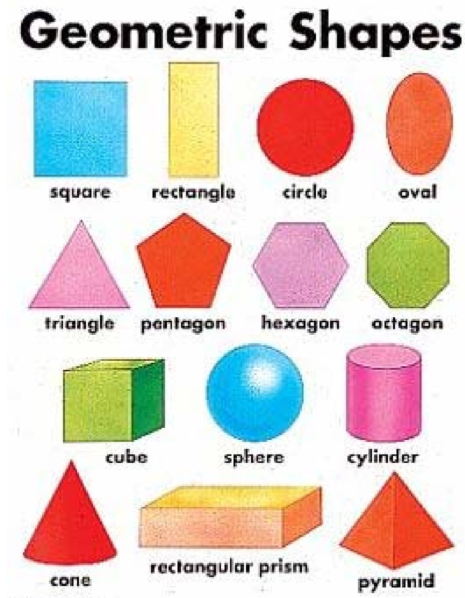
Homework 3

Due 28 February 2006

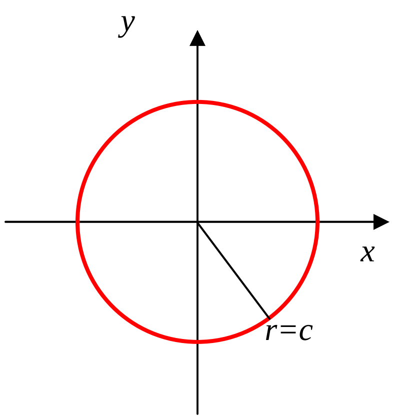
First we will remind ourselves of some shapes and then go over some general ideas.

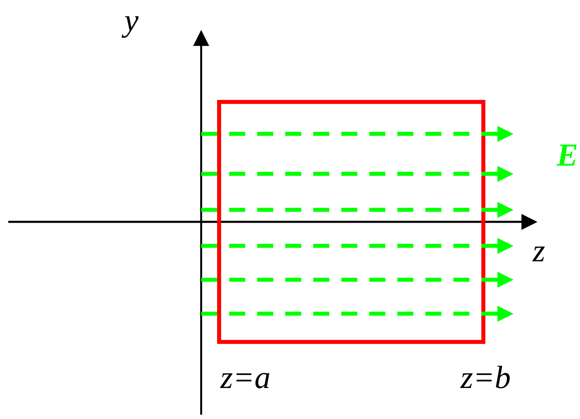


Some general advice on doing basic vector mathematics in Fields and Waves I. *An example is provided in italics.*

- Always draw as many views of each problem as you find necessary to fully understand the configuration.

Assume that we have the following electric field: $\vec{E}(z) = \hat{z}E_o e^{-z/d}$ *where* $\hat{z}$ *is the unit vector in the z direction, only for positive values of z. For negative values, the field is zero. Assume also that we want to find the flux of this vector through the closed cylindrical surface defined by* $a \le z \le b$ *and* $0 \le r \le c$ *where* $a \le d \le b$*. We should begin by drawing the cylinder and adding vectors to show the direction of the E field.*





- Always write out the full expression for the line, surface or volume element before attempting any integrals. Then, for line or surface integrals, take any dot products before doing anything else. This will usually reduce the problem to a more manageable scalar integral.

For a closed cylinder, there are two surface elements. On the cylindrical side of the cylinder: $d\vec{S} = \hat{r}(rd\phi dz)$ *On the flat ends of the cylinder:* $d\vec{S} = \pm\hat{z}(rdrd\phi)$ *where the positive sign goes with the end at the larger value of z and the negative sign at the other end. Then the full integral to be evaluated is:*

$$\oint \vec{E}\cdot d\vec{S} = r\int_0^{2\pi} d\phi \int_a^b dz \hat{r}\cdot\hat{z}E_o e^{-z/d} + \int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}E_o e^{-b/d} - \int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}E_o e^{-a/d}$$

Note that the first term is trivially zero since the unit vectors are orthogonal: $\hat{r}\cdot\hat{z} = 0$

- Simplify the mathematical expressions before you try to solve them. Usually the math, once simplified, will be relatively simple.

Most of these integrals are quite simple since parts are either easy to do or many terms are constants. Consider the second term. $\int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}E_o e^{-b/d}$ *First move all constants outside the integral sign.* $\int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}E_o e^{-b/d} = E_o e^{-b/d}\int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}$ *Then simplify using* $\hat{z}\cdot\hat{z} = 1$ $\int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}E_o e^{-b/d} = E_o e^{-b/d}\left\{\int_0^{2\pi} d\phi\right\}\left\{\int_0^c rdr\right\}$ *where we have also separated the two integrals since they are independent of one another. The integral in the first bracket is* $\left\{\int_0^{2\pi} d\phi\right\} = 2\pi$ *while the integral in the second bracket is* $\left\{\int_0^c rdr\right\} = \frac{c^2}{2}$*. Thus, the surface integral for the flat surface at z = b is* $\int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}E_o e^{-b/d} = E_o e^{-b/d}2\pi\frac{c^2}{2} = \pi c^2 E_o e^{-b/d}$ *and the integral for the flat surface at z = a is* $-\int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}E_o e^{-a/d} = -\pi c^2 E_o e^{-a/d}$.

Thus, the total surface integral is given by $\oint \vec{E}\cdot d\vec{S} = 0 - \pi c^2 E_o e^{-a/d} + \pi c^2 E_o e^{-b/d}$

- When doing surface integrals, it is usually possible to check one's answer against Maxwell's equations or, if the integrals are used to find a field expression, the differential forms of Maxwell's equations can be used to check answers.

