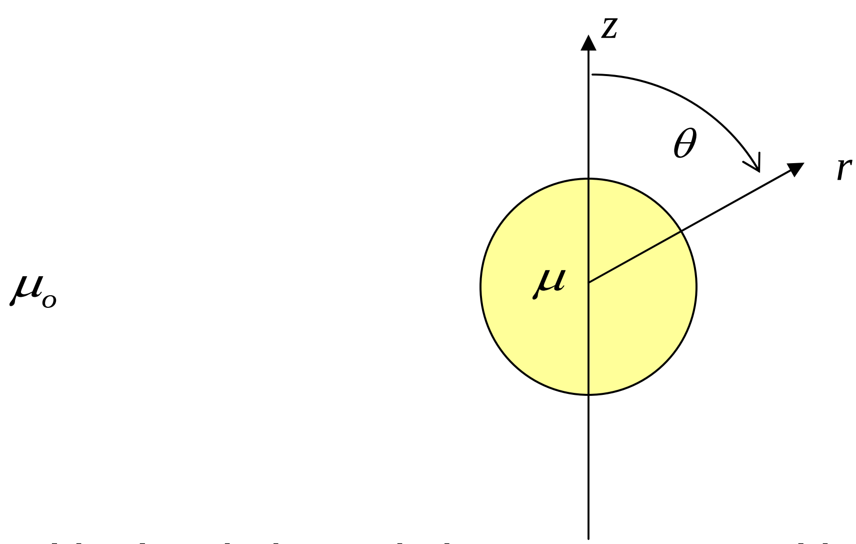
Homework 6

Due 4 April 2005

1. Boundary Conditions

If a sphere of magnetic material with permeability μ is placed in a uniform magnetic field $\vec{B}_o = B_o\hat{z}$, the field inside the sphere will also be uniform, but with a different magnitude $\vec{B}_1 = B_1\hat{z}$. The field outside the sphere will be modified somewhat by an additional dipole field $\vec{B}_2 = B_2(\hat{r}2\cos\theta + \hat{\theta}\sin\theta)\frac{a^3}{r^3}$.



Find the relationship between the three constants B_o, B_1, B_2 and the permeability of the sphere μ and free space μ_o using the two boundary conditions for the magnetic field. That is, find B_1 and B_2 in terms of B_o, μ and μ_o. *Hint: write the field inside and outside the sphere in the same coordinate system and then apply the boundary conditions.*

To match the boundary conditions, it is necessary to write all field expressions in the same coordinate system.

At the surface, r = a, the field inside the magnetic material is given by

$$\vec{B}_{in} = \vec{B}_1 = B_1\hat{z} = B_1\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right)$$

while the field outside the magnetic material is given by

$$\vec{B}_{out} = \vec{B}_o + \vec{B}_2 = B_0\left(\hat{r}\cos\theta - \hat{\theta}\sin\theta\right) + B_2\left(\hat{r}2\cos\theta + \hat{\theta}\sin\theta\right) = (B_o + 2B_2)\cos\theta\hat{r} + (B_0 + B_2)\sin\theta\hat{\theta}$$

The normal component of B is continuous, which is the radial component in this case.

$$B_1 = B_o + 2B_2$$

The tangential component of H is continuous, which is the theta component in this case.

$$-\frac{B_1}{\mu} = \frac{\left(-B_o + B_2\right)}{\mu_o} \quad or \quad B_1 = \frac{\mu\left(B_o - B_2\right)}{\mu_o}$$

