Problem Solution #4

1.

a) $Q = \int \rho_v dv = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^{0.03} 5r^3 r^2 dr = 2\pi \times 2 \times 5 \times \left.\frac{r^6}{6}\right|_0^{0.03} = 7.634\times 10^{-9} C$

b)

$$Q = \int \rho_s ds = 0.03^2 \times \int_0^{2\pi} d\phi \int_0^{\pi} 0.002\cos^2\theta \sin\theta d\theta = 0.03^2 \times 2\pi \times 0.002 \times \left.\frac{-\cos^3\theta}{3}\right|_1^{-1} = 7.54\times 10^{-6} C$$

2.

$$\int \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}\int \rho d^3 r$$

$$2\pi r l E = \frac{1}{\varepsilon_0}\times 2\pi \times 0.005 l \int_0^r r' \times r' dr'$$

$$\vec{E} = \frac{1.67\times 10^{-3} r^2}{\varepsilon_0}\hat{r}$$

3.

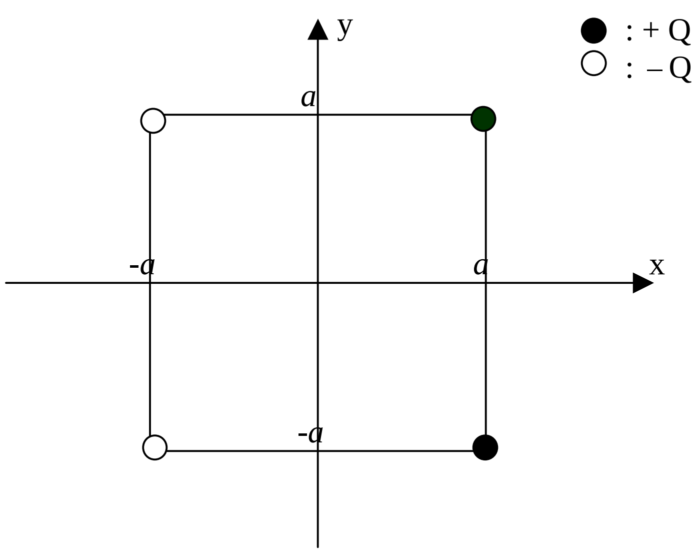
a) $\vec{D} = xy^2 z^3 \hat{x}$,

$$Q = \int \vec{D}\cdot d\vec{a} = \int_0^2\int_0^2 2y^2 z^3 dydz = 2\times \left.\frac{y^3}{3}\right|_0^2 \times \left.\frac{z^4}{4}\right|_0^2 = \frac{64}{3} \approx 21.33 \text{ C}$$

b) $\rho_v = \nabla\cdot\vec{D} = \frac{\partial}{\partial x}\left(xy^2 z^3\right) = y^2 z^3 \quad \text{C/m}^3$

c) $Q = \int \rho_v dv = \int_0^2\int_0^2\int_0^2 y^2 z^3 dxdydz = x\big|_0^2 \left.\frac{y^3}{3}\right|_0^2 \left.\frac{z^4}{4}\right|_0^2 = \frac{64}{3} \approx 21.33 \text{ C}$

4.



a) Electric potential at $(x,0)$

$$V(x,0)=\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{4}\frac{Q_i}{\sqrt{(x-x_i)^2+y_i^2}}$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{(a/2-x)^2+a^2/4}}+\frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{(a/2-x)^2+a^2/4}}$$

$$+\frac{1}{4\pi\varepsilon_0}\frac{-Q}{\sqrt{(-a/2-x)^2+a^2/4}}+\frac{1}{4\pi\varepsilon_0}\frac{-Q}{\sqrt{(-a/2-x)^2+a^2/4}}$$

$$=\frac{1}{2\pi\varepsilon_0}\left\{\frac{Q}{\sqrt{(a/2-x)^2+a^2/4}}-\frac{Q}{\sqrt{(a/2+x)^2+a^2/4}}\right\}$$