From Maxwell's Equations $\oint \vec{E}\cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon}$ *where* $Q_{encl} = \int \rho_v dv$ *is the charge enclosed by the volume. We were not given the charge density, but we can figure it out from*

$$\frac{\rho_v}{\varepsilon} = \nabla\cdot\vec{E} = \frac{\partial}{\partial z}E_z = \frac{\partial}{\partial z}E_o e^{-z/d} = E_o\left(-\frac{1}{d}\right)e^{-z/d}$$ *Now we can evaluate*

$$Q_{encl} = \int \rho_v dv = \varepsilon\left(-\frac{E_o}{d}\right)\int_a^b dz e^{-z/d}\int_0^{2\pi} d\phi\int_0^c rdr = \varepsilon\left(-\frac{E_o}{d}\right)(-d)\left(e^{-b/d} - e^{-a/d}\right)2\pi\frac{c^2}{2}$$ *which simplifies to* $\frac{Q_{encl}}{\varepsilon} = (E_o)\left(e^{-b/d} - e^{-a/d}\right)\pi c^2$ *which is what we obtained above.*

To evaluate integrals, use Maple or go to http://eqworld.ipmnet.ru/en/auxiliary/aux-integrals.htm to find a table of many integrals. One page you may find useful at this URL is for Integrals with square root of $x^2 + a^2$.

Example:

$$\int_0^{\pi} \left(\sin^2 x\right) dx = ?$$

For this integral, we can use equation 8 of the Integrals with **sin**

$$\int \sin^2 ax\, dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int_0^{\pi} \left(\sin^2 x\right) dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi} = \frac{\pi}{2} - \frac{\sin 2\pi}{4} - \frac{0}{2} + \frac{\sin 0}{4} = \frac{\pi}{2}$$

