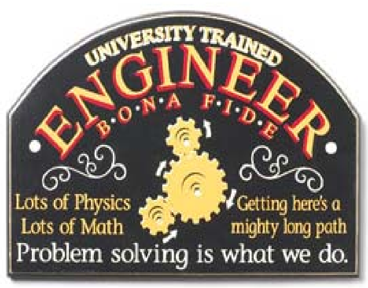
Quiz 2

VECTOR CALCULUS
ELECTROSTATICS
CURRENT & RESISTANCE

Name ____*Solution*_____________

Section ________________________



Notes:
1. In the multiple choice questions, each question may have more than one correct answer; circle all correct answers.
2. For multiple choice questions, you may add some comments to justify your answer.
3. Make sure your calculator is set to perform trigonometric functions in radians & not degrees.
4. Draw pictures for each problem to be sure that you understand the problem statement.

1. (4 Pts)	____________
2. (4 Pts)	____________
3. (4 Pts)	____________
4. (8 Pts)	____________
5. (8 Pts)	____________
6. (8 Pts)	____________
7. (8 Pts)	____________
8. (10 Pts)	____________
9. (24 Pts)	____________
10. (12 Pts)	____________
11. (10 Pts)	____________
12. (Ex Cred)	____________
Total (100 Pts)	____________

Some Comments and Helpful Info:
In this test, we use two types of notation for unit vectors. Keep in mind that

$\hat{a}_x = \hat{x}$ $\hat{a}_y = \hat{y}$ $\hat{a}_z = \hat{z}$ $\hat{a}_r = \hat{r}$ $\hat{a}_\phi = \hat{\phi}$ $\hat{a}_\theta = \hat{\theta}$

Also, sometimes R is used for spherical radius instead of r, so R is another term that gets used for more than one purpose. Pay attention to the context of the questions to minimize problems.
The general form of Ohm's Law is given by $\vec{J} = \sigma\vec{E}$. The resistance of a material with conductivity σ is given by $R = \dfrac{l}{\sigma A}$.

Quiz 2

1. Solving for the Electric Field (4 Points)

If you determine the electric field in some region using Gauss' Law in integral form, which of the following is ***always*** useful for checking your answer?

a. Applying Gauss' Law in Differential Form *Divergence gives the charge distribution*

b. Looking it up on Google *not all problems are online*

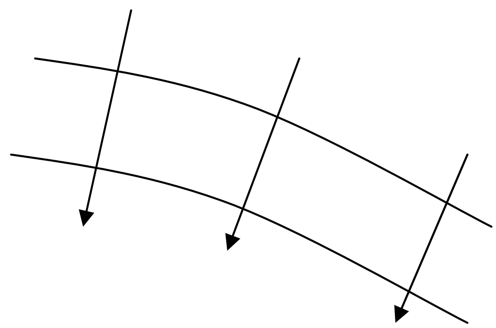
c. Checking the answers in the back of the book *not all problems have solutions*

d. Checking the form of the solution against the electric field due to a point charge. *The field of a point charge shows the form of the solution* $\vec{E} = \frac{q}{4\pi\varepsilon_o r^2}\hat{r}$

e. Asking the clerk at Stewarts. *not many EE students work there*

2. Equipotentials and Electric Field Lines (4 Points)

Electric Field lines and equipotential surfaces are



a. Always parallel to one another

b. Never parallel to one another

c. Always perpendicular to one another

d. Never perpendicular to one another

3. Laplace's Equation (4 Points)

Which of the following statements are correct?

a. The value of a function that is a solution to Laplace's equation is equal to the average of the values at its nearest neighboring points.

