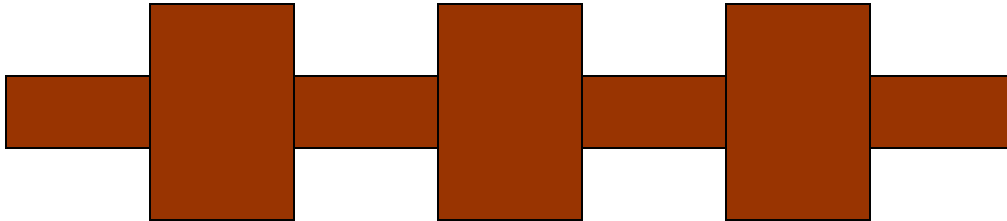


## Homework 5

Due 21 March 2006

## 1. Resistance Measurement

A resistor is deposited on the surface of a printed circuit board in the pattern shown below. The thickness of the layer is  $0.2\text{mm}$ .



The width of the narrow regions is  $3\text{mm}$  while the wider regions are  $9\text{mm}$  in width. Each region is  $6\text{mm}$  long, so the total length of the resistor is  $42\text{mm}$ . Note, these are not really typical dimensions.

- a. Determine the resistance of the region if the resistive material is nichrome. Data on this material can be found at [http://www.8886.co.uk/ref/resistivity\\_values.htm](http://www.8886.co.uk/ref/resistivity_values.htm) First, model the total resistor as one resistor with the same total length whose width is equal to the average width, taking into account all seven regions.



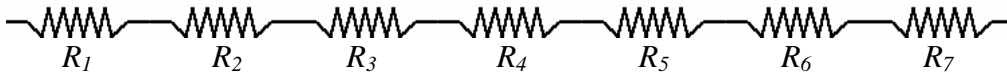
From the reference, the resistivity of nichrome is  $1.1 \times 10^{-6} \Omega - m$ . We use conductivity,

which is  $\sigma = \frac{1}{\rho} = \frac{1}{1.1 \times 10^{-6}} = 9.09 \times 10^5$ . The average width of the seven regions is

$$w_{ave} = \frac{4(3) + 3(9)}{7} = \frac{39}{7} = 5.57\text{mm}. \text{ The resistance is given by}$$

$$R = \frac{\text{length}}{\sigma(\text{area})} = \frac{0.042}{(9.09 \times 10^5)(0.002)(0.00557)} = 41.5\text{m}\Omega$$

Next, model the total resistor as 7 resistors in series, each with a uniform current density.



The resistance of the narrow regions is

$$R = \frac{\text{length}}{\sigma(\text{area})} = \frac{0.006}{(9.09 \times 10^5)(0.002)(0.003)} = 11\text{m}\Omega \text{ The resistance of the wider regions is}$$

$$R = \frac{\text{length}}{\sigma(\text{area})} = \frac{0.006}{(9.09 \times 10^5)(0.002)(0.009)} = 4\text{m}\Omega$$

The total resistance is  $R = 56\text{m}\Omega$

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b. The current does really flow uniformly since it does not turn sharply at the boundary between regions. Rather, the flow pattern looks something like a standing wave. To better model the actual current, assume that the width varies with position according to

$$w(z) = w_o \left( 1 - 0.5 \sin \frac{\pi z}{d} \right) \text{ where } w_o \text{ and } d \text{ are both } 6\text{mm. (The expression for } w(z) \text{ was}$$

*originally incorrect. The 2 in the numerator of the argument for the sine function should not have been there. The period of the width should match that of structure above, which means that it repeats every 12mm not every 6mm. This is the extra credit question.)* This will produce a smoothly varying width that is similar to the step changes in the diagram above. For a variable width resistor, we need to use an expression like the one on page

$$\text{VIII-4 of } \textit{Connor \& Salon} \text{ (as indicated in the reading for Lecture 14): } R = \int \frac{dl}{\sigma(l)S(l)}$$

which allows both the conductivity and the area of the resistor to vary along its length. For this problem, we have constant conductivity (all of the resistor material is nichrome) and the thickness is also constant so only the width varies. Determine the resistance using this model.

