| *Cartesian* | | *Cylindrical* | | *Spherical* |
|---|---|---|---|---|
| $x$ | $=$ | $\mathrm{r}\cos\phi$ | $=$ | $r\sin\theta\cos\phi$ |
| $y$ | $=$ | $\mathrm{r}\sin\phi$ | $=$ | $r\sin\theta\sin\phi$ |
| $z$ | $=$ | $z$ | $=$ | $r\cos\theta$ |
| $\mathbf{i}_x$ | $=$ | $\cos\phi\mathbf{i}_r - \sin\phi\mathbf{i}_\phi$ | $=$ | $\sin\theta\cos\phi\mathbf{i}_r + \cos\theta\cos\phi\mathbf{i}_\theta - \sin\phi\mathbf{i}_\phi$ |
| $\mathbf{i}_y$ | $=$ | $\sin\phi\mathbf{i}_r + \cos\phi\mathbf{i}_\phi$ | $=$ | $\sin\theta\sin\phi\mathbf{i}_r + \cos\theta\sin\phi\mathbf{i}_\theta + \cos\phi\mathbf{i}_\phi$ |
| $\mathbf{i}_z$ | $=$ | $\mathbf{i}_z$ | $=$ | $\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_\theta$ |

| *Cylindrical* | | *Cartesian* | | *Spherical* |
|---|---|---|---|---|
| $\mathrm{r}$ | $=$ | $\sqrt{x^2+y^2}$ | $=$ | $r\sin\theta$ |
| $\phi$ | $=$ | $\tan^{-1} y/x$ | $=$ | $\phi$ |
| $z$ | $=$ | $z$ | $=$ | $r\cos\theta$ |
| $\mathbf{i}_r$ | $=$ | $\cos\phi\mathbf{i}_x + \sin\phi\mathbf{i}_y$ | $=$ | $\sin\theta\mathbf{i}_r + \cos\theta\mathbf{i}_\theta$ |
| $\mathbf{i}_\phi$ | $=$ | $-\sin\phi\mathbf{i}_x + \cos\phi\mathbf{i}_y$ | $=$ | $\mathbf{i}_\phi$ |
| $\mathbf{i}_z$ | $=$ | $\mathbf{i}_z$ | $=$ | $\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_\theta$ |

| *Spherical* | | *Cartesian* | | *Cylindrical* |
|---|---|---|---|---|
| $r$ | $=$ | $\sqrt{x^2+y^2+z^2}$ | $=$ | $\sqrt{\mathrm{r}^2+z^2}$ |
| $\theta$ | $=$ | $\cos^{-1}\dfrac{z}{\sqrt{x^2+y^2+z^2}}$ | $=$ | $\cos^{-1}\dfrac{z}{\sqrt{\mathrm{r}^2+z^2}}$ |
| $\phi$ | $=$ | $\cot^{-1} x/y$ | $=$ | $\phi$ |
| $\mathbf{i}_r$ | $=$ | $\sin\theta\cos\phi\mathbf{i}_x + \sin\theta\sin\phi\mathbf{i}_y + \cos\theta\mathbf{i}_z$ | $=$ | $\sin\theta\mathbf{i}_r + \cos\theta\mathbf{i}_z$ |
| $\mathbf{i}_\theta$ | $=$ | $\cos\theta\cos\phi\mathbf{i}_x + \cos\theta\sin\phi\mathbf{i}_y - \sin\theta\mathbf{i}_z$ | $=$ | $\cos\theta\mathbf{i}_r - \sin\theta\mathbf{i}_z$ |
| $\mathbf{i}_\phi$ | $=$ | $-\sin\phi\mathbf{i}_x + \cos\phi\mathbf{i}_y$ | $=$ | $\mathbf{i}_\phi$ |

Geometric relations between coordinates and unit vectors for Cartesian, cylir drical, and spherical coordinate systems.

*Cartesian Coordinates* $(x, y, z)$

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{i}_y + \frac{\partial f}{\partial z}\mathbf{i}_z$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{i}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{i}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{i}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

*Cylindrical Coordinates* $(\mathrm{r}, \phi, \mathrm{z})$

$$\nabla f = \frac{\partial f}{\partial \mathrm{r}}\mathbf{i}_\mathrm{r} + \frac{1}{\mathrm{r}}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi + \frac{\partial f}{\partial \mathrm{z}}\mathbf{i}_\mathrm{z}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}(\mathrm{r}A_\mathrm{r}) + \frac{1}{\mathrm{r}}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_\mathrm{z}}{\partial \mathrm{z}}$$

$$\nabla \times \mathbf{A} = \mathbf{i}_\mathrm{r}\left(\frac{1}{\mathrm{r}}\frac{\partial A_\mathrm{z}}{\partial \phi} - \frac{\partial A_\phi}{\partial \mathrm{z}}\right) + \mathbf{i}_\phi\left(\frac{\partial A_\mathrm{r}}{\partial \mathrm{z}} - \frac{\partial A_\mathrm{z}}{\partial \mathrm{r}}\right) + \mathbf{i}_\mathrm{z}\frac{1}{\mathrm{r}}\left[\frac{\partial(\mathrm{r}A_\phi)}{\partial \mathrm{r}} - \frac{\partial A_\mathrm{r}}{\partial \phi}\right]$$

$$\nabla^2 f = \frac{1}{\mathrm{r}}\frac{\partial}{\partial \mathrm{r}}\left(\mathrm{r}\frac{\partial f}{\partial \mathrm{r}}\right) + \frac{1}{\mathrm{r}^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial \mathrm{z}^2}$$

*Spherical Coordinates* $(r, \theta, \phi)$

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \mathbf{i}_r\frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi}\right] + \mathbf{i}_\theta\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial(rA_\phi)}{\partial r}\right] + \mathbf{i}_\phi\frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta}\right]$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

# ELECTROMAGNETIC FIELD THEORY:

## *a problem solving approach*

**MARKUS ZAHN**
**Massachusetts Institute of Technology**

*to my parents*

## PREFACE

Electromagnetic field theory is often the least popular course in the electrical engineering curriculum. Heavy reliance on vector and integral calculus can obscure physical phenomena so that the student becomes bogged down in the mathematics and loses sight of the applications. This book instills problem solving confidence by teaching through the use of a large number of worked examples. To keep the subject exciting, many of these problems are based on physical processes, devices, and models.

This text is an introductory treatment on the junior level for a two-semester electrical engineering course starting from the Coulomb–Lorentz force law on a point charge. The theory is extended by the continuous superposition of solutions from previously developed simpler problems leading to the general integral and differential field laws. Often the same problem is solved by different methods so that the advantages and limitations of each approach becomes clear. Sample problems and their solutions are presented for each new concept with great emphasis placed on classical models of such physical phenomena as polarization, conduction, and magnetization. A large variety of related problems that reinforce the text material are included at the end of each chapter for exercise and homework.

It is expected that students have had elementary courses in calculus that allow them to easily differentiate and integrate simple functions. The text tries to keep the mathematical development rigorous but simple by typically describing systems with linear, constant coefficient differential and difference equations.

The text is essentially subdivided into three main subject areas: (1) charges as the source of the electric field coupled to polarizable and conducting media with negligible magnetic field; (2) currents as the source of the magnetic field coupled to magnetizable media with electromagnetic induction generating an electric field; and (3) electrodynamics where the electric and magnetic fields are of equal importance resulting in radiating waves. Wherever possible, electrodynamic solutions are examined in various limits to illustrate the appropriateness of the previously developed quasi-static circuit theory approximations.

Many of my students and graduate teaching assistants have helped in checking the text and exercise solutions and have assisted in preparing some of the field plots.

**Markus Zahn**

## A NOTE TO THE STUDENT

In this text I have tried to make it as simple as possible for an interested student to learn the difficult subject of electromagnetic field theory by presenting many worked examples emphasizing physical processes, devices, and models. The problems at the back of each chapter are grouped by chapter sections and extend the text material. To avoid tedium, most integrals needed for problem solution are supplied as hints. The hints also often suggest the approach needed to obtain a solution easily. Answers to selected problems are listed at the back of this book.

## A NOTE TO THE INSTRUCTOR

An Instructor's Manual with solutions to all exercise problems at the end of chapters is available from the author for the cost of reproduction and mailing. Please address requests on University or Company letterhead to:

Prof. Markus Zahn
Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
Cambridge, MA 01239

## CONTENTS

# ELECTROMAGNETIC FIELD THEORY:

*a problem solving approach*

# chapter 1

*review of vector analysis*

Electromagnetic field theory is the study of forces between charged particles resulting in energy conversion or signal transmission and reception. These forces vary in magnitude and direction with time and throughout space so that the theory is a heavy user of vector, differential, and integral calculus. This chapter presents a brief review that highlights the essential mathematical tools needed throughout the text. We isolate the mathematical details here so that in later chapters most of our attention can be devoted to the applications of the mathematics rather than to its development. Additional mathematical material will be presented as needed throughout the text.

## 1-1 COORDINATE SYSTEMS

A coordinate system is a way of uniquely specifying the location of any position in space with respect to a reference origin. Any point is defined by the intersection of three mutually perpendicular surfaces. The coordinate axes are then defined by the normals to these surfaces at the point. Of course the solution to any problem is always independent of the choice of coordinate system used, but by taking advantage of symmetry, computation can often be simplified by proper choice of coordinate description. In this text we only use the familiar rectangular (Cartesian), circular cylindrical, and spherical coordinate systems.

### 1-1-1 Rectangular (Cartesian) Coordinates

The most common and often preferred coordinate system is defined by the intersection of three mutually perpendicular planes as shown in Figure 1-1*a*. Lines parallel to the lines of intersection between planes define the coordinate axes $(x, y, z)$, where the $x$ axis lies perpendicular to the plane of constant $x$, the $y$ axis is perpendicular to the plane of constant $y$, and the $z$ axis is perpendicular to the plane of constant $z$. Once an origin is selected with coordinate $(0, 0, 0)$, any other point in the plane is found by specifying its $x$-directed, $y$-directed, and $z$-directed distances from this origin as shown for the coordinate points located in Figure 1-1*b*.

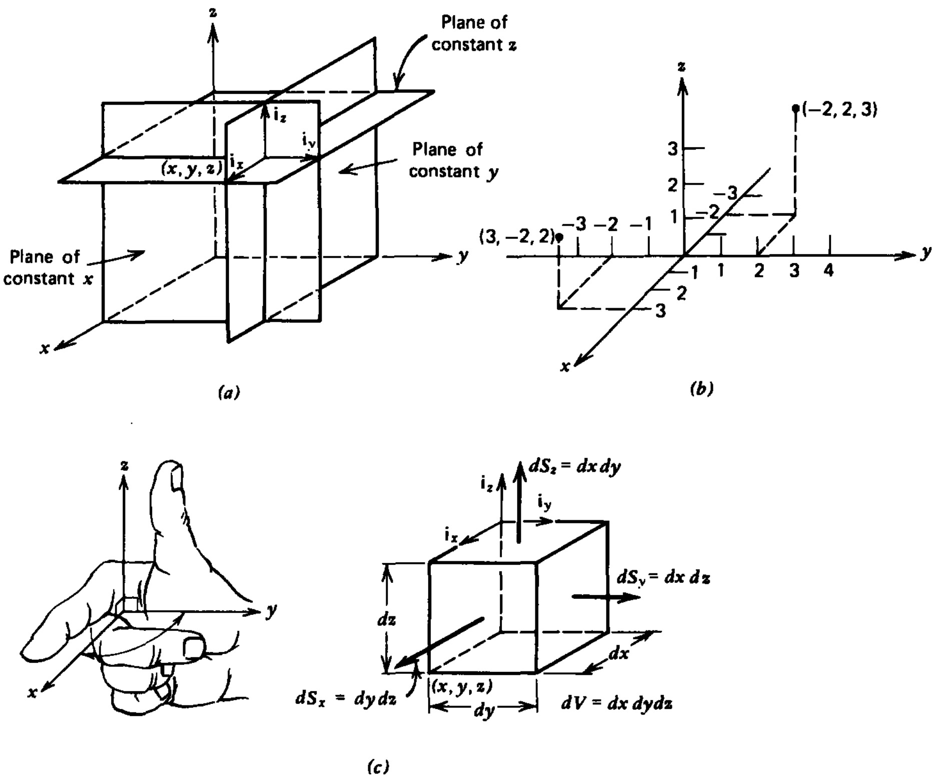



Figure 1-1 Cartesian coordinate system. (*a*) Intersection of three mutually perpendicular planes defines the Cartesian coordinates $(x, y, z)$. (*b*) A point is located in space by specifying its $x$-, $y$- and $z$-directed distances from the origin. (*c*) Differential volume and surface area elements.

By convention, a right-handed coordinate system is always used whereby one curls the fingers of his or her right hand in the direction from $x$ to $y$ so that the forefinger is in the $x$ direction and the middle finger is in the $y$ direction. The thumb then points in the $z$ direction. This convention is necessary to remove directional ambiguities in theorems to be derived later.

Coordinate directions are represented by unit vectors $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$, each of which has a unit length and points in the direction along one of the coordinate axes. Rectangular coordinates are often the simplest to use because the unit vectors always point in the same direction and do not change direction from point to point.

A rectangular differential volume is formed when one moves from a point $(x, y, z)$ by an incremental distance $dx$, $dy$, and $dz$ in each of the three coordinate directions as shown in

Figure 1-1c. To distinguish surface elements we subscript the area element of each face with the coordinate perpendicular to the surface.

### 1-1-2 Circular Cylindrical Coordinates

The cylindrical coordinate system is convenient to use when there is a line of symmetry that is defined as the $z$ axis. As shown in Figure 1-2*a*, any point in space is defined by the intersection of the three perpendicular surfaces of a circular cylinder of radius r, a plane at constant $z$, and a plane at constant angle $\phi$ from the $x$ axis.

The unit vectors $\mathbf{i}_r$, $\mathbf{i}_\phi$ and $\mathbf{i}_z$ are perpendicular to each of these surfaces. The direction of $\mathbf{i}_z$ is independent of position, but unlike the rectangular unit vectors the direction of $\mathbf{i}_r$ and $\mathbf{i}_\phi$ change with the angle $\phi$ as illustrated in Figure 1-2*b*. For instance, when $\phi=0$ then $\mathbf{i}_r=\mathbf{i}_x$ and $\mathbf{i}_\phi=\mathbf{i}_y$, while if $\phi=\pi/2$, then $\mathbf{i}_r=\mathbf{i}_y$ and $\mathbf{i}_\phi=-\mathbf{i}_x$.

By convention, the triplet (r, $\phi$, $z$) must form a right-handed coordinate system so that curling the fingers of the right hand from $\mathbf{i}_r$ to $\mathbf{i}_\phi$ puts the thumb in the $z$ direction.

A section of differential size cylindrical volume, shown in Figure 1-2*c*, is formed when one moves from a point at coordinate (r, $\phi$, $z$) by an incremental distance $dr$, r $d\phi$, and $dz$ in each of the three coordinate directions. The differential volume and surface areas now depend on the coordinate r as summarized in Table 1-1.

**Table 1-1 Differential lengths, surface area, and volume elements for each geometry. The surface element is subscripted by the coordinate perpendicular to the surface**

| CARTESIAN | CYLINDRICAL | SPHERICAL |
|---|---|---|
| $\mathbf{dl}=dx\,\mathbf{i}_x+dy\,\mathbf{i}_y+dz\,\mathbf{i}_z$ | $\mathbf{dl}=dr\,\mathbf{i}_r+r\,d\phi\,\mathbf{i}_\phi+dz\,\mathbf{i}_z$ | $\mathbf{dl}=dr\,\mathbf{i}_r+r\,d\theta\,\mathbf{i}_\theta$ $+r\sin\theta\,d\phi\,\mathbf{i}_\phi$ |
| $dS_x=dy\,dz$ | $dS_r=r\,d\phi\,dz$ | $dS_r=r^2\sin\theta\,d\theta\,d\phi$ |
| $dS_y=dx\,dz$ | $dS_\phi=dr\,dz$ | $dS_\theta=r\sin\theta\,dr\,d\phi$ |
| $dS_z=dx\,dy$ | $dS_z=r\,dr\,d\phi$ | $dS_\phi=r\,dr\,d\theta$ |
| $dV=dx\,dy\,dz$ | $dV=r\,dr\,d\phi\,dz$ | $dV=r^2\sin\theta\,dr\,d\theta\,d\phi$ |

### 1-1-3 Spherical Coordinates

A spherical coordinate system is useful when there is a point of symmetry that is taken as the origin. In Figure 1-3*a* we see that the spherical coordinate ($r$, $\theta$, $\phi$) is obtained by the intersection of a sphere with radius $r$, a plane at constant

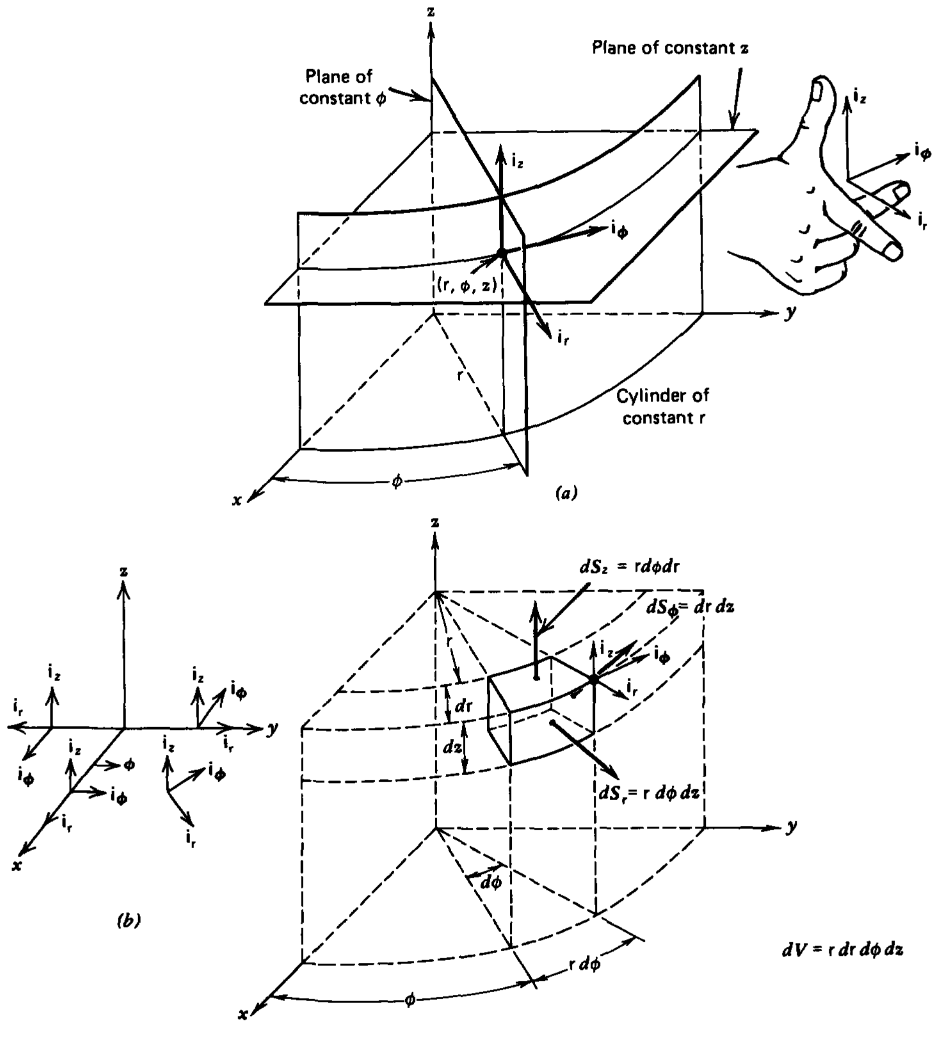
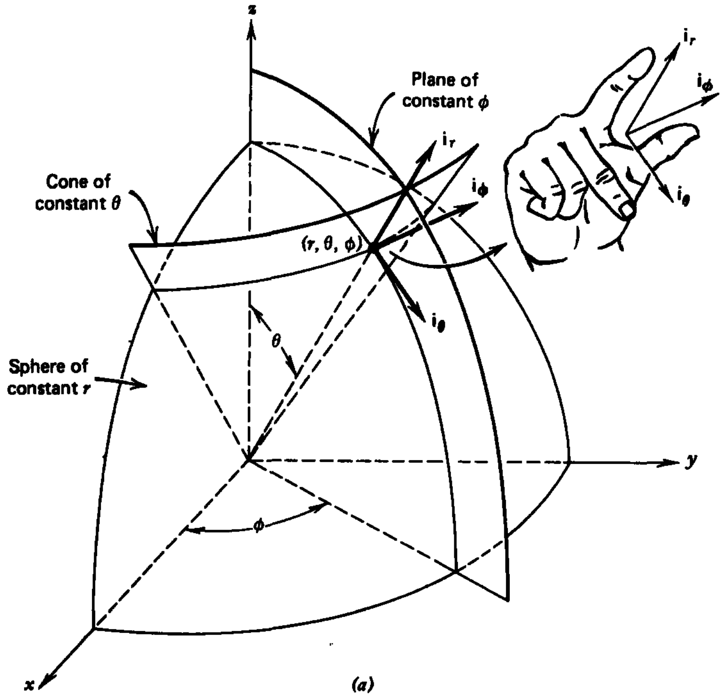



Figure 1-2 Circular cylindrical coordinate system. (*a*) Intersection of planes of constant z and $\phi$ with a cylinder of constant radius r defines the coordinates (r, $\phi$, z). (*b*) The direction of the unit vectors $\mathbf{i}_r$ and $\mathbf{i}_\phi$ vary with the angle $\phi$. (*c*) Differential volume and surface area elements.

angle $\phi$ from the $x$ axis as defined for the cylindrical coordinate system, and a cone at angle $\theta$ from the $z$ axis. The unit vectors $\mathbf{i}_r$, $\mathbf{i}_\theta$ and $\mathbf{i}_\phi$ are perpendicular to each of these surfaces and change direction from point to point. The triplet $(r, \theta, \phi)$ must form a right-handed set of coordinates.

