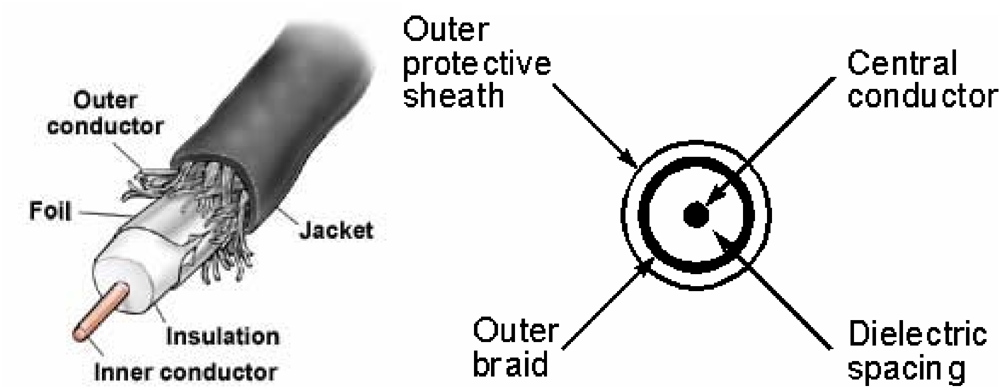
Homework 4

Due 7 March 2005

Note: It is very important that you fully understand the method for determining the capacitance of standard structures like parallel plates, coax and two-wire lines. The first two are discussed in examples 4-11 and 4-12 of Ulaby and in the lecture slides. The two-wire line is also discussed in the lecture slides. Also, read over the handout found at http://hibp.ecse.rpi.edu/%7Econnor/education/Fields/gauss_law.pdf. There are no questions of this type in the homework, since they are done in examples. However, you can expect to see similar questions on tests. If you do not understand this material, ask questions in lecture and during the Tuesday studio sessions.

1. Short Questions



a. When we were studying transmission lines, we learned that commercial coaxial cable specifications generally include velocity and capacitance per unit length. We now know how to determine capacitance and the velocity is determined by the material properties. Assume that we have a cable with velocity equal to 2/3 the speed of light and that the capacitance is 80 pF/m. Determine the inductance and the characteristic impedance.

The inductance per unit length is given by $l = \frac{1}{v^2 c} = \frac{1}{(2x10^8)^2(80x10^{-12})} = 0.3 \, {}^{\mu H}\!/_{m}$

The characteristic impedance $Z_o = \frac{1}{vc} = 62.5 \, {}^{\Omega}\!/_{m}$

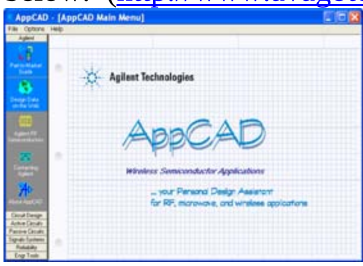
b. Assume we are free to redesign the cable in part a and we would like its characteristic impedance to be 50 Ohms. To make this happen, will we have to increase or decrease the capacitance per unit length of the cable? *Increase, since the capacitance is in the denominator.*

We can accomplish the change we desire in Z_o by using a different insulator (change ε) and/or changing the dimensions of the cable. Should ε be increased or decreased?

The capacitance for a coax is given by $c = \frac{2\pi\varepsilon}{\ln\frac{b}{a}}$ *so increasing* ε *increase the capacitance.* Should the radius of the inner conductor be increased or decreased? *Show your work.*

Increasing the inner diameter will increase the capacitance since it will decrease the ln term in the denominator of the capacitance.

c. Download and install the AppCAD design tool. The first window looks like the one below. (http://www.avagotech.com/pc/downloadDocument.do?id=4219)



Select passive circuits, which will bring up a menu with several kinds of transmission lines on it. Scroll down to the Round Coax and select it. When the Round Coax window appears, change the length units to millimeters. Change the outer diameter to 5mm. Select Polyethylene as the insulator material. (The default is free space.) Adjust the inner diameter until you obtain the Z_o you calculated in part a. Using the formula for the capacitance per unit length of a coaxial cable, show that 80pF/m is the correct value for this cable. Then, adjust the inner diameter to obtain $Z_o = 50\Omega$. Is your answer consistent with your response to part b? What is the inner diameter? If you have a screen capture routine, include a copy of the final window with your answer.

