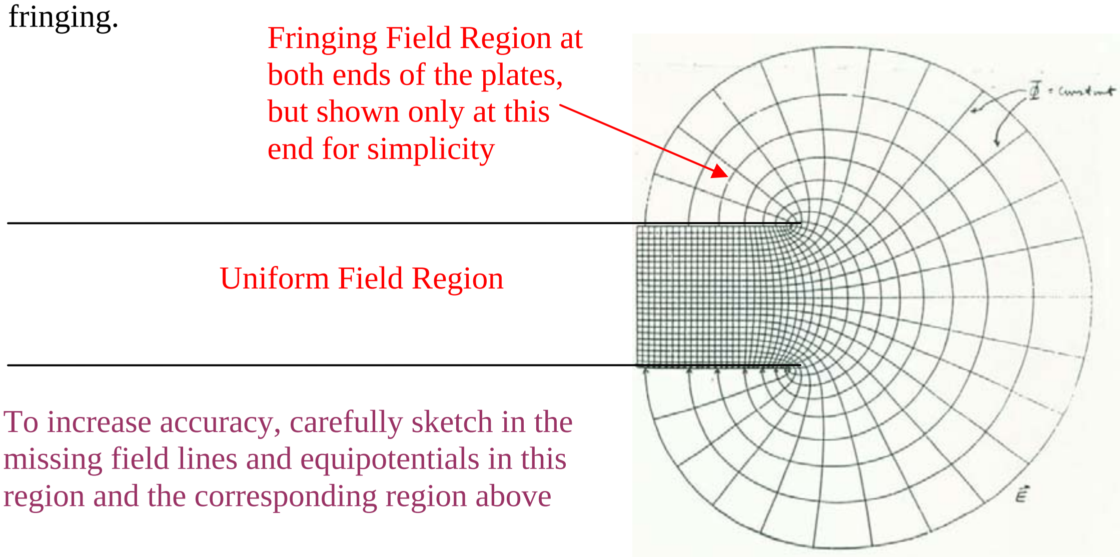
Homework 5

Due 22 March 2005

1. **Resistance Measurement** It is possible to determine the salt content in a body of water by the resistance between two conducting parallel plates immersed in the water. To assure that the analysis is as accurate as possible, we want to include the effects of fringing.



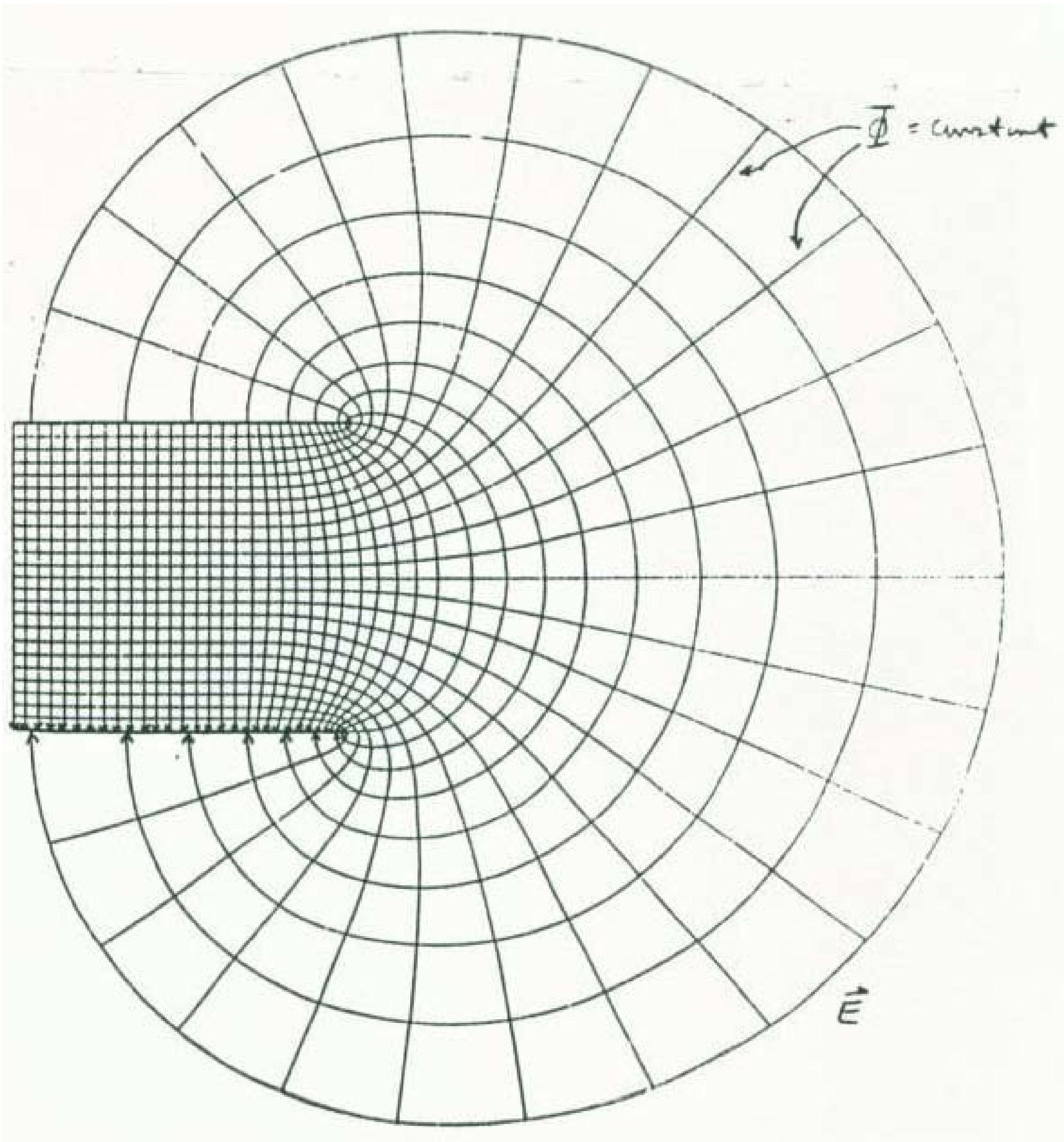
Some background from http://lakeaccess.org/russ/conductivity.htm