Thus, $B_o + 2B_2 = \dfrac{\mu\left(B_o - B_2\right)}{\mu_o}$ *and*

$$B_2 = \frac{\mu - \mu_o}{\mu + 2\mu_o} B_o$$

$$B_1 = \frac{3\mu}{\mu + 2\mu_o} B_o$$

At this point it is always good to check one's algebra.
For the normal component

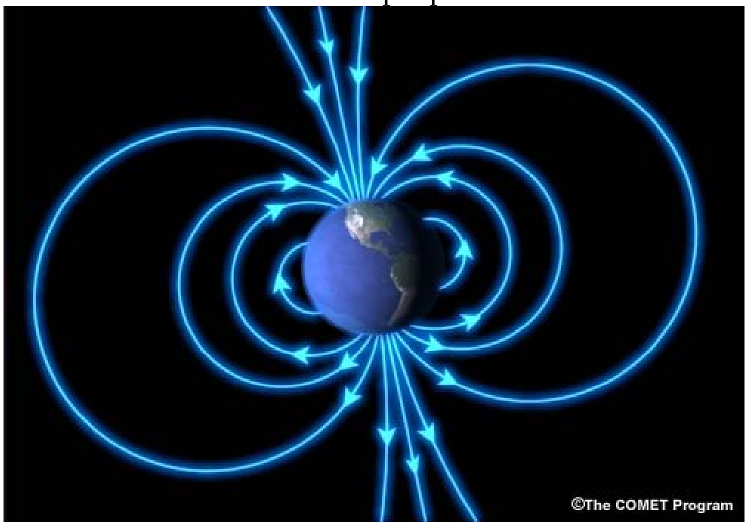
$$B_1 = \frac{3\mu}{\mu + 2\mu_o} B_o = B_o + 2\frac{\mu - \mu_o}{\mu + 2\mu_o} B_o = \frac{3\mu}{\mu + 2\mu_o} B_o$$

$$-\frac{B_1}{\mu} = -\frac{3\mu}{\mu + 2\mu_o}\frac{B_o}{\mu} = \frac{\left(-B_o + B_2\right)}{\mu_o} = \frac{\left(-B_o + \dfrac{\mu - \mu_o}{\mu + 2\mu_o} B_o\right)}{\mu_o} = \frac{B_o}{\mu_o}\left(-1 + \frac{\mu - \mu_o}{\mu + 2\mu_o}\right) = \frac{B_o}{\mu_o}\frac{-3\mu_o}{\mu + 2\mu_o}$$

$$-\frac{B_1}{\mu} = -\frac{3}{\mu + 2\mu_o} B_o = \frac{\left(-B_o + B_2\right)}{\mu_o} = B_o \frac{-3}{\mu + 2\mu_o}$$

Homework 6

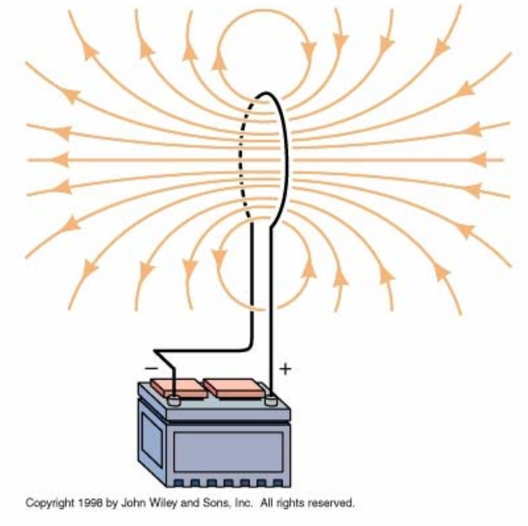
There are many examples of magnetic fields that can be modeled as a dipole. One of the most important is the field of the earth. A simple picture of the field is shown below.



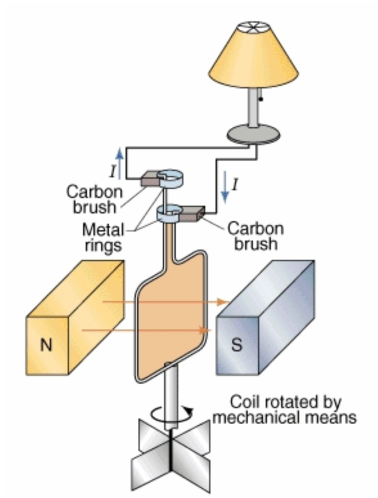
From this picture, is the magnetic pole at the geographic north a north magnetic pole or a south magnetic pole?