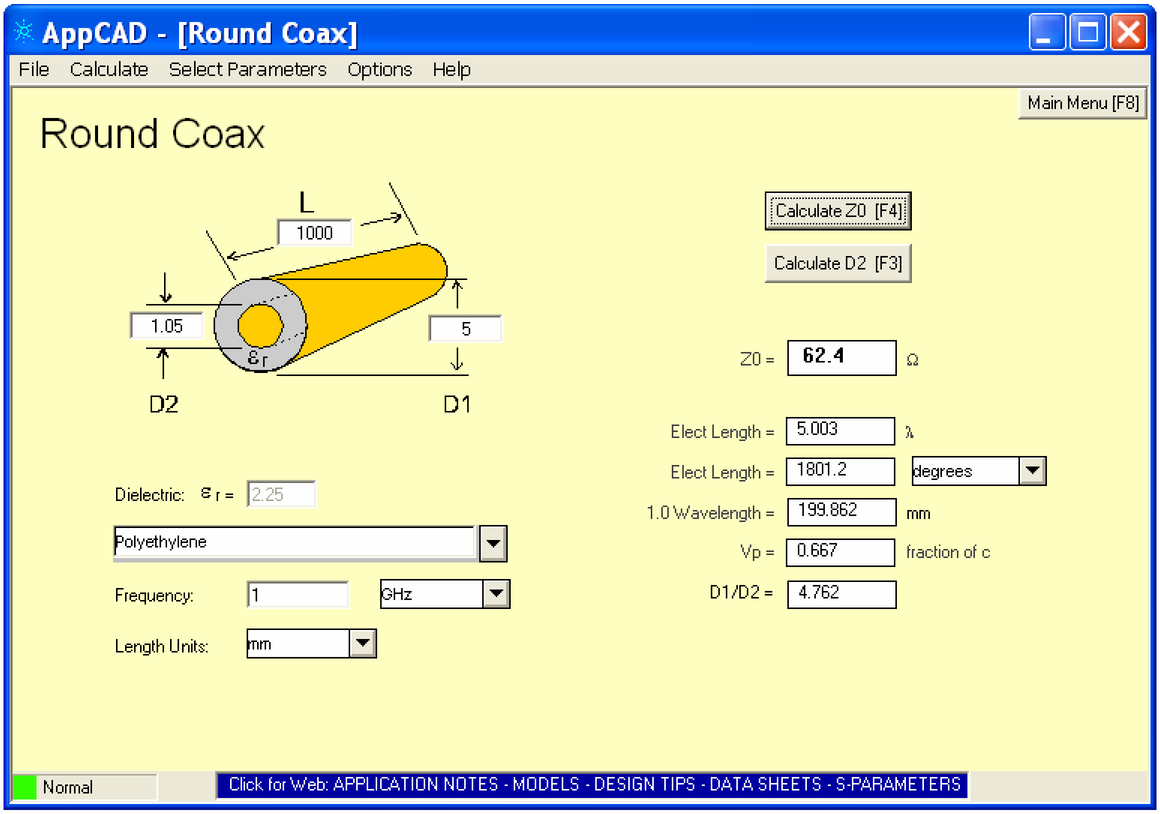
The capacitance per unit length, for the conditions found with AppCAD will be

$$c = \frac{2\pi\varepsilon}{\ln\frac{b}{a}} = \frac{2\pi(2.25)\varepsilon_o}{\ln\frac{5}{1.05}} = 78\,{}^{pF}\!/_{m}$$ *which is as accurate as we have to be.*

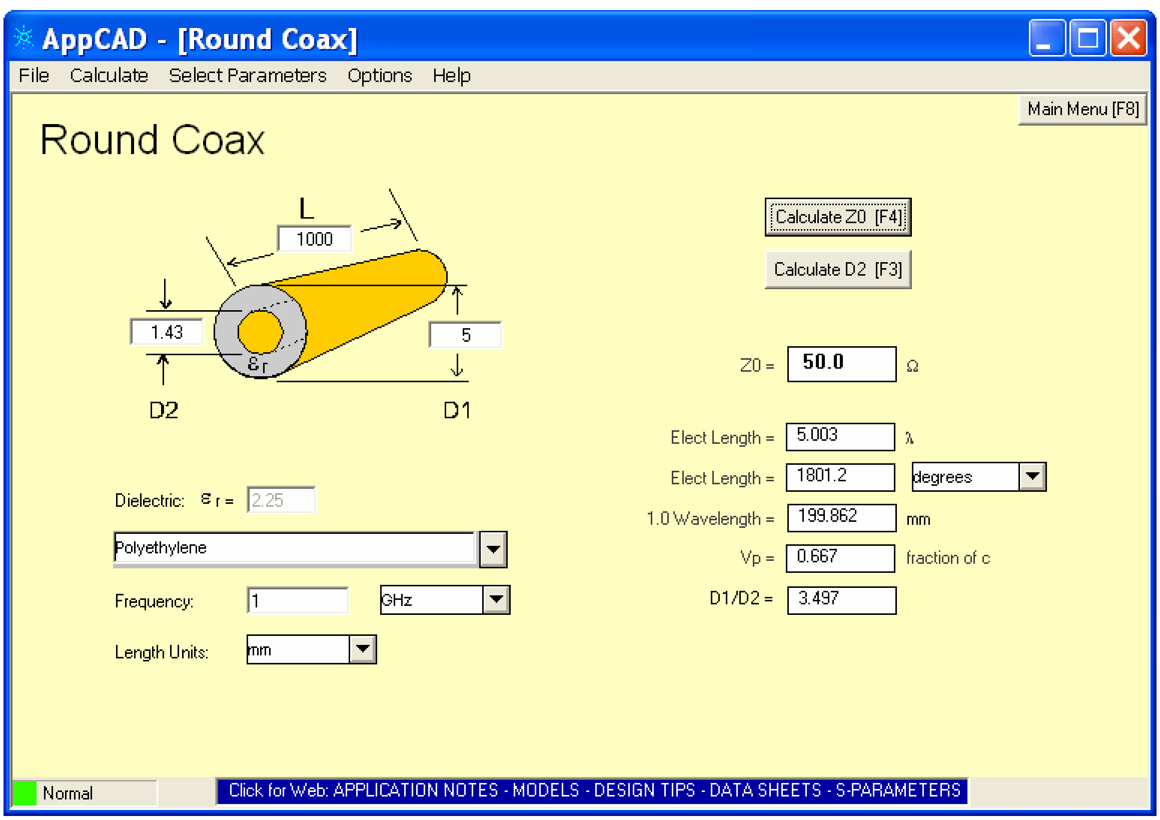
The capacitance per unit length for the lower Z_o *case is*

$$c = \frac{2\pi\varepsilon}{\ln\frac{b}{a}} = \frac{2\pi(2.25)\varepsilon_o}{\ln\frac{5}{1.43}} = 100\,{}^{pF}\!/_{m}$$ *If we use this with the same velocity (since the insulator is the same) we obtain* $Z_o = \frac{1}{vc} = 50\,{}^{\Omega}\!/_{m}$ *which is exact in this case.*