Using Maple

```
> int((sin(x))^2,x=0..pi);
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$$-\frac{1}{2}\sin(\pi)\cos(\pi) + \frac{1}{2}\pi$$

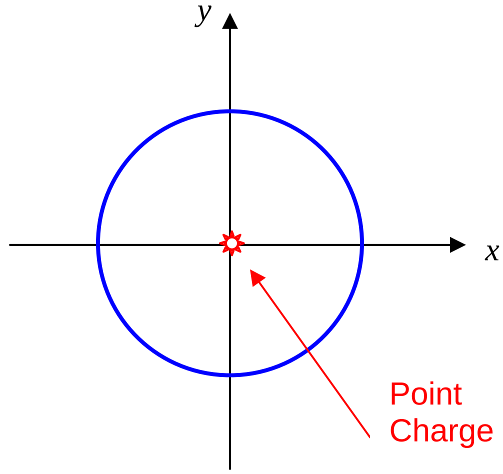
which agrees.

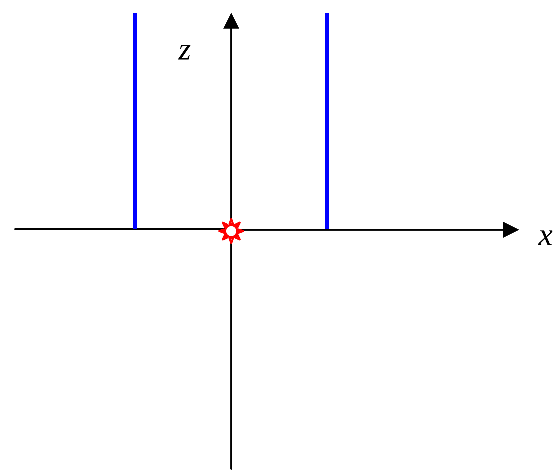
Note: The most mathematically challenging problem in this assignment is problem #1.

Homework 3

1. Flux Integrals

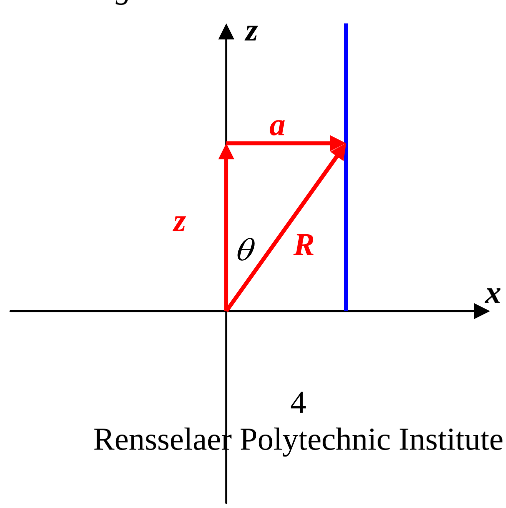
a. The electric field due to a point charge at the origin of a <u>cylindrical</u> coordinate system $(r,\phi,z)=(0,0,0)$ is given by $\vec{E}=\dfrac{q}{4\pi\varepsilon_o R^2}\hat{a}_R$ where $\vec{R}$ is the radial vector in a <u>spherical</u> coordinate system. (Mixed coordinate systems.) Determine the total electric flux $\int \vec{E}\cdot d\vec{S}$ passing through the open <u>cylindrical</u> surface $r=a$, $0\le z<\infty$. Begin by drawing a diagram showing the point charge and the surface in the diagrams below. Also indicate the value of $d\vec{S}$. Recall that the $r = a$ surface goes from $z=0$ to $z=\infty$ and is open at both ends.





On the surface of a cylinder, the surface element is given by $d\vec{S}=\hat{r}rd\phi dz$ where we are using $\hat{r}=\hat{a}_r$ to denote the unit vector in the cylindrical radial direction. Using this notation, the unit vector in the spherical radial direction is $\hat{R}=\hat{a}_R$. It may seem odd to change the notation, for this solution, we take the opportunity to show that there are several choices for unit vector notation. Also, we wanted to use the letter 'a' fewer times to avoid confusion. Given this surface element, the surface integral is now

$\int \vec{E}\cdot d\vec{S}=\iint \dfrac{q}{4\pi\varepsilon_o R^2}\hat{R}\cdot\hat{r}rd\phi dz$ *To evaluate this integral, we need to express everything in cylindrical coordinates. For the spherical radius, we have $R^2=a^2+z^2$ for points on the cylinder of radius = a. The diagram below shows this relationship.*



The dot or inner product of the two unit vectors $\hat{R}\cdot\hat{r} = \sin\theta$ *from the table on page 117 of Ulaby. The sine function also has to be written in cylindrical coordinates.*

$$\hat{R}\cdot\hat{r} = \sin\theta = \frac{a}{\sqrt{a^2+z^2}}$$ *so that*

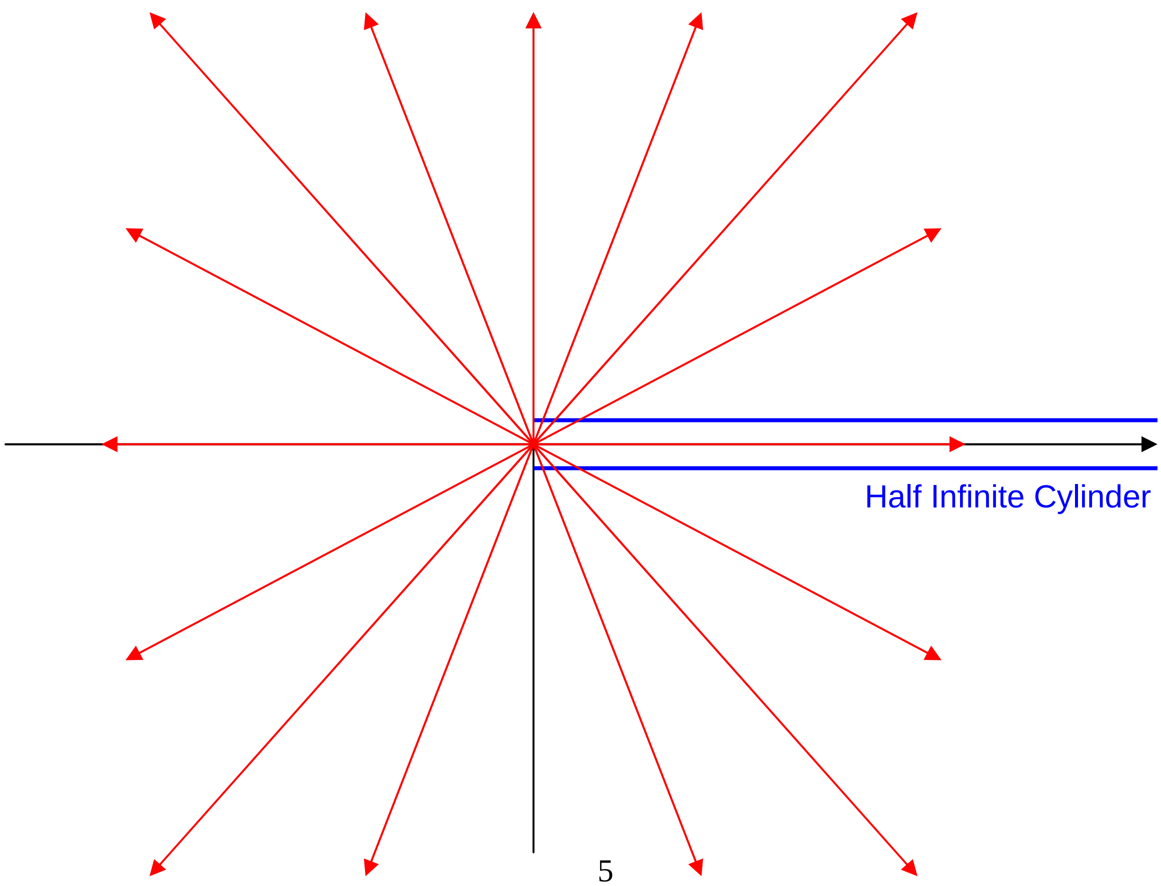
$$\int \vec{E}\cdot d\vec{S} = \frac{qa^2}{4\pi\varepsilon_o}\iint \frac{1}{\left(a^2+z^2\right)^{1.5}}\,d\phi dz = \frac{qa^2 2\pi}{4\pi\varepsilon_o}\int_0^\infty \frac{1}{\left(a^2+z^2\right)^{1.5}}\,dz$$

From the integral tables

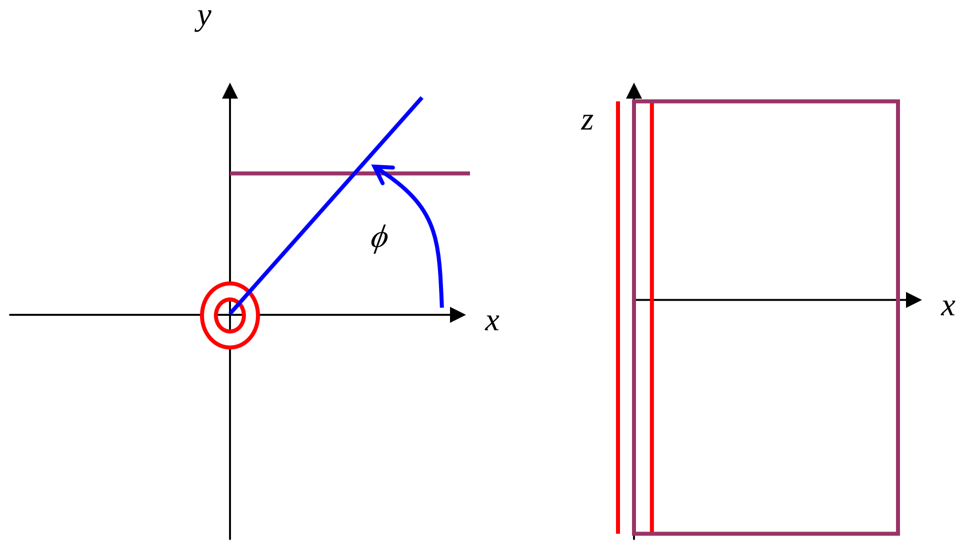
$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$$

so that $\int \vec{E}\cdot d\vec{S} = \frac{qa^2 2\pi}{4\pi\varepsilon_o}\int_0^\infty \frac{1}{\left(a^2+z^2\right)^{1.5}}\,dz = \frac{qa^2}{2\varepsilon_o}\left[\frac{z}{a^2\sqrt{z^2+a^2}}\right]_0^\infty = \frac{qa^2}{2\varepsilon_o}\frac{1}{a^2} = \frac{q}{2\varepsilon_o}$

