Problem Solution #6

Problem 1

Poisson's equation: $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

$$\nabla^2 \vec{A} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \vec{A} = \hat{a}_x \nabla^2 A_x + \hat{a}_y \nabla^2 A_y + \hat{a}_z \nabla^2 A_z$$

Here $\vec{A} = -\frac{\mu_0 J_0}{4}\left(x^2 + y^2\right)\hat{a}_z$ so

$$\nabla^2 \vec{A} = \hat{a}_z \nabla^2 A_z = \hat{a}_z \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[-\frac{\mu_0 J_0}{4}\left(x^2 + y^2\right)\right] = \hat{a}_z \left(-\frac{\mu_0 J_0}{2} - \frac{\mu_0 J_0}{2}\right) = -\mu_0 J_0 \hat{a}_z = -\mu_0 \vec{J}$$

$$\vec{B} = \nabla \times \vec{A} = \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A \end{pmatrix} = \frac{\partial A}{\partial y}\hat{\mathbf{i}} - \frac{\partial A}{\partial x}\hat{\mathbf{j}} = -\frac{\mu_0 J_0}{2} y\hat{\mathbf{i}} + \frac{\mu_0 J_0}{2} x\hat{\mathbf{j}} = \boxed{\frac{\mu_0 J_0}{2}(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}})}$$

$$\vec{B} = \mu_0 \vec{H} \Rightarrow \vec{H} = \frac{\vec{B}}{\mu_0} = \boxed{\frac{J_0}{2}(-y\hat{\mathbf{i}} + x\hat{\mathbf{j}})}$$

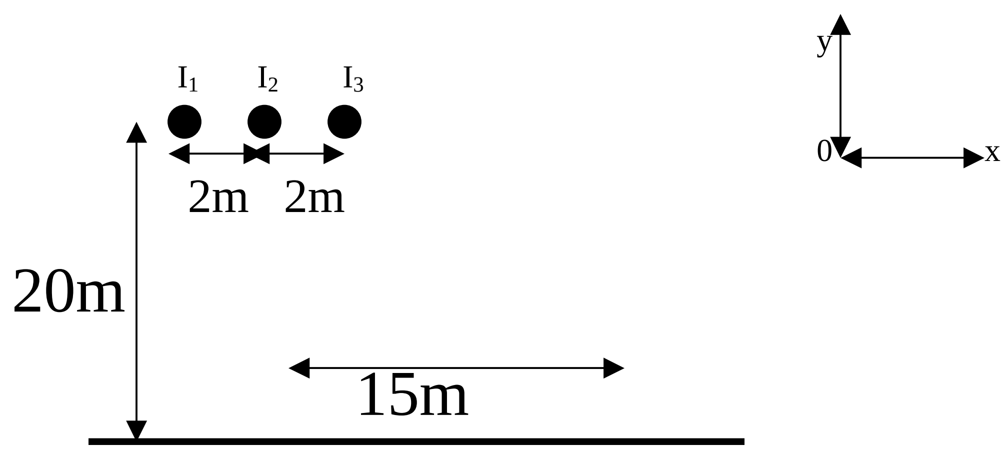
Ampere's law : $\oint_c \vec{H} \cdot d\vec{l} = I$

$$\oint_c \vec{H} \cdot d\vec{l} = \int \nabla \times \vec{H} \cdot d\vec{s} = \int \frac{J_0}{2} \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{pmatrix} \cdot \hat{\mathbf{k}} ds = \int \frac{J_0}{2}\left(1\hat{\mathbf{k}} + 1\hat{\mathbf{k}}\right) \cdot \hat{\mathbf{k}} ds = \int J_0 \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} ds = \int \vec{J} \cdot \hat{\mathbf{k}} ds = I$$

or just try to confirm $\nabla \times \vec{H} = \vec{J}$

Problem 2

a)



$$\oint_c \vec{H} \cdot d\vec{l} = I \quad \Rightarrow \vec{H} = \frac{I}{2\pi r}\hat{\phi}$$