Water Body	EC (uS/cm)	TDS (mg/L)
Lake Superior	97	63
Lake Tahoe	92	64
Lake Mead	850	640
Atlantic Ocean	43000	35000
Great Salt Lake	158000	230000

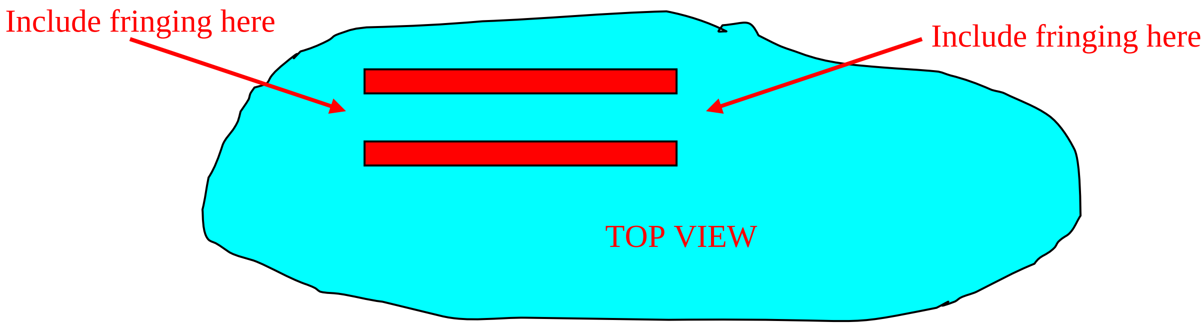
Superior and Tahoe are ultraoligotrophic (nutrient poor) lakes; Mead is an unproductive reservoir (the largest in the U.S.) but has a high TDS (total dissolved salts) due to the salt content of the Colorado River which provides >98% of its water; the Atlantic Ocean overlies the lost Kingdom of Atlantis and possibly Jimmy Hoffa; the Great Salt Lake is an enormous hypersaline lake near Salt Lake City, Utah - it is the relic of what was once a huge inland freshwater sea that dried up, thereby concentrating the remaining salts after the water evaporated.

What in the world are microSiemens per centimeter (µS/cm)?

