Homework #5
Due 4 April

1. **Resistance**

For some background on lightning, please read over the following primer:
http://www.weighing-systems.com/TechnologyCentre/Lightning1.pdf

We will consider a very crude model of a tree to see how it reacts to being struck by lightning. For its conductivity, we will assume it is somewhat poorer than for seawater and use $\sigma = 2\, {}^{S}\!/_{m}$. For the dimensions of a tree, assume the upper trunk has a radius of $a = 10cm$ and it expands toward the ground according to $r(z) = a\left(1+\left(\frac{z}{h}\right)^2\right)$ where $h = 10m$ is the height of the tree and $z = 0$ at the top of the tree and $z = h$ at the bottom. This makes the calculations a little easier even though it seems more natural to assume that $z = 0$ at ground level.

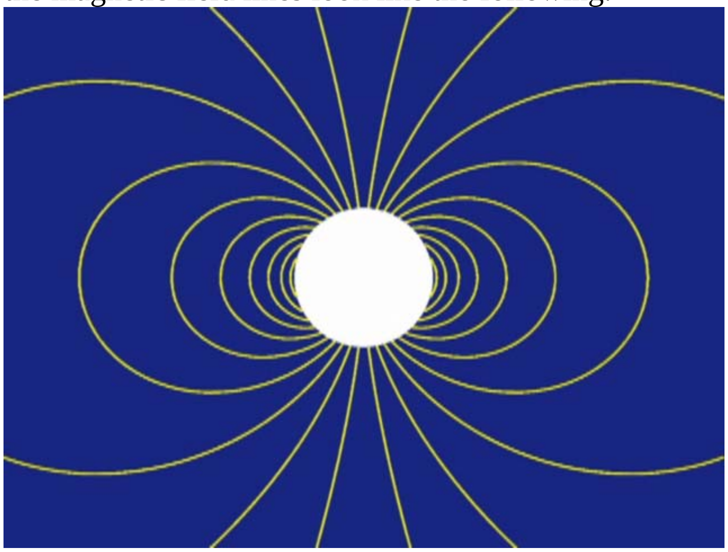
a. Determine the resistance of the tree. *Hint: You can use Maple to evaluate the integral.*
b. Assume that the tree is struck by lightning, which causes a current of *200kA* to flow through the tree for $200\mu s$. What is the voltage drop across the tree and how much energy is deposited in the tree?
c. If half of the volume of the tree is water, is this energy sufficient to boil away all of the water? From the pictures above, you should see that the current does not really flow throughout the entire tree, but we have to make some kind of simplifying assumptions.

2. **Faraday's Law**

Faraday's Law states that $\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{S} = -\frac{d}{dt}\Psi_M$. The left hand side of this equation is the integral of the electric field around a closed path. When we studied statics, we found that integrating the voltage around a closed path produced a zero voltage change because the beginning and ending points were the same. When there is no time variation, the right hand side is zero $\oint \vec{E} \cdot d\vec{l} = 0$. However, when the fields are time-varying, there will be a net voltage created around the closed path if the magnetic flux passing through the surface defined by the path is changing with time. $\Psi_M$ is the magnetic flux passing through the surface. Since we had our lecture on this topic cancelled, we will go through these concepts step-by-step in this problem. We will first find the flux through some surface, cause the flux to change with time and then find the voltage induced. Please read over the slides from lecture 16 on Faraday's Law.

a. **Calculating Flux** – Assume that we have the following magnetic field. Such a field is produced by a current *I* in a circular loop of radius *a* located at the origin of our coordinate system. $\vec{B} \approx \frac{\mu_o a^2 I}{4r^3}\left(\hat{r}2\cos\theta + \hat{\theta}\sin\theta\right)$ where we have noted that this is an approximate expression. It is valid only for *r* >> *a*.

For such a field, the magnetic field lines look like the following.



The empty region in the center shows where the expression is not valid because *r* is too small. Let us determine the flux passing through a circular surface of radius *b* located on the *z* axis at some location $z_o$. The geometry is shown on the next page. Both the cross-sectional view and the 3D view are provided for clarity.

