Fields and Waves

Lesson 1.2

VECTORS and VECTOR CALCULUS

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Today’s Class will focus on:

- vectors - description in 3 coordinate systems

- vector operations - DOT & CROSS PRODUCT

- vector calculus - AREA and VOLUME INTEGRALS
VECTOR NOTATION:

\[ \vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \]

Rectangular or Cartesian Coordinate System

\[ \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \]

Dot Product (SCALAR)

\[ \vec{A} \times \vec{B} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \]

Cross Product (VECTOR)

\[ |\vec{A}| = \left( A_x^2 + A_y^2 + A_z^2 \right)^{1/2} \]

Magnitude of vector
3 PRIMARY COORDINATE SYSTEMS:

- RECTANGULAR
- CYLINDRICAL
- SPHERICAL

Choice is based on symmetry of problem

Examples:
- Sheets - RECTANGULAR
- Wires/Cables - CYLINDRICAL
- Spheres - SPHERICAL
Cylindrical representation uses: $r, \phi, z$

\[ \mathbf{A} = A_r \hat{a}_r + A_\phi \hat{a}_\phi + A_z \hat{a}_z \]

\[ \mathbf{A} \cdot \mathbf{B} = A_r B_r + A_\phi B_\phi + A_z B_z \]

UNIT VECTORS:

\[
\begin{pmatrix}
\hat{a}_r \\
\hat{a}_\phi \\
\hat{a}_z \\
\end{pmatrix}
\]
Spherical representation uses: $r, \theta, \phi$

$$\vec{A} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$$

Dot Product (SCALAR)

$$\vec{A} \cdot \vec{B} = A_r B_r + A_\theta B_\theta + A_\phi B_\phi$$

UNIT VECTORS:

$$\begin{pmatrix} \hat{a}_r & \hat{a}_\theta & \hat{a}_\phi \end{pmatrix}$$
The Unit Vectors imply:

- $\hat{a}_x$ points in the direction of increasing $x$
- $\hat{a}_y$ points in the direction of increasing $y$
- $\hat{a}_z$ points in the direction of increasing $z$
VECTOR REPRESENTATION: UNIT VECTORS

Cylindrical Coordinate System

The Unit Vectors imply:
- \( \hat{a}_r \) points in the direction of increasing \( r \)
- \( \hat{a}_\phi \) points in the direction of increasing \( \phi \)
- \( \hat{a}_z \) points in the direction of increasing \( z \)
Spherical Coordinate System

The Unit Vectors imply:

- \( \hat{a}_r \) points in the direction of increasing \( r \)
- \( \hat{a}_\theta \) points in the direction of increasing \( \theta \)
- \( \hat{a}_\phi \) points in the direction of increasing \( \phi \)
VECTOR REPRESENTATION: UNIT VECTORS

Summary

RECTANGULAR Coordinate Systems

\[
\begin{pmatrix}
\hat{a}_x & \hat{a}_y & \hat{a}_z
\end{pmatrix}
\]

CYLINDRICAL Coordinate Systems

\[
\begin{pmatrix}
\hat{a}_r & \hat{a}_\phi & \hat{a}_z
\end{pmatrix}
\]

SPHERICAL Coordinate Systems

\[
\begin{pmatrix}
\hat{a}_r & \hat{a}_\theta & \hat{a}_\phi
\end{pmatrix}
\]

NOTE THE ORDER!

\[ r, \phi, z \quad r, \theta, \phi \]

Note: We do not emphasize transformations between coordinate systems

Do Problem 1
1. Rectangular Coordinates:

When you move a small amount in \( x \)-direction, the distance is \( dx \).

In a similar fashion, you generate \( dy \) and \( dz \).

Generate: \( (dx, dy, dz) \)

2. Cylindrical Coordinates:

Distance = \( r \, d\phi \)

Differential Distances:

\( (dr, rd\phi, dz) \)
3. Spherical Coordinates:

Distance = $r \sin \theta \, d\phi$

Differential Distances:

( $dr$, $r d\theta$, $r \sin \theta \, d\phi$ )
METRIC COEFFICIENTS

Representation of differential length $dl$ in coordinate systems:

- **rectangular**
  $$dl = dx \cdot \hat{a}_x + dy \cdot \hat{a}_y + dz \cdot \hat{a}_z$$

- **cylindrical**
  $$dl = dr \cdot \hat{a}_r + r d\phi \cdot \hat{a}_\phi + dz \cdot \hat{a}_z$$

- **spherical**
  $$dl = dr \cdot \hat{a}_r + r d\theta \cdot \hat{a}_\theta + r \sin \theta d\phi \cdot \hat{a}_\phi$$
• integration over 2 “delta” distances

Example:

\[
\text{AREA} = \int_{2}^{3} \int_{6}^{7} dy \cdot dx = 16
\]

Note that: \( z = \text{constant} \)

In this course, area & surface integrals will be on similar types of surfaces e.g. \( r = \text{constant} \) or \( \phi = \text{constant} \) or \( \theta = \text{constant} \) et c.…. 
PROBLEM 2

PROBLEM 2A  What is constant?

How is the integration performed? What are the differentials?

Representation of differential surface element:

Vector is **NORMAL** to surface

\[ ds = dx \bullet dy \bullet \hat{a}_z. \]
DIFFERENTIALS FOR INTEGRALS

Example of Line differentials

\[ d\vec{l} = dx \cdot \hat{a}_x \quad \text{or} \quad d\vec{l} = dr \cdot \hat{a}_r \quad \text{or} \quad d\vec{l} = rd\phi \cdot \hat{a}_\phi \]

Example of Surface differentials

\[ d\vec{s} = dx \cdot dy \cdot \hat{a}_z \quad \text{or} \quad d\vec{s} = rd\phi \cdot dz \cdot \hat{a}_r \]

Example of Volume differentials

\[ dv = dx \cdot dy \cdot dz \]