The geographic north pole is a magnetic south pole because the B field lines enter there. Also, north pole of a magnet used as a compass points toward a magnetic south pole since magnetic north poles are attracted to magnetic south poles.

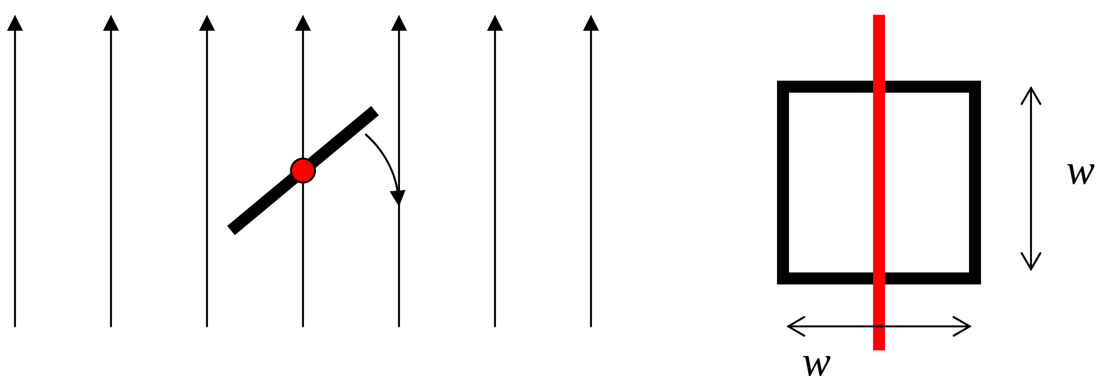
Another example of a dipole field is shown below. This is a single turn loop connected to a car battery. (This information is provided only for background.)



3. Faraday's Law



A simple example of a generator is shown in the figure above. In this application, the coil is caused to spin by some mechanical means (windmill, water wheel, etc). The magnetic flux linked by the coil varies as it rotates through the field of the magnets, which, in turn, induces a voltage in the coil. This configuration is qualitatively similar to the coil in the Beakman's motor. When the coil moves through the magnet in the Beakman's motor, a reverse voltage, called the reverse emf, is induced in the coil. To get a sense of this effect, consider the following. A single turn square coil is caused to rotate in a uniform magnetic field. The magnetic field is z-directed $\vec{B} = B_o\hat{z}$. The axis of the coil aligns with the x axis.



a. The coil is square (*w x w*). Assuming that the coil rotates at a constant angular frequency ω, determine the flux linking the coil as a function of time.

$\Phi = \int \vec{B} \cdot d\vec{S}$ *so we first need to figure out what the surface element is. Assume that the coil is initially horizontal so that the surface element is* $d\vec{S} = \hat{z}dxdy$. *Since the surface area does not change, the area is* w^2. *For it to rotate as a function of time, the direction must be given by* $\hat{z}\cos\omega t + \hat{x}\sin\omega t$ *where we have assumed that the horizontal direction is given by* $\hat{x}$. *Thus, the surface as a function of time is given by* $w^2\left(\hat{z}\cos\omega t + \hat{x}\sin\omega t\right)$.

The flux is then $\psi_m = B_o\hat{z} \cdot w^2\left(\hat{z}\cos\omega t + \hat{x}\sin\omega t\right) = B_o w^2 \cos\omega t$

b. From your answer to part a, determine the voltage induced around the coil.