If we think about this, we can see that this answer makes sense. If the cylinder had extended from $-\infty$ to $+\infty$*, it would have completely enclosed the charge since the open ends of the cylinder are insignificant in size at infinity. Then the answer would have been* $\int \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_o}$*. However, since the cylinder only goes from 0 to* $+\infty$*, it only encloses half of the charge, which accounts for the answer we obtained. In the diagram below, we can see that half of the field lines emanating from the point charge do indeed pass through the half infinite cylinder, if the scale of the geometry becomes infinite.*



b. The magnetic field outside of a long straight wire of cylindrical radius $r = a$, carrying a current I is given by $\vec{B} = \frac{\mu_o I}{2\pi r}\hat{a}_\phi$. Determine the total magnetic flux passing through the surface defined by $y = b$ and $0 \le x \le \infty$. Note that we are again mixing coordinate systems, in that we have a field specified in cylindrical coordinates and we are asking for the flux through a rectangular surface. Begin by drawing a picture of the surface in the two planes below. Also indicate the value of $d\vec{S}$. Next express the field in rectangular coordinates. Then, finally, set up the integral and evaluate it.



First consider the surface element. $d\vec{S} = \hat{y}dxdz$ *Note that we are actually free to choose the direction of the surface as up or down. Up was chosen since it is in the positive y direction. The answer to this question can have either a positive or negative sign. Either is OK. Next we must write the magnetic field in rectangular coordinates. First consider the unit vector.* $\vec{B} = \frac{\mu_o I}{2\pi r}\hat{\phi} = \frac{\mu_o I}{2\pi r}\left(-\hat{x}\sin\phi + \hat{y}\cos\phi\right)$ *Now we have to rewrite both the trig functions and r.* $\sin\phi = \frac{y}{r}$ *and* $\cos\phi = \frac{x}{r}$ *Then,*

$$\vec{B} = \frac{\mu_o I}{2\pi r}\left(-\hat{x}\frac{y}{r} + \hat{y}\frac{x}{r}\right) = \frac{\mu_o I}{2\pi\left(x^2+y^2\right)}\left(-\hat{x}y + \hat{y}x\right)$$ *and*

$$\vec{B}\cdot d\vec{S} = \frac{\mu_o I}{2\pi\left(x^2+y^2\right)}\left(-\hat{x}y + \hat{y}x\right)\cdot\hat{y}dxdz = \frac{\mu_o Ix}{2\pi\left(x^2+y^2\right)}dxdz$$. *Finally,*

$$\int \vec{B}\cdot d\vec{S} = \iint \frac{\mu_o Ix}{2\pi\left(x^2+b^2\right)}dxdz = \frac{\mu_o I}{2\pi}\int_{-\infty}^{+\infty}dz\int_0^\infty \frac{x}{x^2+b^2}dx$$

where we will have to use the integral tables (or Maple) again.

$$\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln(x^2+a^2)$$

Thus, $\int \vec{B}\cdot d\vec{S} = \frac{\mu_o I}{2\pi}\int_{-\infty}^{+\infty} dz \int_0^{\infty} \frac{x}{x^2+b^2}dx = \frac{\mu_o I}{2\pi}[z]_{-\infty}^{+\infty}\frac{1}{2}\left[\ln\left(x^2+b^2\right)\right]_0^{\infty} = \infty$

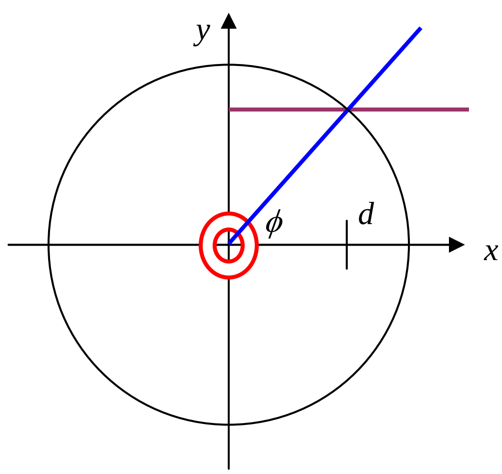
The following is a discussion of this result:
This is one of those things we have to deal with when considering the magnetic field of a long straight wire. The flux integral will be infinite for an infinite surface. For a finite, surface, we can get finite flux. For a wire, we usually calculate the flux per unit length. This takes care of the z integral. $\int_0^1 dz = 1$ *This was the easy one. What about the other integral? The good news is that an isolated long straight wire does not exist. All wires carrying current are part of a system with at least one additional wire carrying a return current. The coaxial cable is an excellent example. If we look at the geometry of a coax and redraw the diagrams above, we see that we obtain a finite integral, since there will only be magnetic field in the region between the center and outer conductors. Now, instead of integrating from 0 to* $+\infty$, *we integrate from 0 to d, which has a finite result.*

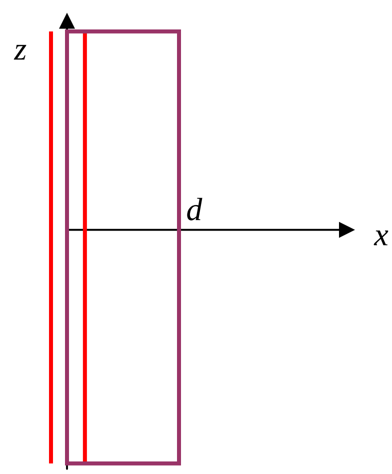
$$\int \vec{B}\cdot d\vec{S} = \frac{\mu_o I}{2\pi}\int_0^1 dz \int_0^d \frac{x}{x^2+b^2}dx = \frac{\mu_o I}{2\pi}[z]_0^1\frac{1}{2}\left[\ln\left(x^2+b^2\right)\right]_0^d = \frac{\mu_o I}{2\pi}\frac{1}{2}1\left[\ln\left(d^2+b^2\right)-\ln b^2\right]$$

Note also that, for the case where b = 0, the surface being integrated over is the $\phi = 0$ *surface. This then becomes* $\int \vec{B}\cdot d\vec{S} = \frac{\mu_o I}{2\pi}\frac{1}{2}\left[\ln d^2 - \ln 0\right] = -\infty$ *so we also cannot integrate from the origin and obtain a finite result. Again, the real world saves us since we should really integrate from the outer edge of the wire, which will be at a radius of say r= a. Then we obtain the following*

$$\int \vec{B}\cdot d\vec{S} = \frac{\mu_o I}{2\pi}\frac{1}{2}\left[\ln d^2 - \ln a^2\right] = \frac{\mu_o I}{2\pi}\frac{1}{2}\left[2\ln d - 2\ln a\right] = \frac{\mu_o I}{2\pi}\left[\ln d - \ln a\right] = \frac{\mu_o I}{2\pi}\ln\frac{d}{a}$$

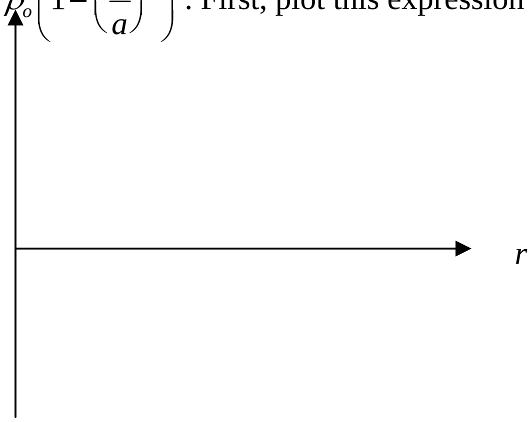
This result is exactly what is obtained in equation (5.98) of Ulaby for the magnetic flux between the two conductors of a coaxial cable.



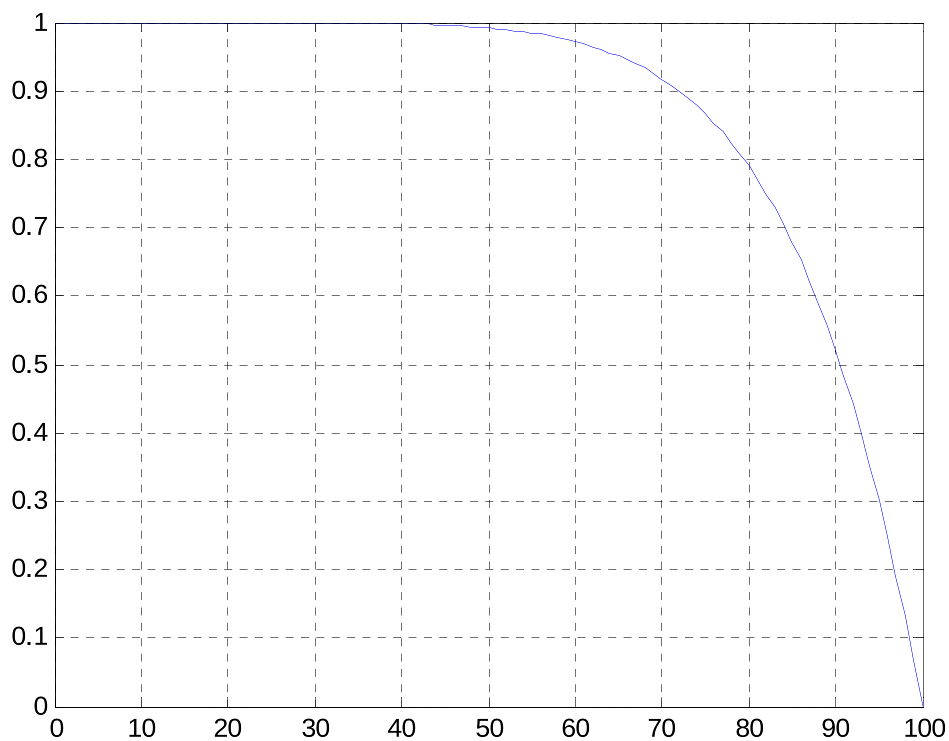


2. The Electric Field due to a Volume Charge Distribution

a. Assume that there is a volume charge distribution in the cylindrical region $0 \le r \le a$ given by $\rho = \rho_o \left(1 - \left(\frac{r}{a}\right)^7\right)$. First, plot this expression as a function of r.



Matlab was used to plot the charge distribution as a function of r, which is shown below. Note that it looks something like a uniform charge distribution, except that it is a more rounded at r = a. We had to choose a = 100 for this plot, but that does not matter.



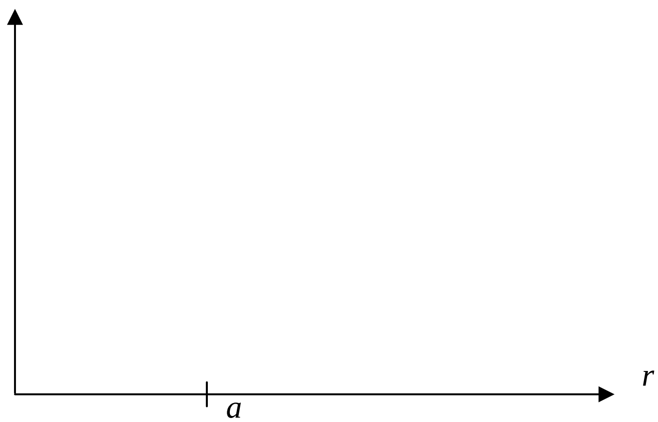
Next, determine the total amount of charge per unit length in this distribution. Repeat for a uniform distribution in the same region, that is, for $\rho = \rho_o$. Compare your results for the two cases.