Homework 4



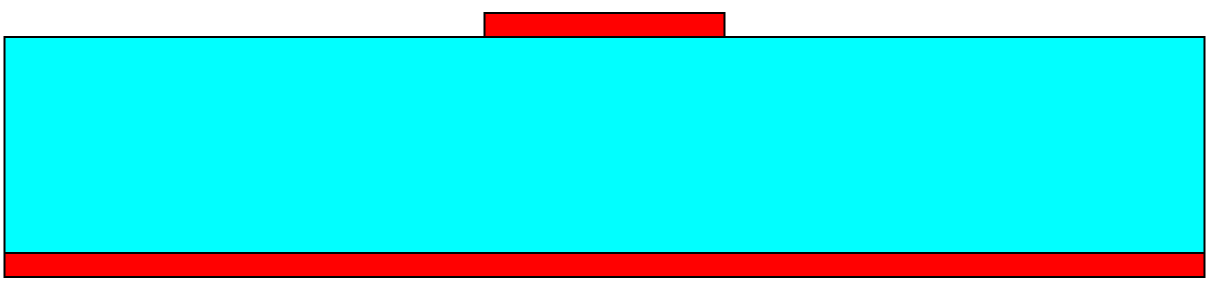
AppCAD - [Round Coax]
File Calculate Select Parameters Options Help
Main Menu [F8]
Round Coax
L
1000
1.05
5
εr
D2
D1
Calculate Z0 [F4]
Calculate D2 [F3]
Z0 = 62.4 Ω
Elect Length = 5.003 λ
Elect Length = 1801.2 degrees
1.0 Wavelength = 199.862 mm
Vp = 0.667 fraction of c
D1/D2 = 4.762
Dielectric: εr = 2.25
Polyethylene
Frequency: 1 GHz
Length Units: mm
Normal
Click for Web: APPLICATION NOTES - MODELS - DESIGN TIPS - DATA SHEETS - S-PARAMETERS