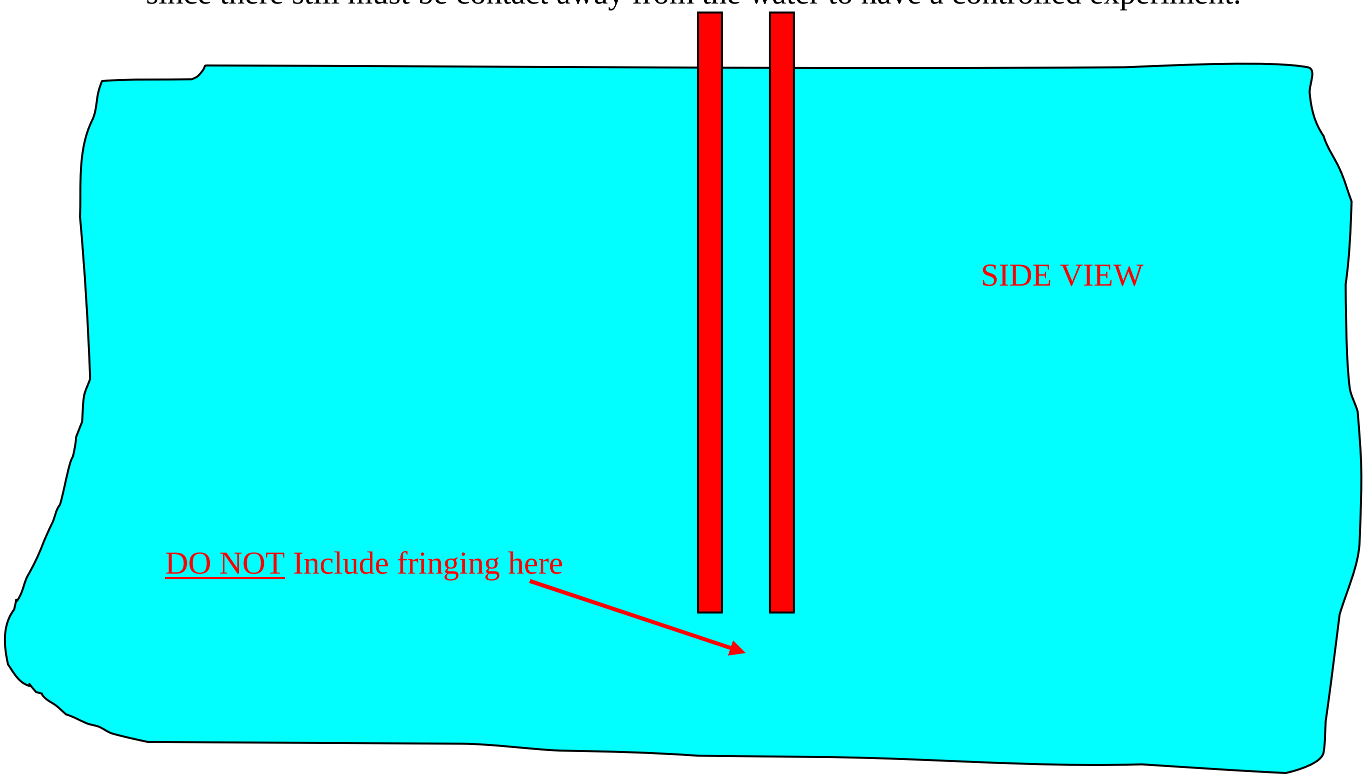
These are the units for electrical conductivity (EC). The sensor simply consists of two metal electrodes that are exactly 1.0 cm apart and protrude into the water. A constant voltage (V) is applied across the electrodes. An electrical current (I) flows through the water due to this voltage and is proportional to the concentration of dissolved ions in the water - the more ions, the more conductive the water resulting in a higher electrical current which is measured electronically. Distilled or deionized water has very few dissolved ions and so there is almost no current flow across the gap (low EC). As an aside, fisheries biologists who electroshock know that if the water is too soft (low EC) it is difficult to electroshock to stun fish for monitoring their abundance and distribution. Up until about the late 1970's the units of EC were micromhos per centimeter (µmhos/cm) after which they were changed to microSiemens/cm (1µS/cm = 1 µmho/cm). You will find both sets of units in the published scientific literature although their numerical values are identical. Interestingly, the unit "mhos" derives from the standard name for electrical resistance reflecting the inverse relationship between resistance and conductivity - the higher the resistance of the water, the lower its conductivity. This also follows from Ohm's Law, V = I x R where R is the resistance of the centimeter of water. Since the electrical current flow (I) increases with increasing temperature, the EC values are automatically corrected to a standard value of 25°C and the values are then technically referred to as specific electrical conductivity.