The resistance is  $R = \int \frac{dz}{\sigma S(z)} = \frac{1}{\sigma T} \int_0^{\text{length}} \frac{dz}{w(z)}$  so we should first evaluate the integral.

$$\int_0^{\text{length}} \frac{dz}{w(z)} = \int_0^{0.042} \frac{dz}{w_o \left( 1 - 0.5 \sin \frac{\pi z}{d} \right)} = 8.08 \text{ where we have used Maple}$$

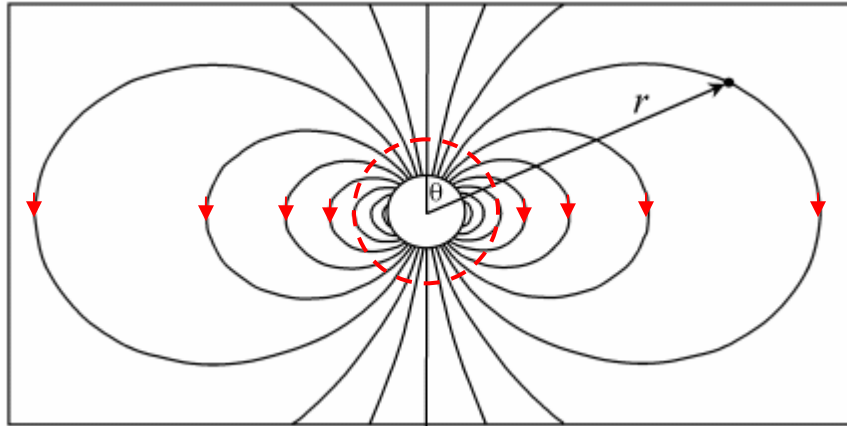
**> int(1/(0.006\*(1-0.5\*sin(3.14159\*z/.006))),z=0..0.042);**  
8.467805188

$$\text{Then, } R = \frac{1}{\sigma T} \int_0^{\text{length}} \frac{dz}{w(z)} = \frac{8.47}{(9.09 \times 10^5)(0.002)} = 46.6\text{m}\Omega$$

Note that the three results are reasonably close. None is probably perfectly accurate since we never solved for the exact current distribution.

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**2. Properties of Magnetic Fields** The following expression characterizes the magnetic field of a dipole, but is valid only at distances large compared with the radius of the dipole. Using Maxwell's equations in differential form for magnetic fields, you are to demonstrate that this is indeed a correct solution. We can use the form of the field expression given in Ulaby section 5-2.2.  $\vec{B} = \frac{\mu_0 m}{4\pi R^3} (\hat{R} 2 \cos \theta + \hat{\theta} \sin \theta)$  where  $m = I\pi a^2$  is the magnetic moment of the current loop with radius  $a$ .



The above figure should help to understand this configuration.

- a. Show that the divergence and curl of this expression have the expected values far from the dipole. That is, evaluate the expressions and explain why your answer is correct.

*In spherical coordinates, the divergence is given by*

$$\nabla \cdot \vec{B} = \frac{1}{R^2} \frac{\partial}{\partial R} R^2 B_R + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} B_\theta \sin \theta$$

*where we have dropped the last term since it is zero. The curl is given by*

$$\nabla \times \vec{B} = \frac{1}{R} \hat{\phi} \left( \frac{\partial}{\partial R} R B_\theta - \frac{\partial}{\partial \theta} B_R \right)$$

*where again we have dropped all of the zero terms.*

*Evaluating these expressions:*

$$\nabla \cdot \vec{B} = \frac{1}{R^2} \frac{\partial}{\partial R} \frac{1}{R} (2 \cos \theta) + \frac{1}{R^4 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta)^2 = \frac{-2}{R^4} (\cos \theta) + \frac{1}{R^4 \sin \theta} (2 \cos \theta \sin \theta) = 0$$

*The divergence of B must always be zero, so this is the correct answer.*

*The curl should also be zero since there is no current in the region where the expression is valid (far from the coil).*

$$\nabla \times \vec{B} = \frac{1}{R} \hat{\phi} \left( \frac{\partial}{\partial R} \frac{1}{R^2} \sin \theta - \frac{\partial}{\partial \theta} \frac{1}{R^3} 2 \cos \theta \right) = \hat{\phi} \left( \frac{-2}{R^4} \sin \theta - \frac{1}{R^4} (-2 \sin \theta) \right) = 0$$

*Thus, the dipole field expression has the correct values for divergence and curl.*

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- b. Evaluate the flux of the magnetic field through a sphere of radius  $b \gg a$ . That is evaluate the integral  $\oint \vec{B} \cdot d\vec{S} = ?$  for the surface shown in red above. First, provide the expression for the surface element  $d\vec{S}$  and then evaluate the integral. Explain why your answer is correct.