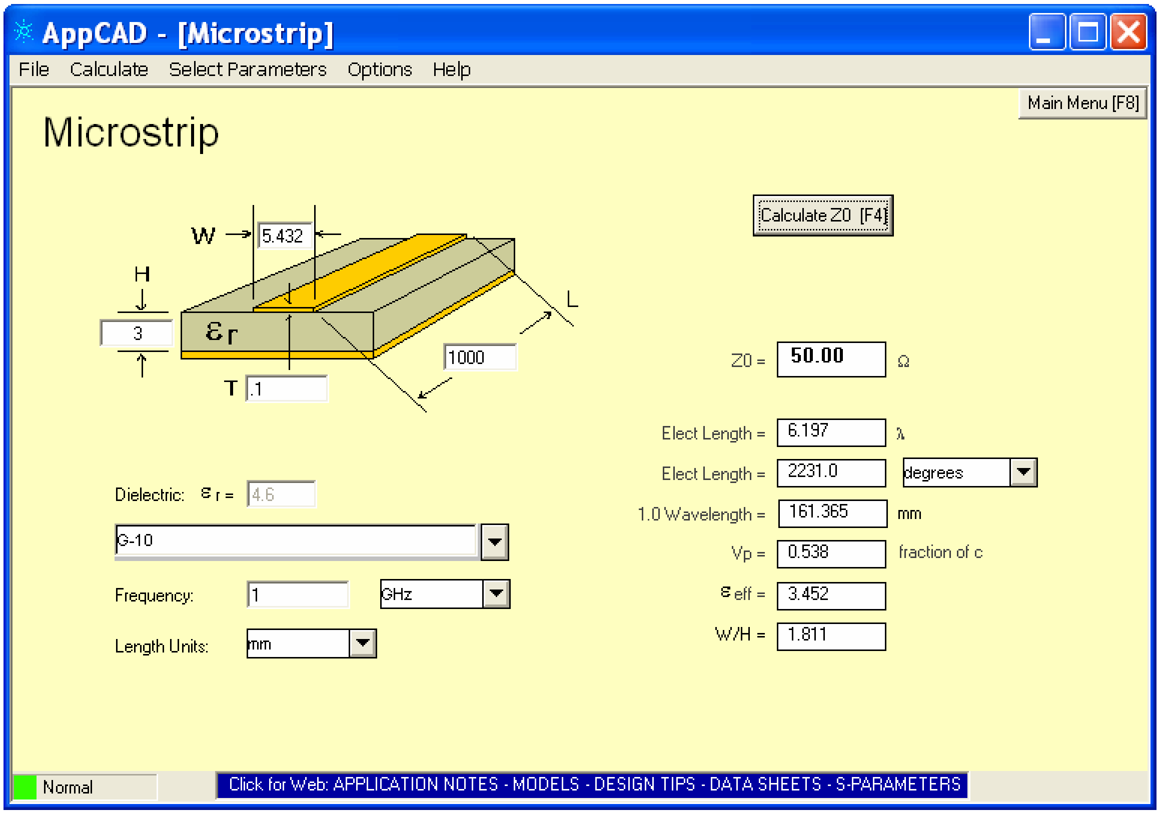


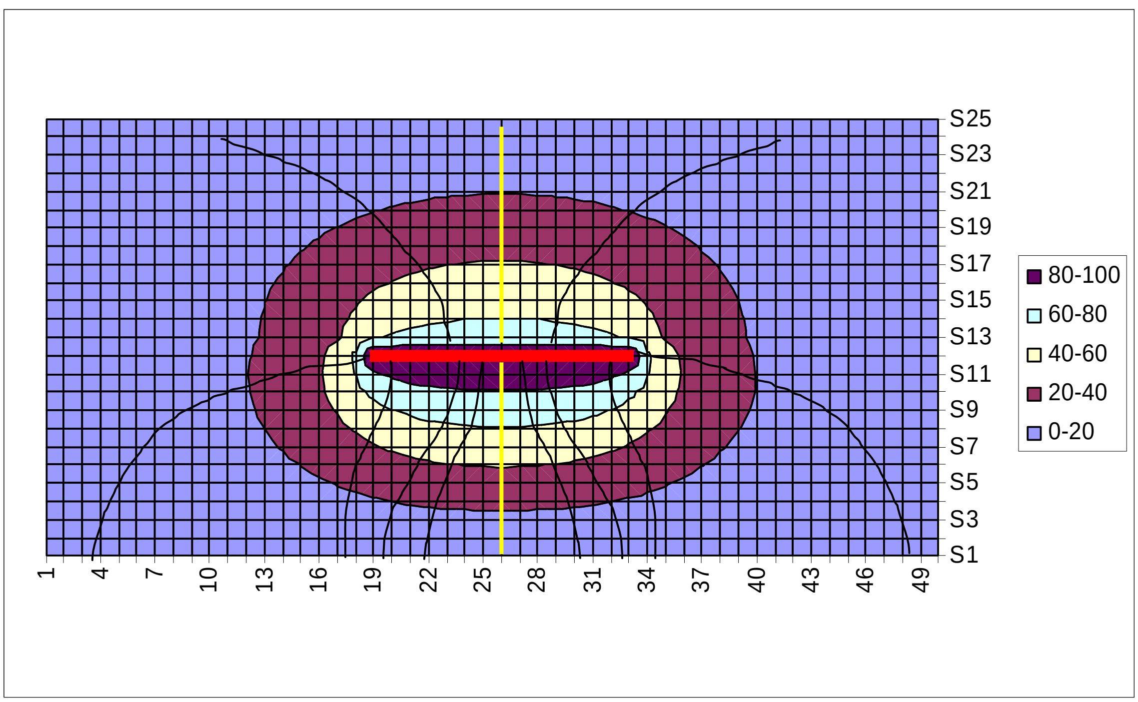
AppCAD - [Round Coax]
File Calculate Select Parameters Options Help
Main Menu [F8]
Round Coax
L
1000
1.43
5
εr
D2
D1
Calculate Z0 [F4]
Calculate D2 [F3]
Z0 = 50.0 Ω
Elect Length = 5.003 λ
Elect Length = 1801.2 degrees
1.0 Wavelength = 199.862 mm
Vp = 0.667 fraction of c
D1/D2 = 3.497
Dielectric: εr = 2.25
Polyethylene
Frequency: 1 GHz
Length Units: mm
Normal
Click for Web: APPLICATION NOTES - MODELS - DESIGN TIPS - DATA SHEETS - S-PARAMETERS

d. Return to the main AppCAD menu. Select the Microstrip. Again change the dimensions to millimeters. Select G-10 as the insulator. G-10 is an excellent machinable plastic that can be used in a wide variety of applications. Select a height H of 3mm and a thickness T of the top strip of 0.1mm. Find the width W of the strip that results in $Z_o = 50\Omega$. On the diagram below, sketch 5 equipotentials and several electric field lines. Be careful to account for the conducting (red) and dielectric (blue) material boundaries in your diagram. (It might be good to review example 4-10 and the similar example from the lecture slides.) Using the information in your diagram, explain why AppCAD indicates that the effective dielectric constant ε_{eff} is less than the G-10 dielectric constant $\varepsilon_r = 4.6$.

The field lines and equipotentials on the next page show that there is field both above and below the strip. Thus, part of the field is in free space and part is in the G-10 so the effective or average permittivity will be between 1 and 4.6.