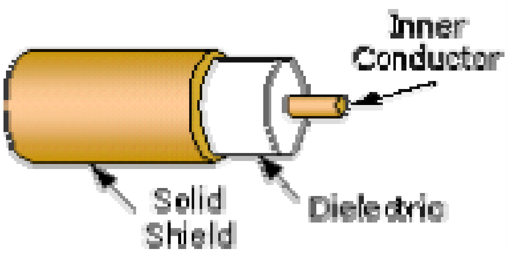
The voltage or emf is given by (see page 235 of Ulaby)

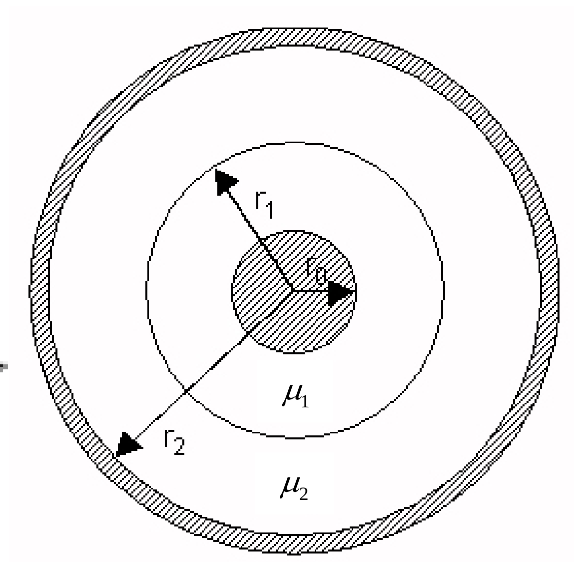
$$V_{emf}^{tr} = -N\frac{d\Phi}{dt} = -B_o w^2\left(-\omega\sin\omega t\right) = B_o w^2 \omega \sin\omega t$$

where we have used N = 1 since this is a single turn coil

2. Inductance

A coaxial cable has an inductance per unit length, which we will determine in the problem using the energy method. The cable geometry is shown below. For this case, we have created a cable with two different materials between the inner and outer conductors. The inner conductor radius is r_o and the outer conductor radius is r_2. The radius of the boundary between the two magnetic materials is r_1.





a. Assume that the current carried by the inner and outer conductors is I_o. Using Ampere's Law, determine the magnetic field intensity $\vec{H}$ in the region between the two conductors ($r_o \le r \le r_2$). Then determine the magnetic flux density $\vec{B}$ in the two regions $r_o \le r \le r_1$ and $r_1 \le r \le r_2$.

From Ampere's Law, $\oint \vec{H}\cdot d\vec{l} = I_{encl} = I_o$ *for* $r_o \le r \le r_2$. *For a loop of radius r where* $r_o \le r \le r_2$, *the left hand side of Ampere's Law is* $\oint \vec{H}\cdot d\vec{l} = H_\phi 2\pi r$ *so that the magnetic field is given by* $\vec{H} = \hat{\phi}H_\phi = \hat{\phi}\frac{I_o}{2\pi r}$

The flux density is determined from this expression by multiplying by the local value of μ. $\vec{B}_1 = \mu_1\vec{H} = \hat{\phi}\frac{\mu_1 I_o}{2\pi r}$ *for* $r_o \le r \le r_1$

& $\vec{B}_2 = \mu_2 \vec{H} = \hat{\phi} \dfrac{\mu_2 I_o}{2\pi r}$ *for* $r_1 \le r \le r_2$

b. Using your expressions from part a, determine the magnetic field energy stored in a unit length of the region $r_o \le r \le r_2$. This is the energy stored external to the current carrying wires.