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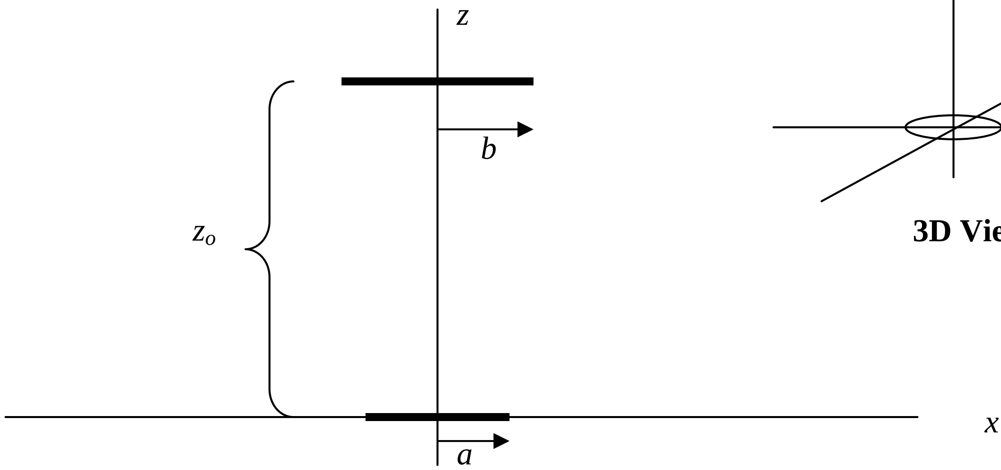
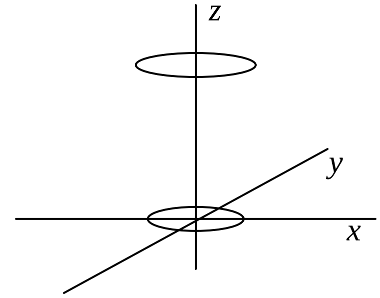
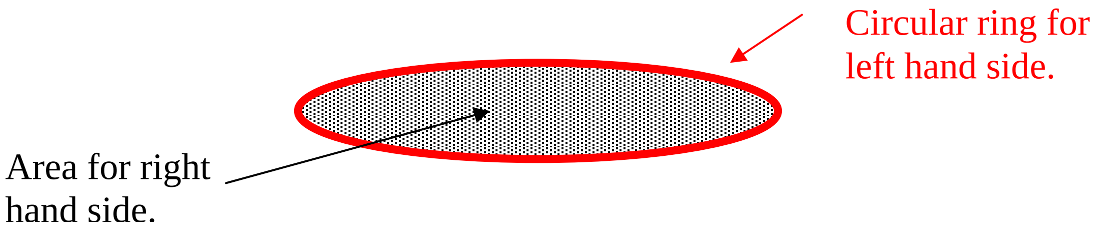



**3D View**

Evaluate $\Psi_M = \int \vec{B} \cdot d\vec{S}$ to find the flux passing through the upper ring due to the magnetic field from the dipole located at the origin. Begin by first finding the vector surface element $d\vec{S}$ for the upper ring. Then take the dot product with the magnetic field and then integrate over the surface of the ring. *Hint: the direction of the surface is $\hat{z}$ so you will have to also rewrite the magnetic field expression in rectangular coordinates before you take the dot product.*

b. **Time-Varying Current** Now assume that the current varies with time sinusoidally. $I = I_o \sin \omega t$ Find $\frac{d}{dt}\Psi_M$ for the flux passing through the upper ring. $\frac{d}{dt}\Psi_M = \frac{d}{dt}\int \vec{B}(t) \cdot d\vec{S}$ From this expression and Faraday's Law, also find the voltage that would be induced around the ring.