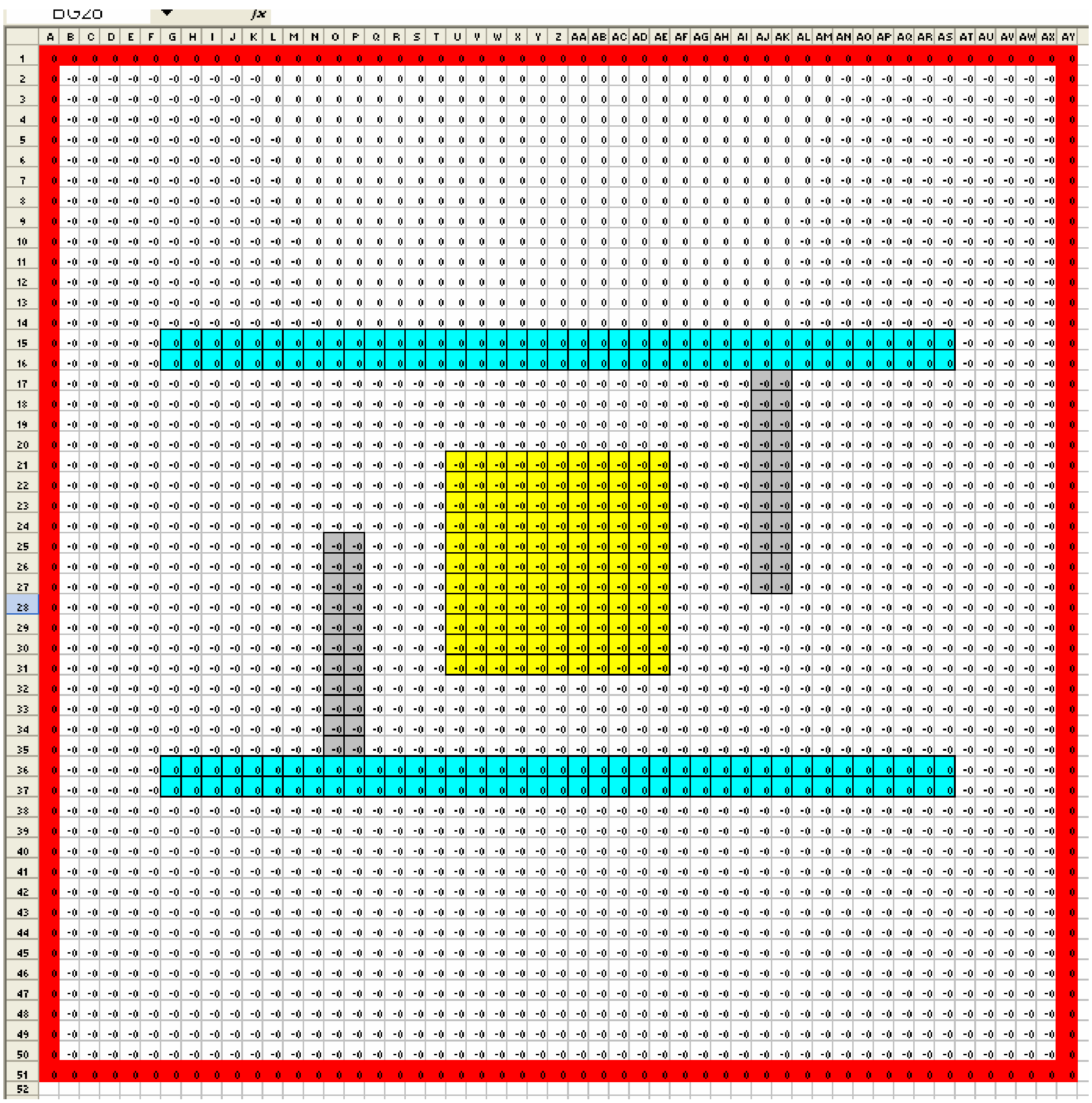






The field lines above were drawn by hand, so they are not perfect. If they were, they would be perpendicular to each of the equipotentials.

2. Using a Spreadsheet to Find Capacitance: Assume that you have the following two dimensional configuration of conductors and dielectrics. The outer (red) surface is grounded. The inner (yellow) square region is some unknown dielectric material. The upper (blue) region is a conductor connected to some positive voltage. The lower (blue) region is a conductor connected to some negative voltage. We will assume for simplicity that the positive and negative voltages have the same magnitude.
Reference:
http://hibp.ecse.rpi.edu/%7Econnor/education/Fields/SpreadsheetLaplace.PDF



With the exception of the two gray areas, the remainder (white) of the region is a known dielectric material (could be air, for example). The outer box is *50mm* by *50mm*. All other dimensions can be determined by counting cells in the figure. *Hint: Using Excel, it is possible to obtain such square cells by selecting the column width as 3 and the row height as 20.*

For the solution, refer to the posted spreadsheet file. Note that the two conductors for the capacitor are the upper and lower plates, not the outer grounded shell. The outer shell is just part of the system, like the diamond shaped object in the problem from HW3.

Case 1: Assume that the two gray areas, the inner (yellow) square region and the (white) background region are all an insulating material with $\varepsilon = \varepsilon_o$ (probably free space).

a. Use the spreadsheet method to find the capacitance per unit length of this shielded, two wire transmission line.
b. Produce a plot showing at least 8 equipotentials. Sketch a representative set of electric field lines on this plot.
c. Find the charge per unit length.

The average normal electric field for the top plate is equal to 1274 V/m. Thus, the charge density is $\rho_s = \varepsilon_o E_n = 1274\varepsilon_o$ *The charge per unit length requires that we use the entire surface of the top plate since that is what we used for the average electric field. The surface is 80mm wide. Thus, the charge per unit length is* $0.08\rho_s = (0.08)1274\varepsilon_o = 102\varepsilon_o$ *The capacitance per unit length is given by the charge divided by the voltage difference or* $c/l = \frac{102\varepsilon_o}{20} = 5.05\varepsilon_o$

Case 2: Assume the same conditions as Case 1, except that the inner (yellow) square region is now filled with water ($\varepsilon = 81\varepsilon_o$).

d. Use the spreadsheet method to find the capacitance per unit length of this shielded, two wire transmission line.
e. Produce a plot showing at least 8 equipotentials. Sketch a representative set of electric field lines on this plot.
f. Find the charge per unit length.

The average normal electric field for the top plate is equal to 1455 V/m. Thus, the charge density is $\rho_s = \varepsilon_o E_n = 1455\varepsilon_o$ *The charge per unit length requires that we use the entire surface of the top plate since that is what we used for the average electric field. The surface is 80mm wide. Thus, the charge per unit length is* $0.08\rho_s = (0.08)1455\varepsilon_o = 116\varepsilon_o$ *The capacitance per unit length is given by the charge divided by the voltage difference or* $c/l = \frac{116\varepsilon_o}{20} = 5.8\varepsilon_o$

Case 3: Assume the same conditions as Case 1, except that the two gray areas are now conductors connected to the top and bottom plates, respectively. That is, the upper gray area is at the potential of the upper plate and the lower gray area is at the potential of the lower plate.

g. Use the spreadsheet method to find the capacitance per unit length of this shielded, two wire transmission line.
h. Produce a plot showing at least 8 equipotentials. Sketch a representative set of electric field lines on this plot.
i. Find the charge per unit length.