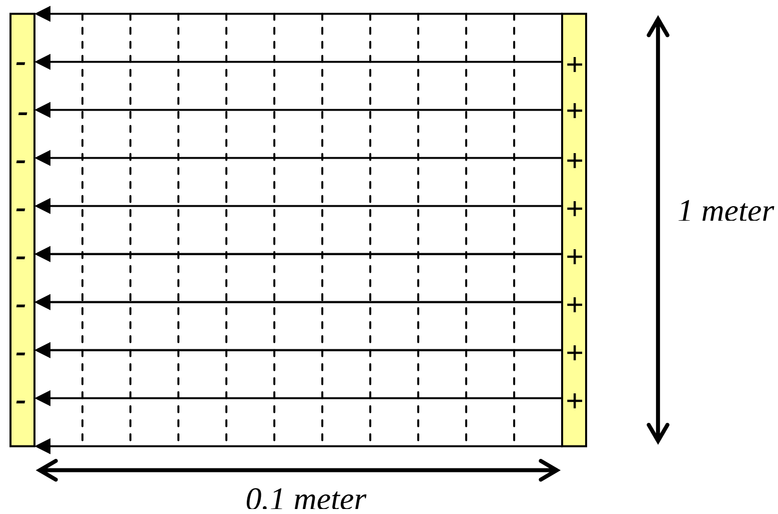
b. The solution to Laplace's equation in some region in space achieves its maximum and minimum values on the boundary of the region. *The voltages in a region are between the voltages on the electrodes*

c. Poisson's Equation becomes Laplace's Equation in any region with no volume charge. *Just set the charge term equal to zero.*

d. Laplace's Equation is a second order partial differential equation. *The Laplacian is a second order operator.*

4. Energy and Capacitance (8 Points)

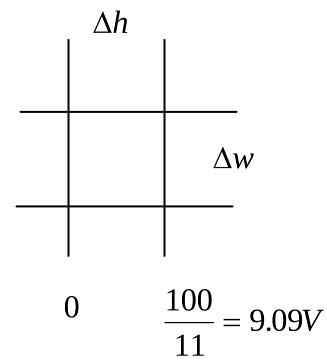
For an air insulated parallel plate capacitor, the field lines and equipotentials can be drawn as shown below. Assume that the capacitor has a depth of 1 meter into the page.



a. What is the capacitance of this parallel plate structure?

$C = \frac{\varepsilon_o A}{d} = \frac{\varepsilon_o}{0.1} = 10\varepsilon_o$ *This is the best form for the solution. A = 1 and d = 0.1*

b. If the voltage on the right plate is *100 V* and the left plate is grounded, how much energy is stored in one of the small cells shown? Each cell is *1 meter* deep.



ΔV *for each step in potential is* $\frac{100}{11} = 9.09V$

$\Delta w = \frac{1meter}{9} = 0.111m$

$\vec{E} = \frac{100V}{0.1m} = 1000\frac{V}{m}$

$\Delta h = \frac{0.1m}{11} = 0.00909m$

$W_e = \frac{1}{2}\varepsilon_o E^2 \cdot (volume) = \frac{1}{2}\varepsilon_o (1000)^2 \cdot (\Delta w \Delta h) = 505\varepsilon_o$

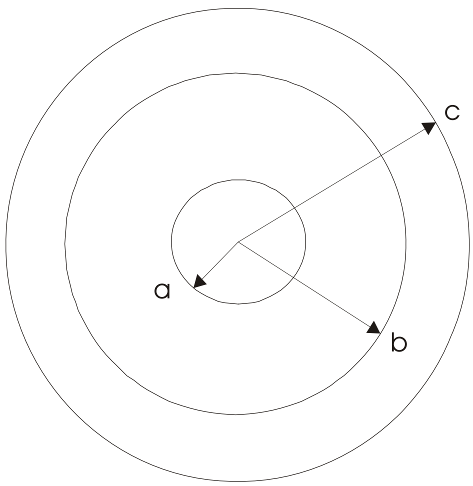
Any answer around 500 ε_o *is OK.*

Can also do this by using $W_e = \frac{1}{2}CV^2$ *and note that each cell has 1/99 of this energy.*

This gives us $W_e = \frac{1}{2}\frac{10\varepsilon_o 100^2}{99} = 505\varepsilon_o$ *which agrees with the other method.*