First, for a uniform distribution, the total charge per unit length is given by $Q_{total} = \int_0^1 dz \int_0^{2\pi} d\phi \int_0^a \rho_o r dr = \pi a^2 \rho_o$ *Note that the z and* ϕ *parts of the integral just become* 2π. *For the given charge distribution* $\int \rho dv = \int \rho_o \left(1 - \left(\frac{r}{a}\right)^7\right) dv = \rho_o 2\pi \int drr \left(1 - \left(\frac{r}{a}\right)^7\right)$

$$\int drr \left(1 - \left(\frac{r}{a}\right)^7\right) = \frac{a^2}{2} - \frac{a^9}{9a^7} = \frac{a^2}{2} - \frac{a^2}{9} = \frac{7a^2}{18}$$

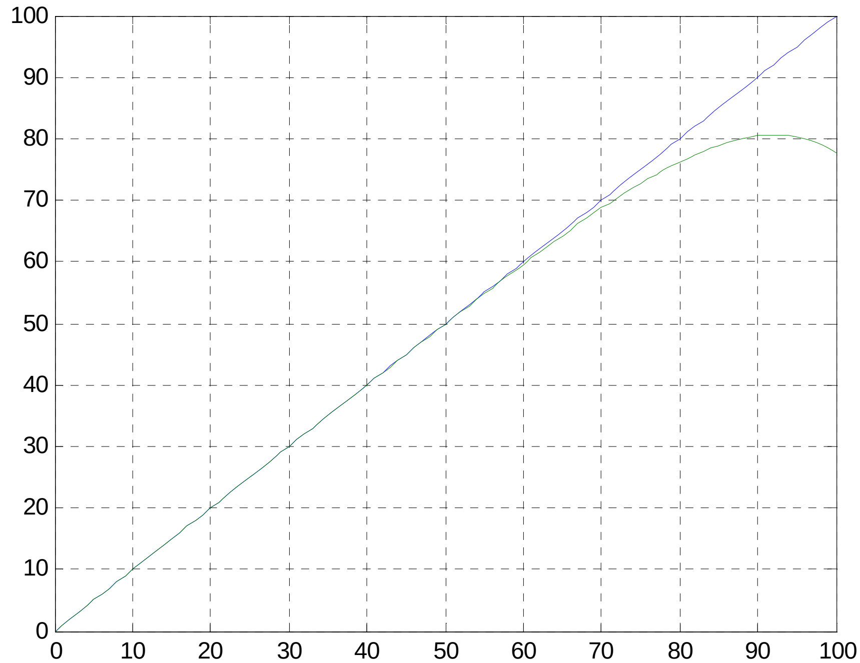
$\int \rho dv = \pi a^2 \rho_o \frac{7}{9}$ *which is* $\frac{7}{9}$ *as much as the uniform distribution. This also shows that the given distribution is indeed pretty close to uniform.*

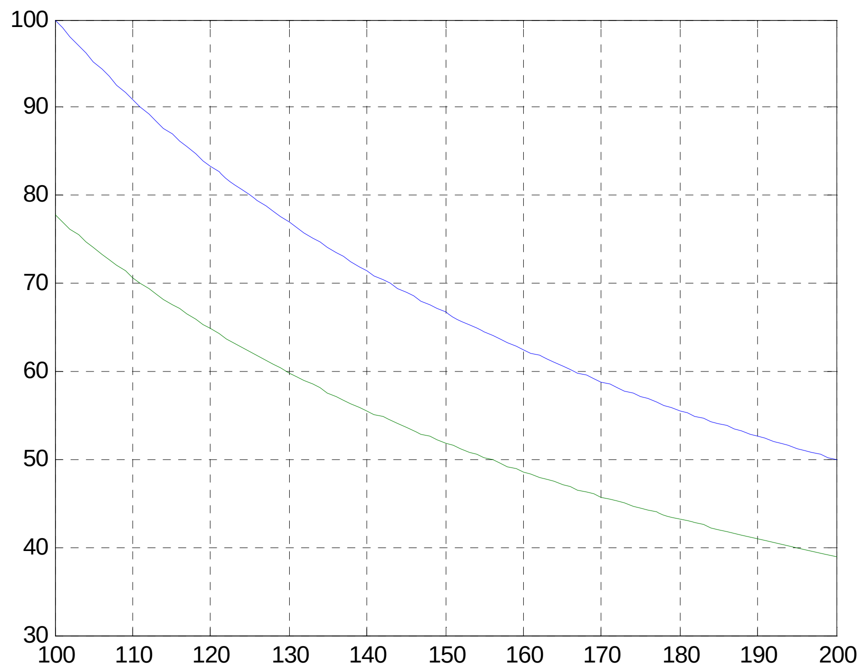
b. Using Gauss' Law in integral form, determine the electric field $\vec{E}$ for all values of radius for both charge distributions. Plot the magnitude of the electric field as a function of radius $\left|\vec{E}\right|$.



To solve for the electric field using Gauss' Law, we need the total charge and the charge enclosed by a Gaussian surface with a radius smaller than the charge distribution (0<r<a). We can use the integrals above and just replace the upper limit a by r. Then $\int \rho dv = \int \rho_o \left(1 - \left(\frac{r}{a}\right)^7\right) dv = \rho_o \left(\pi r^2 - 2\pi \frac{r^9}{9a^7}\right) = \pi r^2 \rho_o \left(1 - \frac{2r^7}{9a^7}\right)$ *and* $\int \rho dv = \int \rho_o dv = \pi r^2 \rho_o$ *for the two charge distributions. The left hand side of Gauss' Law is* $\oint \vec{D} \cdot d\vec{S} = D_r 2\pi r = \varepsilon_o E_r 2\pi r$ *for all radii. Thus, for the uniform charge distribution* $E_r(r) = \frac{\rho_o r}{2\varepsilon_o}$ *inside the charge and* $E_r(r) = \frac{\rho_o a^2}{2\varepsilon_o r}$ *outside the charge. For the*

given distribution $E_r(r)=\dfrac{\rho_o r}{2\varepsilon_o}\left(1-\dfrac{2}{9}\dfrac{r^7}{a^7}\right)$ *inside the charge and* $E_r(r)=\dfrac{\rho_o a^2}{2\varepsilon_o r}\left(\dfrac{7}{9}\right)$. *The two field expressions in each region are very similar to one another. Plotting them using Matlab, we see the following:*





Again, the non-uniform distribution is more rounded.
c. Use Gauss' Law in differential form to check your answer for both cases.

Inside the two charge distributions $\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon_o}$ *while outside* $\nabla\cdot\vec{E}=0$. *Thus, we need to take the divergence of the four expressions, which is given by* $\nabla\cdot\vec{E}=\frac{1}{r}\frac{\partial}{\partial r}\left(rE_r\right)$. *For the uniform charge and* $r\geq a$, $\nabla\cdot\vec{E}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\rho_o a^2}{2\varepsilon_o r}\right)=0$. *For* $r\leq a$

$\nabla\cdot\vec{E}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\rho_o r}{2\varepsilon_o}\right)=\frac{\rho_o}{2\varepsilon_o r}2r=\frac{\rho_o}{\varepsilon_o}$. *For the given charge distribution and* $r\geq a$