The average normal electric field for the top plate is equal to 1364 V/m. Thus, the charge density is $\rho_s = \varepsilon_o E_n = 1364\varepsilon_o$ *The charge per unit length requires that we use the entire surface of the top plate since that is what we used for the average electric field. The surface is 101mm wide. Thus, the charge per unit length is*
$0.101\rho_s = (0.101)1364\varepsilon_o = 138\varepsilon_o$
The capacitance per unit length is given by the charge divided by the voltage difference or $c/l = \dfrac{138\varepsilon_o}{20} = 6.9\varepsilon_o$

Case 4: Assume the same conditions as Case 3, except that the inner (yellow) square region is now filled with water ($\varepsilon = 81\varepsilon_o$).

j. Use the spreadsheet method to find the capacitance per unit length of this shielded, two wire transmission line.
k. Produce a plot showing at least 8 equipotentials. Sketch a representative set of electric field lines on this plot.
l. Find the charge per unit length.

The average normal electric field for the top plate is equal to 1553 V/m. Thus, the charge density is $\rho_s = \varepsilon_o E_n = 1553\varepsilon_o$ *The charge per unit length requires that we use the entire surface of the top plate since that is what we used for the average electric field. The surface is 101mm wide. Thus, the charge per unit length is*
$0.101\rho_s = (0.101)1553\varepsilon_o = 157\varepsilon_o$
The capacitance per unit length is given by the charge divided by the voltage difference or $c/l = \dfrac{157\varepsilon_o}{20} = 7.8\varepsilon_o$

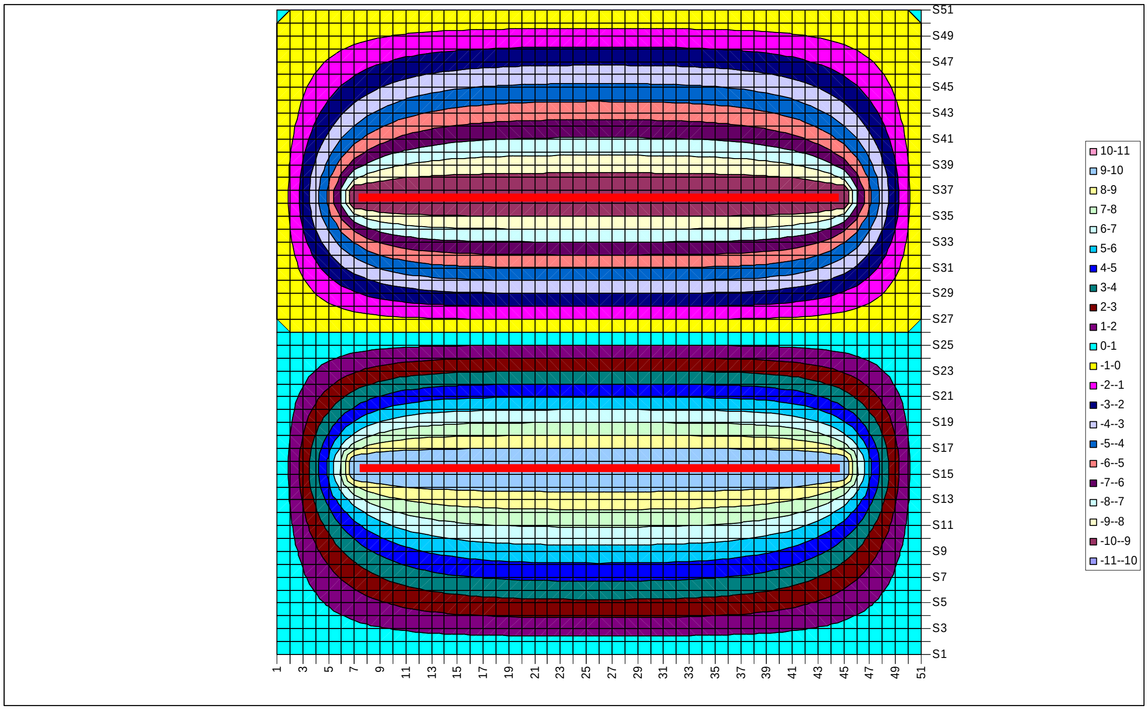
The purpose of these configurations is to determine when water is present in the inner (yellow) square region. For example, one might want to do this in a basement that is prone to flooding during heavy rains. Compare the percentage capacitance change observed with the two conductor configurations. If we also knew the different costs associated with the two configurations, we could decide which is preferable.