$$H_{1x} = H_1 \frac{20}{r} = \frac{20 I_1}{2\pi r^2} = -\frac{20\times 250}{2\pi\left(17^2+20^2\right)} \qquad H_{1y} = H_1 \frac{17}{r} = \frac{17 I_1}{2\pi r^2} = -\frac{17\times 250}{2\pi\left(17^2+20^2\right)}$$

$$H_{2x} = H_2 \frac{20}{r} = \frac{20 I_2}{2\pi r^2} = \frac{20\times 500}{2\pi\left(15^2+20^2\right)} \qquad H_{2y} = H_2 \frac{15}{r} = \frac{15 I_2}{2\pi r^2} = \frac{15\times 250}{2\pi\left(15^2+20^2\right)}$$

$$H_{3x} = H_3 \frac{20}{r} = \frac{20 I_3}{2\pi r^2} = -\frac{20\times 250}{2\pi\left(13^2+20^2\right)} \qquad H_{3y} = H_3 \frac{13}{r} = \frac{13 I_3}{2\pi r^2} = -\frac{13\times 250}{2\pi\left(13^2+20^2\right)}$$

$$H_x = H_{1x} + H_{2x} + H_{3x} \approx \boxed{-0.007 A/m} \qquad H_y = H_{1y} + H_{2y} + H_{3y} \approx \boxed{0.019 A/m}$$

$$\boxed{\vec{H} = -0.007\hat{x} + 0.019\hat{y} (A/m)}$$

b) $\vec{H} = \dfrac{Ia^2}{2(a^2+z^2)^{3/2}}\hat{z} = \dfrac{1\times 0.02^2}{2(0.02^2+0.5^2)^{3/2}}\hat{z} \approx 1.6\hat{z}(mA/m)$

Problem 3

Boundary condition: $\mu_1 H_{1n} = \mu_2 H_{2n}$ and $H_{1t} = H_{2t}$

Then $\mu_1 H_1 \cos\theta_1 = \mu_2 H_2 \cos\theta_2$, $\mu_2 H_2 \cos\theta_2 = \mu_3 H_3 \cos\theta_3$, and

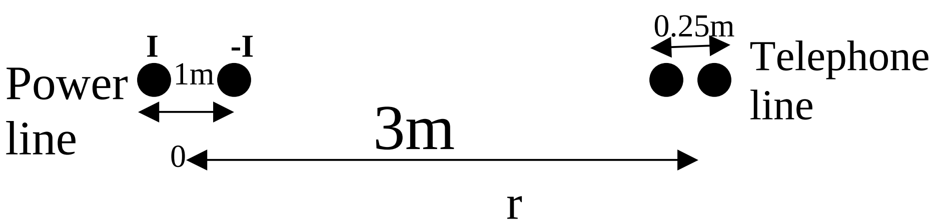
$H_1 \sin\theta_1 = H_2 \sin\theta_2$, $H_2 \sin\theta_2 = H_3 \sin\theta_3$

so $\boxed{\theta_3 = \tan^{-1}\left(\dfrac{\mu_1 \tan\theta_1}{\mu_3}\right)}$

These show that the angle θ_3 is independent of μ_2.

Problem 4

Set the center of the power line as the origin,



$$\oint_c \vec{B}\cdot d\vec{l} = \mu_0 I \Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

$$B_+ = \frac{\mu_0 I}{2\pi\left(0.5+r\right)} \qquad B_- = -\frac{\mu_0 I}{2\pi\left(r-0.5\right)}$$

$$\Phi = \int \vec{B}\cdot d\vec{s}$$

$$\Phi = \frac{I\mu_0}{2\pi}\int_{3-\frac{0.25}{2}}^{3+\frac{0.25}{2}}\left(\frac{1}{r+0.5} - \frac{1}{r-0.5}\right)dr$$

$$= \frac{I\mu_0}{2\pi}\left\{In\left(r+0.5\right) - In\left(r+0.5\right)\right\}\Big|_{2.875}^{3.125} \approx \boxed{-5.72\times 10^{-7} Wb}$$