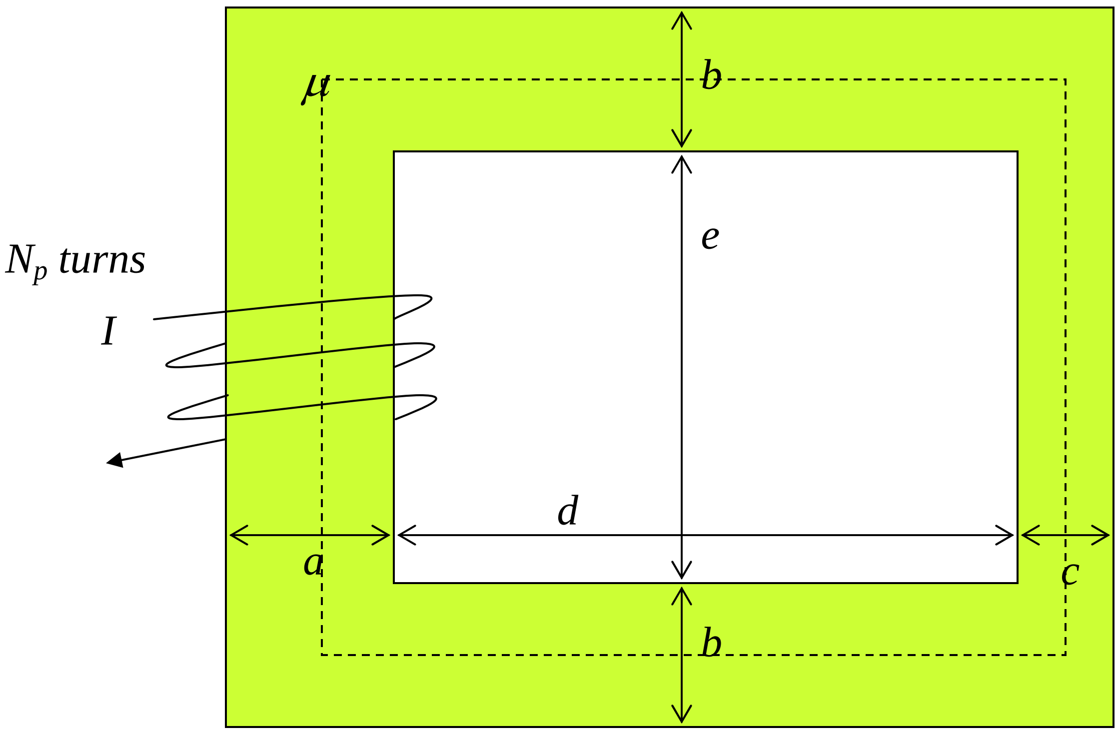
$$W_m = \frac{1}{2}\int \vec{B}\cdot\vec{H}dv = \frac{1}{2}(2\pi)(1)\left(\frac{I_o}{2\pi}\right)^2 \left(\mu_1 \int_{r_o}^{r_1}\frac{dr}{r} + \mu_2 \int_{r_1}^{r_2}\frac{dr}{r}\right) = \frac{1}{2}\frac{I_o^{\,2}}{2\pi}\left(\mu_1 \ln\frac{r_1}{r_o} + \mu_2 \ln\frac{r_2}{r_1}\right)$$

c. Using the total energy stored external to the conductors, find the external inductance per unit length for this coaxial cable.

$$W_m = \frac{1}{2}\int \vec{B}\cdot\vec{H}dv = \frac{1}{2}L_e I^2 = \frac{1}{2}\frac{I_o^{\,2}}{2\pi}\left(\mu_1 \ln\frac{r_1}{r_o} + \mu_2 \ln\frac{r_2}{r_1}\right) \textit{ so that}$$

$$L_e = \frac{1}{2\pi}\left(\mu_1 \ln\frac{r_1}{r_o} + \mu_2 \ln\frac{r_2}{r_1}\right)$$

3. Magnetic Circuits



a. A magnetic core with the geometry shown has N_p windings wrapped around its left post. The core has a rectangular cross section with depth *w*. The permeability of the core is μ. Using the magnetic circuit technique, find the reluctance and inductance of this configuration and the total energy stored for a current *I* in the coil. Note that the total width of the core is *a+d+c* and the total height is *2b+e*.

