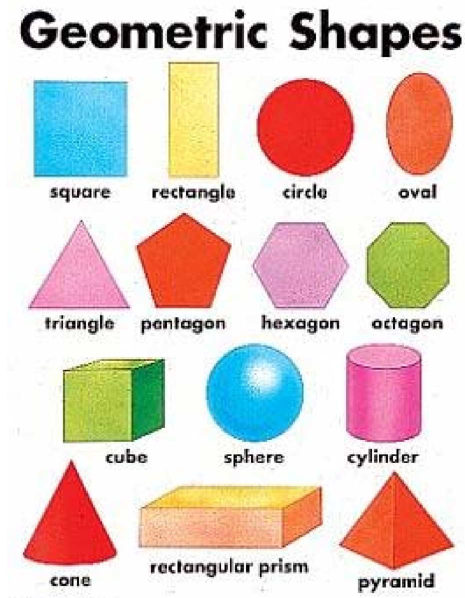
Homework 3

Due 28 February 2006

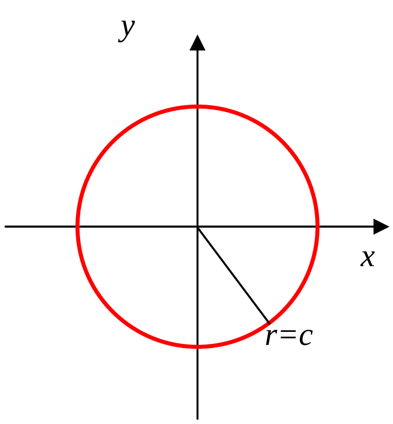
First we will remind ourselves of some shapes and then go over some general ideas.

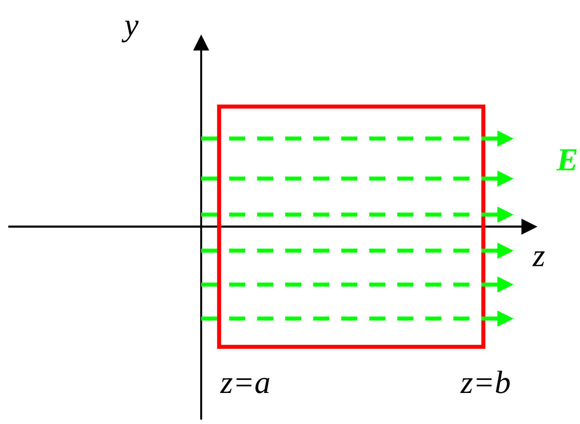


Some general advice on doing basic vector mathematics in Fields and Waves I. *An example is provided in italics.*

- Always draw as many views of each problem as you find necessary to fully understand the configuration.

Assume that we have the following electric field: $\vec{E}(z) = \hat{z}E_o e^{-z/d}$ *where* $\hat{z}$ *is the unit vector in the z direction, only for positive values of z. For negative values, the field is zero. Assume also that we want to find the flux of this vector through the closed cylindrical surface defined by* $a \le z \le b$ *and* $0 \le r \le c$ *where* $a \le d \le b$. *We should begin by drawing the cylinder and adding vectors to show the direction of the E field.*





- Always write out the full expression for the line, surface or volume element before attempting any integrals. Then, for line or surface integrals, take any dot products before doing anything else. This will usually reduce the problem to a more manageable scalar integral.

For a closed cylinder, there are two surface elements. On the cylindrical side of the cylinder: $d\vec{S} = \hat{r}(rd\phi dz)$ *On the flat ends of the cylinder:* $d\vec{S} = \pm\hat{z}(rdrd\phi)$ *where the positive sign goes with the end at the larger value of z and the negative sign at the other end. Then the full integral to be evaluated is:*

$$\oint \vec{E}\cdot d\vec{S} = r\int_0^{2\pi} d\phi \int_a^b dz\hat{r}\cdot\hat{z}E_o e^{-z/d} + \int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}E_o e^{-b/d} - \int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}E_o e^{-a/d}$$

Note that the first term is trivially zero since the unit vectors are orthogonal: $\hat{r}\cdot\hat{z} = 0$

- Simplify the mathematical expressions before you try to solve them. Usually the math, once simplified, will be relatively simple.