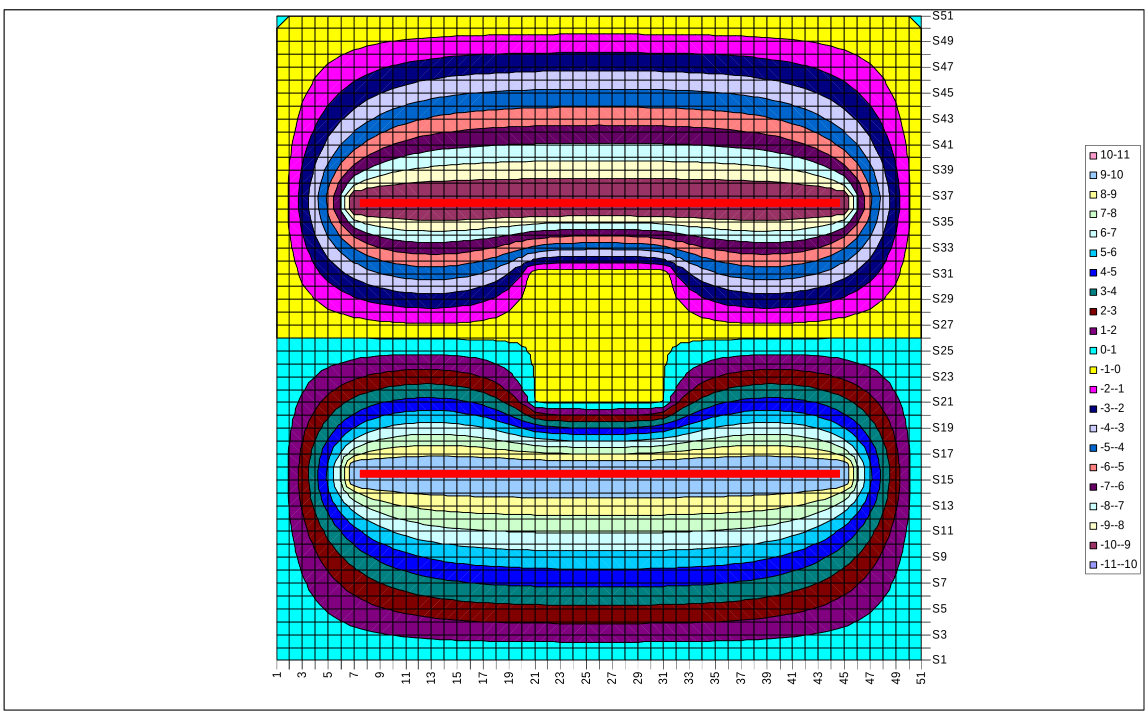
For the first two cases, the change is 0.75/5.05 or 15%

For the second two cases the change is 0.9/6.9 or 13%

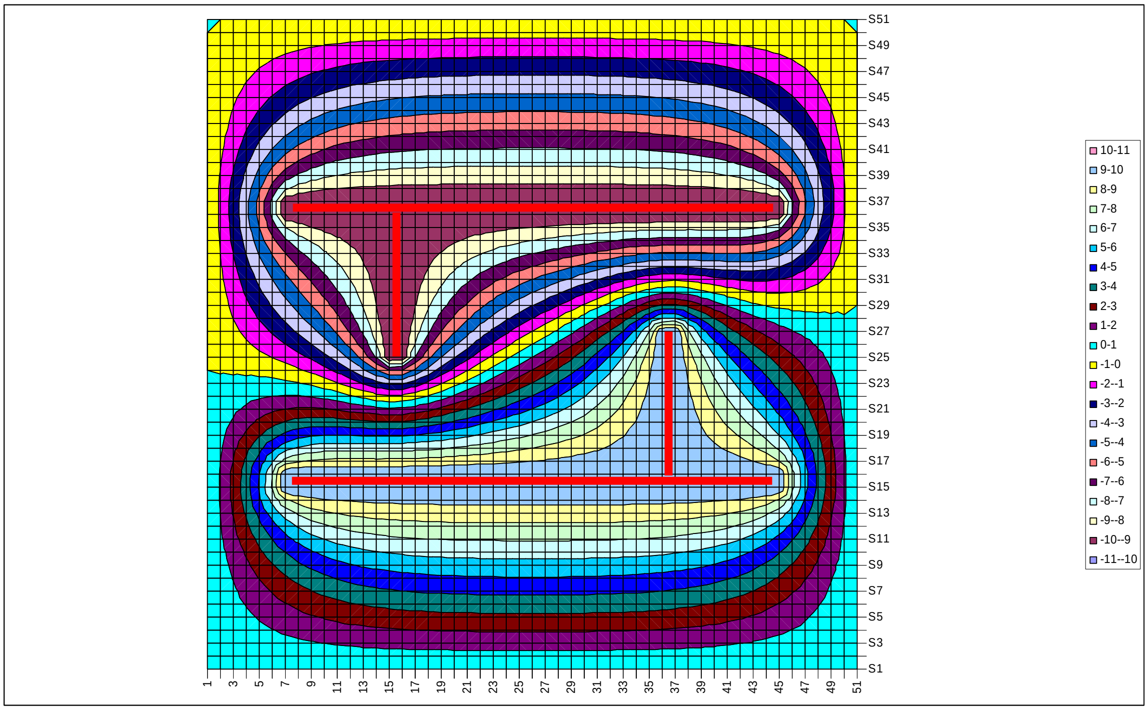
It looks like the simpler system is better based on percentage, but the absolute change is greater for the more complex system.