Note that there are 24 equipotential steps between the two plates in this flux plot. As noted below, the separation between the plates will be 1.2 cm so that it is easier to figure out the size of each little box since we do not show all of the boxes in the uniform field region.



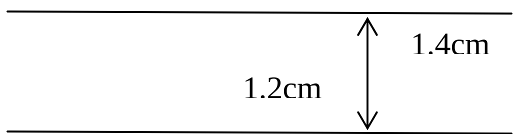
Two capacitor plates are immersed in the bodies of water and the resistance between them is measured to determine the salt content. The plates are only partially submerged, since there still must be contact away from the water to have a controlled experiment.



Assume that the field pattern around the plates looks like the pattern given above and that the plates are immersed 50 cm into the water, the plates are 5 cm wide and separated by 1.2 cm. Treat the problem as two dimensional. That is, ignore the fringing at the bottom edge of the plates, only considering it on the two sides, since the plates are much longer than they are wide. Using the graphical information given, and the conductivity data for the water, determine the resistance that would be observed in Lake Superior and the Great Salt Lake. To increase your accuracy, you should sketch some additional field lines and equipotentials for the remainder of the fringing field outside of the plates. Also determine the resistance assuming no fringing as a comparison. Note that the separation is 1.2 cm to make it easy to figure out the size of each little box in the flux plot. Also, be sure that you incorporate the depth of the cells properly when you determine resistance.

Before beginning the real problem, let us do the comparison calculation. The resistance of the parallel plate structure, using ideal, non-fringing conditions, will be given by the following expression. $R = \frac{l}{\sigma A} = \frac{0.012}{\sigma(0.5)(.05)} = \frac{0.48}{\sigma}$. *The conductivity given in the table needs to be converted to S/m. For Lake Superior,*
$\sigma = \frac{97\mu S}{cm}\frac{100cm}{m}\frac{0.000001S}{\mu S} = 9.7x10^{-3}\frac{S}{m}$ *and for Salt Lake*
$\sigma = \frac{158000\mu S}{cm}\frac{100cm}{m}\frac{0.000001S}{\mu S} = 15.8\frac{S}{m}$. *Thus, the resistance in the two cases is*
$R = \frac{0.48}{9.7x10^{-3}} = 49.5\Omega$ *and* $R = \frac{0.48}{15.8} = 0.03\Omega$

To complete the picture that goes with the case with fringing, we need a properly scaled drawing.



The space in the middle with no lines drawn is thus equal to 5-2.8=2.2cm. In this region, we assume that all is uniform and the resistance is given by the ideal expression. To complete the flux tubes, two more should be drawn on each side or three total. Either is a reasonable choice, since we are only guessing. Probably the better choice is 3 more tubes total. (I am not able to draw the tubes in this solution.) The total number of tubes in the drawing are, thus, 38+38+3=79. The number of steps between the two plates is 24. Each little square has a resistance of $R = \frac{h}{\sigma dh} = \frac{1}{d\sigma}$ *where d is the depth of 50cm or 0.5m.*

The total resistance is then $R = \frac{m}{n}\frac{1}{\sigma d}$ *where m is the number of steps between the plates (24) and n is the number of tubes (79). Thus, the total resistance is given by the parallel combination of* $R = \frac{m}{n}\frac{1}{\sigma d}$ *and* $R_{uniform}$ *where the latter term is for the uniform region in the middle. In general, then,* $R_{total} = \frac{\frac{m}{n}\frac{1}{\sigma d}\frac{l}{\sigma wd}}{\frac{m}{n}\frac{1}{\sigma d}+\frac{l}{\sigma wd}}$ *where w is the width of the uniform region (2.2cm) and l is the separation between the plates (1.2cm). The total resistance is then* $R_{total} = \frac{\frac{24}{79}\frac{1}{\sigma 0.5}\frac{0.012}{\sigma(0.022)0.5}}{\frac{24}{79}\frac{1}{\sigma 0.5}+\frac{0.012}{\sigma(0.022)0.5}} = \frac{0.39}{\sigma}$, *which is just a little smaller than the ideal case. This makes sense because the current should flow in such a way as to somewhat* *reduce the resistance. If the resistance was lower with the current only flowing directly across the plates, it would do it. Thus, the resistance in the two cases will be*

$$R_{total} = \frac{0.39}{9.7x10^{-3}} = 40.2\Omega \;\; and \;\; R_{total} = \frac{0.39}{15.8} = 0.025\Omega$$