5. Gauss' Law (8 points)

The electric field for a particular charge distribution in free space is given by
$\vec{E}(r,\theta,\phi) = \hat{a}_r \frac{\rho_o}{\varepsilon_o}\left[\frac{r}{3} - \frac{r^2}{4b}\right]$ for $r \le b$ and $\vec{E}(r,\theta,\phi) = 0$ for $r > b$. Find the total charge enclosed for the two surfaces shown in the figure below: spherical surfaces at $r = a$ and at $r = c$. *Be sure that you explain your method, since there is more than one way to do this problem. If you have time, you might try using more than one method to check your answer. Hint: Begin by writing Gauss' Law in integral form.*



$$\oint \vec{D}\cdot d\vec{S} = \varepsilon_o \oint \vec{E}\cdot d\vec{S} = Q_{encl}$$

$$\vec{D}(r,\theta,\phi) = \hat{a}_r \rho_o\left[\frac{r}{3} - \frac{r^2}{4b}\right] \textit{for } r \le b \textit{ and } \vec{D}(r,\theta,\phi) = 0 \textit{ for } r \le b$$

Q_{encl} *for r=c:* $\oint \vec{D}\cdot d\vec{S} = 0 = Q_{encl}$

for r=a: $\oint \vec{D}\cdot d\vec{S} = \oint \rho_o\left(\frac{a}{3} - \frac{a^2}{4b}\right) = \rho_o\left(\frac{a}{3} - \frac{a^2}{4b}\right)4\pi a^2 = Q_{encl}$

This was found by evaluating $\oint \vec{D}\cdot d\vec{S} = Q_{encl}$

One could also do this by first finding the charge distributions.

$$\rho_v = \nabla\cdot\vec{D} = \nabla\cdot\hat{r}\rho_o\left(\frac{r}{3} - \frac{r^2}{4b}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\rho_o\left(\frac{r}{3} - \frac{r^2}{4b}\right) = \frac{\rho_o}{r^2}\frac{\partial}{\partial r}\left(\frac{r^3}{3} - \frac{r^4}{4b}\right) = \frac{\rho_o}{r^2}\left(r^2 - \frac{r^3}{b}\right)$$

and $\rho_v = \rho_o\left(1 - \frac{r}{b}\right)$. *Integrating this from 0 to a,*

$Q_{encl} = 4\pi\int_0^a \rho_o\left(1 - \frac{r}{b}\right)r^2 dr = 4\pi\rho_o\left(\frac{a^3}{3} - \frac{a^4}{4b}\right)$ *which is the same answer. The charge at c is also still zero since the surface charge at b must cancel positive charge inside b.*

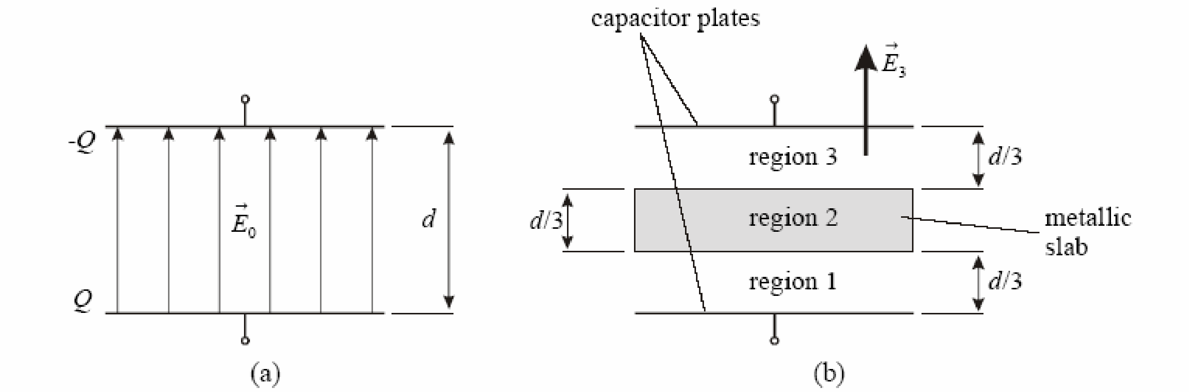
6. Resistance and Conductivity (8 points)

A 100 meter long steel pipe with inner diameter of 10 cm and outer diameter of 11 cm. The conductivity of the steel is 10^6 S/m. The pipe characteristics considered here are typical for pipes used in oil wells. What is the resistance of the pipe? *Be careful of units.*

The resistance expression is given on the front page of this quiz.