The differential-size spherical volume element formed by considering incremental displacements $dr$, $r\,d\theta$, $r \sin\theta\, d\phi$



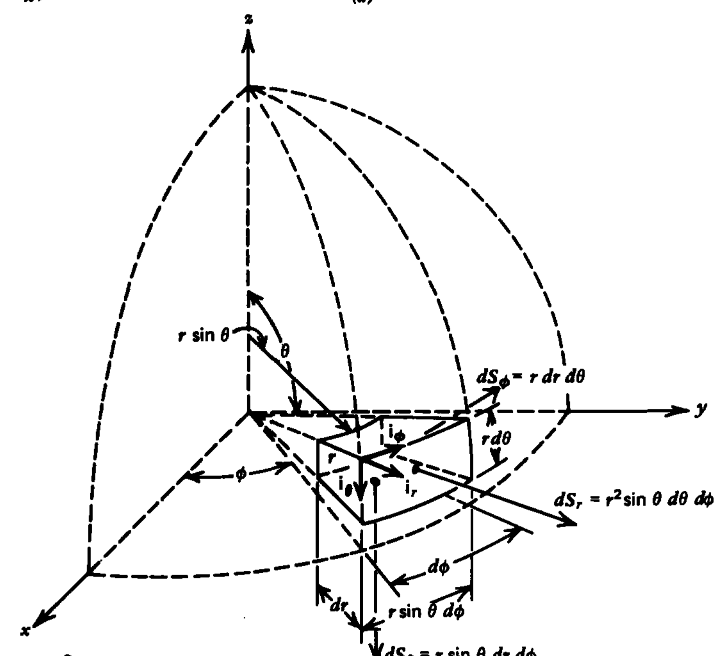




Figure 1-3 Spherical coordinate system. (*a*) Intersection of plane of constant angle $\phi$ with cone of constant angle $\theta$ and sphere of constant radius $r$ defines the coordinates $(r, \theta, \phi)$. (*b*) Differential volume and surface area elements.

**Table 1-2 Geometric relations between coordinates and unit vectors for Cartesian, cylindrical, and spherical coordinate systems***

| | | | | |
|---|---|---|---|---|
| CARTESIAN | | CYLINDRICAL | | SPHERICAL |
| $x$ | = | $\mathrm{r}\cos\phi$ | = | $r\sin\theta\cos\phi$ |
| $y$ | = | $\mathrm{r}\sin\phi$ | = | $r\sin\theta\sin\phi$ |
| $z$ | = | $z$ | = | $r\cos\theta$ |
| $\mathbf{i}_x$ | = | $\cos\phi\,\mathbf{i}_r-\sin\phi\,\mathbf{i}_\phi$ | = | $\sin\theta\cos\phi\,\mathbf{i}_r+\cos\theta\cos\phi\,\mathbf{i}_\theta-\sin\phi\,\mathbf{i}_\phi$ |
| $\mathbf{i}_y$ | = | $\sin\phi\,\mathbf{i}_r+\cos\phi\,\mathbf{i}_\phi$ | = | $\sin\theta\sin\phi\,\mathbf{i}_r+\cos\theta\sin\phi\,\mathbf{i}_\theta+\cos\phi\,\mathbf{i}_\phi$ |
| $\mathbf{i}_z$ | = | $\mathbf{i}_z$ | = | $\cos\theta\,\mathbf{i}_r-\sin\theta\,\mathbf{i}_\theta$ |
| CYLINDRICAL | | CARTESIAN | | SPHERICAL |
| $\mathrm{r}$ | = | $\sqrt{x^2+y^2}$ | = | $r\sin\theta$ |
| $\phi$ | = | $\tan^{-1}\frac{y}{x}$ | = | $\phi$ |
| $z$ | = | $z$ | = | $r\cos\theta$ |
| $\mathbf{i}_r$ | = | $\cos\phi\,\mathbf{i}_x+\sin\phi\,\mathbf{i}_y$ | = | $\sin\theta\,\mathbf{i}_r+\cos\theta\,\mathbf{i}_\theta$ |
| $\mathbf{i}_\phi$ | = | $-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y$ | = | $\mathbf{i}_\phi$ |
| $\mathbf{i}_z$ | = | $\mathbf{i}_z$ | = | $\cos\theta\,\mathbf{i}_r-\sin\theta\,\mathbf{i}_\theta$ |
| SPHERICAL | | CARTESIAN | | CYLINDRICAL |
| $r$ | = | $\sqrt{x^2+y^2+z^2}$ | = | $\sqrt{\mathrm{r}^2+z^2}$ |
| $\theta$ | = | $\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$ | = | $\cos^{-1}\frac{z}{\sqrt{\mathrm{r}^2+z^2}}$ |
| $\phi$ | = | $\cot^{-1}\frac{x}{y}$ | = | $\phi$ |
| $\mathbf{i}_r$ | = | $\sin\theta\cos\phi\,\mathbf{i}_x+\sin\theta\sin\phi\,\mathbf{i}_y+\cos\theta\,\mathbf{i}_z$ | = | $\sin\theta\,\mathbf{i}_r+\cos\theta\,\mathbf{i}_z$ |
| $\mathbf{i}_\theta$ | = | $\cos\theta\cos\phi\,\mathbf{i}_x+\cos\theta\sin\phi\,\mathbf{i}_y-\sin\theta\,\mathbf{i}_z$ | = | $\cos\theta\,\mathbf{i}_r-\sin\theta\,\mathbf{i}_z$ |
| $\mathbf{i}_\phi$ | = | $-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y$ | = | $\mathbf{i}_\phi$ |

* *Note that throughout this text a lower case roman* r *is used for the cylindrical radial coordinate while an italicized* $r$ *is used for the spherical radial coordinate.*

from the coordinate $(r, \theta, \phi)$ now depends on the angle $\theta$ and the radial position $r$ as shown in Figure 1-3*b* and summarized in Table 1-1. Table 1-2 summarizes the geometric relations between coordinates and unit vectors for the three coordinate systems considered. Using this table, it is possible to convert coordinate positions and unit vectors from one system to another.

## 1-2 VECTOR ALGEBRA

### 1-2-1 Scalars and Vectors

A scalar quantity is a number completely determined by its magnitude, such as temperature, mass, and charge, the last being especially important in our future study. Vectors, such as velocity and force, must also have their direction specified and in this text are printed in boldface type. They are completely described by their components along three coordinate directions as shown for rectangular coordinates in Figure 1-4. A vector is represented by a directed line segment in the direction of the vector with its length proportional to its magnitude. The vector

$$\mathbf{A} = A_x\mathbf{i}_x + A_y\mathbf{i}_y + A_z\mathbf{i}_z \qquad (1)$$

in Figure 1-4 has magnitude

$$A = |\mathbf{A}| = [A_x^2 + A_y^2 + A_z^2]^{1/2} \qquad (2)$$

Note that each of the components in (1) ($A_x$, $A_y$, and $A_z$) are themselves scalars. The direction of each of the components is given by the unit vectors. We could describe a vector in any of the coordinate systems replacing the subscripts $(x, y, z)$ by $(r, \phi, z)$ or $(r, \theta, \phi)$; however, for conciseness we often use rectangular coordinates for general discussion.

### 1-2-2 Multiplication of a Vector by a Scalar

If a vector is multiplied by a positive scalar, its direction remains unchanged but its magnitude is multiplied by the

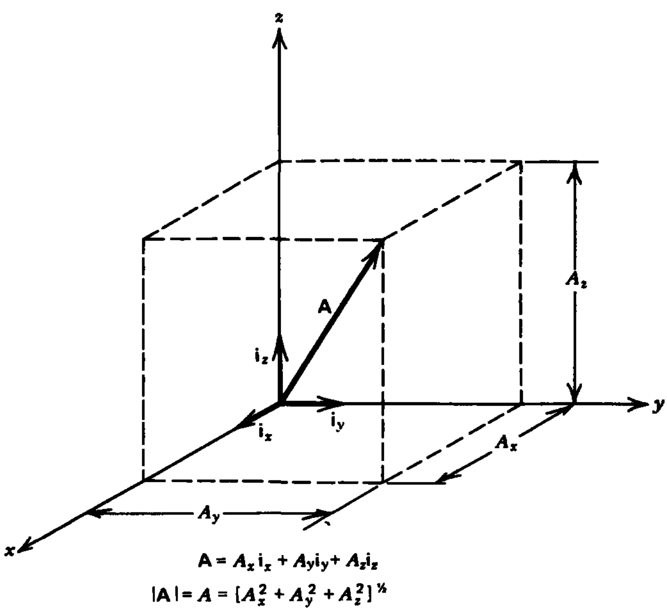



Figure 1-4 A vector is described by its components along the three coordinate directions.

scalar. If the scalar is negative, the direction of the vector is reversed:

$$a\mathbf{A} = aA_x\mathbf{i}_x + aA_y\mathbf{i}_y + aA_z\mathbf{i}_z \tag{3}$$

### 1-2-3 Addition and Subtraction

The sum of two vectors is obtained by adding their components while their difference is obtained by subtracting their components. If the vector **B**

$$\mathbf{B} = B_x\mathbf{i}_x + B_y\mathbf{i}_y + B_z\mathbf{i}_z \tag{4}$$

is added or subtracted to the vector **A** of (1), the result is a new vector **C**:

$$\mathbf{C} = \mathbf{A} \pm \mathbf{B} = (A_x \pm B_x)\mathbf{i}_x + (A_y \pm B_y)\mathbf{i}_y + (A_z \pm B_z)\mathbf{i}_z \tag{5}$$

Geometrically, the vector sum is obtained from the diagonal of the resulting parallelogram formed from **A** and **B** as shown in Figure 1-5*a*. The difference is found by first

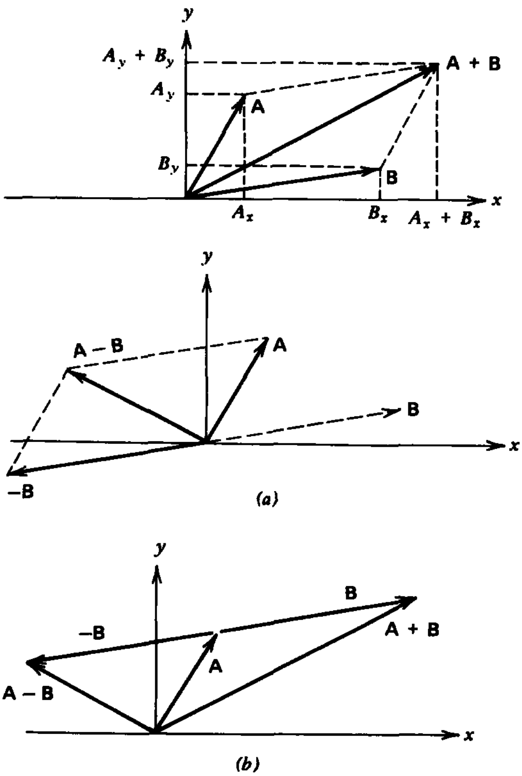



Figure 1-5 The sum and difference of two vectors (*a*) by finding the diagonal of the parallelogram formed by the two vectors, and (*b*) by placing the tail of a vector at the head of the other.

drawing $-\mathbf{B}$ and then finding the diagonal of the parallelogram formed from the sum of $\mathbf{A}$ and $-\mathbf{B}$. The sum of the two vectors is equivalently found by placing the tail of a vector at the head of the other as in Figure 1-5*b*.

Subtraction is the same as addition of the negative of a vector.

## EXAMPLE 1-1 VECTOR ADDITION AND SUBTRACTION

Given the vectors

$$\mathbf{A}=4\mathbf{i}_x+4\mathbf{i}_y, \quad \mathbf{B}=\mathbf{i}_x+8\mathbf{i}_y$$

find the vectors $\mathbf{B}\pm\mathbf{A}$ and their magnitudes. For the geometric solution, see Figure 1-6.

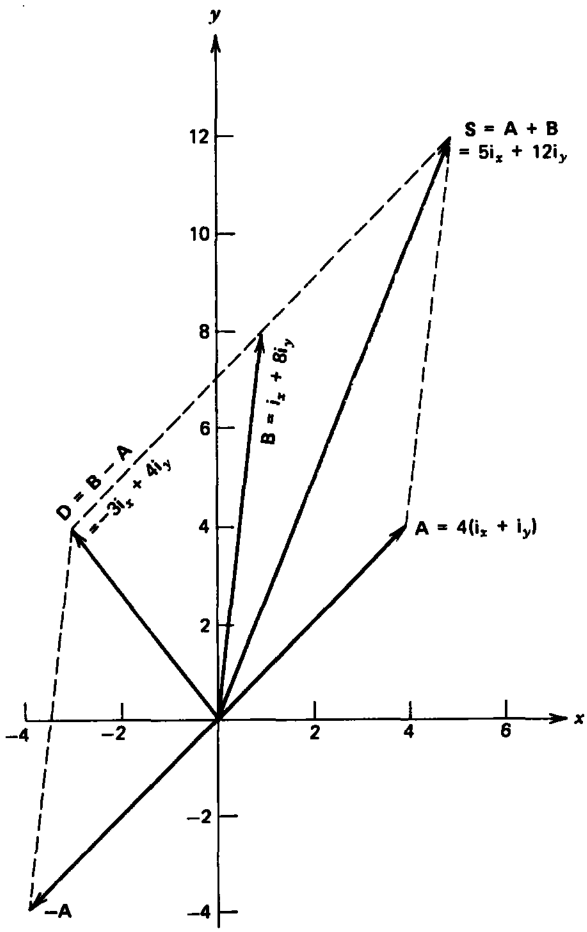



Figure 1-6 The sum and difference of vectors **A** and **B** given in Example 1-1.

**SOLUTION**

---

*Sum*

$$\mathbf{S} = \mathbf{A} + \mathbf{B} = (4+1)\mathbf{i}_x + (4+8)\mathbf{i}_y = 5\mathbf{i}_x + 12\mathbf{i}_y$$
$$S = [5^2 + 12^2]^{1/2} = 13$$

*Difference*

$$\mathbf{D} = \mathbf{B} - \mathbf{A} = (1-4)\mathbf{i}_x + (8-4)\mathbf{i}_y = -3\mathbf{i}_x + 4\mathbf{i}_y$$
$$D = [(-3)^2 + 4^2]^{1/2} = 5$$

### 1-2-4 The Dot (Scalar) Product

The dot product between two vectors results in a scalar and is defined as

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta \tag{6}$$

where $\theta$ is the smaller angle between the two vectors. The term $A \cos\theta$ is the component of the vector $\mathbf{A}$ in the direction of $\mathbf{B}$ shown in Figure 1-7. One application of the dot product arises in computing the incremental work $dW$ necessary to move an object a differential vector distance $\mathbf{dl}$ by a force $\mathbf{F}$. Only the component of force in the direction of displacement contributes to the work

$$dW = \mathbf{F} \cdot \mathbf{dl} \tag{7}$$

The dot product has maximum value when the two vectors are colinear ($\theta = 0$) so that the dot product of a vector with itself is just the square of its magnitude. The dot product is zero if the vectors are perpendicular ($\theta = \pi/2$). These properties mean that the dot product between different orthogonal unit vectors at the same point is zero, while the dot

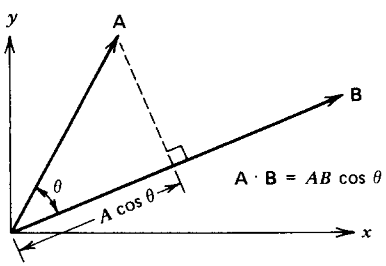



Figure 1-7 The dot product between two vectors.

product between a unit vector and itself is unity

$$\begin{aligned} \mathbf{i}_x \cdot \mathbf{i}_x &= 1, & \mathbf{i}_x \cdot \mathbf{i}_y &= 0 \\ \mathbf{i}_y \cdot \mathbf{i}_y &= 1, & \mathbf{i}_x \cdot \mathbf{i}_z &= 0 \\ \mathbf{i}_z \cdot \mathbf{i}_z &= 1, & \mathbf{i}_y \cdot \mathbf{i}_z &= 0 \end{aligned} \tag{8}$$

Then the dot product can also be written as

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (A_x\mathbf{i}_x + A_y\mathbf{i}_y + A_z\mathbf{i}_z) \cdot (B_x\mathbf{i}_x + B_y\mathbf{i}_y + B_z\mathbf{i}_z) \\ &= A_xB_x + A_yB_y + A_zB_z \end{aligned} \tag{9}$$

From (6) and (9) we see that the dot product does not depend on the order of the vectors

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{10}$$

By equating (6) to (9) we can find the angle between vectors as

$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{AB} \tag{11}$$

Similar relations to (8) also hold in cylindrical and spherical coordinates if we replace $(x, y, z)$ by $(r, \phi, z)$ or $(r, \theta, \phi)$. Then (9) to (11) are also true with these coordinate substitutions.

## EXAMPLE 1-2 DOT PRODUCT

Find the angle between the vectors shown in Figure 1-8,

$$\mathbf{A} = \sqrt{3}\,\mathbf{i}_x + \mathbf{i}_y, \quad \mathbf{B} = 2\mathbf{i}_x$$

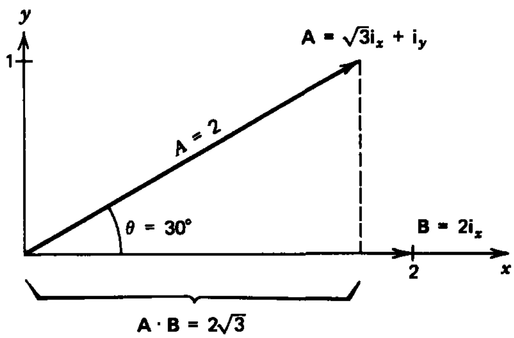



Figure 1-8 The angle between the two vectors **A** and **B** in Example 1-2 can be found using the dot product.

**SOLUTION**

From (11)

$$\cos\theta = \frac{A_x B_x}{[A_x^2 + A_y^2]^{1/2} B_x} = \frac{\sqrt{3}}{2}$$

$$\theta = \cos^{-1}\frac{\sqrt{3}}{2} = 30°$$

### 1-2-5 The Cross (Vector) Product

The cross product between two vectors $\mathbf{A}\times\mathbf{B}$ is defined as a vector perpendicular to both $\mathbf{A}$ and $\mathbf{B}$, which is in the direction of the thumb when using the right-hand rule of curling the fingers of the right hand from $\mathbf{A}$ to $\mathbf{B}$ as shown in Figure 1-9. The magnitude of the cross product is

$$|\mathbf{A}\times\mathbf{B}| = AB\sin\theta \tag{12}$$

where $\theta$ is the enclosed angle between $\mathbf{A}$ and $\mathbf{B}$. Geometrically, (12) gives the area of the parallelogram formed with $\mathbf{A}$ and $\mathbf{B}$ as adjacent sides. Interchanging the order of $\mathbf{A}$ and $\mathbf{B}$ reverses the sign of the cross product:

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \tag{13}$$

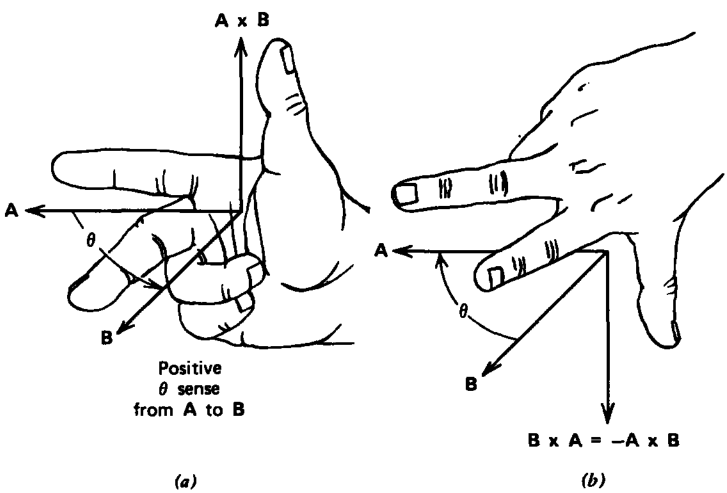



**Figure 1-9** (*a*) The cross product between two vectors results in a vector perpendicular to both vectors in the direction given by the right-hand rule. (*b*) Changing the order of vectors in the cross product reverses the direction of the resultant vector.