Most of these integrals are quite simple since parts are either easy to do or many terms are constants. Consider the second term. $\int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}E_o e^{-b/d}$ *First move all constants outside the integral sign.* $\int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}E_o e^{-b/d} = E_o e^{-b/d}\int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}$ *Then simplify using* $\hat{z}\cdot\hat{z} = 1$ $\int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}E_o e^{-b/d} = E_o e^{-b/d}\left\{\int_0^{2\pi} d\phi\right\}\left\{\int_0^c rdr\right\}$ *where we have also separated the two integrals since they are independent of one another. The integral in the first bracket is* $\left\{\int_0^{2\pi} d\phi\right\} = 2\pi$ *while the integral in the second bracket is* $\left\{\int_0^c rdr\right\} = \frac{c^2}{2}$*. Thus, the surface integral for the flat surface at z = b is* $\int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}E_o e^{-b/d} = E_o e^{-b/d} 2\pi\frac{c^2}{2} = \pi c^2 E_o e^{-b/d}$ *and the integral for the flat surface at z = a is* $-\int_0^{2\pi} d\phi \int_0^c rdr\hat{z}\cdot\hat{z}E_o e^{-a/d} = -\pi c^2 E_o e^{-a/d}$.

Thus, the total surface integral is given by $\oint \vec{E}\cdot d\vec{S} = 0 - \pi c^2 E_o e^{-a/d} + \pi c^2 E_o e^{-b/d}$

- When doing surface integrals, it is usually possible to check one's answer against Maxwell's equations or, if the integrals are used to find a field expression, the differential forms of Maxwell's equations can be used to check answers.

From Maxwell's Equations $\oint \vec{E}\cdot d\vec{S} = \frac{Q_{encl}}{\varepsilon}$ *where* $Q_{encl} = \int \rho_v dv$ *is the charge enclosed by the volume. We were not given the charge density, but we can figure it out from*

$\frac{\rho_v}{\varepsilon} = \nabla\cdot\vec{E} = \frac{\partial}{\partial z}E_z = \frac{\partial}{\partial z}E_o e^{-z/d} = E_o\left(-\frac{1}{d}\right)e^{-z/d}$ *Now we can evaluate*

$Q_{encl} = \int \rho_v dv = \varepsilon\left(-\frac{E_o}{d}\right)\int_a^b dze^{-z/d}\int_0^{2\pi} d\phi\int_0^c rdr = \varepsilon\left(-\frac{E_o}{d}\right)(-d)\left(e^{-b/d} - e^{-a/d}\right)2\pi\frac{c^2}{2}$ *which simplifies to* $\frac{Q_{encl}}{\varepsilon} = (E_o)\left(e^{-b/d} - e^{-a/d}\right)\pi c^2$ *which is what we obtained above.*

To evaluate integrals, use Maple or go to http://eqworld.ipmnet.ru/en/auxiliary/aux-integrals.htm to find a table of many integrals. One page you may find useful at this URL is for Integrals with square root of $x^2 + a^2$.

Example:

$$\int_0^{\pi} \left(\sin^2 x\right) dx = ?$$

For this integral, we can use equation 8 of the *Integrals with* ***sin***

$$\int \sin^2 ax\, dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int_0^{\pi} \left(\sin^2 x\right) dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi} = \frac{\pi}{2} - \frac{\sin 2\pi}{4} - \frac{0}{2} + \frac{\sin 0}{4} = \frac{\pi}{2}$$

Using Maple

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> int((sin(x))^2,x=0..pi);
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$$-\frac{1}{2}\sin(\pi)\cos(\pi) + \frac{1}{2}\pi$$

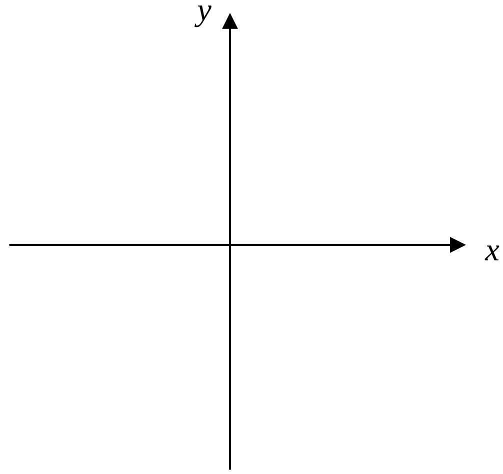
which agrees.