$$R = \frac{l}{\sigma A} = \frac{100m}{10^6 \pi\left(\left(\frac{0.11}{2}\right)^2 - \left(\frac{0.1}{2}\right)^2\right)} = \frac{100m}{10^6 \pi(0.000525)} = 60m\Omega$$

7. Capacitance (8 points)
An air-filled parallel-plate capacitor is charged and its terminals are left open. The charges of the lower and upper plates are Q and $-Q$, respectively, the fringing effects can be neglected, and the electric field intensity vector between the plates is $\boldsymbol{E}_0$ [Figure (a)]. An uncharged metallic slab is then inserted between the plates, without touching the plates by hands or any other conducting body [Figure (b)].



a) The electric field intensity vector in region 3 between the slab and the upper plate in the new stage is

i) $\vec{E}_3 = 0$
ii) $\vec{E}_3 = -\vec{E}_o$
iii) $\vec{E}_3 = \vec{E}_o/3$
iv) $\vec{E}_3 = 3\vec{E}_o/2$

v) $\vec{E}_3 = \vec{E}_o$ *since the metal slab is uncharged, it must have the same* ρ_s *on the top and bottom and equal in magnitude to the* ρ_s *on the capacitor plates.*

b) If the capacitance in figure (a) is C_a, the capacitance of figure (b) is

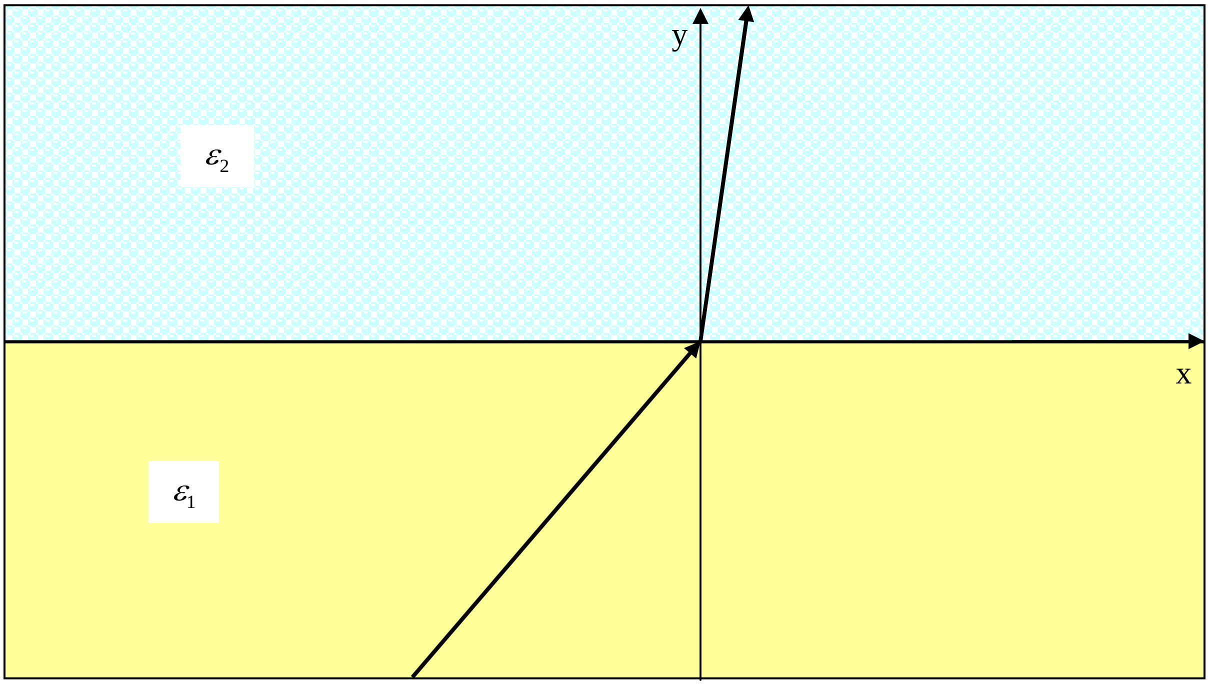
i) $C_b = 0$
ii) $C_b = C_a$
iii) $C_b = 3C_a$

iv) $C_b = \left(2/3\right)C_a$

v) $C_b = \left(3/2\right)C_a$ *The E field is the same, but the voltage is* $2/3$ *of its previous value.*