The cross product is zero for colinear vectors ($\theta = 0$) so that the cross product between a vector and itself is zero and is maximum for perpendicular vectors ($\theta = \pi/2$). For rectangular unit vectors we have

$$\begin{aligned} &\mathbf{i}_x \times \mathbf{i}_x = 0, \quad &&\mathbf{i}_x \times \mathbf{i}_y = \mathbf{i}_z, \quad &&\mathbf{i}_y \times \mathbf{i}_x = -\mathbf{i}_z \\ &\mathbf{i}_y \times \mathbf{i}_y = 0, \quad &&\mathbf{i}_y \times \mathbf{i}_z = \mathbf{i}_x, \quad &&\mathbf{i}_z \times \mathbf{i}_y = -\mathbf{i}_x \\ &\mathbf{i}_z \times \mathbf{i}_z = 0, \quad &&\mathbf{i}_z \times \mathbf{i}_x = \mathbf{i}_y, \quad &&\mathbf{i}_x \times \mathbf{i}_z = -\mathbf{i}_y \end{aligned} \tag{14}$$

These relations allow us to simply define a right-handed coordinate system as one where

$$\mathbf{i}_x \times \mathbf{i}_y = \mathbf{i}_z \tag{15}$$

Similarly, for cylindrical and spherical coordinates, right-handed coordinate systems have

$$\mathbf{i}_r \times \mathbf{i}_\phi = \mathbf{i}_z, \quad \mathbf{i}_r \times \mathbf{i}_\theta = \mathbf{i}_\phi \tag{16}$$

The relations of (14) allow us to write the cross product between $\mathbf{A}$ and $\mathbf{B}$ as

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (A_x \mathbf{i}_x + A_y \mathbf{i}_y + A_z \mathbf{i}_z) \times (B_x \mathbf{i}_x + B_y \mathbf{i}_y + B_z \mathbf{i}_z) \\ &= \mathbf{i}_x (A_y B_z - A_z B_y) + \mathbf{i}_y (A_z B_x - A_x B_z) + \mathbf{i}_z (A_x B_y - A_y B_x) \end{aligned} \tag{17}$$

which can be compactly expressed as the determinantal expansion

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \det \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \mathbf{i}_x (A_y B_z - A_z B_y) + \mathbf{i}_y (A_z B_x - A_x B_z) + \mathbf{i}_z (A_x B_y - A_y B_x) \end{aligned} \tag{18}$$

The cyclical and orderly permutation of $(x, y, z)$ allows easy recall of (17) and (18). If we think of $xyz$ as a three-day week where the last day $z$ is followed by the first day $x$, the days progress as

$$\underline{xyz}\,x\,\underline{yzx}\,y\,\underline{zxy}\,z \cdots \tag{19}$$

where the three possible positive permutations are underlined. Such permutations of $xyz$ in the subscripts of (18) have positive coefficients while the odd permutations, where $xyz$ do not follow sequentially

$$xzy, \; yxz, \; zyx \tag{20}$$

have negative coefficients in the cross product.

In (14)–(20) we used Cartesian coordinates, but the results remain unchanged if we sequentially replace $(x, y, z)$ by the cylindrical coordinates $(r, \phi, z)$ or the spherical coordinates $(r, \theta, \phi)$.

## EXAMPLE 1-3 CROSS PRODUCT

Find the unit vector $\mathbf{i}_n$ perpendicular in the right-hand sense to the vectors shown in Figure 1-10.

$$\mathbf{A} = -\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z, \quad \mathbf{B} = \mathbf{i}_x - \mathbf{i}_y + \mathbf{i}_z$$

What is the angle between **A** and **B**?

### SOLUTION

---

The cross product $\mathbf{A} \times \mathbf{B}$ is perpendicular to both **A** and **B**

$$\mathbf{A} \times \mathbf{B} = \det \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2(\mathbf{i}_x + \mathbf{i}_y)$$

The unit vector $\mathbf{i}_n$ is in this direction but it must have a magnitude of unity

$$\mathbf{i}_n = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \frac{1}{\sqrt{2}}(\mathbf{i}_x + \mathbf{i}_y)$$

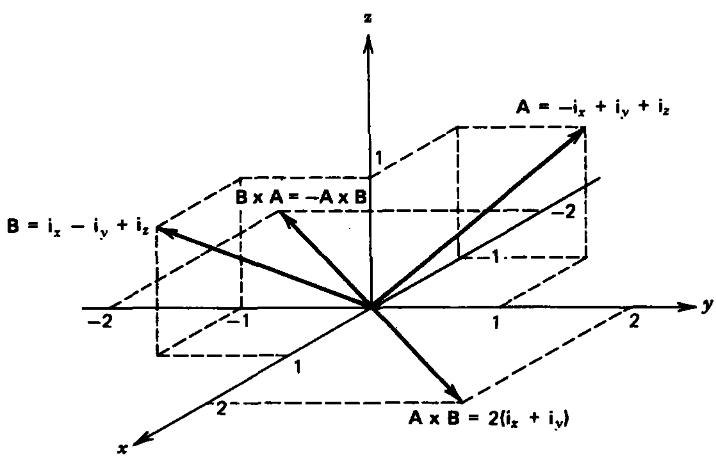



Figure 1-10 The cross product between the two vectors in Example 1-3.

The angle between **A** and **B** is found using (12) as

$$\sin\theta = \frac{|\mathbf{A}\times\mathbf{B}|}{AB} = \frac{2\sqrt{2}}{\sqrt{3}\sqrt{3}}$$
$$= \tfrac{2}{3}\sqrt{2} \Rightarrow \theta = 70.5° \text{ or } 109.5°$$

The ambiguity in solutions can be resolved by using the dot product of (11)

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{AB} = \frac{-1}{\sqrt{3}\sqrt{3}} = -\tfrac{1}{3} \Rightarrow \theta = 109.5°$$

## 1-3 THE GRADIENT AND THE DEL OPERATOR

### 1-3-1 The Gradient

Often we are concerned with the properties of a scalar field $f(x, y, z)$ around a particular point. The chain rule of differentiation then gives us the incremental change $df$ in $f$ for a small change in position from $(x, y, z)$ to $(x+dx, y+dy, z+dz)$:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{1}$$

If the general differential distance vector **dl** is defined as

$$\mathbf{dl} = dx\,\mathbf{i}_x + dy\,\mathbf{i}_y + dz\,\mathbf{i}_z \tag{2}$$

(1) can be written as the dot product:

$$df = \left(\frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{i}_y + \frac{\partial f}{\partial z}\mathbf{i}_z\right)\cdot\mathbf{dl}$$
$$= \operatorname{grad} f\cdot\mathbf{dl} \tag{3}$$

where the spatial derivative terms in brackets are defined as the gradient of $f$:

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{i}_y + \frac{\partial f}{\partial z}\mathbf{i}_z \tag{4}$$

The symbol $\nabla$ with the gradient term is introduced as a general vector operator, termed the del operator:

$$\nabla = \mathbf{i}_x\frac{\partial}{\partial x} + \mathbf{i}_y\frac{\partial}{\partial y} + \mathbf{i}_z\frac{\partial}{\partial z} \tag{5}$$

By itself the del operator is meaningless, but when it premultiplies a scalar function, the gradient operation is defined. We will soon see that the dot and cross products between the del operator and a vector also define useful operations.

With these definitions, the change in $f$ of (3) can be written as

$$df = \nabla f \cdot \mathbf{dl} = |\nabla f|\, dl \cos\theta \tag{6}$$

where $\theta$ is the angle between $\nabla f$ and the position vector $\mathbf{dl}$. The direction that maximizes the change in the function $f$ is when $\mathbf{dl}$ is colinear with $\nabla f(\theta = 0)$. The gradient thus has the direction of maximum change in $f$. Motions in the direction along lines of constant $f$ have $\theta = \pi/2$ and thus by definition $df = 0$.

### 1-3-2 Curvilinear Coordinates

#### (a) Cylindrical

The gradient of a scalar function is defined for any coordinate system as that vector function that when dotted with $\mathbf{dl}$ gives $df$. In cylindrical coordinates the differential change in $f(\mathrm{r}, \phi, z)$ is

$$df = \frac{\partial f}{\partial \mathrm{r}}\, d\mathrm{r} + \frac{\partial f}{\partial \phi}\, d\phi + \frac{\partial f}{\partial z}\, dz \tag{7}$$

The differential distance vector is

$$\mathbf{dl} = d\mathrm{r}\mathbf{i}_\mathrm{r} + \mathrm{r}\, d\phi\, \mathbf{i}_\phi + dz\, \mathbf{i}_z \tag{8}$$

so that the gradient in cylindrical coordinates is

$$df = \nabla f \cdot \mathbf{dl} \Rightarrow \nabla f = \frac{\partial f}{\partial \mathrm{r}}\mathbf{i}_\mathrm{r} + \frac{1}{\mathrm{r}}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi + \frac{\partial f}{\partial z}\mathbf{i}_z \tag{9}$$

#### (b) Spherical

Similarly in spherical coordinates the distance vector is

$$\mathbf{dl} = dr\, \mathbf{i}_r + r\, d\theta\, \mathbf{i}_\theta + r \sin\theta\, d\phi\, \mathbf{i}_\phi \tag{10}$$

with the differential change of $f(r, \theta, \phi)$ as

$$df = \frac{\partial f}{\partial r}\, dr + \frac{\partial f}{\partial \theta}\, d\theta + \frac{\partial f}{\partial \phi}\, d\phi = \nabla f \cdot \mathbf{dl} \tag{11}$$

Using (10) in (11) gives the gradient in spherical coordinates as

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{i}_\theta + \frac{1}{r \sin\theta}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi \tag{12}$$

**EXAMPLE 1-4 GRADIENT**

Find the gradient of each of the following functions where $a$ and $b$ are constants:

(a) $f = ax^2y + by^3z$

**SOLUTION**

---

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{i}_y + \frac{\partial f}{\partial z}\mathbf{i}_z$$
$$= 2axy\mathbf{i}_x + (ax^2 + 3by^2z)\mathbf{i}_y + by^3\mathbf{i}_z$$

(b) $f = ar^2 \sin\phi + brz \cos 2\phi$

**SOLUTION**

---

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi + \frac{\partial f}{\partial z}\mathbf{i}_z$$
$$= (2ar\sin\phi + bz\cos 2\phi)\mathbf{i}_r$$
$$+ (ar\cos\phi - 2bz\sin 2\phi)\mathbf{i}_\phi + br\cos 2\phi\mathbf{i}_z$$

(c) $f = \frac{a}{r} + br\sin\theta\cos\phi$

**SOLUTION**

---

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{i}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi$$
$$= \left(-\frac{a}{r^2} + b\sin\theta\cos\phi\right)\mathbf{i}_r + b\cos\theta\cos\phi\mathbf{i}_\theta - b\sin\phi\mathbf{i}_\phi$$

### 1-3-3 The Line Integral

In Section 1-2-4 we motivated the use of the dot product through the definition of incremental work as depending only on the component of force **F** in the direction of an object's differential displacement **dl**. If the object moves along a path, the total work is obtained by adding up the incremental works along each small displacement on the path as in Figure 1-11. If we break the path into $N$ small displacements

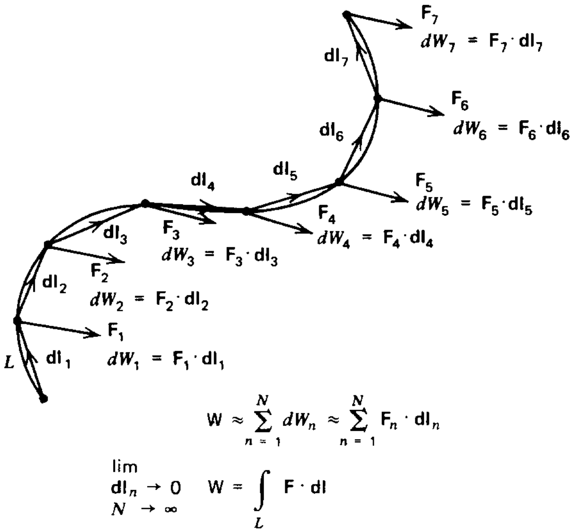



Figure 1-11 The total work in moving a body over a path is approximately equal to the sum of incremental works in moving the body each small incremental distance **dl**. As the differential distances approach zero length, the summation becomes a line integral and the result is exact.

$\mathbf{dl}_1, \mathbf{dl}_2, \ldots, \mathbf{dl}_N$, the work performed is approximately

$$\begin{aligned} W &\approx \mathbf{F}_1 \cdot \mathbf{dl}_1 + \mathbf{F}_2 \cdot \mathbf{dl}_2 + \mathbf{F}_3 \cdot \mathbf{dl}_3 + \cdots + \mathbf{F}_N \cdot \mathbf{dl}_N \\ &\approx \sum_{n=1}^{N} \mathbf{F}_n \cdot \mathbf{dl}_n \end{aligned} \tag{13}$$

The result becomes exact in the limit as $N$ becomes large with each displacement $\mathbf{dl}_n$ becoming infinitesimally small:

$$W = \lim_{\substack{N\to\infty \\ \mathbf{dl}_n \to 0}} \sum_{n=1}^{N} \mathbf{F}_n \cdot \mathbf{dl}_n = \int_L \mathbf{F} \cdot \mathbf{dl} \tag{14}$$

In particular, let us integrate (3) over a path between the two points $a$ and $b$ in Figure 1-12a:

$$\int_a^b df = f|_b - f|_a = \int_a^b \nabla f \cdot \mathbf{dl} \tag{15}$$

Because $df$ is an exact differential, its line integral depends only on the end points and not on the shape of the contour itself. Thus, all of the paths between $a$ and $b$ in Figure 1-12$a$ have the same line integral of $\nabla f$, no matter what the function $f$ may be. If the contour is a closed path so that $a = b$, as in

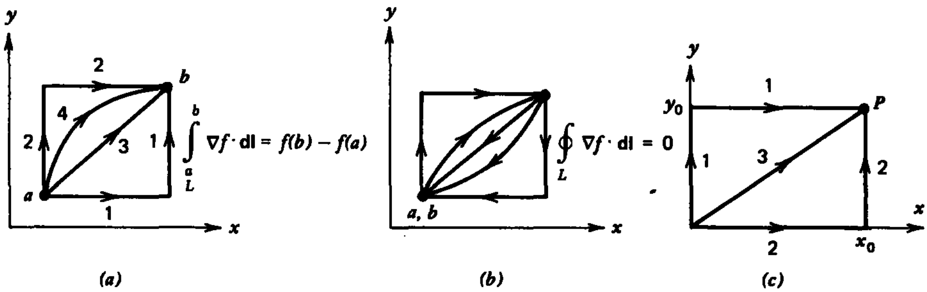



Figure 1-12 The component of the gradient of a function integrated along a line contour depends only on the end points and not on the contour itself. (*a*) Each of the contours have the same starting and ending points at *a* and *b* so that they all have the same line integral of $\nabla f$. (*b*) When all the contours are closed with the same beginning and ending point at *a*, the line integral of $\nabla f$ is zero. (*c*) The line integral of the gradient of the function in Example (1-5) from the origin to the point *P* is the same for all paths.

Figure 1-12*b*, then (15) is zero:

$$\oint_L \nabla f \cdot \mathbf{dl} = f|_a - f|_a = 0 \tag{16}$$

where we indicate that the path is closed by the small circle in the integral sign $\oint$. The line integral of the gradient of a function around a closed path is zero.

## EXAMPLE 1-5 LINE INTEGRAL

For $f = x^2 y$, verify (15) for the paths shown in Figure 1-12*c* between the origin and the point *P* at $(x_0, y_0)$.

### SOLUTION

---

The total change in $f$ between 0 and $P$ is

$$\int_0^P df = f|_P - f|_0 = x_0^2 y_0$$

From the line integral along path 1 we find

$$\int_0^P \nabla f \cdot \mathbf{dl} = \int_{\substack{y=0 \\ x=0}}^{y_0} \cancelto{0}{\underbrace{\frac{\partial f}{\partial y}}_{x^2}}\, dy + \int_{\substack{x=0 \\ y=y_0}}^{x_0} \underbrace{\frac{\partial f}{\partial x}}_{2xy}\, dx = x_0^2 y_0$$

Similarly, along path 2 we also obtain

$$\int_0^P \nabla f \cdot d\mathbf{l} = \int_{\substack{x=0\\y=0}}^{x_0} \underbrace{\cancel{\frac{\partial f}{\partial x}}}_{2xy}{}^0\, dx + \int_{\substack{y=0\\x=x_0}}^{y_0} \underbrace{\frac{\partial f}{\partial y}}_{x^2} dy = x_0^2 y_0$$

while along path 3 we must relate $x$ and $y$ along the straight line as

$$y = \frac{y_0}{x_0} x \Rightarrow dy = \frac{y_0}{x_0} dx$$

to yield

$$\int_0^P \nabla f \cdot d\mathbf{l} = \int_0^P \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) = \int_{x=0}^{x_0} \frac{3y_0 x^2}{x_0} dx = x_0^2 y_0$$

## 1-4 FLUX AND DIVERGENCE

If we measure the total mass of fluid entering the volume in Figure 1-13 and find it to be less than the mass leaving, we know that there must be an additional source of fluid within the pipe. If the mass leaving is less than that entering, then

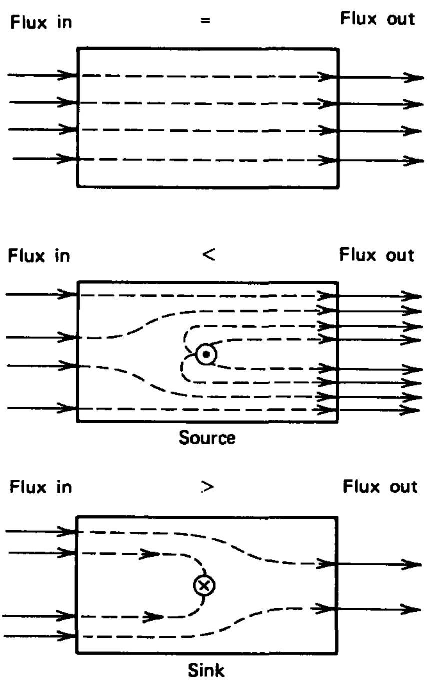



Figure 1-13 The net flux through a closed surface tells us whether there is a source or sink within an enclosed volume.

there is a sink (or drain) within the volume. In the absence of sources or sinks, the mass of fluid leaving equals that entering so the flow lines are continuous. Flow lines originate at a source and terminate at a sink.

### 1-4-1 Flux

We are illustrating with a fluid analogy what is called the flux $\Phi$ of a vector $\mathbf{A}$ through a closed surface:

$$\Phi = \oint_S \mathbf{A} \cdot \mathbf{dS} \tag{1}$$

The differential surface element $\mathbf{dS}$ is a vector that has magnitude equal to an incremental area on the surface but points in the direction of the outgoing unit normal $\mathbf{n}$ to the surface $S$, as in Figure 1-14. Only the component of $\mathbf{A}$ perpendicular to the surface contributes to the flux, as the tangential component only results in flow of the vector $\mathbf{A}$ along the surface and not through it. A positive contribution to the flux occurs if $\mathbf{A}$ has a component in the direction of $\mathbf{dS}$ out from the surface. If the normal component of $\mathbf{A}$ points into the volume, we have a negative contribution to the flux.

If there is no source for $\mathbf{A}$ within the volume $V$ enclosed by the surface $S$, all the flux entering the volume equals that leaving and the net flux is zero. A source of $\mathbf{A}$ within the volume generates more flux leaving than entering so that the flux is positive ($\Phi > 0$) while a sink has more flux entering than leaving so that $\Phi < 0$.

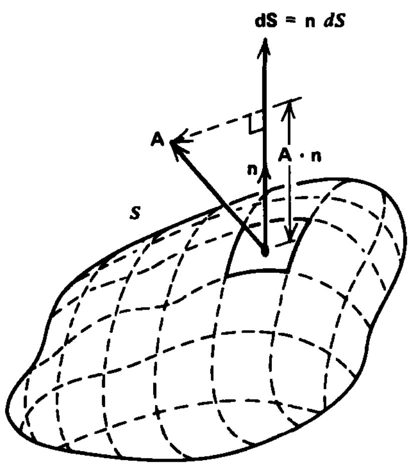



Figure 1-14 The flux of a vector $\mathbf{A}$ through the closed surface $S$ is given by the surface integral of the component of $\mathbf{A}$ perpendicular to the surface $S$. The differential vector surface area element $\mathbf{dS}$ is in the direction of the unit normal $\mathbf{n}$.