$\nabla\cdot\vec{E}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\rho_o a^2}{2\varepsilon_o r}\frac{7}{9}\right)=0$. *For* $r\leq a$ $\nabla\cdot\vec{E}=\frac{\rho_o}{2\varepsilon_o}\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\left(1-\frac{2}{9}\frac{r^7}{a^7}\right)\right)$ *which simplifies to* $\nabla\cdot\vec{E}=\frac{\rho_o}{2\varepsilon_o}\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\left(1-\frac{2}{9}\frac{r^7}{a^7}\right)\right)=\frac{\rho_o}{2\varepsilon_o}\frac{1}{r}\left(\left(2r-2\frac{r^8}{a^7}\right)\right)=\frac{\rho_o}{\varepsilon_o}\left(1-\frac{r^7}{a^7}\right)$ *so all four expressions check out.*

3. Electric Scalar Potential

For both cases in problem 2, find the electric scalar potential as a function of position $V=V(r)$ and then also evaluate the potential at the origin $V=V(0)$. Assume that $V(b)=0$ so voltages are referenced to $r = b$.

To determine the potential as a function of position, we need to evaluate the integral $V=V(r)=-\int_b^r E_r(r)dr$ *where we have assumed that the voltage is zero at r = b. For the uniform charge distribution and* $r\geq a$, $V(r)=-\int_b^r\frac{\rho_o a^2}{2\varepsilon_o r}dr=\frac{\rho_o a^2}{2\varepsilon_o}\ln\frac{b}{r}=\frac{Q_{total}}{2\pi\varepsilon_o}\ln\frac{b}{r}$ *where we have put the solution in the form of a line charge to check it against that result. For* $r\leq a$, $V(r)=\frac{\rho_o a^2}{2\varepsilon_o}\ln\frac{b}{a}-\int_a^r\frac{\rho_o r}{2\varepsilon_o}dr=\frac{\rho_o a^2}{2\varepsilon_o}\ln\frac{b}{a}+\frac{\rho_o}{\varepsilon_o}\left(a^2-r^2\right)$. *The voltage at the origin is then* $V(0)=\frac{\rho_o a^2}{2\varepsilon_o}\ln\frac{b}{a}+\frac{\rho_o}{\varepsilon_o}\left(a^2\right)$. *For the given non-uniform charge and* $r\geq a$, $V=V(r)=-\int_b^r\frac{\rho_o a^2}{2\varepsilon_o r}\left(\frac{7}{9}\right)dr=\frac{\rho_o a^2}{2\varepsilon_o}\left(\frac{7}{9}\right)\ln\frac{b}{r}$ *which again looks just like the other result except for being a little smaller. For* $r\leq a$, *things are a bit more complex.*

$$V = V(r) = V(a) - \int_a^r \frac{\rho_o r}{2\varepsilon_o}\left(1 - \frac{2}{9}\frac{r^7}{a^7}\right)dr = \frac{\rho_o a^2}{2\varepsilon_o}\left(\frac{7}{9}\right)\ln\frac{b}{a} - \frac{\rho_o}{2\varepsilon_o}\int_a^r \left(r - \frac{2}{9}\frac{r^8}{a^7}\right)dr$$

$$V(r) = \frac{\rho_o a^2}{2\varepsilon_o}\left(\frac{7}{9}\right)\ln\frac{b}{a} - \frac{\rho_o}{2\varepsilon_o}\left(\frac{r^2}{2} - \frac{a^2}{2} - \frac{2r^9}{81a^7} + \frac{2a^2}{81}\right) \text{ so that at } r = 0,$$

$$.V(0) = \frac{\rho_o a^2}{2\varepsilon_o}\left(\frac{7}{9}\right)\ln\frac{b}{a} + \frac{\rho_o}{2\varepsilon_o}\left(\frac{a^2}{2} - \frac{2a^2}{81}\right)$$

4. Charge on a Capacitor Plate

First, before addressing a specific configuration, address the following general question. Assume you are given the voltages at 9 nearby points in space. The points are given by the values of *(x,y) = (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), & (3,3),* where the units are assumed to be centimeters. The voltages at these nine locations are *V= 0, 0, 0, 5, 10, 15, 10, 25, & 40.* For clarity, these points are shown below.

