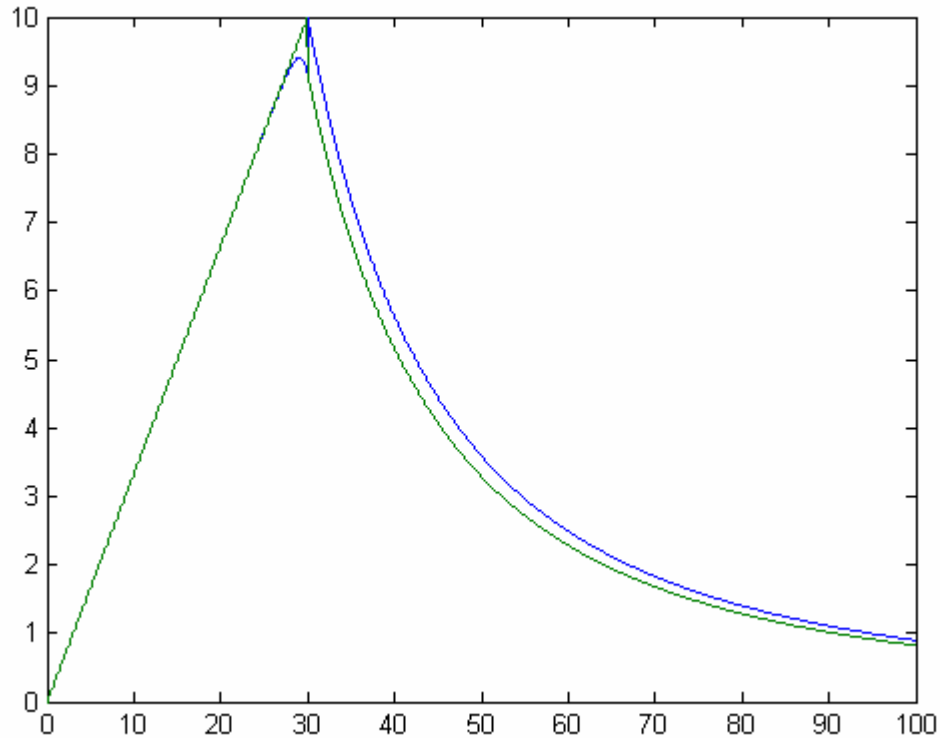
*The two are identical for most of the region inside the charge and then deviate slightly near the edge of the charge where the given distribution begins to drop off.*

c. Use Gauss' Law in differential form to check your answer for both cases.

*Inside the two charge distributions $\nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_o}$ while outside $\nabla \cdot \vec{E} = 0$.  Thus, we need to*

*take the divergence of the four expressions, which is given by $\nabla \cdot \vec{E} = \dfrac{1}{r^2} \dfrac{\partial}{\partial r}\left(r^2 E_r\right)$.  For*

*the uniform charge and $r \geq a$, $\nabla \cdot \vec{E} = \dfrac{1}{r^2} \dfrac{\partial}{\partial r}\left(r^2 \dfrac{\rho_o a^3}{3\varepsilon_o r^2}\right) = 0$. For $r \leq a$*

$$\nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\rho_o r}{3\varepsilon_o}\right) = \frac{\rho_o}{3\varepsilon_o r^2} 3r^2 = \frac{\rho_o}{\varepsilon_o}.$$ *For the given charge distribution and $r \geq a$*

$$\nabla \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\rho_o a^3}{3\varepsilon_o r^2}\frac{11}{12}\right) = 0.$$ *For $r \leq a$* $\nabla \cdot \vec{E} = \dfrac{\rho_o}{3\varepsilon_o} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^3\left(1 - \dfrac{1}{12}\dfrac{r^{33}}{a^{33}}\right)\right)$ *which*

*simplifies to*

$$\nabla \cdot \vec{E} = \frac{\rho_o}{3\varepsilon_o}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\left(1 - \frac{1}{12}\frac{r^{33}}{a^{33}}\right)\right) = \frac{\rho_o}{3\varepsilon_o}\frac{1}{r^2}\left(\left(3r^2 - \frac{36}{12}\frac{r^{35}}{a^{33}}\right)\right) = \frac{\rho_o}{\varepsilon_o}\left(1 - \frac{r^{33}}{a^{33}}\right)$$ *so all four*

*expressions check out.*

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## 3. Electric Scalar Potential

For both cases in problem 2, find the electric scalar potential as a function of position $V = V(r)$ and then also evaluate the potential at the origin $V = V(0)$.