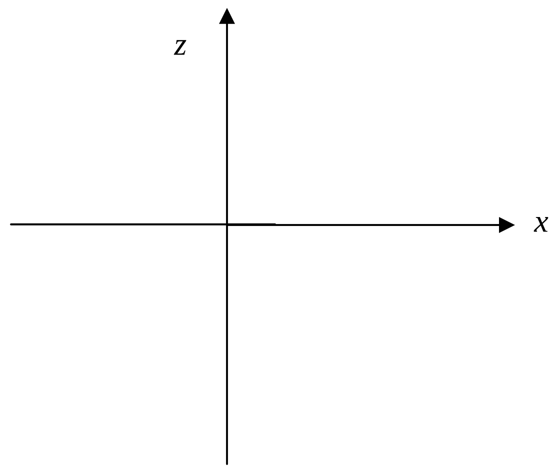
Note: The most mathematically challenging problem in this assignment is problem #1.

Homework 3

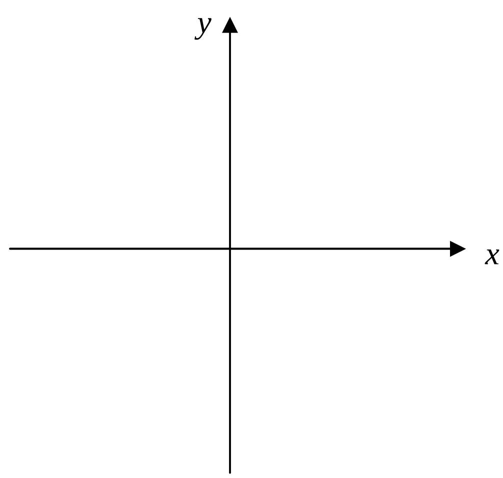
1. Flux Integrals

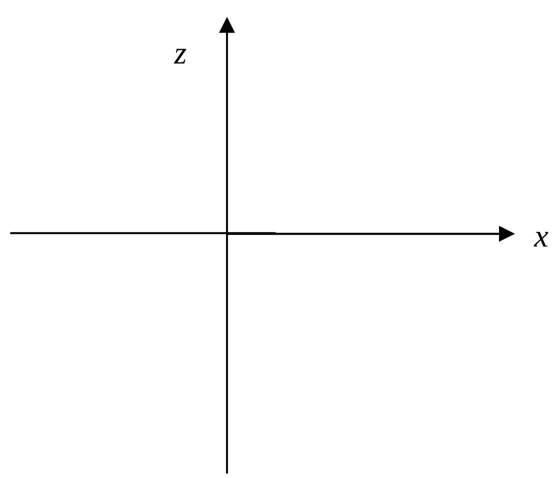
a. The electric field due to a point charge at the origin of a <u>cylindrical</u> coordinate system $(r,\phi,z)=(0,0,0)$ is given by $\vec{E}=\dfrac{q}{4\pi\varepsilon_o R^2}\hat{a}_R$ where $\vec{R}$ is the radial vector in a <u>spherical</u> coordinate system. (Mixed coordinate systems.) Determine the total electric flux $\int \vec{E}\cdot d\vec{S}$ passing through the open <u>cylindrical</u> surface $r=a$, $0\le z<\infty$. Begin by drawing a diagram showing the point charge and the surface in the diagrams below. Also indicate the value of $d\vec{S}$. Recall that the $r = a$ surface goes from $z=0$ to $z=\infty$ and is open at both ends.





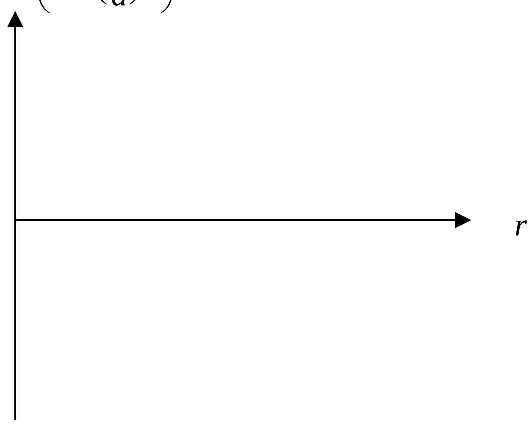
b. The magnetic field outside of a long straight wire of cylindrical radius $r = a$, carrying a current I is given by $\vec{B}=\dfrac{\mu_o I}{2\pi r}\hat{a}_\phi$. Determine the total magnetic flux passing through the surface defined by $y=b$ and $0\le x\le\infty$. Note that we are again mixing coordinate systems, in that we have a field specified in cylindrical coordinates and we are asking for the flux through a rectangular surface. Begin by drawing a picture of the surface in the two planes below. Also indicate the value of $d\vec{S}$. Next express the field in rectangular coordinates. Then, finally, set up the integral and evaluate it.