EC Meters

The numbers used for the resistance measurement in problem 1 are not realistic, since no one wants such a large meter. However, they make for a more interesting problem for us. Just so that you know a bit more about the actual devices used in the field based on conductivity measurements, here is some additional information. First, some commercially available devices:

BlueLab Truncheon (New Zealand) http://www.bluelabassist.tx.co.nz/index.html

Oakton Instruments (Illinois) http://www.4oakton.com/

Links from GlobalSpec
http://search.globalspec.com/Industrial/tds/conductivity_and_TDS

Using EC Meters
http://www.gemplers.com/a/pages/iecmeters.asp

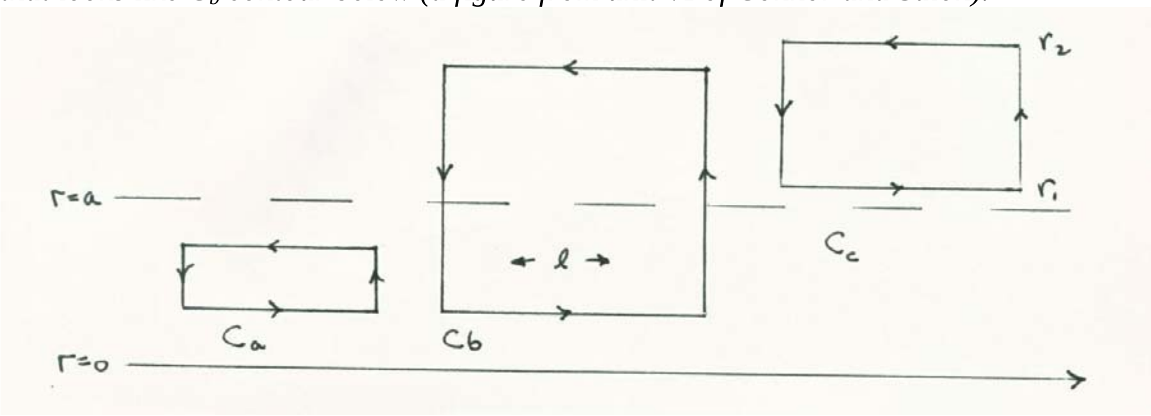
2. **Properties of Magnetic Fields** The following expressions characterize magnetic fields. Using Maxwell's equations in either differential or integral form, or Laplace's or Poisson's equations for magnetic fields, demonstrate that these are indeed correct solutions. Draw a picture to go with each question to show that you understand the configuration.

a. $\vec{B} = \frac{\mu_o I}{2\pi r}\hat{a}_\phi$ for $r>a$ and $\vec{B} = \frac{\mu_o I r}{2\pi a^2}\hat{a}_\phi$ for $r<a$ are the magnetic fields for a cylindrical wire of radius a carrying a current I uniformly distributed throughout its cross section.

Take the divergence of each expression and you will see that it is equal to zero as $\nabla \cdot \vec{B} = \frac{\partial B_\phi}{r\partial\phi} = 0$ *says it should. The curl in each case should give a term that includes the current since* $\nabla \times \vec{B} = \hat{a}_z \frac{1}{r}\frac{\partial}{\partial r}\left(rB_\phi\right) = \mu_o \vec{J}$