Since Q is also the same, $C = \frac{1}{2/3} = 3/2$ *of before.*

8. Boundary Conditions (10 points)



The electric field in region 1 is $\vec{E}_1 = E_o\left(\hat{a}_x + \hat{a}_y\right)$. The electric field in region 2 is $\vec{E}_2 = E_o\left(\hat{a}_x + 3\hat{a}_y\right)$.

a. Assuming that one of these regions is free space, what is the dielectric constant ε of the other region? (6 Points)

$$E_{t1} = E_{t2} \qquad\qquad D_{n1} = D_{n2}$$

$$\hat{x}: \quad E_o = E_o \qquad\qquad \hat{y}: \quad \varepsilon_1 E_{n1} = \varepsilon_2 E_{n2}$$

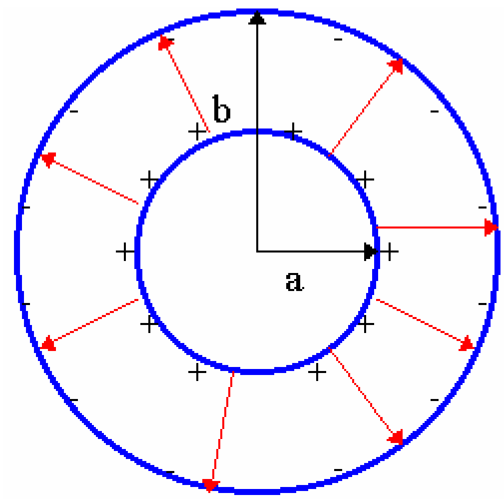
$$\varepsilon_1 E_o = 3\varepsilon_2 E_o$$

$$\Rightarrow \quad \varepsilon_1 = 3\varepsilon_2 = 3\varepsilon_o$$

b. Identify which region is free space (air), region 1 or region 2. (4 Points) *The field line is always closer to normal in the region with the smaller* ε *which is* ***region 2***. *We also know this from the fact that* ε *cannot be less than* ε_o.

9. Gauss' Law, Materials and Capacitance (24 points)

The diagram of a spherical capacitor shown below was found at a website for an electromagnetic theory course at another university. The inner conductor, radius = a, has a total positive charge on it +Q while the outer conductor, radius = b, has a total negative charge –Q. The plus (+) and minus (-) signs indicate where the charge is found. Also included in this figure are some typical, if badly drawn, electric field lines.



A spherical capacitor

Air insulated

Inner radius = a

Outer radius = b

Total charge Q on each conductor

a. What are the surface charge densities at r = a and r = b, ρ_{sa} & ρ_{sb} ?

$$\rho_{sa} = \frac{Q}{4\pi a^2} \qquad \rho_{sb} = \frac{-Q}{4\pi b^2}$$

b. Given that this capacitor is air insulated ($\varepsilon = \varepsilon_o$), determine the electric flux density D everywhere in space in terms of parameters like Q, ε_o and the dimensions of the capacitor. *It may not be necessary to use all of these terms.*

This configuration behaves like a point charge ➡ $\vec{E} = \hat{r}\dfrac{Q}{4\pi\varepsilon_o r^2}$

$$\vec{D} = \varepsilon_o \vec{E} = \hat{r}\frac{Q}{4\pi r^2}$$

c. Determine the electric field E using your answer to part b.

$$\vec{E} = \hat{r}\frac{Q}{4\pi\varepsilon_o r^2}$$

d. Now assume that the air insulator has been replaced by a dielectric with a relative permittivity of $\varepsilon_r = 3$. Write the new expressions for D and E

$$\vec{D} = \hat{r}\frac{Q}{4\pi r^2} \qquad \vec{E} = \hat{r}\frac{Q}{12\pi\varepsilon_o r^2} = \hat{r}\frac{Q}{4\pi\varepsilon r^2}$$

You can complete this problem using either of the general methods for finding capacitance.