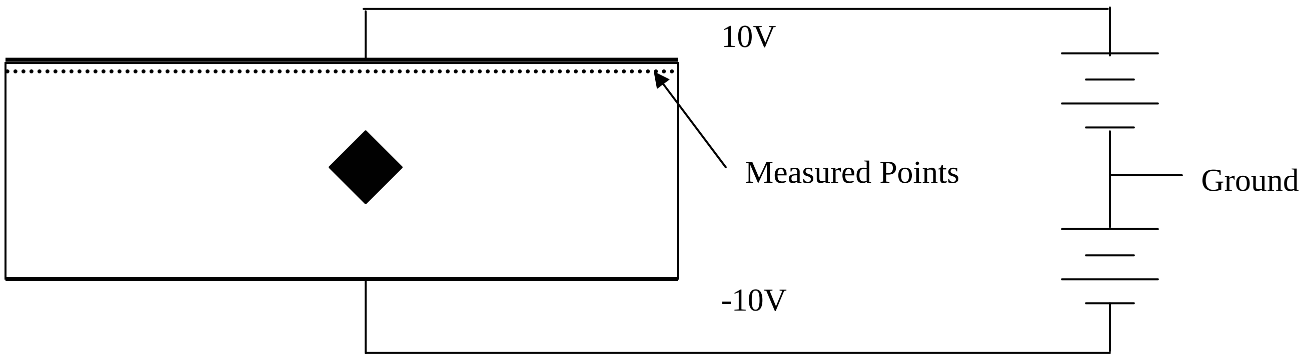
Given this information, determine the best approximate value for the E_x and E_y, the x and y components of the electric field. Be sure that your answers are expressed in SI units.

The best approximation uses all available points. However, there are several reasonable ways to do this. My choice is to evaluate $-\frac{\Delta V}{\Delta x}$ *, for example, on both sides of the center point and then take their average. Thus,* $E_y = -\frac{1}{2}\left(\frac{15-10}{0.01} + \frac{10-5}{0.01}\right) = -500\frac{V}{m}$ *and*

$$E_x = -\frac{1}{2}\left(\frac{25-10}{0.01} + \frac{10-0}{0.01}\right) = -1250\frac{V}{m}$$

Homework 3

A parallel plate capacitor with a metal object embedded in its insulator is connected to 2 10V DC voltage sources, as shown. Little is known about how the capacitor is constructed so we do not know enough to calculate the capacitance from first principles. However, somehow, we are able to measure the voltage at an array of points located *1cm* inside the top surface of the capacitor. The measured voltages are given in the Excel spreadsheet *HW3-s06.xls* found next to this assignment on the *Handout* webpage. Each plate of the capacitor is *24cm* by *24cm*. The voltages are measured at the edge points and points every *cm* to form the *24x24* grid given in the spreadsheet.



a. Determine the average value of the electric field in the region between the measured points and the top plate.

The average value of the voltage for the measured points is 9.05V. This average can easily be found using the average function in Excel. The average difference between the top plate and the measured points is 0.95V. Again this can easily be found using Excel. This difference voltage divided by 0.01m give the average E field of 95V/m.

b. Assume the dielectric constant in the region where the voltage is measured is $\varepsilon = \varepsilon_r \varepsilon_o = 3\varepsilon_o$. Determine the average value of the electric flux density.

The average flux density is obtained by multiplying the electric field by the dielectric constant. Thus $D = 95 \times 3\varepsilon_o = 285\varepsilon_o = 3x10^{-8}$

c. Determine the total charge on the top plate.

The total charge is given by the charge density (which is equal to the flux density in this case) times the area. The area is 5.8x10^{-2}*m*^{2} *so that the charge is* $16.4\varepsilon_o = 1.73x10^{-9}$

d. Find the capacitance.

The capacitance is given by the charge divided by the total voltage or

$$C = \frac{16.4\varepsilon_o}{20} = 0.82\varepsilon_o = \frac{1.73x10^{-9}}{20} = 8.65x10^{-11} = 86.5pF$$

A sanity check for this result is to look at the ideal capacitor with no diamond shaped blob inside. The formula for capacitance is $C = \frac{3\varepsilon_o A}{d} = 0.72\varepsilon_o$ *which is very close to this value and smaller, as it should be. (Why should it be smaller?)*