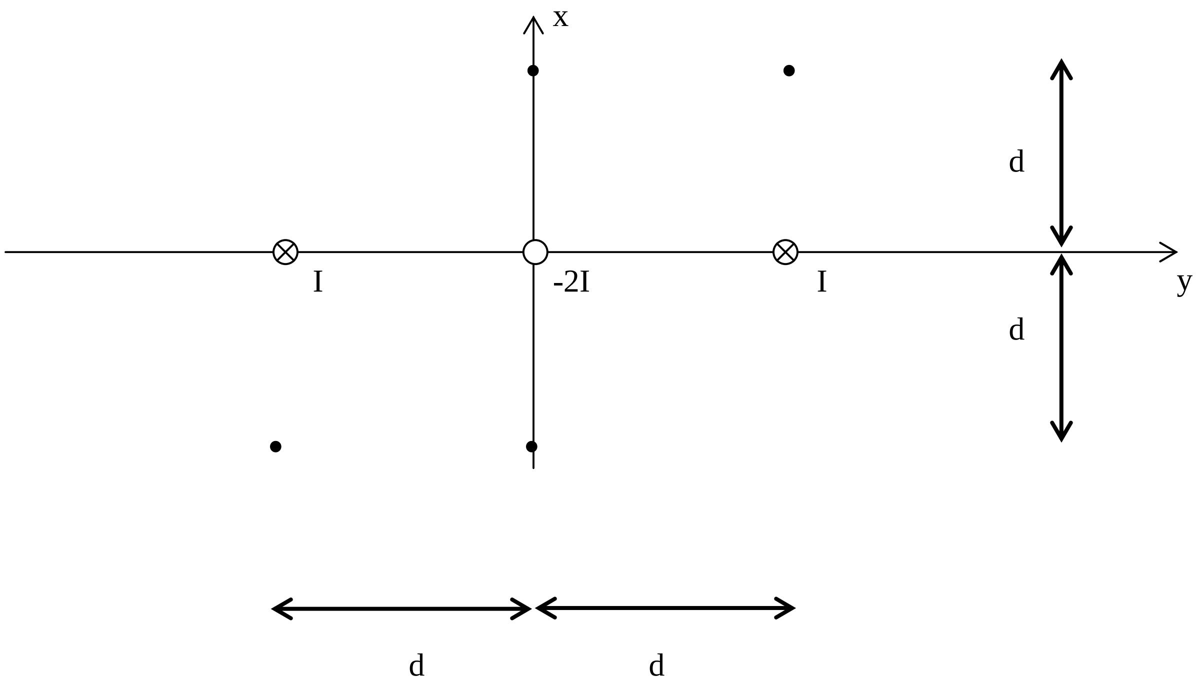
b. $\vec{B} = \frac{\mu_o NI}{d}\hat{a}_z$ for $r<a$ and $\vec{B} = 0$ for $r>a$ are the magnetic fields for a cylindrical solenoid of radius a, length d, wound with N turns of wire carrying a current I. The current is carried in a single layer of wire, whose thickness is so small that it can be neglected.

For this problem, both the divergence and curl of the magnetic field are trivially zero. $\nabla \cdot \vec{B} = \frac{\partial B_z}{\partial z} = 0$ $\nabla \times \vec{B} = \hat{a}_\phi\left(-\frac{\partial B_z}{\partial r}\right) = \mu_o \vec{J}$ *There is no volume current in this case. The current exists only on the surface where the wires are wound. The integral form of Ampere's Law needs to be applied to find out how much current is enclosed by a path that looks like* C_b *contour below (a figure from unit VI of Connor and Salon).*



The integral along the sides and top give zero. The integral along the bottom is B times the length. Take a length of l, then the integral is $\frac{\mu_o NI}{d} l$. *If we take the full length of the solenoid, then we get* $\mu_o NI$ *which is the total current enclosed by the loop, as it needs to be. Thus, the expressions are correct.*

3. **Field Direction** The three parallel current-carrying wires shown below will produce a net magnetic field at each of the four points indicated. Determine the direction of the magnetic flux density $\vec{B}$ and the magnetic vector potential $\vec{A}$ at each of the four points. Assume that the currents marked as ⊗ are in the z-direction and those marked with ○ are in the negative z-direction. The other two axes are shown.



Do the vector potential first since it will always be in the direction of the current. If the direction out of the page is z, then A will be either in the z or negative z direction. Thus, in general, we say it is in the z direction.

For B, the direction is a bit harder to come up with. At the point on the x axis above the currents, the B field will be in the y direction, since the vertical components from the two outer currents (I) will cancel. The trickiest points are the points above the right current and below the left current (which are really the same question because of symmetry). To answer this question, we need to break up the field into x and y components. $\vec{B} = \frac{\mu_o I}{2\pi r}\hat{a}_\phi$ *for each of the currents. This question is longer than I had planned, so I am dropping it from this assignment, except as extra credit.*