*To determine the potential as a function of position, we need to evaluate the integral*

$V = V(r) = -\int_{\infty}^{r} E_r(r)dr$ *where we have assumed that the voltage is zero at infinity. For*

*the uniform charge distribution and* $r \geq a$, $V(r) = -\int_{\infty}^{r} \frac{\rho_o a^3}{3\varepsilon_o r^2}dr = \frac{\rho_o a^3}{3\varepsilon_o r} = \frac{Q_{total}}{4\pi\varepsilon_o r}$ *where*

*we have put the solution in the form of a point charge to check it against that result. For*

$r \leq a$, $V(r) = \frac{\rho_o a^2}{3\varepsilon_o} - \int_{a}^{r} \frac{\rho_o r}{3\varepsilon_o}dr = \frac{\rho_o a^2}{3\varepsilon_o} + \frac{\rho_o}{6\varepsilon_o}\left(a^2 - r^2\right)$. *The voltage at the origin is then*

$V(0) = \frac{\rho_o a^2}{3\varepsilon_o} + \frac{\rho_o}{6\varepsilon_o}\left(a^2\right) = \frac{\rho_o a^2}{2\varepsilon_o}$. *For the given non-uniform charge and* $r \geq a$,

$V = V(r) = -\int_{\infty}^{r} \frac{\rho_o a^3}{3\varepsilon_o r^2}\left(\frac{11}{12}\right)dr = \frac{\rho_o a^3}{3\varepsilon_o r}\left(\frac{11}{12}\right)$ *which again looks just like the other result*

*except for being a little smaller. For* $r \leq a$, *things are a bit more complex.*

$V = V(r) = V(a) - \int_{\infty}^{r} \frac{\rho_o r}{3\varepsilon_o}\left(1 - \frac{1}{12}\frac{r^{33}}{a^{33}}\right)dr = \frac{\rho_o a^2}{3\varepsilon_o}\left(\frac{11}{12}\right) - \frac{\rho_o}{3\varepsilon_o}\int_{a}^{r}\left(r - \frac{1}{12}\frac{r^{34}}{a^{33}}\right)dr$

$V(r) = \frac{\rho_o a^2}{3\varepsilon_o}\left(\frac{11}{12}\right) - \frac{\rho_o}{3\varepsilon_o}\left(\frac{r^2}{2} - \frac{a^2}{2} - \frac{r^{35}}{35a^{33}} + \frac{a^2}{35}\right)$ *so that at r = 0,*

$V(0) = \frac{\rho_o a^2}{3\varepsilon_o}\left(\frac{11}{12}\right) + \frac{\rho_o}{3\varepsilon_o}\left(\frac{a^2}{2} - \frac{a^2}{35}\right)$.

### 4. Charge on a Capacitor Plate

A parallel plate capacitor is connected to a 100V DC voltage source, as shown. Little is known about how the capacitor is constructed so we do not know enough to calculate the capacitance from first principles. However, somehow, we are able to measure the voltage at an array of points located *1mm* inside the top surface of the capacitor. The measured voltages are given in the Excel spreadsheet *vdata.xls* found next to this assignment on the *Handout* webpage. Each plate of the capacitor is *30cm* by *30cm*. The voltages are measured at the edge points and points every *cm* to form the *31x31* grid given in the spreadsheet.
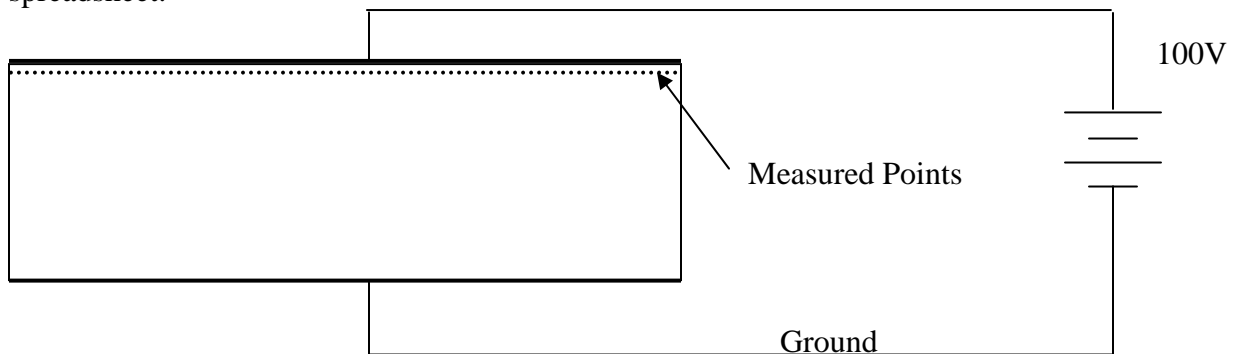


a.  Determine the average value of the electric field in the region between the measured points and the top plate.

*The average voltage is about 90V. Thus, the average electric field is (100-90)/(.001)=10kV/m.*

b.  Assume the dielectric constant in the region where the voltage is measured is $\varepsilon = \varepsilon_r \varepsilon_o = 4\varepsilon_o$. Determine the average value of the electric flux density.

*The electric flux density is $\vec{D} = \varepsilon \vec{E} = 4\varepsilon_o \vec{E}$*

c.  Determine the total charge on the top plate.

*The charge is $Q = \int \vec{D} \cdot d\vec{S} = 4\varepsilon_o ES = 0.36\varepsilon_o E$*

d.  Find the capacitance.

$$C = \frac{Q}{V} = \frac{0.36\varepsilon_o E}{V} = \frac{0.36\varepsilon_o 10000}{100} = 36\varepsilon_o$$

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