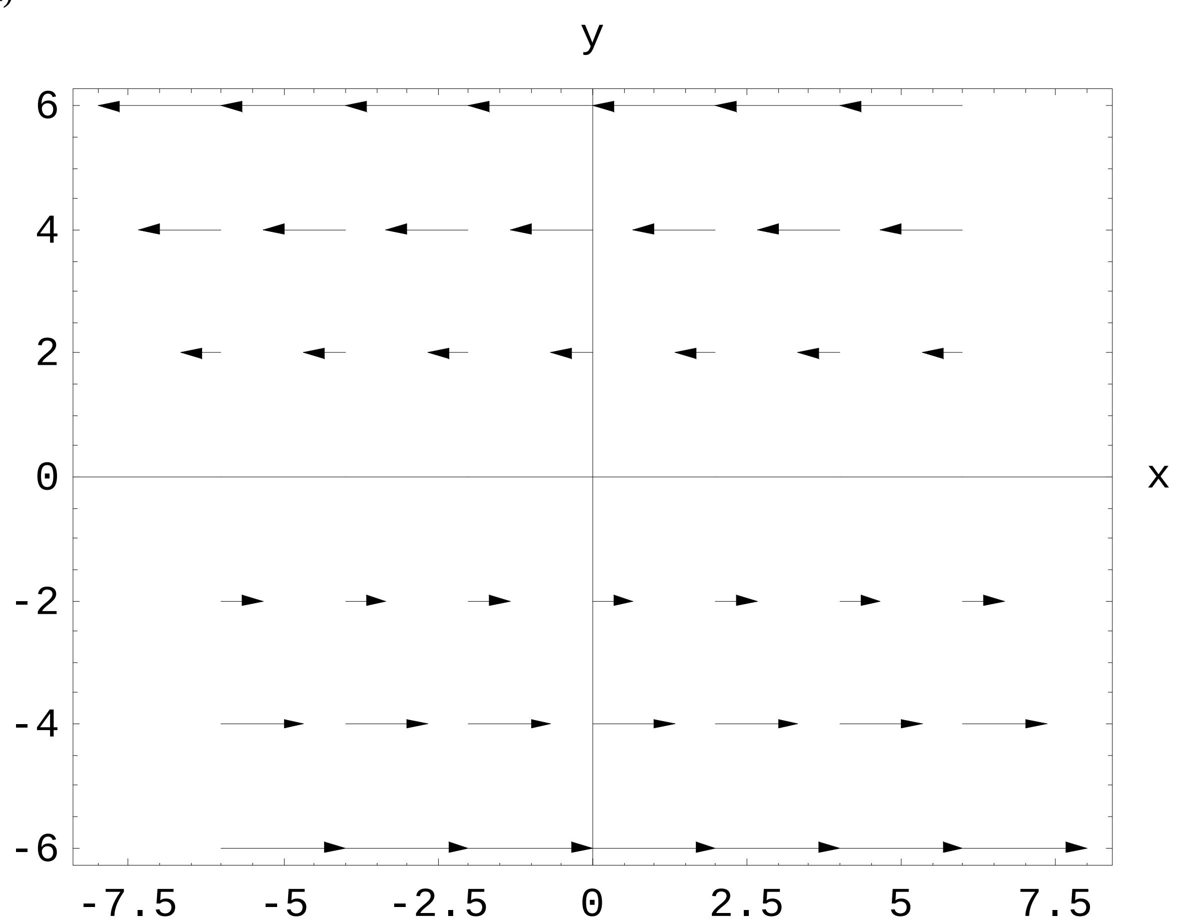
b) substitute $x=a/2$ into the above expression for the potential