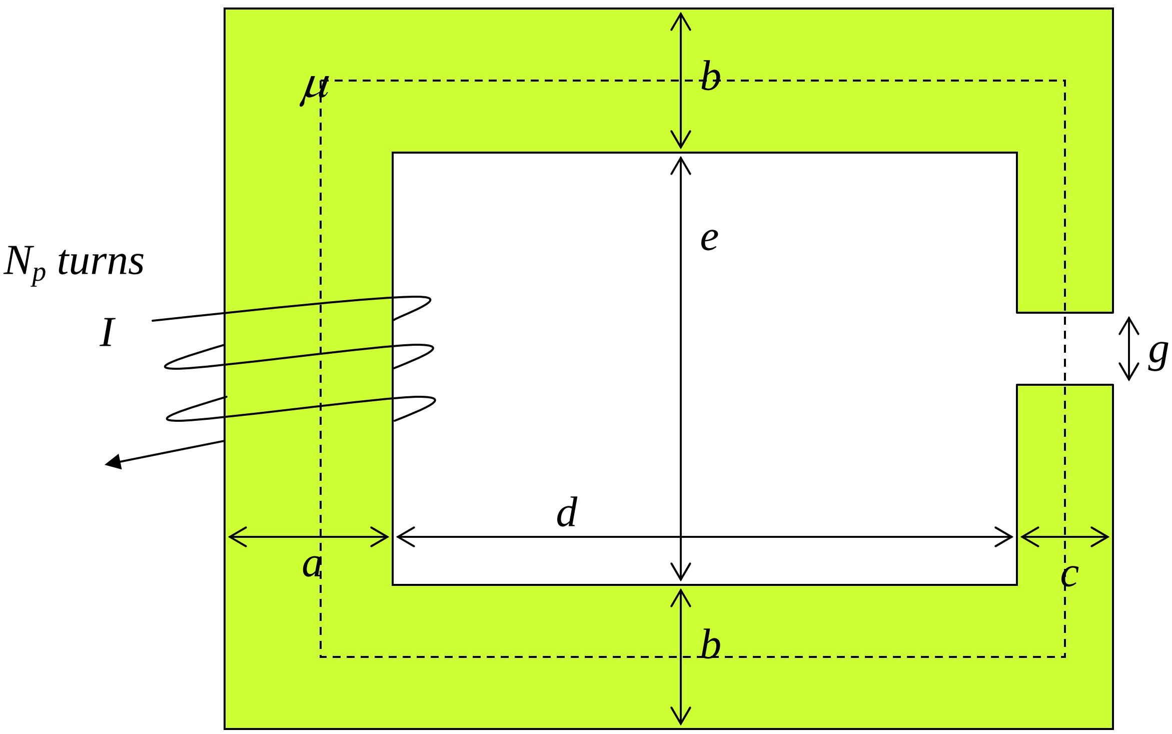
The reluctance of the core has four parts, one for each leg.

$$R = \frac{l}{\mu A} = \frac{b+e}{\mu a w} + \frac{d + \frac{a}{2} + \frac{c}{2}}{\mu b w} + \frac{e+b}{\mu c w} + \frac{d + \frac{a}{2} + \frac{c}{2}}{\mu b w} = \frac{b+e}{\mu a w} + \frac{2d+a+c}{\mu b w} + \frac{e+b}{\mu c w}$$

The flux is $\Phi = \frac{N_p I}{R}$ *and the inductance is* $L = \frac{N_p \Phi}{I}$. *Thus,*

$$L = \frac{N_p^{\,2}}{R} = \frac{N_p^{\,2} \mu w}{\frac{b+e}{a} + \frac{2d+a+c}{b} + \frac{e+b}{c}}$$

The total stored energy is $W_m = \frac{1}{2} L I^2$



b. A small gap of height *g* is cut into the right leg of the core. Repeat the calculations of part a for the reluctance, inductance and energy. Is the energy larger or smaller for this configuration?