Thus we see that the sign and magnitude of the net flux relates the quantity of a field through a surface to the sources or sinks of the vector field within the enclosed volume.

### 1-4-2 Divergence

We can be more explicit about the relationship between the rate of change of a vector field and its sources by applying (1) to a volume of differential size, which for simplicity we take to be rectangular in Figure 1-15. There are three pairs of plane parallel surfaces perpendicular to the coordinate axes so that (1) gives the flux as

$$\Phi = \int_1 A_x(x)\,dy\,dz - \int_{1'} A_x(x-\Delta x)\,dy\,dz$$
$$+ \int_2 A_y(y+\Delta y)\,dx\,dz - \int_{2'} A_y(y)\,dx\,dz$$
$$+ \int_3 A_z(z+\Delta z)\,dx\,dy - \int_{3'} A_z(z)\,dx\,dy \qquad (2)$$

where the primed surfaces are differential distances behind the corresponding unprimed surfaces. The minus signs arise because the outgoing normals on the primed surfaces point in the negative coordinate directions.

Because the surfaces are of differential size, the components of **A** are approximately constant along each surface so that the surface integrals in (2) become pure

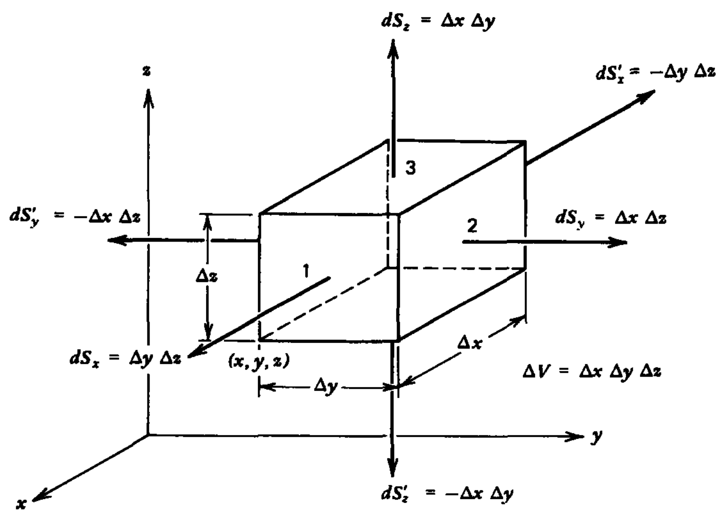



Figure 1-15 Infinitesimal rectangular volume used to define the divergence of a vector.

multiplications of the component of **A** perpendicular to the surface and the surface area. The flux then reduces to the form

$$\Phi \approx \left(\frac{[A_x(x) - A_x(x-\Delta x)]}{\Delta x} + \frac{[A_y(y+\Delta y) - A_y(y)]}{\Delta y} + \frac{[A_z(z+\Delta z) - A_z(z)]}{\Delta z}\right) \Delta x\, \Delta y\, \Delta z \tag{3}$$

We have written (3) in this form so that in the limit as the volume becomes infinitesimally small, each of the bracketed terms defines a partial derivative

$$\lim_{\substack{\Delta x \to 0\\ \Delta y \to 0\\ \Delta z \to 0}} \Phi = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) \Delta V \tag{4}$$

where $\Delta V = \Delta x\, \Delta y\, \Delta z$ is the volume enclosed by the surface $S$.

The coefficient of $\Delta V$ in (4) is a scalar and is called the divergence of **A**. It can be recognized as the dot product between the vector del operator of Section 1-3-1 and the vector **A**:

$$\operatorname{div} \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{5}$$

### 1-4-3 Curvilinear Coordinates

In cylindrical and spherical coordinates, the divergence operation is not simply the dot product between a vector and the del operator because the directions of the unit vectors are a function of the coordinates. Thus, derivatives of the unit vectors have nonzero contributions. It is easiest to use the generalized definition of the divergence independent of the coordinate system, obtained from (1)–(5) as

$$\nabla \cdot \mathbf{A} = \lim_{\Delta V \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta V} \tag{6}$$

**(a) Cylindrical Coordinates**

In cylindrical coordinates we use the small volume shown in Figure 1-16$a$ to evaluate the net flux as

$$\begin{aligned}\Phi = \oint_S \mathbf{A} \cdot d\mathbf{S} &= \int_1 (\mathrm{r}+\Delta \mathrm{r}) A_{\mathrm{r}|_{\mathrm{r}+\Delta \mathrm{r}}}\, d\phi\, dz - \int_{1'} \mathrm{r} A_{\mathrm{r}|_{\mathrm{r}}}\, d\phi\, dz \\ &+ \int_2 A_{\phi|_{\phi+\Delta\phi}}\, d\mathrm{r}\, dz - \int_{2'} A_{\phi|_{\phi}}\, d\mathrm{r}\, dz \\ &+ \int_3 \mathrm{r} A_{z|_{z+\Delta z}}\, d\mathrm{r}\, d\phi - \int_{3'} \mathrm{r} A_{z|_{z}}\, d\mathrm{r}\, d\phi \end{aligned} \tag{7}$$

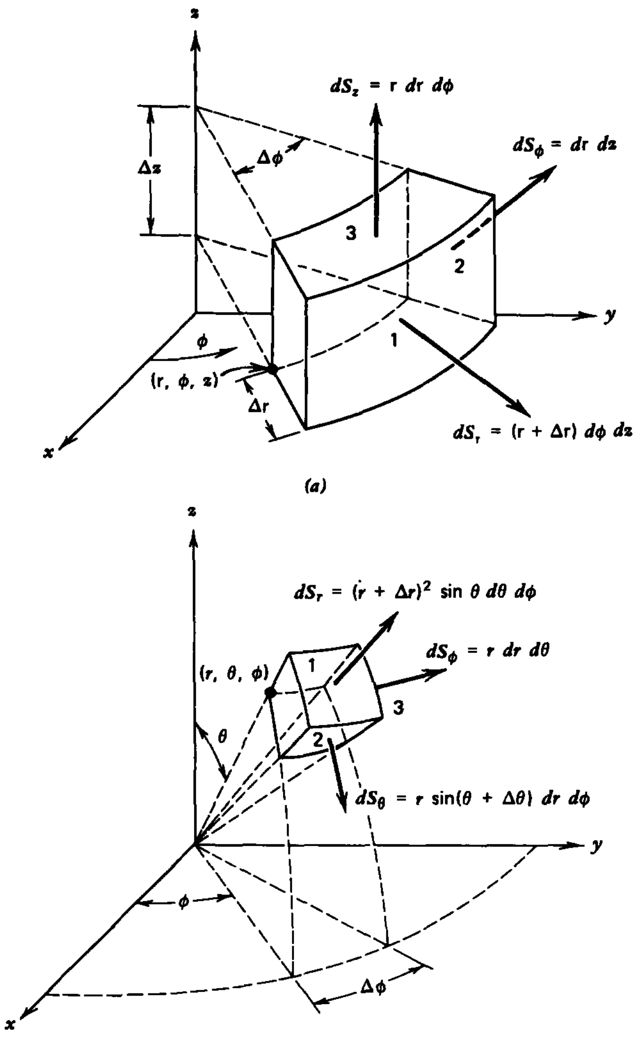



Figure 1-16 Infinitesimal volumes used to define the divergence of a vector in (*a*) cylindrical and (*b*) spherical geometries.

Again, because the volume is small, we can treat it as approximately rectangular with the components of **A** approximately constant along each face. Then factoring out the volume $\Delta V = r\Delta r\,\Delta\phi\,\Delta z$ in (7),

$$\Phi \approx \left(\frac{[(r+\Delta r)A_{r|_{r+\Delta r}} - rA_{r|_r}]}{r\,\Delta r} + \frac{[A_{\phi|_{\phi+\Delta\phi}} - A_{\phi|_\phi}]}{r\,\Delta\phi} + \frac{[A_{z|_{z+\Delta z}} - A_{z|_z}]}{\Delta z}\right) r\Delta r\,\Delta\phi\,\Delta z \tag{8}$$

lets each of the bracketed terms become a partial derivative as the differential lengths approach zero and (8) becomes an exact relation. The divergence is then

$$\nabla \cdot \mathbf{A} = \lim_{\substack{\Delta r \to 0 \\ \Delta\phi \to 0 \\ \Delta z \to 0}} \frac{\oint_S \mathbf{A} \cdot \mathbf{dS}}{\Delta V} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{9}$$

**(b) Spherical Coordinates**

Similar operations on the spherical volume element $\Delta V = r^2 \sin\theta\, \Delta r\, \Delta\theta\, \Delta\phi$ in Figure 1-16*b* defines the net flux through the surfaces:

$$\begin{aligned}\Phi &= \oint_S \mathbf{A} \cdot \mathbf{dS} \\ &\approx \left( \frac{[(r+\Delta r)^2 A_{r|_{r+\Delta r}} - r^2 A_{r|_r}]}{r^2\, \Delta r} \right. \\ &\quad + \frac{[A_{\theta|_{\theta+\Delta\theta}} \sin(\theta+\Delta\theta) - A_{\theta|_\theta} \sin\theta]}{r \sin\theta\, \Delta\theta} \\ &\quad \left. + \frac{[A_{\phi|_{\phi+\Delta\phi}} - A_{\phi|_\phi}]}{r \sin\theta\, \Delta\phi} \right) r^2 \sin\theta\, \Delta r\, \Delta\theta\, \Delta\phi \end{aligned} \tag{10}$$

The divergence in spherical coordinates is then

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \lim_{\substack{\Delta r \to 0 \\ \Delta\theta \to 0 \\ \Delta\phi \to 0}} \frac{\oint_S \mathbf{A} \cdot \mathbf{dS}}{\Delta V} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}\end{aligned} \tag{11}$$

### 1-4-4 The Divergence Theorem

If we now take many adjoining incremental volumes of any shape, we form a macroscopic volume $V$ with enclosing surface $S$ as shown in Figure 1-17*a*. However, each interior common surface between incremental volumes has the flux leaving one volume (positive flux contribution) just entering the adjacent volume (negative flux contribution) as in Figure 1-17*b*. The net contribution to the flux for the surface integral of (1) is zero for all interior surfaces. Nonzero contributions to the flux are obtained only for those surfaces which bound the outer surface $S$ of $V$. Although the surface contributions to the flux using (1) cancel for all interior volumes, the flux obtained from (4) in terms of the divergence operation for

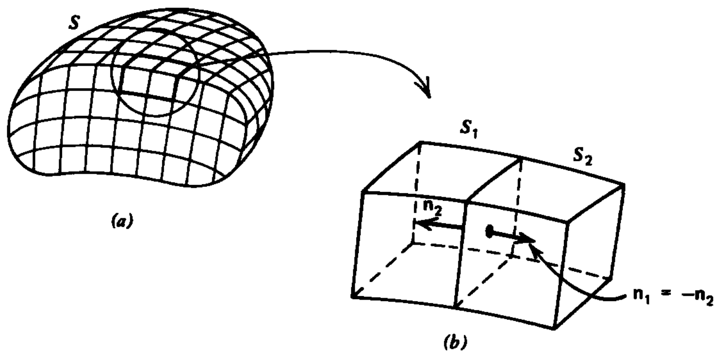



Figure 1-17 Nonzero contributions to the flux of a vector are only obtained across those surfaces that bound the outside of a volume. (a) Within the volume the flux leaving one incremental volume just enters the adjacent volume where (*b*) the outgoing normals to the common surface separating the volumes are in opposite directions.

each incremental volume add. By adding all contributions from each differential volume, we obtain the divergence theorem:

$$\Phi = \oint_S \mathbf{A} \cdot \mathbf{dS} = \lim_{\substack{N \to \infty \\ \Delta V_n \to 0}} \sum_{n=1}^{\infty} (\nabla \cdot \mathbf{A}) \Delta V_n = \int_V \nabla \cdot \mathbf{A} \, dV \quad (12)$$

where the volume $V$ may be of macroscopic size and is enclosed by the outer surface $S$. This powerful theorem converts a surface integral into an equivalent volume integral and will be used many times in our development of electromagnetic field theory.

## EXAMPLE 1-6 THE DIVERGENCE THEOREM

Verify the divergence theorem for the vector

$$\mathbf{A} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z = r\mathbf{i}_r$$

by evaluating both sides of (12) for the rectangular volume shown in Figure 1-18.

### SOLUTION

---

The volume integral is easier to evaluate as the divergence of $\mathbf{A}$ is a constant

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 3$$

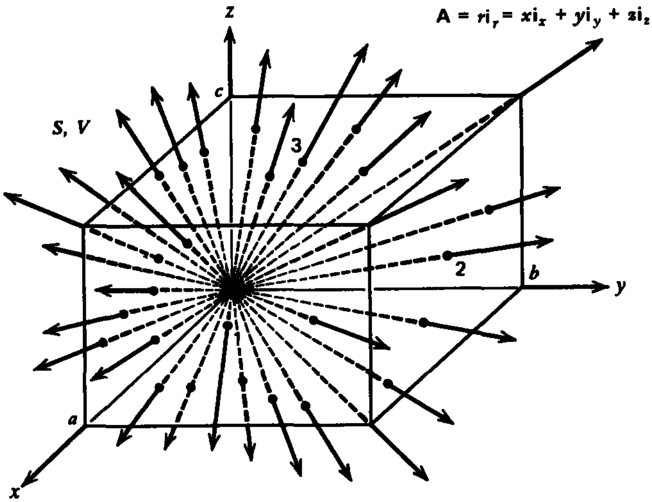



Figure 1-18 The divergence theorem is verified in Example 1-6 for the radial vector through a rectangular volume.

(In spherical coordinates $\nabla \cdot \mathbf{A} = (1/r^2)(\partial/\partial r)(r^3) = 3$) so that the volume integral in (12) is

$$\int_V \nabla \cdot \mathbf{A}\, dV = 3abc$$

The flux passes through the six plane surfaces shown:

$$\Phi = \oint_S \mathbf{A} \cdot \mathbf{dS} = \int_1 \underbrace{A_x(a)}_{a}\, dy\, dz - \int_{1'} \underbrace{A_x(0)}_{0}\, dy\, dz$$

$$+ \int_2 \underbrace{A_y(b)}_{b}\, dx\, dz - \int_{2'} \underbrace{A_y(0)}_{0}\, dx\, dz$$

$$+ \int_3 \underbrace{A_z(c)}_{c}\, dx\, dy - \int_{3'} \underbrace{A_z(0)}_{0}\, dx\, dy = 3abc$$

which verifies the divergence theorem.

## 1.5 THE CURL AND STOKES' THEOREM

### 1-5-1 Curl

We have used the example of work a few times previously to motivate particular vector and integral relations. Let us do so once again by considering the line integral of a vector around a closed path called the circulation:

$$C=\oint_L \mathbf{A}\cdot d\mathbf{l} \tag{1}$$

where if $C$ is the work, $\mathbf{A}$ would be the force. We evaluate (1) for the infinitesimal rectangular contour in Figure 1-19*a*:

$$C=\int_{\underset{1}{x}}^{x+\Delta x} A_x(y)\,dx+\int_{\underset{2}{y}}^{y+\Delta y} A_y(x+\Delta x)\,dy+\int_{\underset{3}{x+\Delta x}}^{x} A_x(y+\Delta y)\,dx$$

$$+\int_{\underset{4}{y+\Delta y}}^{y} A_y(x)\,dy \tag{2}$$

The components of $\mathbf{A}$ are approximately constant over each differential sized contour leg so that (2) is approximated as

$$C\approx\left(\frac{[A_x(y)-A_x(y+\Delta y)]}{\Delta y}+\frac{[A_y(x+\Delta x)-A_y(x)]}{\Delta x}\right)\Delta x\,\Delta y \tag{3}$$

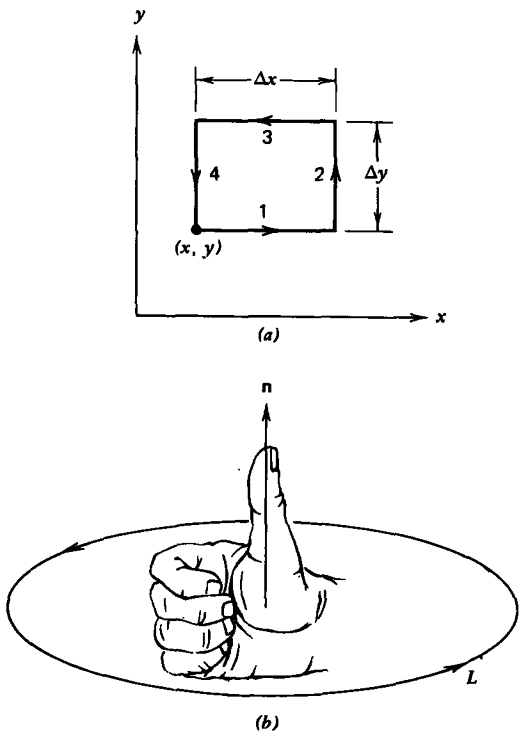



Figure 1-19 (*a*) Infinitesimal rectangular contour used to define the circulation. (*b*) The right-hand rule determines the positive direction perpendicular to a contour.

where terms are factored so that in the limit as $\Delta x$ and $\Delta y$ become infinitesimally small, (3) becomes exact and the bracketed terms define partial derivatives:

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta S_z=\Delta x\,\Delta y}} C=\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\Delta S_z \tag{4}$$

The contour in Figure 1-19*a* could just have as easily been in the $xz$ or $yz$ planes where (4) would equivalently become

$$C=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\Delta S_x \quad (yz \text{ plane})$$

$$C=\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\Delta S_y \quad (xz \text{ plane}) \tag{5}$$

by simple positive permutations of $x$, $y$, and $z$.

The partial derivatives in (4) and (5) are just components of the cross product between the vector del operator of Section 1-3-1 and the vector **A**. This operation is called the curl of **A** and it is also a vector:

$$\operatorname{curl}\mathbf{A}=\nabla\times\mathbf{A}=\det\begin{vmatrix}\mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}$$

$$=\mathbf{i}_x\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\mathbf{i}_y\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)$$

$$+\mathbf{i}_z\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right) \tag{6}$$

The cyclical permutation of $(x, y, z)$ allows easy recall of (6) as described in Section 1-2-5.

In terms of the curl operation, the circulation for any differential sized contour can be compactly written as

$$C=(\nabla\times\mathbf{A})\cdot\mathbf{dS} \tag{7}$$

where $\mathbf{dS}=\mathbf{n}\,dS$ is the area element in the direction of the normal vector **n** perpendicular to the plane of the contour in the sense given by the right-hand rule in traversing the contour, illustrated in Figure 1-19*b*. Curling the fingers on the right hand in the direction of traversal around the contour puts the thumb in the direction of the normal **n**.

For a physical interpretation of the curl it is convenient to continue to use a fluid velocity field as a model although the general results and theorems are valid for any vector field. If

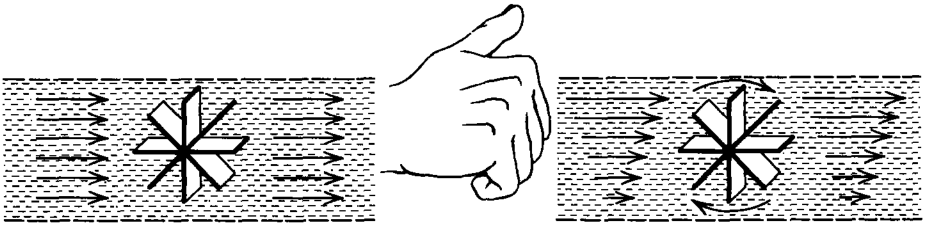



Figure 1-20 A fluid with a velocity field that has a curl tends to turn the paddle wheel. The curl component found is in the same direction as the thumb when the fingers of the right hand are curled in the direction of rotation.

a small paddle wheel is imagined to be placed without disturbance in a fluid flow, the velocity field is said to have circulation, that is, a nonzero curl, if the paddle wheel rotates as illustrated in Figure 1-20. The curl component found is in the direction of the axis of the paddle wheel.

### 1-5-2 The Curl for Curvilinear Coordinates

A coordinate independent definition of the curl is obtained using (7) in (1) as

$$(\nabla\times\mathbf{A})_n=\lim_{dS_n\to 0}\frac{\oint_L \mathbf{A}\cdot d\mathbf{l}}{dS_n}\tag{8}$$

where the subscript $n$ indicates the component of the curl perpendicular to the contour. The derivation of the curl operation (8) in cylindrical and spherical coordinates is straightforward but lengthy.