Option 1:
e. Assuming that the outer conductor is grounded, determine the potential on the inner conductor V(a) using your expression for E from part d.

$$V(a) = -\int_b^a \frac{Q}{4\pi\varepsilon r^2}dr = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right) \textit{ where we have assumed that } V(b) = 0$$

f. Using the voltage method, find the capacitance of this structure with $\varepsilon_r = 3$.

$$C = \frac{Q}{V(a)} = \frac{4\pi\varepsilon}{\left(\frac{1}{a} - \frac{1}{b}\right)} = \frac{4\pi\varepsilon ab}{(b-a)}$$

Option 2:
e. Find the energy stored in the region between r = a and r = b using your expression for D and E from part d.

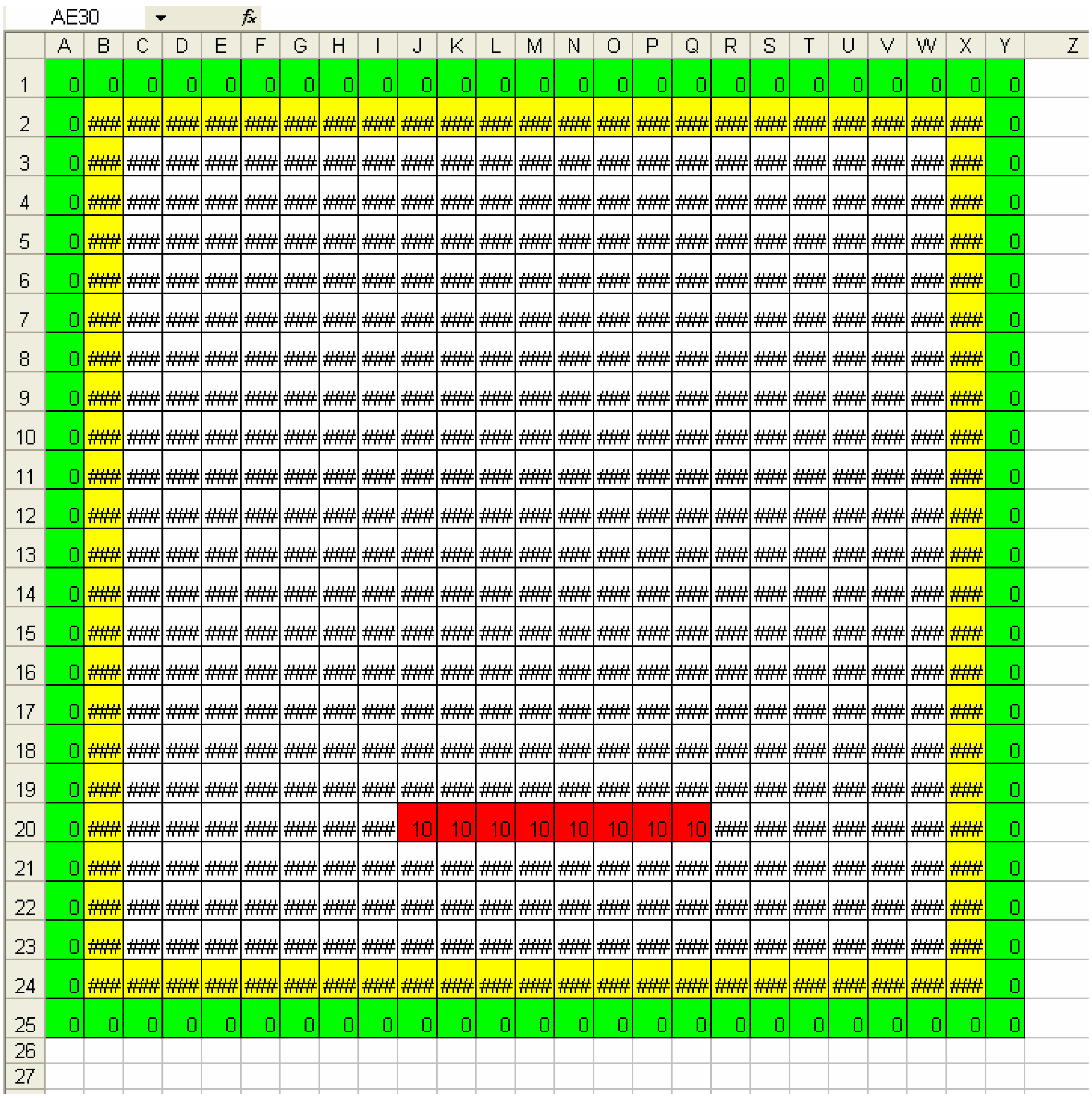
$$W_e = \frac{1}{2}\int \vec{D}\cdot\vec{E}dv = \frac{1}{2}\int \varepsilon E^2 dv = \frac{1}{2}\varepsilon\int\left(\frac{Q}{4\pi\varepsilon r^2}\right)^2 r^2 \sin\theta dr d\theta d\phi = \frac{Q^2}{8\pi\varepsilon}\int r^2 dr = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon}\left(-\frac{1}{b} + \frac{1}{a}\right)$$