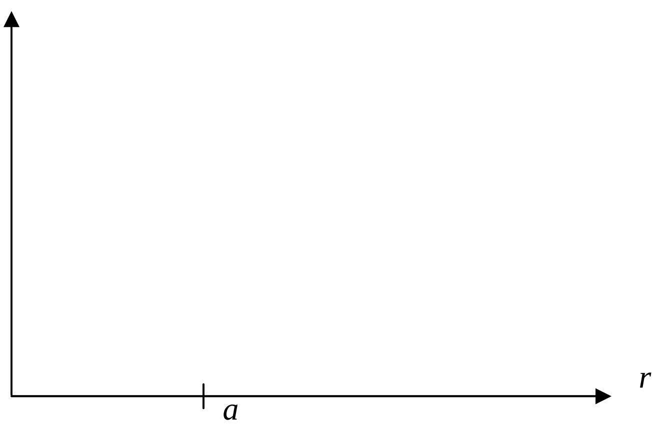
2. The Electric Field due to a Volume Charge Distribution

a. Assume that there is a volume charge distribution in the cylindrical region $0 \le r \le a$ given by $\rho = \rho_o \left(1 - \left(\frac{r}{a}\right)^7\right)$. First, plot this expression as a function of *r*.



Next, determine the total amount of charge per unit length in this distribution. Repeat for a uniform distribution in the same region, that is, for $\rho = \rho_o$. Compare your results for the two cases.

b. Using Gauss' Law in integral form, determine the electric field $\vec{E}$ for all values of radius for both charge distributions. Plot the magnitude of the electric field as a function of radius $|\vec{E}|$.



c. Use Gauss' Law in differential form to check your answer for both cases.

3. Electric Scalar Potential

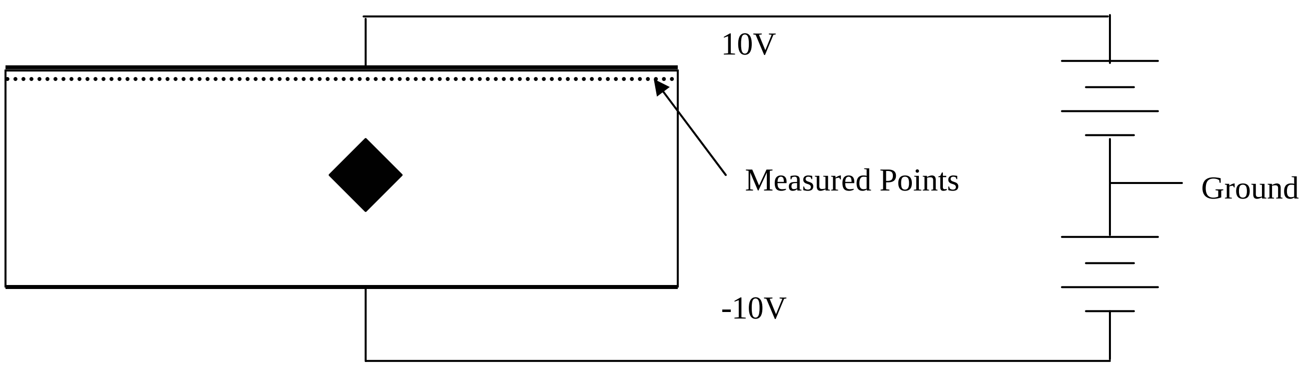
For both cases in problem 2, find the electric scalar potential as a function of position $V = V(r)$ and then also evaluate the potential at the origin $V = V(0)$. Assume that $V(b) = 0$ so voltages are referenced to $r = b$.