**(a) Cylindrical Coordinates**

To express each of the components of the curl in cylindrical coordinates, we use the three orthogonal contours in Figure 1-21. We evaluate the line integral around contour $a$:

$$\begin{aligned}\oint_a \mathbf{A}\cdot d\mathbf{l}&=\int_z^{z-\Delta z} A_z(\phi)\,dz+\int_\phi^{\phi+\Delta\phi} A_\phi(z-\Delta z)\,\mathrm{r}\,d\phi\\&\quad+\int_{z-\Delta z}^{z} A_z(\phi+\Delta\phi)\,dz+\int_{\phi+\Delta\phi}^{\phi} A_\phi(z)\,\mathrm{r}\,d\phi\\&\approx\left(\frac{[A_z(\phi+\Delta\phi)-A_z(\phi)]}{\mathrm{r}\Delta\phi}-\frac{[A_\phi(z)-A_\phi(z-\Delta z)]}{\Delta z}\right)\mathrm{r}\,\Delta\phi\,\Delta z\end{aligned}\tag{9}$$

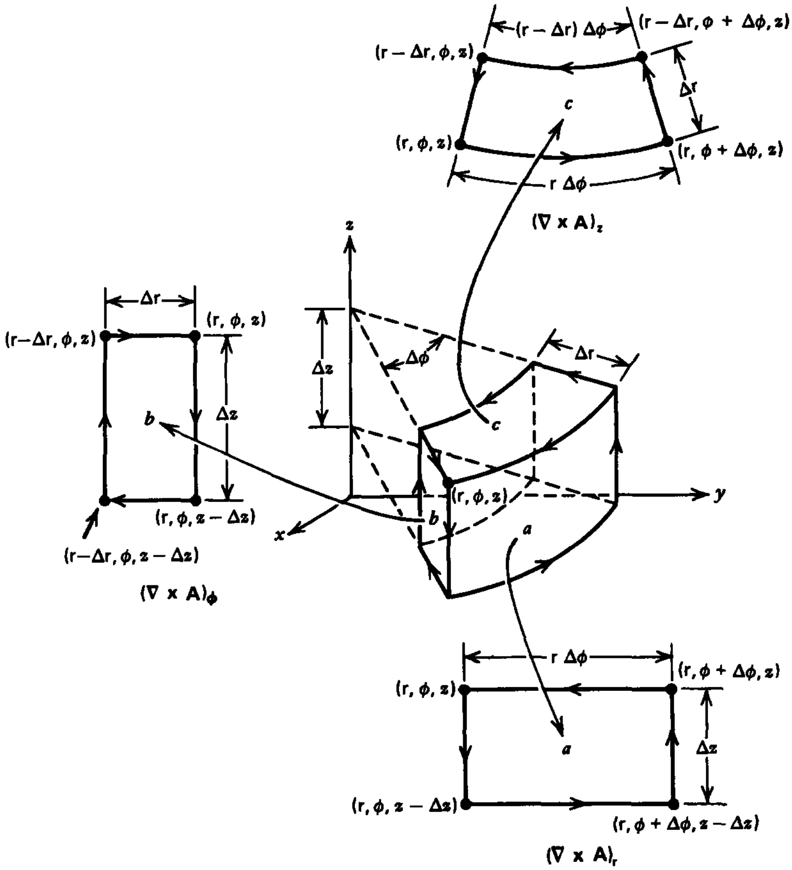



Figure 1-21 Incremental contours along cylindrical surface area elements used to calculate each component of the curl of a vector in cylindrical coordinates.

to find the radial component of the curl as

$$(\nabla\times\mathbf{A})_r=\lim_{\substack{\Delta\phi\to 0\\ \Delta z\to 0}}\frac{\oint_a \mathbf{A}\cdot d\mathbf{l}}{r\,\Delta\phi\,\Delta z}=\frac{1}{r}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\tag{10}$$

We evaluate the line integral around contour $b$:

$$\oint_b \mathbf{A}\cdot d\mathbf{l}=\int_{r-\Delta r}^{r}A_r(z)\,dr+\int_{z}^{z-\Delta z}A_z(r)\,dz+\int_{r}^{r-\Delta r}A_r(z-\Delta z)\,dr$$
$$+\int_{z-\Delta z}^{z}A_z(r-\Delta r)\,dz$$
$$\approx\left(\frac{[A_r(z)-A_r(z-\Delta z)]}{\Delta z}-\frac{[A_z(r)-A_z(r-\Delta r)]}{\Delta r}\right)\Delta r\,\Delta z\tag{11}$$

to find the $\phi$ component of the curl,

$$(\nabla\times\mathbf{A})_\phi=\lim_{\substack{\Delta r\to 0\\ \Delta z\to 0}}\frac{\oint_b \mathbf{A}\cdot d\mathbf{l}}{\Delta r\,\Delta z}=\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right) \tag{12}$$

The $z$ component of the curl is found using contour $c$:

$$\begin{aligned}\oint_c \mathbf{A}\cdot d\mathbf{l}&=\int_{r-\Delta r}^{r} A_{r|_\phi}\,dr+\int_{\phi}^{\phi+\Delta\phi} rA_{\phi|_r}\,d\phi+\int_{r}^{r-\Delta r} A_{r|_{\phi+\Delta\phi}}\,dr\\&\quad+\int_{\phi+\Delta\phi}^{\phi}(r-\Delta r)A_{\phi|_{r-\Delta r}}\,d\phi\\&\approx\left(\frac{[rA_{\phi|_r}-(r-\Delta r)A_{\phi|_{r-\Delta r}}]}{r\Delta r}-\frac{[A_{r|_{\phi+\Delta\phi}}-A_{r|_\phi}]}{r\Delta\phi}\right)r\,\Delta r\,\Delta\phi\end{aligned} \tag{13}$$

to yield

$$(\nabla\times\mathbf{A})_z=\lim_{\substack{\Delta r\to 0\\ \Delta\phi\to 0}}\frac{\oint_c \mathbf{A}\cdot d\mathbf{l}}{r\,\Delta r\,\Delta\phi}=\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi)-\frac{\partial A_r}{\partial\phi}\right) \tag{14}$$

The curl of a vector in cylindrical coordinates is thus

$$\begin{aligned}\nabla\times\mathbf{A}&=\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\mathbf{i}_r+\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)\mathbf{i}_\phi\\&\quad+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi)-\frac{\partial A_r}{\partial\phi}\right)\mathbf{i}_z\end{aligned} \tag{15}$$

**(b) Spherical Coordinates**

Similar operations on the three incremental contours for the spherical element in Figure 1-22 give the curl in spherical coordinates. We use contour $a$ for the radial component of the curl:

$$\begin{aligned}\oint_a \mathbf{A}\cdot d\mathbf{l}&=\int_{\phi}^{\phi+\Delta\phi} A_{\phi|_\theta} r\sin\theta\,d\phi+\int_{\theta}^{\theta-\Delta\theta} rA_{\theta|_{\phi+\Delta\phi}}\,d\theta\\&\quad+\int_{\phi+\Delta\phi}^{\phi} r\sin(\theta-\Delta\theta)A_{\phi|_{\theta-\Delta\theta}}\,d\phi+\int_{\theta-\Delta\theta}^{\theta} rA_{\theta|_\phi}\,d\theta\\&\approx\left(\frac{[A_{\phi|_\theta}\sin\theta-A_{\phi|_{\theta-\Delta\theta}}\sin(\theta-\Delta\theta)]}{r\sin\theta\,\Delta\theta}\right.\\&\quad\left.-\frac{[A_{\theta|_{\phi+\Delta\phi}}-A_{\theta|_\phi}]}{r\sin\theta\,\Delta\phi}\right)r^2\sin\theta\,\Delta\theta\,\Delta\phi\end{aligned} \tag{16}$$

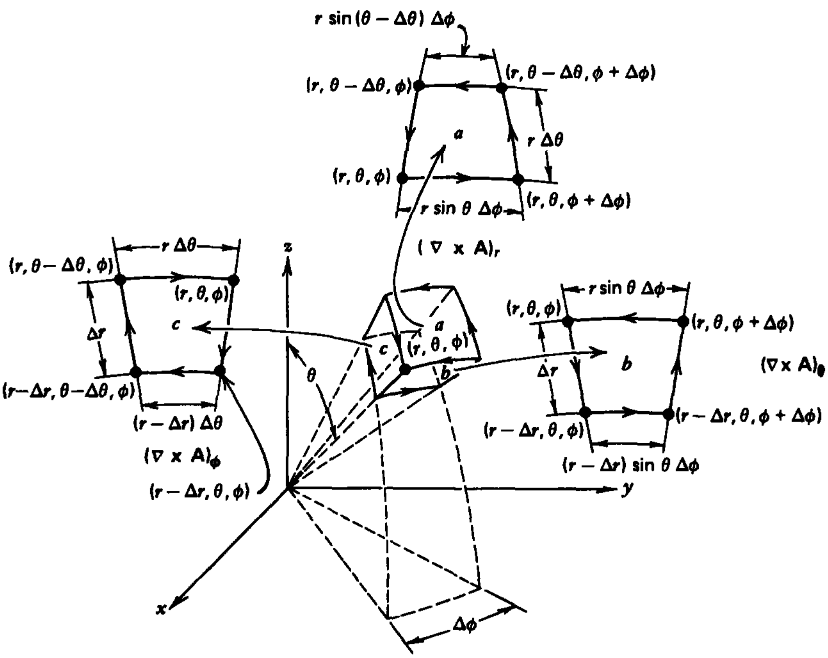



Figure 1-22 Incremental contours along spherical surface area elements used to calculate each component of the curl of a vector in spherical coordinates.

to obtain

$$(\nabla\times\mathbf{A})_r=\lim_{\substack{\Delta\theta\to 0\\ \Delta\phi\to 0}}\frac{\oint_a \mathbf{A}\cdot d\mathbf{l}}{r^2\sin\theta\,\Delta\theta\,\Delta\phi}=\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{\partial A_\theta}{\partial\phi}\right) \tag{17}$$

The $\theta$ component is found using contour $b$:

$$\begin{aligned}\oint_b \mathbf{A}\cdot d\mathbf{l}&=\int_r^{r-\Delta r} A_r|_\phi\,dr+\int_\phi^{\phi+\Delta\phi}(r-\Delta r)A_\phi|_{r-\Delta r}\sin\theta\,d\phi\\ &\quad+\int_{r-\Delta r}^{r} A_r|_{\phi+\Delta\phi}\,dr+\int_{\phi+\Delta\phi}^{\phi} rA_\phi|_r\sin\theta\,d\phi\\ &\approx\left(\frac{[A_r|_{\phi+\Delta\phi}-A_r|_\phi]}{r\sin\theta\,\Delta\phi}\right.\\ &\qquad\left.-\frac{[rA_\phi|_r-(r-\Delta r)A_\phi|_{r-\Delta r}]}{r\,\Delta r}\right)r\sin\theta\,\Delta r\,\Delta\phi\end{aligned} \tag{18}$$

as

$$(\nabla\times\mathbf{A})_\theta=\lim_{\substack{\Delta r\to 0\\ \Delta\phi\to 0}}\frac{\oint_b \mathbf{A}\cdot\mathbf{dl}}{r\sin\theta\,\Delta r\,\Delta\phi}=\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}(rA_\phi)\right)\tag{19}$$

The $\phi$ component of the curl is found using contour $c$:

$$\begin{aligned}\oint_c \mathbf{A}\cdot\mathbf{dl}&=\int_{\theta-\Delta\theta}^{\theta} rA_{\theta|_r}\,d\theta+\int_r^{r-\Delta r}A_{r|_\theta}\,dr\\&\quad+\int_\theta^{\theta-\Delta\theta}(r-\Delta r)A_{\theta|_{r-\Delta r}}\,d\theta+\int_{r-\Delta r}^{r}A_{r|_{\theta-\Delta\theta}}\,dr\\&\approx\left(\frac{[rA_{\theta|_r}-(r-\Delta r)A_{\theta|_{r-\Delta r}}]}{r\,\Delta r}-\frac{[A_{r|_\theta}-A_{r|_{\theta-\Delta\theta}}]}{r\,\Delta\theta}\right)r\,\Delta r\,\Delta\theta\end{aligned}\tag{20}$$

as

$$(\nabla\times\mathbf{A})_\phi=\lim_{\substack{\Delta r\to 0\\ \Delta\theta\to 0}}\frac{\oint_c \mathbf{A}\cdot\mathbf{dl}}{r\,\Delta r\,\Delta\theta}=\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right)\tag{21}$$

The curl of a vector in spherical coordinates is thus given from (17), (19), and (21) as

$$\begin{aligned}\nabla\times\mathbf{A}&=\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{\partial A_\theta}{\partial\phi}\right)\mathbf{i}_r\\&\quad+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}(rA_\phi)\right)\mathbf{i}_\theta\\&\quad+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right)\mathbf{i}_\phi\end{aligned}\tag{22}$$

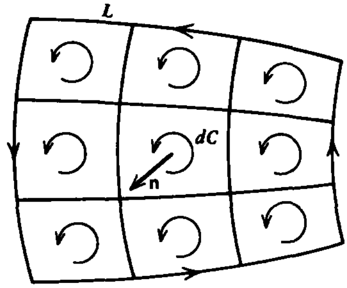
### 1-5-3 Stokes' Theorem

We now piece together many incremental line contours of the type used in Figures 1-19–1-21 to form a macroscopic surface $S$ like those shown in Figure 1-23. Then each small contour generates a contribution to the circulation

$$dC=(\nabla\times\mathbf{A})\cdot\mathbf{dS}\tag{23}$$

so that the total circulation is obtained by the sum of all the small surface elements

$$C=\int_S(\nabla\times\mathbf{A})\cdot\mathbf{dS}\tag{24}$$



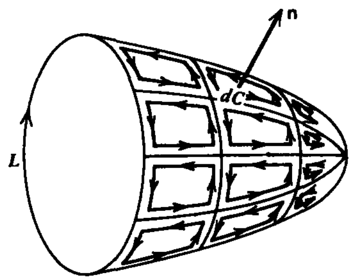




Figure 1-23 Many incremental line contours distributed over any surface, have nonzero contribution to the circulation only along those parts of the surface on the boundary contour $L$.

Each of the terms of (23) are equivalent to the line integral around each small contour. However, all interior contours share common sides with adjacent contours but which are twice traversed in opposite directions yielding no net line integral contribution, as illustrated in Figure 1-23. Only those contours with a side on the open boundary $L$ have a nonzero contribution. The total result of adding the contributions for all the contours is Stokes' theorem, which converts the line integral over the bounding contour $L$ of the outer edge to a surface integral over any area $S$ bounded by the contour

$$\oint_L \mathbf{A} \cdot \mathbf{dl} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{dS} \tag{25}$$

Note that there are an infinite number of surfaces that are bounded by the same contour $L$. Stokes' theorem of (25) is satisfied for all these surfaces.

## EXAMPLE 1-7 STOKES' THEOREM

Verify Stokes' theorem of (25) for the circular bounding contour in the $xy$ plane shown in Figure 1-24 with a vector

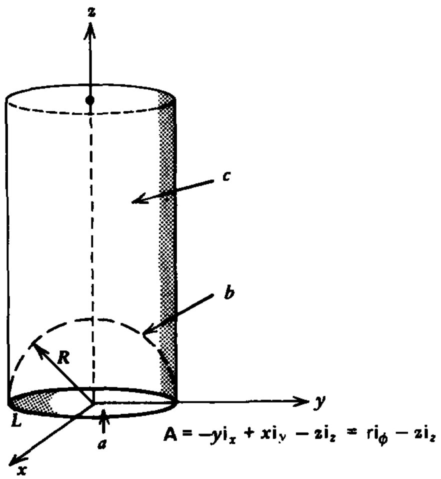



Figure 1-24 Stokes' theorem for the vector given in Example 1-7 can be applied to any surface that is bounded by the same contour $L$.

field

$$\mathbf{A} = -y\mathbf{i}_x + x\mathbf{i}_y - z\mathbf{i}_z = r\mathbf{i}_\phi - z\mathbf{i}_z$$

Check the result for the (a) flat circular surface in the $xy$ plane, (b) for the hemispherical surface bounded by the contour, and (c) for the cylindrical surface bounded by the contour.

## SOLUTION

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For the contour shown

$$\mathbf{dl} = R\, d\phi\, \mathbf{i}_\phi$$

so that

$$\mathbf{A} \cdot \mathbf{dl} = R^2\, d\phi$$

where on $L$, $r = R$. Then the circulation is

$$C = \oint_L \mathbf{A} \cdot \mathbf{dl} = \int_0^{2\pi} R^2\, d\phi = 2\pi R^2$$

The $z$ component of $\mathbf{A}$ had no contribution because $\mathbf{dl}$ was entirely in the $xy$ plane.

The curl of $\mathbf{A}$ is

$$\nabla \times \mathbf{A} = \mathbf{i}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = 2\mathbf{i}_z$$

(a) For the circular area in the plane of the contour, we have that

$$\int_S (\nabla\times\mathbf{A})\cdot\mathbf{dS}=2\int_S dS_z=2\pi R^2$$

which agrees with the line integral result.

(b) For the hemispherical surface

$$\int_S(\nabla\times\mathbf{A})\cdot\mathbf{dS}=\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}2\mathbf{i}_z\cdot\mathbf{i}_rR^2\sin\theta\,d\theta\,d\phi$$

From Table 1-2 we use the dot product relation

$$\mathbf{i}_z\cdot\mathbf{i}_r=\cos\theta$$

which again gives the circulation as

$$C=\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}R^2\sin 2\theta\,d\theta\,d\phi=-2\pi R^2\frac{\cos 2\theta}{2}\bigg|_{\theta=0}^{\pi/2}=2\pi R^2$$

(c) Similarly, for the cylindrical surface, we only obtain nonzero contributions to the surface integral at the upper circular area that is perpendicular to $\nabla\times\mathbf{A}$. The integral is then the same as part (a) as $\nabla\times\mathbf{A}$ is independent of $z$.

### 1-5-4 Some Useful Vector Identities

The curl, divergence, and gradient operations have some simple but useful properties that are used throughout the text.

**(a) The Curl of the Gradient is Zero [$\nabla\times(\nabla f)=0$]**

We integrate the normal component of the vector $\nabla\times(\nabla f)$ over a surface and use Stokes' theorem

$$\int_S\nabla\times(\nabla f)\cdot\mathbf{dS}=\oint_L\nabla f\cdot\mathbf{dl}=0 \tag{26}$$

where the zero result is obtained from Section 1-3-3, that the line integral of the gradient of a function around a closed path is zero. Since the equality is true for any surface, the vector coefficient of $\mathbf{dS}$ in (26) must be zero

$$\nabla\times(\nabla f)=0$$

The identity is also easily proved by direct computation using the determinantal relation in Section 1-5-1 defining the curl operation:

$$\nabla\times(\nabla f)=\det\begin{vmatrix}\mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z}\end{vmatrix}$$

$$=\mathbf{i}_x\left(\frac{\partial^2 f}{\partial y\partial z}-\frac{\partial^2 f}{\partial z\partial y}\right)+\mathbf{i}_y\left(\frac{\partial^2 f}{\partial z\partial x}-\frac{\partial^2 f}{\partial x\partial z}\right)+\mathbf{i}_z\left(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\right)=0 \tag{28}$$

Each bracketed term in (28) is zero because the order of differentiation does not matter.

**(b) The Divergence of the Curl of a Vector is Zero [$\nabla\cdot(\nabla\times\mathbf{A})=0$]**

One might be tempted to apply the divergence theorem to the surface integral in Stokes' theorem of (25). However, the divergence theorem requires a closed surface while Stokes' theorem is true in general for an open surface. Stokes' theorem for a closed surface requires the contour $L$ to shrink to zero giving a zero result for the line integral. The divergence theorem applied to the closed surface with vector $\nabla\times\mathbf{A}$ is then

$$\oint_S \nabla\times\mathbf{A}\cdot d\mathbf{S}=0\Rightarrow\int_V \nabla\cdot(\nabla\times\mathbf{A})\,dV=0\Rightarrow\nabla\cdot(\nabla\times\mathbf{A})=0 \tag{29}$$

which proves the identity because the volume is arbitrary.

More directly we can perform the required differentiations

$$\begin{aligned}\nabla\cdot(\nabla\times\mathbf{A})&\\ &=\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\\ &=\left(\frac{\partial^2 A_z}{\partial x\partial y}-\frac{\partial^2 A_z}{\partial y\partial x}\right)+\left(\frac{\partial^2 A_x}{\partial y\partial z}-\frac{\partial^2 A_x}{\partial z\partial y}\right)+\left(\frac{\partial^2 A_y}{\partial z\partial x}-\frac{\partial^2 A_y}{\partial x\partial z}\right)=0\end{aligned} \tag{30}$$

where again the order of differentiation does not matter.