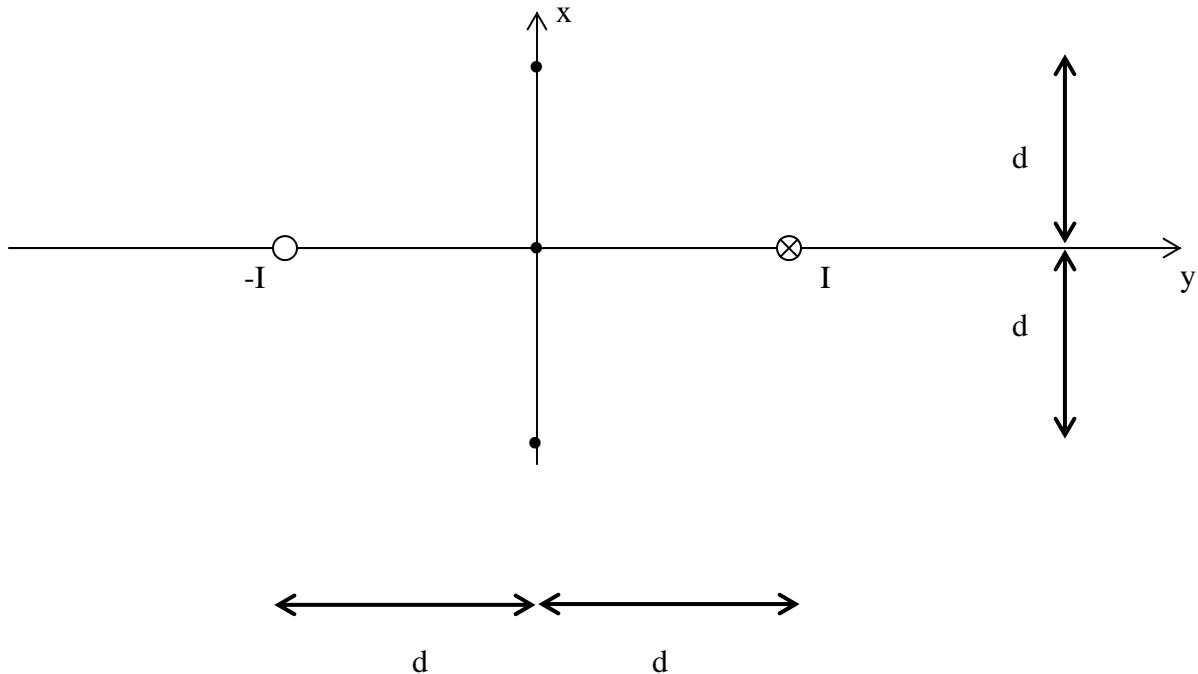
$d\vec{S} = \hat{R}R^2 \sin\theta d\theta d\phi$  since the surface is in the radial direction. The surface integral is then

$$\oint \vec{B} \cdot d\vec{S} = \frac{\mu_o m}{4\pi R^3} \oint 2 \cos\theta R^2 \sin\theta d\theta d\phi = \frac{\mu_o m}{2\pi R} \oint \cos\theta \sin\theta d\theta d\phi = \frac{\mu_o m}{2\pi R} 2\pi \int_0^\pi \cos\theta \sin\theta d\theta = 0$$

*This is the expected result since the flux passing out of the surface in the upper part of the sphere passes back into the region in the lower part of the sphere. (Red arrows  $\blacktriangledown$  have been added to the field diagram above to show the direction of the flux.) Thus, the net flux is zero.*

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3. **Field Direction** The two parallel current-carrying wires shown below will produce a net magnetic field at each of the three points indicated. Determine the direction of the magnetic flux density  $\vec{B}$  at each of the three points. Assume that the currents marked as  $\otimes$  are in the  $z$ -direction and those marked with  $\circ$  are in the negative  $z$ -direction. The other two axes are shown.



Find the expression for the magnetic flux density  $\vec{B}(x)$  everywhere on the  $x$ -axis.

Remember that this expression is a vector, so you will need both the magnitude and the direction. Your result should be a function of  $x$ .

*For a long straight wire centered at the origin, the magnetic field is given by*

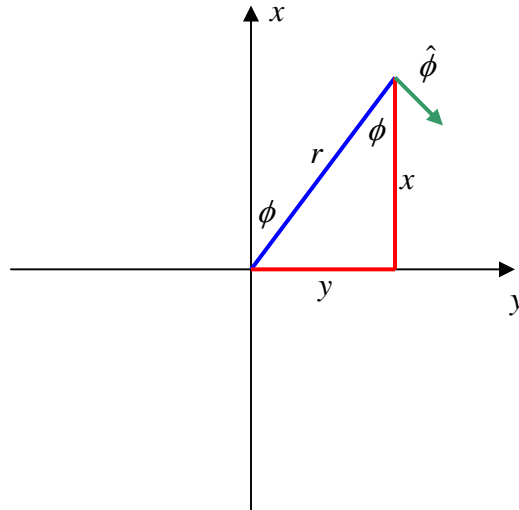
$$\vec{B}(r, \phi, z) = \hat{\phi} \frac{\mu_0 I}{2\pi r}$$