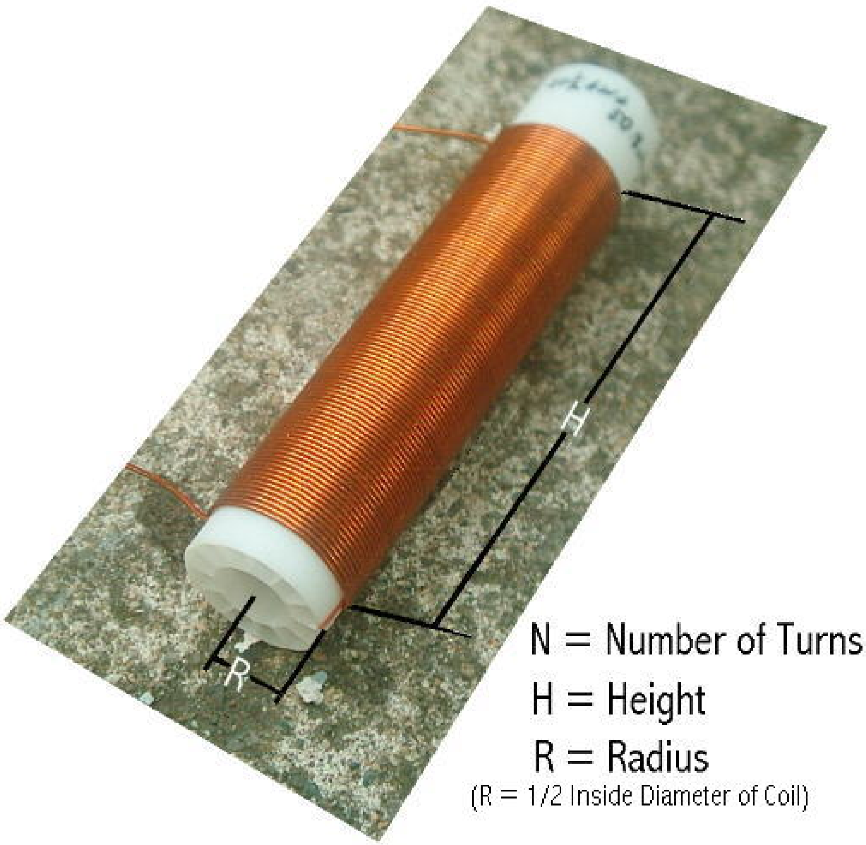


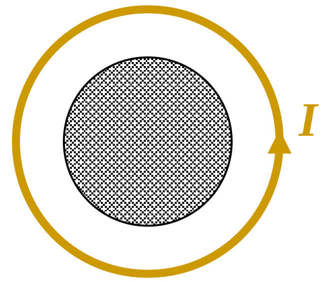

c. **Moving Loop** There is a second way to have the flux passing through the loop change with time. If the loop is moving away from the origin, the amount of flux passing through it will get smaller as time increases. Assume that the current is again constant, but that the upper loop is moving in the *z* direction at a velocity $v_o$. Find the changing flux $\frac{d}{dt}\Psi_M$ and the induced voltage around the loop.

$$\frac{d}{dt}\Psi_M = \frac{d}{dt}\int \vec{B} \cdot d\vec{S} = \int \vec{B} \cdot \frac{d\vec{S}}{dt} = \oint \left(\vec{v} \times \vec{B}\right) \cdot d\vec{l}$$

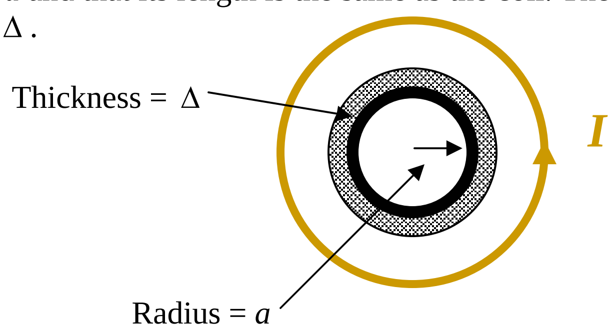
3. **Application of Faraday's Law**

Assume that we create a time varying magnetic field using a simple solenoid configuration. That is, we have a solenoid of radius $R$ and length $H$ wound with $N$ turns of wire (in a single layer) carrying a current $I$. The end view is also shown with the assumed current direction.




a. Assuming that we can neglect fringing effects, determine the magnetic field in the region inside the solenoid (the hole inside the plastic tube).
b. We now slide a hollow conducting cylinder inside the hole. Assume that the cylinder radius is $a$ and that its length is the same as the coil. The cylinder thickness is $\Delta$.



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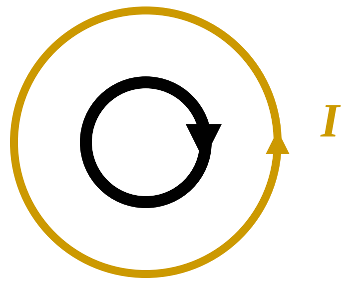
For this configuration, determine the flux passing through inside of the cylindrical conductor. $\Psi_M = \int \vec{B} \cdot d\vec{S}$ . Assume that the current is sinusoidal $I = I_o \sin \omega t$ and find the time rate of change of the flux $\frac{d}{dt}\Psi_M = \frac{d}{dt}\int \vec{B}(t) \cdot d\vec{S}$ .

c. From the time rate of change of the flux, find the voltage induced around the cylinder. This voltage will drive a current in the opposite direction to the current in the solenoid. To find this current, we must first find the resistance of the cylinder.




Assuming that the cylinder has conductivity $\sigma$, find the resistance to current flow in the direction shown. From this resistance and the induced voltage, find the induced current.

d. The conducting cylinder is also an inductor in addition to being a resistor. Find its inductance by first evaluating the flux that would be produced if it carries a constant current $I_o$. Then determine the inductance $L$ from the flux.

To get a sense of what happens when a real cylinder is placed inside a solenoid, we will use some real parameters to evaluate both $R$ and $L$. We will use the characteristics of a typical soda can. Assume a thickness of 0.2mm, aluminum conductivity (look it up), a length of 12cm and a radius of 3cm. Determine both $R$ and $L$. Since the cylinder is both a resistor and an inductor, any current induced in it will decay. What is the time constant of this decay? Evaluate the number.