The reluctance of the core now has five parts, one for each leg and one for the gap.

$$R = \frac{l}{\mu A} = \frac{b+e}{\mu aw} + \frac{d+\frac{a}{2}+\frac{c}{2}}{\mu bw} + \frac{e+b-g}{\mu cw} + \frac{g}{\mu_o cw} + \frac{d+\frac{a}{2}+\frac{c}{2}}{\mu bw} = \frac{b+e}{\mu aw} + \frac{2d+a+c}{\mu bw} + \frac{e+b}{\mu cw}$$

$$R = \frac{l}{\mu A} = \frac{b+e}{\mu aw} + \frac{2d+a+c}{\mu bw} + \frac{e+b-g}{\mu cw} + \frac{g}{\mu_o cw} \approx \frac{g}{\mu_o cw}$$

The flux is $\Phi = \frac{N_p I}{R}$ *and the inductance is* $L = \frac{N_p \Phi}{I}$. *Thus,*

$$L = \frac{N_p^{\,2}}{R} = \frac{N_p^{\,2} w}{\frac{b+e}{\mu a} + \frac{2d+a+c}{\mu b} + \frac{e+b-g}{\mu c} + \frac{g}{\mu_o c}} \approx N_p^{\,2} w\left(\frac{\mu_o c}{g}\right) \quad \text{since } \mu >> \mu_o$$

The total stored energy is $W_m = \frac{1}{2} L I^2$

c. Evaluate your answers to parts a and b for *$a = b = w = 2cm$, $c = 1cm$, $d = e = 3cm$, $g = 0.5cm$*, $\mu = 1000\mu_o$, and *$N_p = 1000$.*

$$R = \frac{b+e}{\mu aw} + \frac{2d+a+c}{\mu bw} + \frac{e+b}{\mu cw} = \frac{0.05}{1000\mu_o(0.02)(0.02)} + \frac{0.09}{1000\mu_o(0.02)(0.02)} + \frac{0.05}{1000\mu_o(0.01)(0.02)} =$$

$R = \frac{0.6}{\mu_o}$ *for no gap. When there is a gap, the reluctance is given by*

$$R = \frac{0.05}{1000\mu_o(0.02)(0.02)} + \frac{0.09}{1000\mu_o(0.02)(0.02)} + \frac{0.05-0.005}{1000\mu_o(0.01)(0.02)} + \frac{0.005}{\mu_o(0.01)(0.02)}$$

$R = \frac{0.23}{\mu_o(0.4)} + \frac{25}{\mu_o} = \frac{25.575}{\mu_o} \approx \frac{25}{\mu_o}$ *as expected.*

Then the Inductance is

$L = \frac{N_p^{\ 2}}{R} = \frac{\mu_o N_p^{\ 2}}{0.6}$ *and* $L = \frac{N_p^{\ 2}}{R} = \frac{\mu_o N_p^{\ 2}}{25.575}$ *for the two cases.*

The energy is given by $W_m = \frac{1}{2} I^2 \frac{\mu_o N_p^{\ 2}}{0.6}$ *and* $W_m = \frac{1}{2} I^2 \frac{\mu_o N_p^{\ 2}}{25.575}$

Note that the energy and inductance change a great deal with the gap added. Also, since this is an approximate method, it does not really make sense to use all the significant digits of the full solution. the approximate solution for the gap is sufficient. Also note that the gap very much dominates the inductance and energy.

Note that this problem shows that we can detect whether or not the gap is open or closed from the change in reluctance and inductance. There may not be a lot of applications for this specific configuration, but it shows the basic principle. A practical application of this idea is a gear tooth sensor. Since gears can be made from magnetic (ferrous) materials, they can be part of a magnetic circuit. The configuration below shows such a sensor made by Honeywell that uses a Hall Sensor to detect changes in the magnetic field that occur when a gear tooth passes nearby. It is easier to measure the magnetic field change than the change in inductance.