*To add the contributions from two such wires, as in this case, we*

*should re-write the expression in rectangular coordinates, taking into account the location of the wire. For the wire at the left (current coming out of the page in the negative  $z$  direction), we need to convert both the radial term ( $r$ ) and the unit vector ( $\hat{\phi}$ ) to their corresponding expressions in rectangular coordinates. We can do this by using the expressions from Ulaby for converting between coordinate systems, but here we want to try a more intuitive approach. First, let us do the easier of the two ( $r$ ). In the expression for  $\vec{B}(r, \phi, z)$ , the radius represents the distance from the axis of the wire to the observation point (the location where we want to know the field). If the wire is*

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located on the  $z$ -axis, we will have that  $r = \sqrt{x^2 + y^2}$ . For the left wire, we have to translate this expression a distance  $d$  in the negative  $y$  direction and, thus,  $r = \sqrt{x^2 + (y + d)^2}$  which makes sense because for  $x = 0$  and  $y = -d$ , we have  $r = 0$ . To convert the unit vector, it is easiest to draw a diagram showing the relationship between the cylindrical and rectangular coordinates, again starting with the wire at the origin.



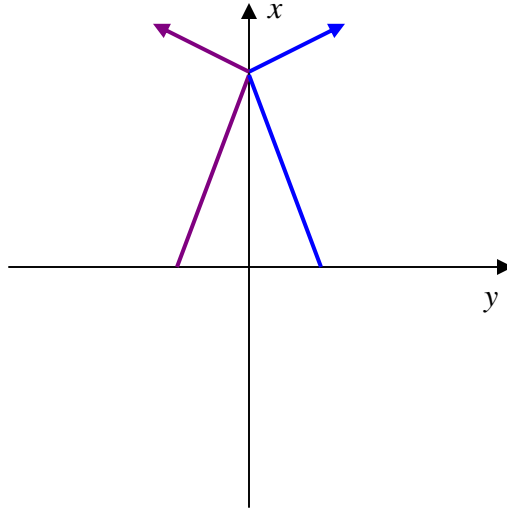
Note that the orientation of the  $x$  and  $y$  axes have been chosen so that positive  $z$  is into the page. On the diagram, we have indicated the direction of the unit vector  $\hat{\phi}$  with the green arrow. Note that when  $\phi = 0$ , the  $\hat{\phi}$  unit vector is in the  $\hat{y}$  direction and when  $\phi = \pi/2$ , the  $\hat{\phi}$  unit vector is in the  $-\hat{x}$  direction (note the sign). Thus, we can write the relationship between the unit vectors as  $\hat{\phi} = -\hat{x} \sin \phi + \hat{y} \cos \phi$  where  $\sin \phi = \frac{y}{r}$  and  $\cos \phi = \frac{x}{r}$ . Thus,

$\hat{\phi} = -\hat{x} \frac{y}{r} + \hat{y} \frac{x}{r}$  We now have enough information to re-write the magnetic field

expression  $\vec{B}(x, y, z) = \left( -\hat{x} \frac{y}{r} + \hat{y} \frac{x}{r} \right) \frac{\mu_0 I}{2\pi r}$  where we have left the radial term ( $r$ ) in for

convenience. Since we are only concerned with the value of the magnetic field on the  $x$  axis, we should next simplify the expressions for the combined contributions from both wires. The diagram on the next page shows how the vector fields add. The radial distance and field direction for the wire at the left are shown in violet and the radial distance and the field direction for the wire at the right are shown in blue. Note that the  $y$  directed components will cancel since they are in opposite directions (currents are in opposite directions) and the  $x$  directed terms add. Thus, we can drop the  $y$  directed terms. The total  $x$  directed term will just be double the contribution for either wire. Thus, we do not have to write an expression for the field from the other wire. Finally, using the expression for the radius ( $r$ ), we have that  $\vec{B}(x) = \left( \hat{x} \frac{d}{x^2 + d^2} \right) \frac{\mu_0 I}{\pi}$  where we have also used the fact that  $y = -d$ .

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*This expression for the magnetic field is the answer to the last part of this problem. For the other parts, we only need to use the value of  $x$ .*

$$\text{At } x = 0: \vec{B}(x) = \vec{B}(0) = \left( \hat{x} \frac{1}{d} \right) \frac{\mu_o I}{\pi} = \hat{x} \frac{\mu_o I}{\pi d}$$

$$\text{At } x = d: \vec{B}(d) = \left( \hat{x} \frac{1}{2d} \right) \frac{\mu_o I}{\pi} = \hat{x} \frac{\mu_o I}{2\pi d}$$

$$\text{At } x = -d: \vec{B}(-d) = \left( \hat{x} \frac{1}{2d} \right) \frac{\mu_o I}{\pi} = \hat{x} \frac{\mu_o I}{2\pi d} \text{ which is the same as the previous expression.}$$