## PROBLEMS

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Section 1-1

1. Find the area of a circle in the $xy$ plane centered at the origin using:

(a) rectangular coordinates $x^2+y^2=a^2$ (**Hint:**

$$\int\sqrt{a^2-x^2}\,dx=\tfrac{1}{2}[x\sqrt{a^2-x^2}+a^2\sin^{-1}(x/a)])$$

(b) cylindrical coordinates $r = a$.
Which coordinate system is easier to use?

2. Find the volume of a sphere of radius $R$ centered at the origin using:

(a) rectangular coordinates $x^2+y^2+z^2=R^2$ (**Hint:**
$$\int \sqrt{a^2-x^2}\,dx = \tfrac{1}{2}[x\sqrt{a^2-x^2}+a^2\sin^{-1}(x/a)])$$
(b) cylindrical coordinates $r^2+z^2=R^2$;
(c) spherical coordinates $r = R$.
Which coordinate system is easiest?

Section 1-2
3. Given the three vectors

$$\mathbf{A} = 3\mathbf{i}_x + 2\mathbf{i}_y - \mathbf{i}_z$$
$$\mathbf{B} = 3\mathbf{i}_x - 4\mathbf{i}_y - 5\mathbf{i}_z$$
$$\mathbf{C} = \mathbf{i}_x - \mathbf{i}_y + \mathbf{i}_z$$

find the following:

(a) $\mathbf{A}\pm\mathbf{B}, \mathbf{B}\pm\mathbf{C}, \mathbf{A}\pm\mathbf{C}$
(b) $\mathbf{A}\cdot\mathbf{B}, \mathbf{B}\cdot\mathbf{C}, \mathbf{A}\cdot\mathbf{C}$
(c) $\mathbf{A}\times\mathbf{B}, \mathbf{B}\times\mathbf{C}, \mathbf{A}\times\mathbf{C}$
(d) $(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}$, $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ [Are they equal?]
(e) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$, $\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ [Are they equal?]
(f) What is the angle between $\mathbf{A}$ and $\mathbf{C}$ and between $\mathbf{B}$ and $\mathbf{A}\times\mathbf{C}$?

4. Given the sum and difference between two vectors,

$$\mathbf{A}+\mathbf{B} = -\mathbf{i}_x + 5\mathbf{i}_y - 4\mathbf{i}_z$$
$$\mathbf{A}-\mathbf{B} = 3\mathbf{i}_x - \mathbf{i}_y - 2\mathbf{i}_z$$

find the individual vectors $\mathbf{A}$ and $\mathbf{B}$.

5. (a) Given two vectors $\mathbf{A}$ and $\mathbf{B}$, show that the component of $\mathbf{B}$ parallel to $\mathbf{A}$ is

$$\mathbf{B}_{\|} = \frac{\mathbf{B}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}}\mathbf{A}$$

(**Hint:** $\mathbf{B}_{\|} = \alpha\mathbf{A}$. What is $\alpha$?)

(b) If the vectors are

$$\mathbf{A} = \mathbf{i}_x - 2\mathbf{i}_y + \mathbf{i}_z$$
$$\mathbf{B} = 3\mathbf{i}_x + 5\mathbf{i}_y - 5\mathbf{i}_z$$

what are the components of $\mathbf{B}$ parallel and perpendicular to $\mathbf{A}$?

$$\mathbf{B} = \mathbf{B}_{\perp} + \mathbf{B}_{\|}$$

6. What are the angles between each of the following vectors:

$$\mathbf{A} = 4\mathbf{i}_x - 2\mathbf{i}_y + 2\mathbf{i}_z$$

$$\mathbf{B} = -6\mathbf{i}_x + 3\mathbf{i}_y - 3\mathbf{i}_z$$

$$\mathbf{C} = \mathbf{i}_x + 3\mathbf{i}_y + \mathbf{i}_z$$

7. Given the two vectors

$$\mathbf{A} = 3\mathbf{i}_x + 4\mathbf{i}_y \quad \text{and} \quad \mathbf{B} = 7\mathbf{i}_x - 24\mathbf{i}_y$$

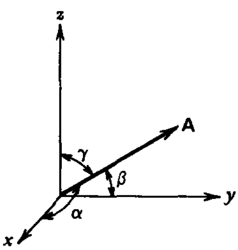
(a) What is their dot product?
(b) What is their cross product?
(c) What is the angle $\theta$ between the two vectors?

8. Given the vector

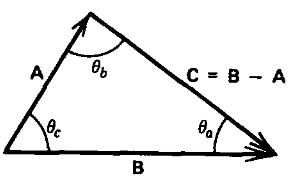
$$\mathbf{A} = A_x\mathbf{i}_x + A_y\mathbf{i}_y + A_z\mathbf{i}_z$$

the directional cosines are defined as the cosines of the angles between $\mathbf{A}$ and each of the Cartesian coordinate axes. Find each of these directional cosines and show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$




9. A triangle is formed by the three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C} = \mathbf{B} - \mathbf{A}$.




(a) Find the length of the vector $\mathbf{C}$ in terms of the lengths of $\mathbf{A}$ and $\mathbf{B}$ and the enclosed angle $\theta_c$. The result is known as the law of cosines. (**Hint:** $\mathbf{C}\cdot\mathbf{C} = (\mathbf{B}-\mathbf{A})\cdot(\mathbf{B}-\mathbf{A})$.)

(b) For the same triangle, prove the law of sines:

$$\frac{\sin\theta_a}{A} = \frac{\sin\theta_b}{B} = \frac{\sin\theta_c}{C}$$

(**Hint:** $\mathbf{B}\times\mathbf{A} = (\mathbf{C}+\mathbf{A})\times\mathbf{A}$.)

10. (a) Prove that the dot and cross can be interchanged in the scalar triple product

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}=(\mathbf{B}\times\mathbf{C})\cdot\mathbf{A}=(\mathbf{C}\times\mathbf{A})\cdot\mathbf{B}$$

(b) Show that this product gives the volume of a parallelepiped whose base is defined by the vectors **A** and **B** and whose height is given by **C**.

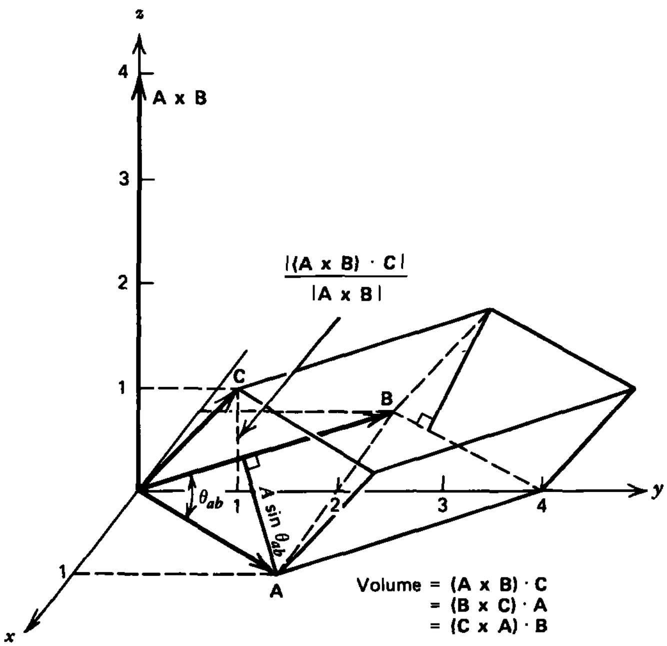
(c) If

$$\mathbf{A}=\mathbf{i}_x+2\mathbf{i}_y,\quad \mathbf{B}=-\mathbf{i}_x+2\mathbf{i}_y,\quad \mathbf{C}=\mathbf{i}_x+\mathbf{i}_z$$

verify the identities of (a) and find the volume of the parallelepiped formed by the vectors.

(d) Prove the vector triple product identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

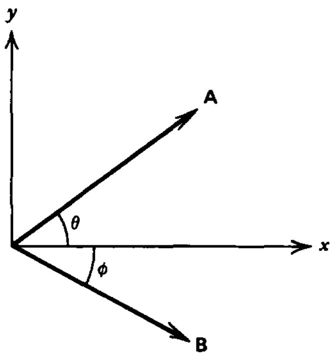



11. (a) Write the vectors **A** and **B** using Cartesian coordinates in terms of their angles $\theta$ and $\phi$ from the $x$ axis.

(b) Using the results of (a) derive the trigonometric expansions

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\sin\phi\cos\theta$$

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$$




Section 1-3

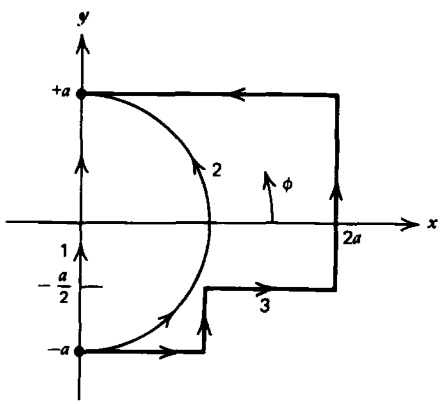
12. Find the gradient of each of the following functions where $a$ and $b$ are constants:

(a) $f = axz + bx^3y$
(b) $f = (a/r)\sin\phi + brz^2\cos 3\phi$
(c) $f = ar\cos\theta + (b/r^2)\sin\phi$

13. Evaluate the line integral of the gradient of the function

$$f = r\sin\phi$$

over each of the contours shown.




Section 1-4

14. Find the divergence of the following vectors:

(a) $\mathbf{A} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z = r\mathbf{i}_r$
(b) $\mathbf{A} = (xy^2z^3)[\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z]$
(c) $\mathbf{A} = r\cos\phi\mathbf{i}_r + [(z/r)\sin\phi)]\mathbf{i}_z$
(d) $\mathbf{A} = r^2\sin\theta\cos\phi\,[\mathbf{i}_r + \mathbf{i}_\theta + \mathbf{i}_\phi]$

15. Using the divergence theorem prove the following integral identities:

(a) $\int_V \nabla f\, dV = \oint_S f\, d\mathbf{S}$

(**Hint:** Let $\mathbf{A} = \mathbf{i}f$, where $\mathbf{i}$ is any *constant* unit vector.)

(b) $\int_V \nabla \times \mathbf{F}\, dV = -\oint_S \mathbf{F} \times \mathbf{dS}$

(**Hint:** Let $\mathbf{A} = \mathbf{i} \times \mathbf{F}$.)

(c) Using the results of (a) show that the normal vector integrated over a surface is zero:

$$\oint_S \mathbf{dS} = 0$$

(d) Verify (c) for the case of a sphere of radius $R$. (**Hint:** $\mathbf{i}_r = \sin\theta\cos\phi\,\mathbf{i}_x + \sin\theta\sin\phi\,\mathbf{i}_y + \cos\theta\,\mathbf{i}_z$.

16. Using the divergence theorem prove Green's theorem

$$\oint_S [f\nabla g - g\nabla f] \cdot \mathbf{dS} = \int_V [f\nabla^2 g - g\nabla^2 f]\, dV$$

(**Hint:** $\nabla \cdot (f\nabla g) = f\nabla^2 g + \nabla f \cdot \nabla g$.)

17. (a) Find the area element $\mathbf{dS}$ (magnitude and direction) on each of the four surfaces of the pyramidal figure shown.

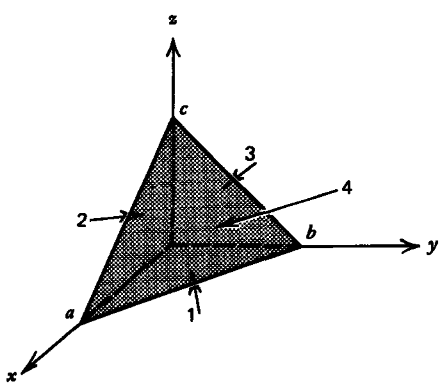
(b) Find the flux of the vector

$$\mathbf{A} = r\mathbf{i}_r = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z$$

through the surface of (a).

(c) Verify the divergence theorem by also evaluating the flux as

$$\Phi = \int_V \nabla \cdot \mathbf{A}\, dV$$




Section 1-5

18. Find the curl of the following vectors:

(a) $\mathbf{A} = x^2y\mathbf{i}_x + y^2z\mathbf{i}_y + xy\mathbf{i}_z$

(b) $\mathbf{A} = r\cos\phi\,\mathbf{i}_z + \dfrac{z\sin\phi}{r}\,\mathbf{i}_r$

(c) $\mathbf{A} = r^2\sin\theta\cos\phi\,\mathbf{i}_r + \dfrac{\cos\theta\sin\phi}{r^2}\,\mathbf{i}_\theta$

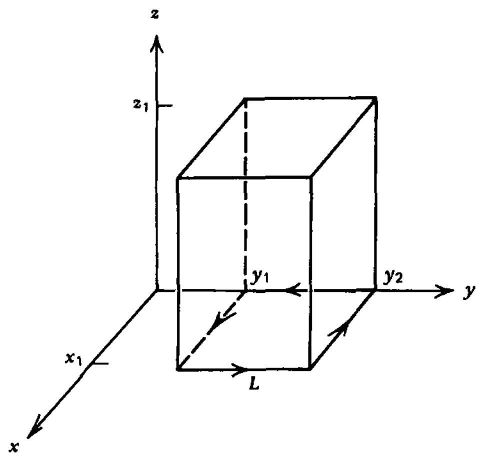
19. Using Stokes' theorem prove that

$$\oint_L f\,\mathbf{dl} = -\int_S \nabla f\times\mathbf{dS}$$

(**Hint:** Let $\mathbf{A} = \mathbf{i}f$, where $\mathbf{i}$ is any *constant* unit vector.)

20. Verify Stokes' theorem for the rectangular bounding contour in the $xy$ plane with a vector field

$$\mathbf{A} = (x+a)(y+b)(z+c)\mathbf{i}_x$$




Check the result for (a) a flat rectangular surface in the $xy$ plane, and (b) for the rectangular cylinder.

21. Show that the order of differentiation for the mixed second derivative

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

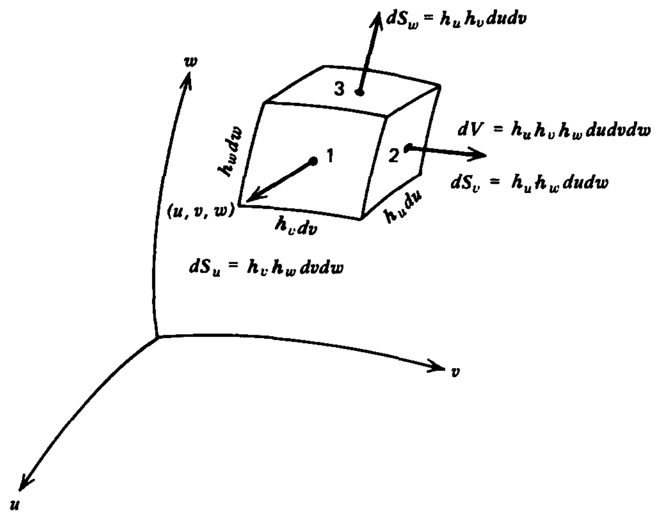
does not matter for the function

$$f = \frac{x^2\ln y}{y}$$

22. Some of the unit vectors in cylindrical and spherical coordinates change direction in space and thus, unlike Cartesian unit vectors, are not constant vectors. This means that spatial derivatives of these unit vectors are generally nonzero. Find the divergence and curl of all the unit vectors.

23. A general right-handed orthogonal curvilinear coordinate system is described by variables $(u, v, w)$, where

$$\mathbf{i}_u \times \mathbf{i}_v = \mathbf{i}_w$$




Since the incremental coordinate quantities $du$, $dv$, and $dw$ do not necessarily have units of length, the differential length elements must be multiplied by coefficients that generally are a function of $u$, $v$, and $w$:

$$dL_u = h_u \, du, \quad dL_v = h_v \, dv, \quad dL_w = h_w \, dw$$

(a) What are the $h$ coefficients for the Cartesian, cylindrical, and spherical coordinate systems?

(b) What is the gradient of any function $f(u, v, w)$?

(c) What is the area of each surface and the volume of a differential size volume element in the $(u, v, w)$ space?

(d) What are the curl and divergence of the vector

$$\mathbf{A} = A_u \mathbf{i}_u + A_v \mathbf{i}_v + A_w \mathbf{i}_w?$$

(e) What is the scalar Laplacian $\nabla^2 f = \nabla \cdot (\nabla f)$?

(f) Check your results of (b)–(e) for the three basic coordinate systems.

24. Prove the following vector identities:

(a) $\nabla(fg) = f\nabla g + g\nabla f$
(b) $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$
(c) $\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + (\mathbf{A} \cdot \nabla)f$
(d) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
(e) $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}$

(f) $\nabla \times (f\mathbf{A}) = \nabla f \times \mathbf{A} + f \nabla \times \mathbf{A}$
(g) $(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A})$
(h) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

25. Two points have Cartesian coordinates $(1, 2, -1)$ and $(2, -3, 1)$.

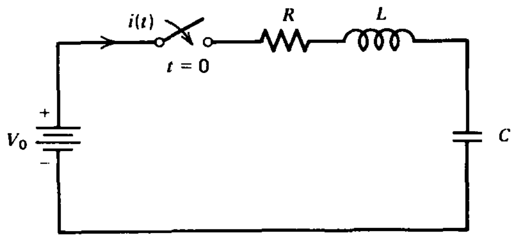
(a) What is the distance between these two points?

(b) What is the unit vector along the line joining the two points?

(c) Find a unit vector in the $xy$ plane perpendicular to the unit vector found in ($b$).

Miscellaneous

26. A series RLC circuit offers a good review in solving linear, constant coefficient ordinary differential equations. A step voltage $V_0$ is applied to the initially unexcited circuit at $t = 0$.




(a) Write a single differential equation for the current.

(b) Guess an exponential solution of the form

$$i(t) = \hat{I}e^{st}$$

and find the natural frequencies of the circuit.

(c) What are the initial conditions? What are the steady-state voltages across each element?

(d) Write and sketch the solution for $i(t)$ when

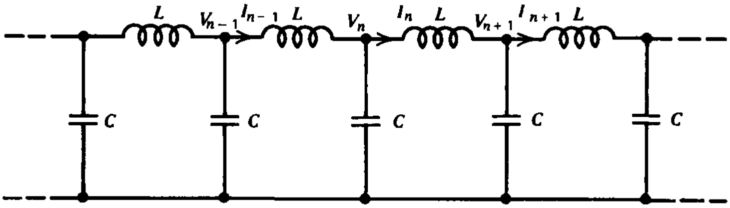
$$\left(\frac{R}{2L}\right)^2 < \frac{1}{LC}, \quad \left(\frac{R}{2L}\right)^2 = \frac{1}{LC}, \quad \left(\frac{R}{2L}\right)^2 > \frac{1}{LC}$$

(e) What is the voltage across each element?

(f) After the circuit has reached the steady state, the terminal voltage is instantly short circuited. What is the short circuit current?

27. Many times in this text we consider systems composed of repetitive sequences of a basic building block. Such discrete element systems are described by difference equations. Consider a distributed series inductance-shunt capacitance system excited by a sinusoidal frequency $\omega$ so that the voltage and current in the $n$th loop vary as

$$i_n = \text{Re}\,(I_n e^{j\omega t}); \quad v_n = \text{Re}(V_n e^{j\omega t})$$




(a) By writing Kirchoff's voltage law for the $n$th loop, show that the current obeys the difference equation

$$I_{n+1}-\left(2-\frac{\omega^2}{\omega_0^2}\right)I_n+I_{n-1}=0$$

What is $\omega_0^2$?

(b) Just as exponential solutions satisfy linear constant coefficient differential equations, power-law solutions satisfy linear constant coefficient difference equations

$$I_n=\hat{I}\lambda^n$$

What values of $\lambda$ satisfy (a)?

(c) The general solution to (a) is a linear combination of all the possible solutions. The circuit ladder that has $N$ nodes is excited in the zeroth loop by a current source

$$i_0=\mathrm{Re}\,(I_0\,e^{j\omega t})$$

Find the general expression for current $i_n$ and voltage $v_n$ for any loop when the last loop $N$ is either open ($I_N=0$) or short circuited ($V_N=0$). (**Hint:** $a+\sqrt{a^2-1}=1/(a-\sqrt{a^2-1})$)

(d) What are the natural frequencies of the system when the last loop is either open or short circuited? (**Hint:** $(1)^{1/(2N)}=e^{j2\pi r/2N},\quad r=1,2,3,\ldots,2N.$)

# chapter 2

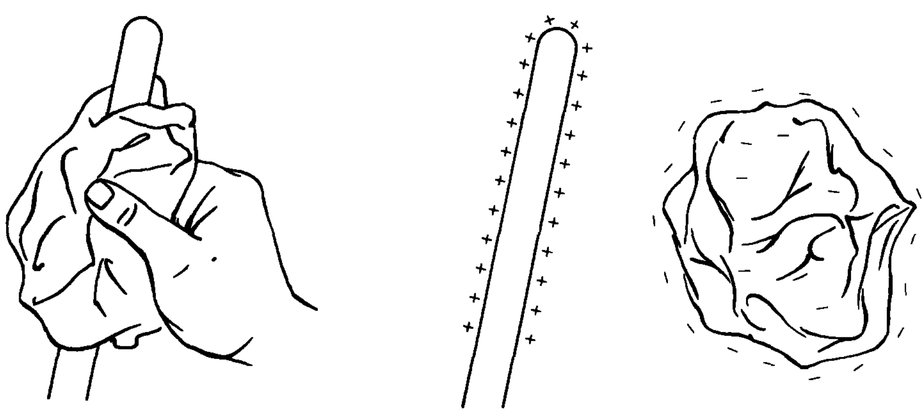
*the electric field*

The ancient Greeks observed that when the fossil resin amber was rubbed, small light-weight objects were attracted. Yet, upon contact with the amber, they were then repelled. No further significant advances in the understanding of this mysterious phenomenon were made until the eighteenth century when more quantitative electrification experiments showed that these effects were due to electric charges, the source of all effects we will study in this text.