4. Charge on a Capacitor Plate

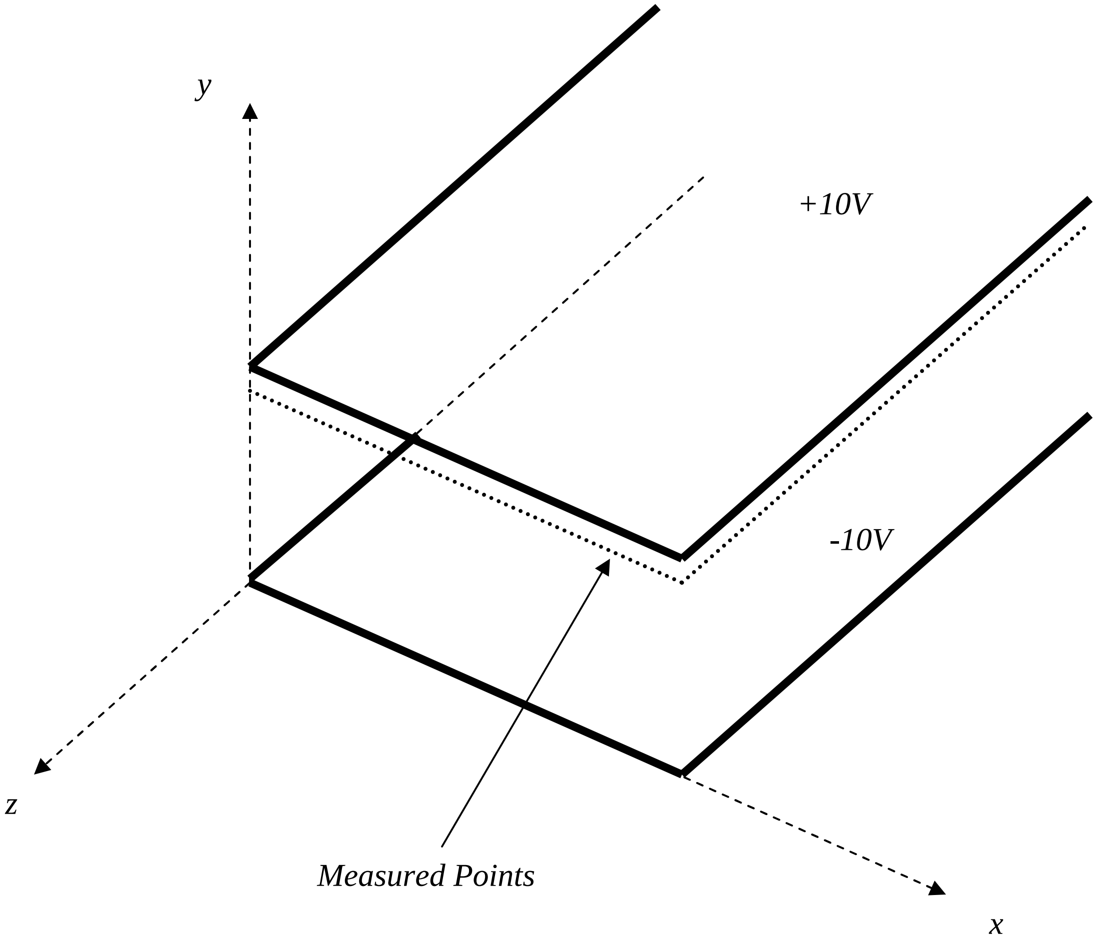
First, before addressing a specific configuration, address the following general question. Assume you are given the voltages at 9 nearby points in space. The points are given by the values of *(x,y) = (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), & (3,3)*, where the units are assumed to be centimeters. The voltages at these nine locations are *V= 0, 0, 0, 5, 10, 15, 10, 25, & 40.* For clarity, these points are shown below.

Given this information, determine the best approximate value for the E_x and E_y, the x and y components of the electric field. Be sure that your answers are expressed in SI units.

A parallel plate capacitor with a metal object embedded in its insulator is connected to 2 10V DC voltage sources, as shown. Little is known about how the capacitor is constructed so we do not know enough to calculate the capacitance from first principles. However, somehow, we are able to measure the voltage at an array of points located *1cm* inside the top surface of the capacitor. The measured voltages are given in the Excel spreadsheet *HW3-s06.xls* found next to this assignment on the *Handout* webpage. Each plate of the capacitor is *24cm* by *24cm.* The voltages are measured at the edge points and points every *cm* to form the *24x24* grid given in the spreadsheet.



a. Determine the average value of the electric field in the region between the measured points and the top plate.
b. Assume the dielectric constant in the region where the voltage is measured is $\varepsilon = \varepsilon_r \varepsilon_o = 3\varepsilon_o$. Determine the average value of the electric flux density.
c. Determine the total charge on the top plate.
d. Find the capacitance.


y
+10V
-10V
z
Measured Points
x