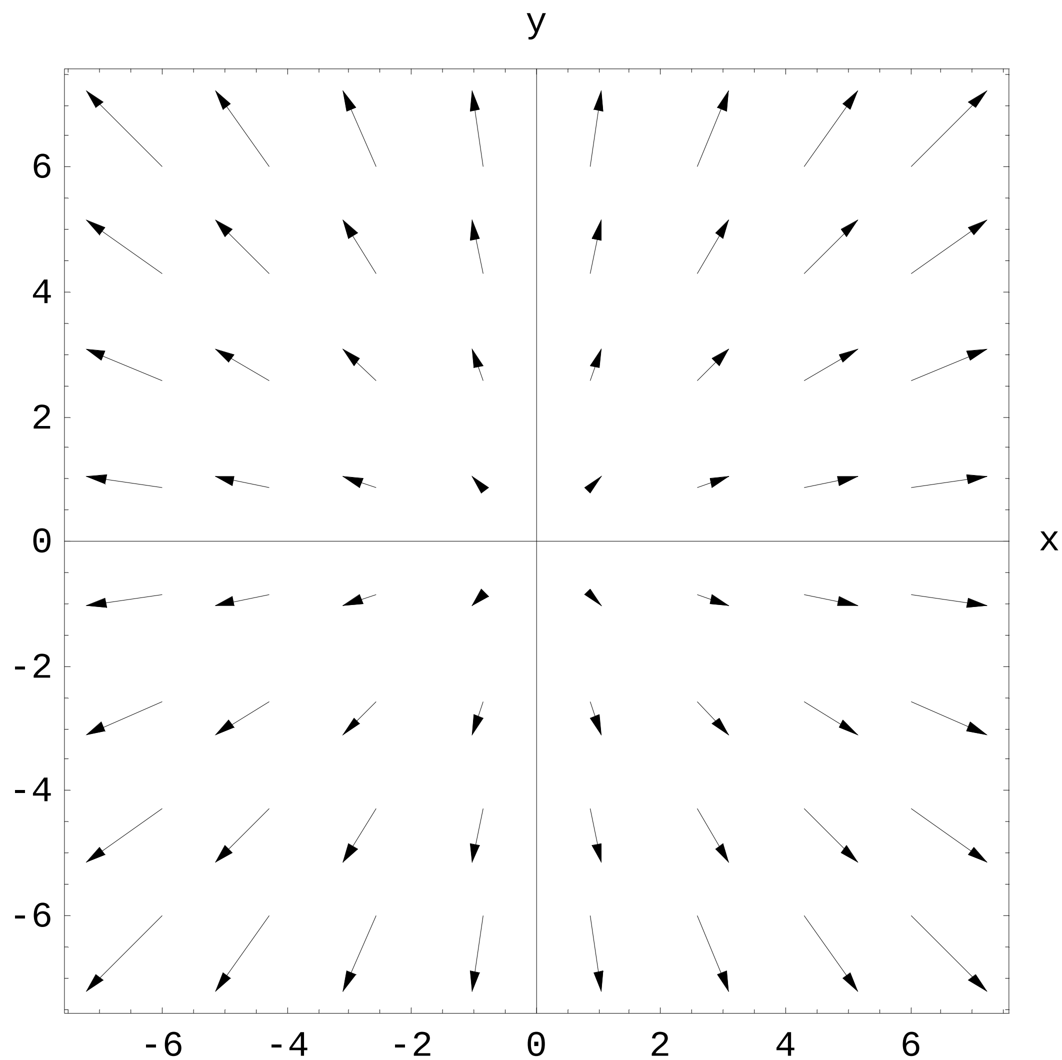
$$V(a/2,0)=\frac{Q}{2\pi\varepsilon_0}\left\{\frac{1}{\sqrt{(a/2-a/2)^2+a^2/4}}-\frac{1}{\sqrt{(a/2+a/2)^2+a^2/4}}\right\}$$

$$=\frac{Q}{\pi\varepsilon_0 a}\left\{1-\frac{\sqrt{5}}{5}\right\}$$

5.

a)





b)

6.

Area of a Triangle via the Cross Product:

Find the area of the triangle defined by

$A\,(0,2,2)$, $B\,(2,-2,2)$, $C\,(1,1,-2)$

The area of the parallelogram with sides AB and AC is equal to the magnitude of the cross product of vectors representing two adjacent sides.

Area (parallelogram) = $\left|\overline{AB}\times\overline{AC}\right|$. The area of the triangle is half of this.

First we need to determine the vectors:

$$\overline{AB} = (2-0)\hat{\mathbf{i}} + (-2-2)\hat{\mathbf{j}} + (2-2)\hat{\mathbf{k}} = 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

$$\overline{AC} = (1-0)\hat{\mathbf{i}} + (1-2)\hat{\mathbf{j}} + (-2-2)\hat{\mathbf{k}} = 1\hat{\mathbf{i}} - 1\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$$

Hence the cross product

$$\overline{AB}\times\overline{AC} = \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -4 & 0 \\ 1 & -1 & -4 \end{pmatrix} = 16\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

Hence the area of the triangle is

Area (triangle) $= \frac{1}{2}\left|16\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + 2\hat{\mathbf{k}}\right| = 9$