## 2-1 ELECTRIC CHARGE

### 2-1-1 Charging by Contact

We now know that all matter is held together by the attractive force between equal numbers of negatively charged electrons and positively charged protons. The early researchers in the 1700s discovered the existence of these two species of charges by performing experiments like those in Figures 2-1 to 2-4. When a glass rod is rubbed by a dry cloth, as in Figure 2-1, some of the electrons in the glass are rubbed off onto the cloth. The cloth then becomes negatively charged because it now has more electrons than protons. The glass rod becomes



(a) (b)

Figure 2-1 A glass rod rubbed with a dry cloth loses some of its electrons to the cloth. The glass rod then has a net positive charge while the cloth has acquired an equal amount of negative charge. The total charge in the system remains zero.

positively charged as it has lost electrons leaving behind a surplus number of protons. If the positively charged glass rod is brought near a metal ball that is free to move as in Figure 2-2*a*, the electrons in the ball near the rod are attracted to the surface leaving uncovered positive charge on the other side of the ball. This is called electrostatic induction. There is then an attractive force of the ball to the rod. Upon contact with the rod, the negative charges are neutralized by some of the positive charges on the rod, the whole combination still retaining a net positive charge as in Figure 2-2*b*. This transfer of charge is called conduction. It is then found that the now positively charged ball is repelled from the similarly charged rod. The metal ball is said to be conducting as charges are easily induced and conducted. It is important that the supporting string not be conducting, that is, insulating, otherwise charge would also distribute itself over the whole structure and not just on the ball.

If two such positively charged balls are brought near each other, they will also repel as in Figure 2-3*a*. Similarly, these balls could be negatively charged if brought into contact with the negatively charged cloth. Then it is also found that two negatively charged balls repel each other. On the other hand, if one ball is charged positively while the other is charged negatively, they will attract. These circumstances are summarized by the simple rules:

Opposite Charges Attract. Like Charges Repel.

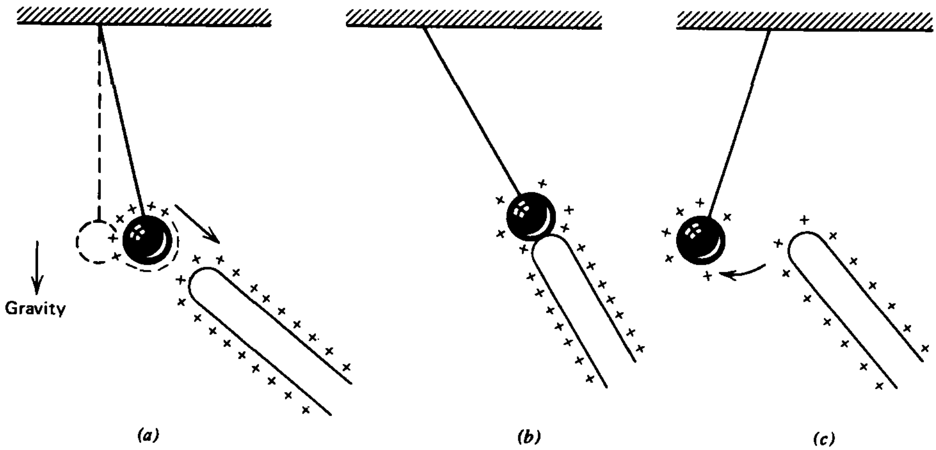



Figure 2-2 (*a*) A charged rod near a neutral ball will induce an opposite charge on the near surface. Since the ball is initially neutral, an equal amount of positive charge remains on the far surface. Because the negative charge is closer to the rod, it feels a stronger attractive force than the repelling force due to the like charges. (*b*) Upon contact with the rod the negative charge is neutralized leaving the ball positively charged. (*c*) The like charges then repel causing the ball to deflect away.

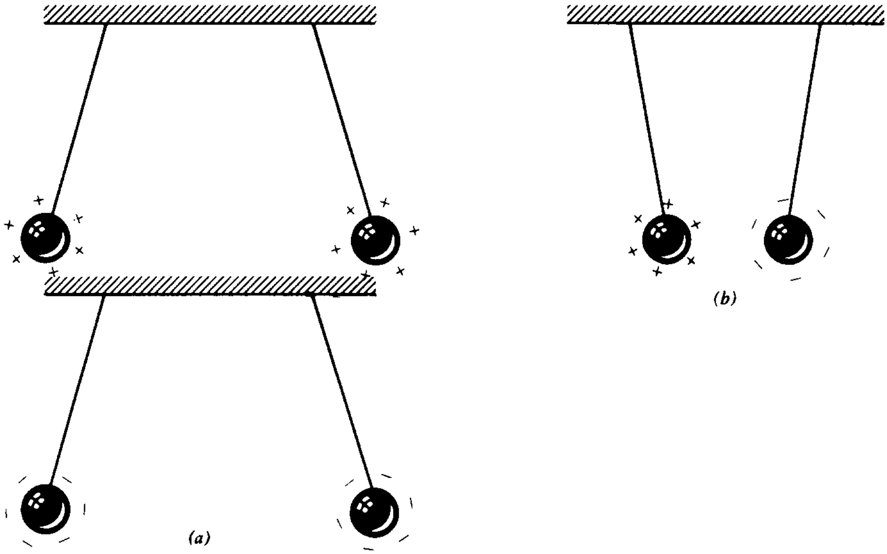



Figure 2-3 (*a*) Like charged bodies repel while (*b*) oppositely charged bodies attract.

In Figure 2-2*a*, the positively charged rod attracts the negative induced charge but repels the uncovered positive charge on the far end of the ball. The net force is attractive because the positive charge on the ball is farther away from the glass rod so that the repulsive force is less than the attractive force.

We often experience nuisance frictional electrification when we walk across a carpet or pull clothes out of a dryer. When we comb our hair with a plastic comb, our hair often becomes charged. When the comb is removed our hair still stands up, as like charged hairs repel one another. Often these effects result in sparks because the presence of large amounts of charge actually pulls electrons from air molecules.

### 2-1-2 Electrostatic Induction

Even without direct contact net charge can also be placed on a body by electrostatic induction. In Figure 2-4*a* we see two initially neutral suspended balls in contact acquiring opposite charges on each end because of the presence of a charged rod. If the balls are now separated, each half retains its net charge even if the inducing rod is removed. The net charge on the two balls is zero, but we have been able to isolate net positive and negative charges on each ball.

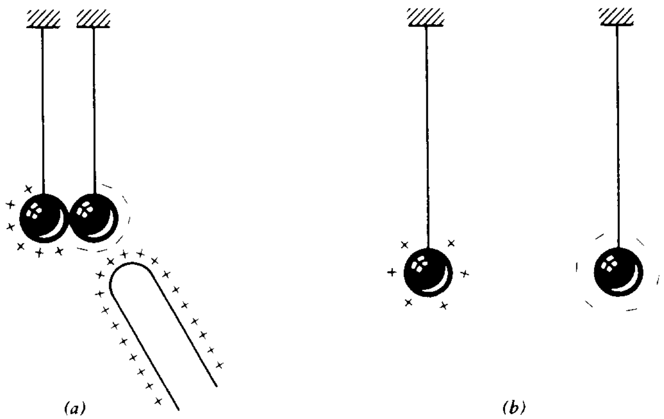



Figure 2-4 A net charge can be placed on a body without contact by electrostatic induction. (*a*) When a charged body is brought near a neutral body, the near side acquires the opposite charge. Being neutral, the far side takes on an equal but opposite charge. (*b*) If the initially neutral body is separated, each half retains its charge.

### 2-1-3 Faraday's "Ice-Pail" Experiment

These experiments showed that when a charged conductor contacted another conductor, whether charged or not, the total charge on both bodies was shared. The presence of charge was first qualitatively measured by an electroscope that consisted of two attached metal foil leaves. When charged, the mutual repulsion caused the leaves to diverge.

In 1843 Michael Faraday used an electroscope to perform the simple but illuminating "ice-pail" experiment illustrated in Figure 2-5. When a charged body is inside a closed isolated conductor, an equal amount of charge appears on the outside of the conductor as evidenced by the divergence of the electroscope leaves. This is true whether or not the charged body has contacted the inside walls of the surrounding conductor. If it has not, opposite charges are induced on the inside wall leaving unbalanced charge on the outside. If the charged body is removed, the charge on the inside and outside of the conductor drops to zero. However, if the charged body does contact an inside wall, as in Figure 2-5*c*, all the charge on the inside wall and ball is neutralized leaving the outside charged. Removing the initially charged body as in Figure 2-5*d* will find it uncharged, while the ice-pail now holds the original charge.

If the process shown in Figure 2-5 is repeated, the charge on the pail can be built up indefinitely. This is the principle of electrostatic generators where large amounts of charge are stored by continuous deposition of small amounts of charge.

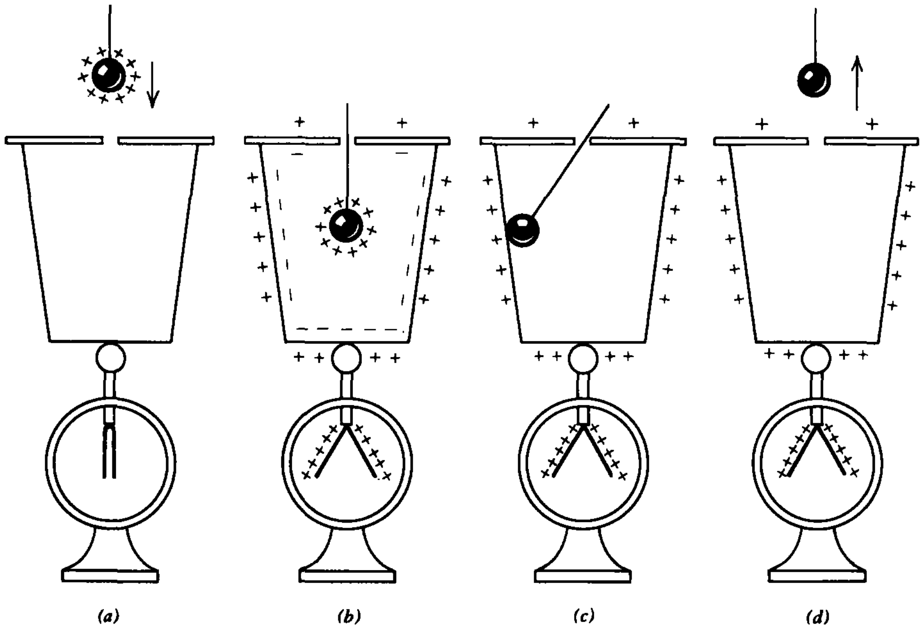



Figure 2-5 Faraday first demonstrated the principles of charge conservation by attaching an electroscope to an initially uncharged metal ice pail. (*a*) When all charges are far away from the pail, there is no charge on the pail nor on the flexible gold leaves of the electroscope attached to the outside of the can, which thus hang limply. (*b*) As a charged ball comes within the pail, opposite charges are induced on the inner surface. Since the pail and electroscope were originally neutral, unbalanced charge appears on the outside of which some is on the electroscope leaves. The leaves being like charged repel each other and thus diverge. (*c*) Once the charged ball is within a closed conducting body, the charge on the outside of the pail is independent of the position of the charged ball. If the charged ball contacts the inner surface of the pail, the inner charges neutralize each other. The outside charges remain unchanged. (*d*) As the now uncharged ball leaves the pail, the distributed charge on the outside of the pail and electroscope remains unchanged.

This large accumulation of charge gives rise to a large force on any other nearby charge, which is why electrostatic generators have been used to accelerate charged particles to very high speeds in atomic studies.

## 2-2 THE COULOMB FORCE LAW BETWEEN STATIONARY CHARGES

### 2-2-1 Coulomb's Law

It remained for Charles Coulomb in 1785 to express these experimental observations in a quantitative form. He used a very sensitive torsional balance to measure the force between two stationary charged balls as a function of their distance apart. He discovered that the force between two small charges $q_1$ and $q_2$ (idealized as point charges of zero size) is proportional to their magnitudes and inversely proportional to the square of the distance $r_{12}$ between them, as illustrated in Figure 2-6. The force acts along the line joining the charges in the same or opposite direction of the unit vector $\mathbf{i}_{12}$ and is attractive if the charges are of opposite sign and repulsive if like charged. The force $\mathbf{F}_2$ on charge $q_2$ due to charge $q_1$ is equal in magnitude but opposite in direction to the force $\mathbf{F}_1$ on $q_1$, the net force on the pair of charges being zero.

$$\mathbf{F}_2 = -\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\mathbf{i}_{12}\ \text{nt}\ [\text{kg}-\text{m}-\text{s}^{-2}] \quad (1)$$

## 2-2-2 Units

The value of the proportionality constant $1/4\pi\varepsilon_0$ depends on the system of units used. Throughout this book we use SI units (Système International d'Unités) for which the base units are taken from the rationalized MKSA system of units where distances are measured in meters (m), mass in kilograms (kg), time in seconds (s), and electric current in amperes (A). The unit of charge is a coulomb where 1 coulomb = 1 ampere-second. The adjective "rationalized" is used because the factor of $4\pi$ is arbitrarily introduced into the proportionality factor in Coulomb's law of (1). It is done this way so as to cancel a $4\pi$ that will arise from other more often used laws we will introduce shortly. Other derived units are formed by combining base units.

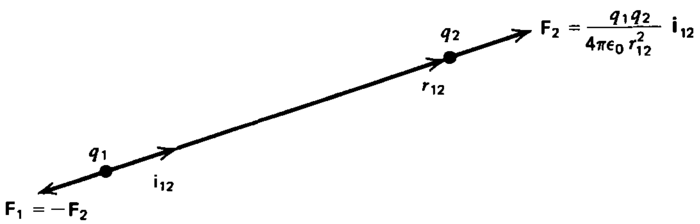



Figure 2-6 The Coulomb force between two point charges is proportional to the magnitude of the charges and inversely proportional to the square of the distance between them. The force on each charge is equal in magnitude but opposite in direction. The force vectors are drawn as if $q_1$ and $q_2$ are of the same sign so that the charges repel. If $q_1$ and $q_2$ are of opposite sign, both force vectors would point in the opposite directions, as opposite charges attract.

The parameter $\varepsilon_0$ is called the permittivity of free space and has a value

$$\varepsilon_0 = (4\pi \times 10^{-7} c^2)^{-1}$$
$$\approx \frac{10^{-9}}{36\pi} \approx 8.8542 \times 10^{-12} \text{ farad/m } [\text{A}^2-\text{s}^4-\text{kg}^{-1}-\text{m}^{-3}] \quad (2)$$

where $c$ is the speed of light in vacuum ($c \approx 3 \times 10^8$ m/sec).

This relationship between the speed of light and a physical constant was an important result of the early electromagnetic theory in the late nineteenth century, and showed that light is an electromagnetic wave; see the discussion in Chapter 7.

To obtain a feel of how large the force in (1) is, we compare it with the gravitational force that is also an inverse square law with distance. The smallest unit of charge known is that of an electron with charge $e$ and mass $m_e$

$$e \approx 1.60 \times 10^{-19} \text{ Coul}, \quad m_e \approx 9.11 \times 10^{-31} \text{ kg}$$

Then, the ratio of electric to gravitational force magnitudes for two electrons is independent of their separation:

$$\frac{\mathbf{F}_e}{\mathbf{F}_g} = -\frac{e^2/(4\pi\varepsilon_0 r^2)}{Gm_e^2/r^2} = -\frac{e^2}{m_e^2}\frac{1}{4\pi\varepsilon_0 G} \approx -4.16 \times 10^{42} \quad (3)$$

where $G = 6.67 \times 10^{-11}$ [$\text{m}^3$-$\text{s}^{-2}$-$\text{kg}^{-1}$] is the gravitational constant. This ratio is so huge that it exemplifies why electrical forces often dominate physical phenomena. The minus sign is used in (3) because the gravitational force between two masses is always attractive while for two like charges the electrical force is repulsive.

### 2-2-3 The Electric Field

If the charge $q_1$ exists alone, it feels no force. If we now bring charge $q_2$ within the vicinity of $q_1$, then $q_2$ feels a force that varies in magnitude and direction as it is moved about in space and is thus a way of mapping out the vector force field due to $q_1$. A charge other than $q_2$ would feel a different force from $q_2$ proportional to its own magnitude and sign. It becomes convenient to work with the quantity of force per unit charge that is called the electric field, because this quantity is independent of the particular value of charge used in mapping the force field. Considering $q_2$ as the test charge, the electric field due to $q_1$ at the position of $q_2$ is defined as

$$\mathbf{E}_2 = \lim_{q_2 \to 0} \frac{\mathbf{F}_2}{q_2} = \frac{q_1}{4\pi\varepsilon_0 r_{12}^2}\,\mathbf{i}_{12} \text{ volts/m } [\text{kg-m-s}^{-3}-\text{A}^{-1}] \quad (4)$$

In the definition of (4) the charge $q_1$ must remain stationary. This requires that the test charge $q_2$ be negligibly small so that its force on $q_1$ does not cause $q_1$ to move. In the presence of nearby materials, the test charge $q_2$ could also induce or cause redistribution of the charges in the material. To avoid these effects in our definition of the electric field, we make the test charge infinitely small so its effects on nearby materials and charges are also negligibly small. Then (4) will also be a valid definition of the electric field when we consider the effects of materials. To correctly map the electric field, the test charge must not alter the charge distribution from what it is in the absence of the test charge.

### 2-2-4 Superposition

If our system only consists of two charges, Coulomb's law (1) completely describes their interaction and the definition of an electric field is unnecessary. The electric field concept is only useful when there are large numbers of charge present as each charge exerts a force on all the others. Since the forces on a particular charge are linear, we can use superposition, whereby if a charge $q_1$ alone sets up an electric field $\mathbf{E}_1$, and another charge $q_2$ alone gives rise to an electric field $\mathbf{E}_2$, then the resultant electric field with both charges present is the vector sum $\mathbf{E}_1+\mathbf{E}_2$. This means that if a test charge $q_p$ is placed at point $P$ in Figure 2-7, in the vicinity of $N$ charges it will feel a force

$$\mathbf{F}_p = q_p\mathbf{E}_P \tag{5}$$

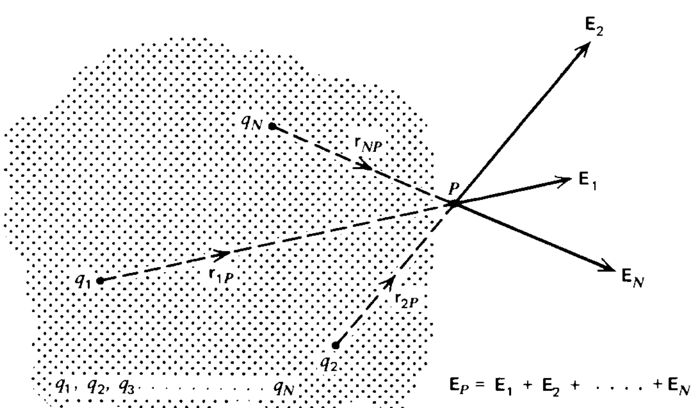



Figure 2-7 The electric field due to a collection of point charges is equal to the vector sum of electric fields from each charge alone.

where $\mathbf{E}_P$ is the vector sum of the electric fields due to all the $N$-point charges,

$$\mathbf{E}_P=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{1P}^2}\mathbf{i}_{1P}+\frac{q_2}{r_{2P}^2}\mathbf{i}_{2P}+\frac{q_3}{r_{3P}^2}\mathbf{i}_{3P}+\cdots+\frac{q_N}{r_{NP}^2}\mathbf{i}_{NP}\right)$$

$$=\frac{1}{4\pi\varepsilon_0}\sum_{n=1}^{N}\frac{q_n}{r_{nP}^2}\mathbf{i}_{nP} \tag{6}$$

Note that $\mathbf{E}_P$ has no contribution due to $q_p$ since a charge cannot exert a force upon itself.