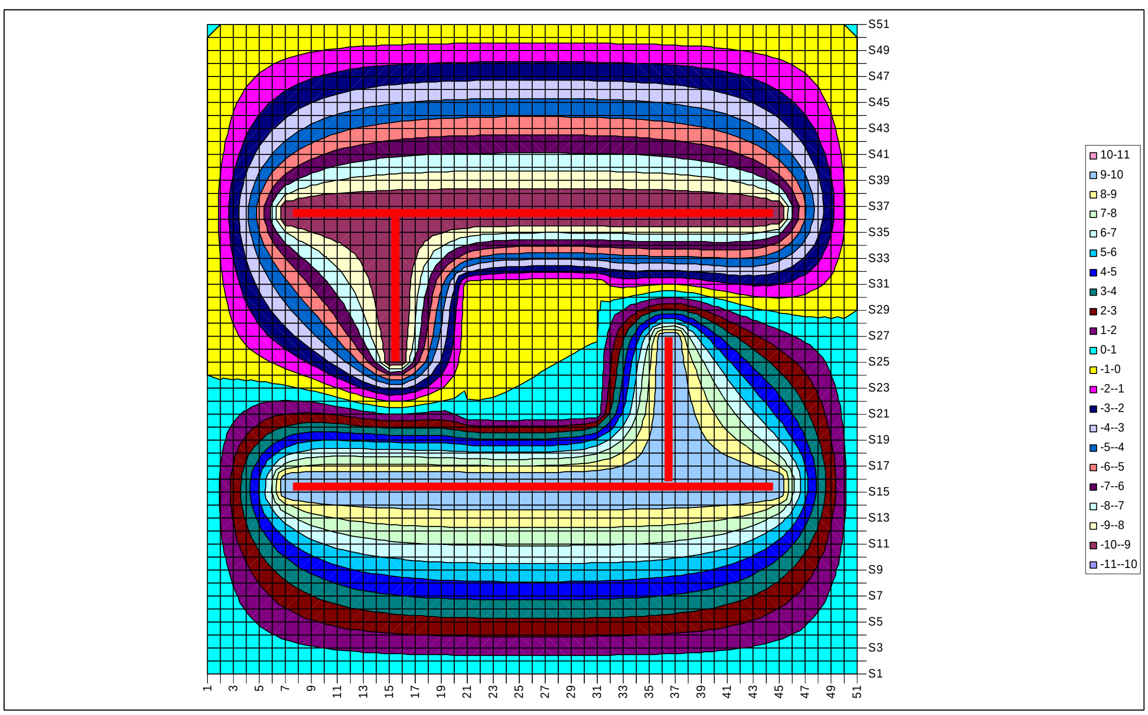
Note, E field lines must still be drawn on these figures.





Homework 4


S51
S49
S47
S45
S43
S41
S39
S37
S35
S33
S31
S29
S27
S25
S23
S21
S19
S17
S15
S13
S11
S9
S7
S5
S3
S1
1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45 47 49 51
10-11
9-10
8-9
7-8
6-7
5-6
4-5
3-4
2-3
1-2
0-1
-1-0
-2--1
-3--2
-4--3
-5--4
-6--5
-7--6
-8--7
-9--8
-10--9
-11--10


S51
S49
S47
S45
S43
S41
S39
S37
S35
S33
S31
S29
S27
S25
S23
S21
S19
S17
S15
S13
S11
S9
S7
S5
S3
S1
1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45 47 49 51
10-11
9-10
8-9
7-8
6-7
5-6
4-5
3-4
2-3
1-2
0-1
-1-0
-2--1
-3--2
-4--3
-5--4
-6--5
-7--6
-8--7
-9--8
-10--9
-11--10

3. Poisson's Equation and Gauss' Law: The electric scalar potential in a cylindrical region $0 \le r \le a$ is given by $V(r) = V_o\left(1 - \frac{r^2}{a^2}\right)$. This region is known to be filled with charge with an unknown density distribution $\rho = \rho(r)$ and is surrounded by a cylindrical conducting shell at $r = a$.

a. Determine the charge distribution responsible for this potential.

To find the charge distribution, one applies Poisson's Equation: $\nabla^2 V(r) = -\frac{\rho_v}{\varepsilon_o}$

$$\frac{\partial}{\partial r}V(r) = \frac{\partial}{\partial r}V_o\left(1 - \frac{r^2}{a^2}\right) = V_o\left(-\frac{2r}{a^2}\right) \qquad r\frac{\partial}{\partial r}V(r) = V_o\left(-\frac{2r^2}{a^2}\right)$$

$$\frac{\partial}{\partial r}r\frac{\partial}{\partial r}V(r) = V_o\left(-\frac{4r}{a^2}\right) \qquad \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}V(r) = V_o\left(-\frac{4}{a^2}\right) = -\frac{4V_o}{a^2} = -\frac{\rho_v}{\varepsilon_o}$$

Thus, the charge density in the region $0 \le r \le a$ *is* $\rho_v = \frac{4\varepsilon_o V_o}{a^2}$

b. Determine the electric field $\vec{E}(r)$ in the region $0 \le r \le a$.

This is determined from $\vec{E}(r) = -\nabla V(r) = -\hat{r}\frac{\partial V(r)}{\partial r} = \hat{r}V_o\frac{2r}{a^2}$

c. Since the electric field must end at the conducting shell, there must be a surface charge distribution on the conductor. Determine the density of this surface charge.

There are at least two ways to determine the surface charge density. Since the voltage is zero at r = a, the conducting shell is grounded. Inside a grounded shell, the total charge must be zero. Thus, one could integrate over the charge distribution to obtain the total positive charge and then set the surface charge equal to the negative of this value divided by the area of the shell. The approach used here is simpler, but we will do both just to check the answer. The surface charge density is equal to the normal component of the electric flux density from the boundary condition. Since the field vector points into the conductor, the charge must be negative.

$$\vec{D}(r)\cdot\hat{n} = D_n = -\rho_s = \varepsilon_o E(a) = \varepsilon_o V_o\frac{2a}{a^2} = \varepsilon_o V_o\frac{2}{a}$$

From this density, the total charge must be equal to $Q_s = -\varepsilon_o V_o\frac{2}{a}2\pi a = -4\pi V_o\varepsilon_o$

We can check this answer by finding the total volume charge

$Q_v = \int \rho_v dv = \pi a^2\frac{\varepsilon_o 4V_o}{a^2} = 4\pi\varepsilon_o V_o$ *so the total charge is zero.*