f. Use the energy method to find the capacitance of this structure with $\varepsilon_r = 3$.

$$W_e = \frac{1}{2}\frac{Q^2}{C} \qquad C = \frac{4\pi\varepsilon}{\left(\frac{1}{a} - \frac{1}{b}\right)} = \frac{4\pi\varepsilon ab}{(b-a)}$$

10. Finite Difference Solution (12 Points)

An Excel spreadsheet was used to determine the capacitance per unit length of the following transmission line configuration. The insulator is characterized by $\varepsilon_r = 2.25$



AE30 fx

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
2	0	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	0	
3	0	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	0	
4	0	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	0	
5	0	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	0	
6	0	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	0	
7	0	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	0	
8	0	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	0	
9	0	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	0	
10	0	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	0	
11	0	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	0	
12	0	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	0	
13	0	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	0	
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24	0	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	###	0	
25	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
26																										
27																										

Note that the grounded outer conductor is a square trough, 24 mm by 24 mm and the inner conductor is a horizontal plate connected to a 10V DC source. From the spreadsheet, it is determined that the capacitance per unit length is approximately

$$C/l = 10\varepsilon_o$$

a. Determine the charge per unit length for this transmission line.

$$\Delta V = 10V \qquad C/l = 10\varepsilon_o \qquad Q/l = C/l\, \Delta V = 100\varepsilon_o$$

b. Note that the voltages in the interior cells are not shown. (The number of significant digits is too large to fit in the cell, so Excel just shows the # sign.) From your knowledge of this method, what is the average voltage in the cells neighboring (just inside) the outer conductor.

Per unit length area of the outer conductor is 24 x 4 = 96mm by 1meter = 0.096m²

Cell spacing is 1mm

$$\rho_s = \frac{100\varepsilon_o}{0.096} = 1042\varepsilon_o = D_n$$

$$E_n = \frac{D_n}{\varepsilon} = \frac{1042\varepsilon_o}{2.25\varepsilon_o} = 463 V/m = \frac{\Delta V}{\Delta x} = \frac{\Delta V}{0.001} \ \ or \ \ \Delta V = 0.463V$$

The average voltage in the cells is 0.463V.

Any answer from 0.4 to 0.5 V is OK.

11. True or False (10 Points)

It has been our experience through many years with this course that there is one electromagnetic fact almost every student can recall from their basic Physics classes.

True or false? There is no electric field inside of a perfect conductor.

12. Extra Credit (5 Points)

It is always a frustration to study for a test and then discover that one of the problems you were sure would be on the test is not included. This is your chance to earn some extra points for your efforts. In the space below, write out a problem you think would be fair to ask on this test, but is not like anything already included. It is not necessary to solve the problem, only to state it clearly. Remember that your new problem must be different and not just a variation of something given above.

No one came up with a really good question. Most people reproduced something from spring or fall 2005. Most were not very complete. Generally, the popular answers were

1. *More vector calculus (taking divergence and curl)*
2. *Solving for the electric field from a charge distribution, usually rectangular*
3. *Solving Poisson's equation from a charge distribution*
4. *Drawing equipotentials and field lines*
5. *Telling a cute story that involved electric fields in some way*
6. *Magnetic field problems (which indeed were not like the other problems in this test).*