## EXAMPLE 2-1 TWO-POINT CHARGES

Two-point charges are a distance $a$ apart along the $z$ axis as shown in Figure 2-8. Find the electric field at any point in the $z=0$ plane when the charges are:

(a) both equal to $q$
(b) of opposite polarity but equal magnitude $\pm q$. This configuration is called an electric dipole.

### SOLUTION

---

(a) In the $z=0$ plane, each point charge alone gives rise to field components in the $\mathbf{i}_r$ and $\mathbf{i}_z$ directions. When both charges are equal, the superposition of field components due to both charges cancel in the $z$ direction but add radially:

$$E_r(z=0)=\frac{q}{4\pi\varepsilon_0}\frac{2\mathrm{r}}{[\mathrm{r}^2+(a/2)^2]^{3/2}}$$

As a check, note that far away from the point charges ($\mathrm{r}\gg a$) the field approaches that of a point charge of value $2q$:

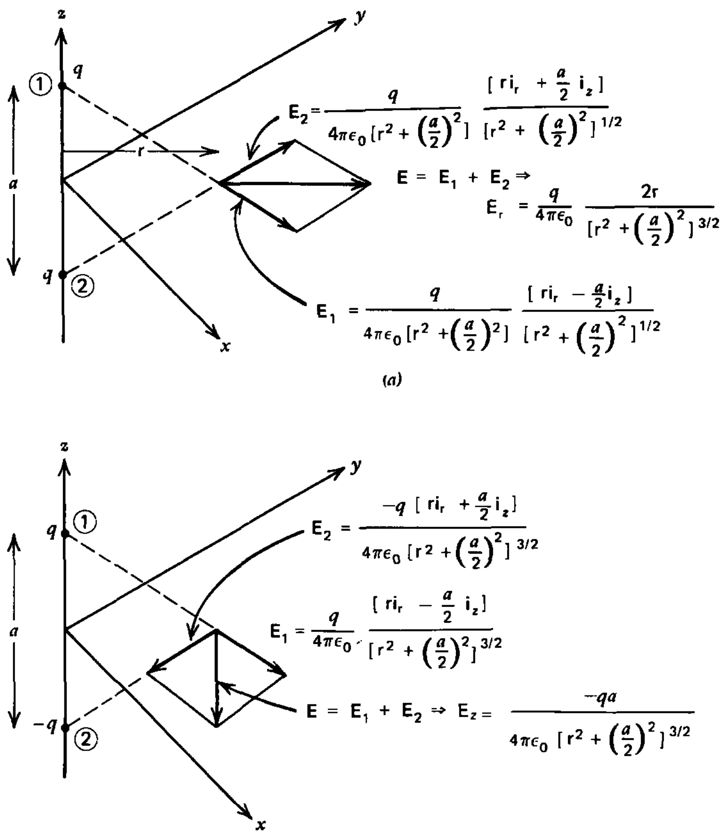
$$\lim_{\mathrm{r}\gg a}E_r(z=0)=\frac{2q}{4\pi\varepsilon_0\mathrm{r}^2}$$

(b) When the charges have opposite polarity, the total electric field due to both charges now cancel in the radial direction but add in the $z$ direction:

$$E_z(z=0)=\frac{-q}{4\pi\varepsilon_0}\frac{a}{[\mathrm{r}^2+(a/2)^2]^{3/2}}$$

Far away from the point charges the electric field dies off as the inverse cube of distance:

$$\lim_{\mathrm{r}\gg a}E_z(z=0)=\frac{-qa}{4\pi\varepsilon_0\mathrm{r}^3}$$




(b)

Figure 2-8 Two equal magnitude point charges are a distance $a$ apart along the $z$ axis. ($a$) When the charges are of the same polarity, the electric field due to each is radially directed away. In the $z = 0$ symmetry plane, the net field component is radial. ($b$) When the charges are of opposite polarity, the electric field due to the negative charge is directed radially inwards. In the $z = 0$ symmetry plane, the net field is now $-z$ directed.

The faster rate of decay of a dipole field is because the net charge is zero so that the fields due to each charge tend to cancel each other out.

## 2-3 CHARGE DISTRIBUTIONS

The method of superposition used in Section 2.2.4 will be used throughout the text in relating fields to their sources. We first find the field due to a single-point source. Because the field equations are linear, the net field due to many point sources is just the superposition of the fields from each source alone. Thus, knowing the electric field for a single-point charge at an arbitrary position immediately gives us the total field for any distribution of point charges.

In typical situations, one coulomb of total charge may be present requiring $6.25\times10^{18}$ elementary charges ($e\approx1.60\times10^{-19}$ coul). When dealing with such a large number of particles, the discrete nature of the charges is often not important and we can consider them as a continuum. We can then describe the charge distribution by its density. The same model is used in the classical treatment of matter. When we talk about mass we do not go to the molecular scale and count the number of molecules, but describe the material by its mass density that is the product of the local average number of molecules in a unit volume and the mass per molecule.

### 2-3-1 Line, Surface, and Volume Charge Distributions

We similarly speak of charge densities. Charges can distribute themselves on a line with line charge density $\lambda$ (coul/m), on a surface with surface charge density $\sigma$ (coul/m$^2$) or throughout a volume with volume charge density $\rho$ (coul/m$^3$).

Consider a distribution of free charge $dq$ of differential size within a macroscopic distribution of line, surface, or volume charge as shown in Figure 2-9. Then, the total charge $q$ within each distribution is obtained by summing up all the differential elements. This requires an integration over the line, surface, or volume occupied by the charge.

$$dq=\begin{cases}\lambda\,dl\\ \sigma\,dS\\ \rho\,dV\end{cases}\Rightarrow q=\begin{cases}\int_L\lambda\,dl & \text{(line charge)}\\ \int_S\sigma\,dS & \text{(surface charge)}\\ \int_V\rho\,dV & \text{(volume charge)}\end{cases}\qquad(1)$$

## EXAMPLE 2-2 CHARGE DISTRIBUTIONS

Find the total charge within each of the following distributions illustrated in Figure 2-10.

(a) Line charge $\lambda_0$ uniformly distributed in a circular hoop of radius $a$.

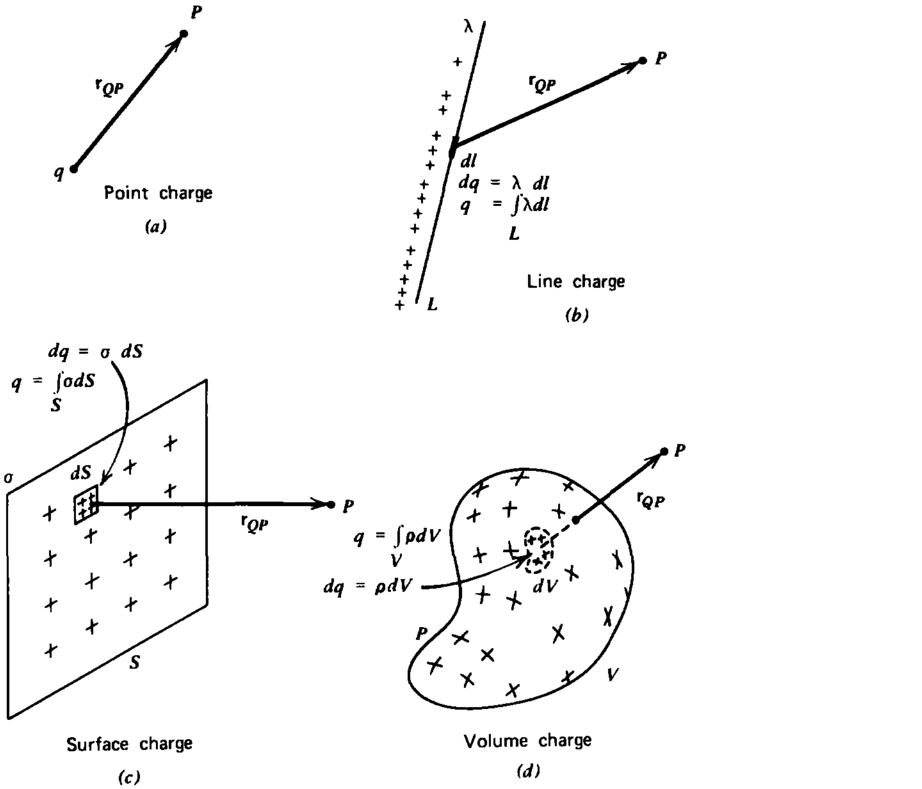



Figure 2-9 Charge distributions. (*a*) Point charge; (*b*) Line charge; (*c*) Surface charge; (*d*) Volume charge.

**SOLUTION**

---

$$q=\int_L \lambda\, dl=\int_0^{2\pi} \lambda_0 a\, d\phi=2\pi a\lambda_0$$

(b) Surface charge $\sigma_0$ uniformly distributed on a circular disk of radius $a$.

**SOLUTION**

---

$$q=\int_S \sigma\, dS=\int_{r=0}^{a}\int_{\phi=0}^{2\pi} \sigma_0 r\, dr\, d\phi=\pi a^2\sigma_0$$

(c) Volume charge $\rho_0$ uniformly distributed throughout a sphere of radius $R$.

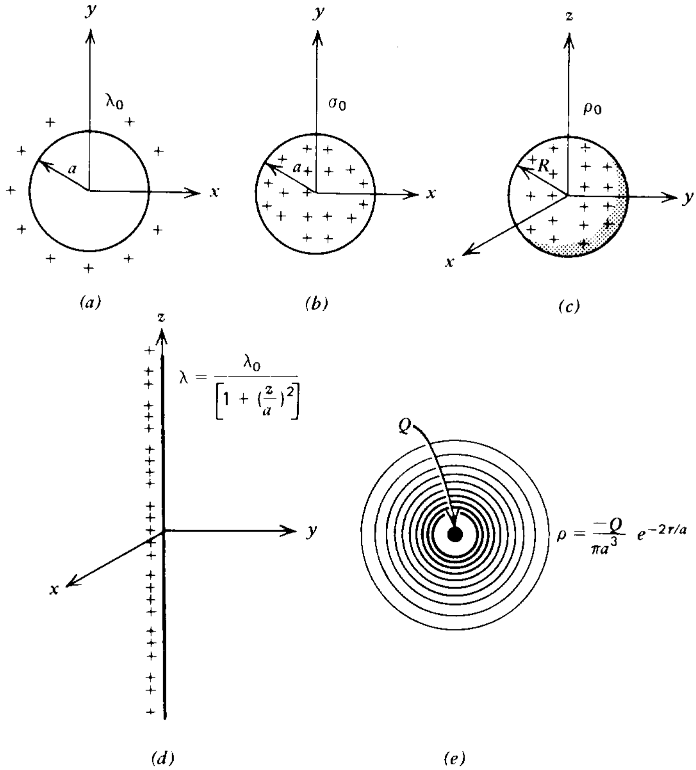



Figure 2-10 Charge distributions of Example 2-2. (*a*) Uniformly distributed line charge on a circular hoop. (*b*) Uniformly distributed surface charge on a circular disk. (*c*) Uniformly distributed volume charge throughout a sphere. (*d*) Nonuniform line charge distribution. (*e*) Smooth radially dependent volume charge distribution throughout all space, as a simple model of the electron cloud around the positively charged nucleus of the hydrogen atom.

**SOLUTION**

$$q = \int_V \rho\, dV = \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \rho_0 r^2 \sin\theta\, dr\, d\theta\, d\phi = \tfrac{4}{3}\pi R^3 \rho_0$$

(d) A line charge of infinite extent in the $z$ direction with charge density distribution

$$\lambda = \frac{\lambda_0}{[1 + (z/a)^2]}$$

**SOLUTION**

$$q = \int_L \lambda\, dl = \int_{-\infty}^{+\infty} \frac{\lambda_0\, dz}{[1 + (z/a)^2]} = \lambda_0 a \tan^{-1} \frac{z}{a} \bigg|_{-\infty}^{+\infty} = \lambda_0 \pi a$$

(e) The electron cloud around the positively charged nucleus $Q$ in the hydrogen atom is simply modeled as the spherically symmetric distribution

$$\rho(r) = -\frac{Q}{\pi a^3} e^{-2r/a}$$

where $a$ is called the Bohr radius.

**SOLUTION**

---

The total charge in the cloud is

$$\begin{aligned}
q &= \int_V \rho\, dV \\
&= -\int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{Q}{\pi a^3} e^{-2r/a} r^2 \sin\theta\, dr\, d\theta\, d\phi \\
&= -\int_{r=0}^{\infty} \frac{4Q}{a^3} e^{-2r/a} r^2\, dr \\
&= -\frac{4Q}{a^3}\left(-\frac{a}{2}\right) e^{-2r/a}\left[r^2 - \frac{a^2}{2}\left(-\frac{2r}{a} - 1\right)\right]\Bigg|_{r=0}^{\infty} \\
&= -Q
\end{aligned}$$

## 2-3-2 The Electric Field Due to a Charge Distribution

Each differential charge element $dq$ as a source at point $Q$ contributes to the electric field at a point $P$ as

$$d\mathbf{E} = \frac{dq}{4\pi\varepsilon_0 r_{QP}^2} \mathbf{i}_{QP} \tag{2}$$

where $r_{QP}$ is the distance between $Q$ and $P$ with $\mathbf{i}_{QP}$ the unit vector directed from $Q$ to $P$. To find the total electric field, it is necessary to sum up the contributions from each charge element. This is equivalent to integrating (2) over the entire charge distribution, remembering that both the distance $r_{QP}$ and direction $\mathbf{i}_{QP}$ vary for each differential element throughout the distribution

$$\mathbf{E} = \int_{\text{all } q} \frac{dq}{4\pi\varepsilon_0 r_{QP}^2} \mathbf{i}_{QP} \tag{3}$$

where (3) is a line integral for line charges ($dq = \lambda\, dl$), a surface integral for surface charges ($dq = \sigma\, dS$), a volume integral for a volume charge distribution ($dq = \rho\, dV$), or in general, a combination of all three.

If the total charge distribution is known, the electric field is obtained by performing the integration of (3). Some general rules and hints in using (3) are:

1. It is necessary to distinguish between the coordinates of the field points and the charge source points. Always integrate over the coordinates of the charges.
2. Equation (3) is a vector equation and so generally has three components requiring three integrations. Symmetry arguments can often be used to show that particular field components are zero.
3. The distance $r_{QP}$ is always positive. In taking square roots, always make sure that the positive square root is taken.
4. The solution to a particular problem can often be obtained by integrating the contributions from simpler differential size structures.

### 2-3-3 Field Due to an Infinitely Long Line Charge

An infinitely long uniformly distributed line charge $\lambda_0$ along the $z$ axis is shown in Figure 2-11. Consider the two symmetrically located charge elements $dq_1$ and $dq_2$ a distance $z$ above and below the point $P$, a radial distance r away. Each charge element alone contributes radial and $z$ components to the electric field. However, just as we found in Example 2-1$a$, the two charge elements together cause equal magnitude but oppositely directed $z$ field components that thus cancel leaving only additive radial components:

$$dE_r = \frac{\lambda_0\, dz}{4\pi\varepsilon_0(z^2+r^2)}\cos\theta = \frac{\lambda_0 r\, dz}{4\pi\varepsilon_0(z^2+r^2)^{3/2}} \tag{4}$$

To find the total electric field we integrate over the length of the line charge:

$$\begin{aligned} E_r &= \frac{\lambda_0 r}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{dz}{(z^2+r^2)^{3/2}} \\ &= \frac{\lambda_0 r}{4\pi\varepsilon_0}\frac{z}{r^2(z^2+r^2)^{1/2}}\bigg|_{z=-\infty}^{+\infty} \\ &= \frac{\lambda_0}{2\pi\varepsilon_0 r} \end{aligned} \tag{5}$$

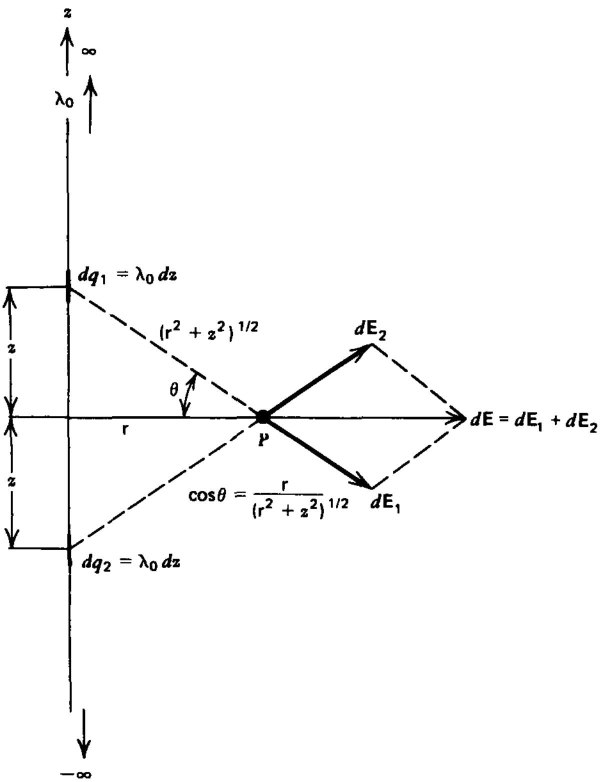
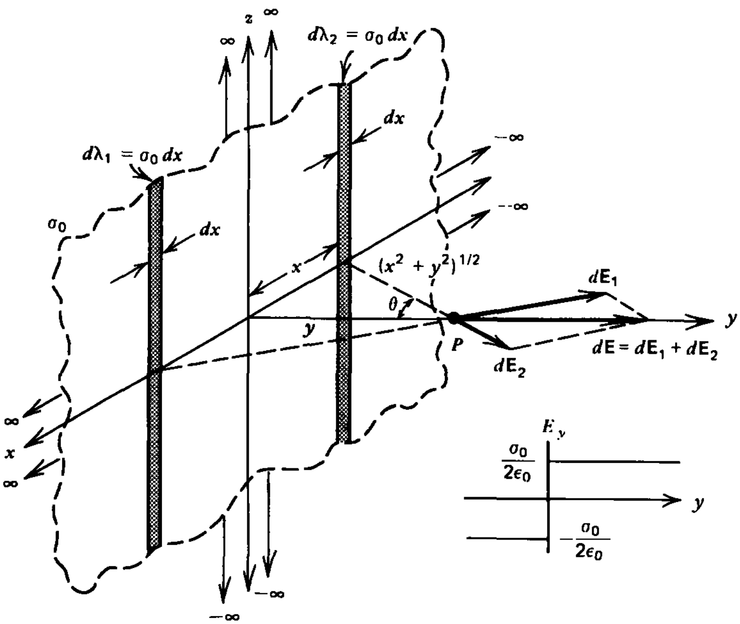



Figure 2-11 An infinitely long uniform distribution of line charge only has a radially directed electric field because the $z$ components of the electric field are canceled out by symmetrically located incremental charge elements as also shown in Figure 2-8*a*.

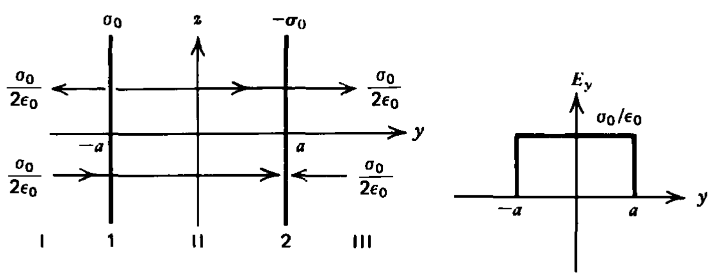
## 2-3-4 Field Due to Infinite Sheets of Surface Charge

### (a) Single Sheet

A surface charge sheet of infinite extent in the $y=0$ plane has a uniform surface charge density $\sigma_0$ as in Figure 2-12*a*. We break the sheet into many incremental line charges of thickness $dx$ with $d\lambda = \sigma_0\,dx$. We could equivalently break the surface into incremental horizontal line charges of thickness $dz$. Each incremental line charge alone has a radial field component as given by (5) that in Cartesian coordinates results in $x$ and $y$ components. Consider the line charge $d\lambda_1$, a distance $x$ to the left of $P$, and the symmetrically placed line charge $d\lambda_2$ the same distance $x$ to the right of $P$. The $x$ components of the resultant fields cancel while the $y$




(a)




(b)

Figure 2-12 (a) The electric field from a uniformly surface charged sheet of infinite extent is found by summing the contributions from each incremental line charge element. Symmetrically placed line charge elements have $x$ field components that cancel, but $y$ field components that add. (b) Two parallel but oppositely charged sheets of surface charge have fields that add in the region between the sheets but cancel outside. (c) The electric field from a volume charge distribution is obtained by summing the contributions from each incremental surface charge element.

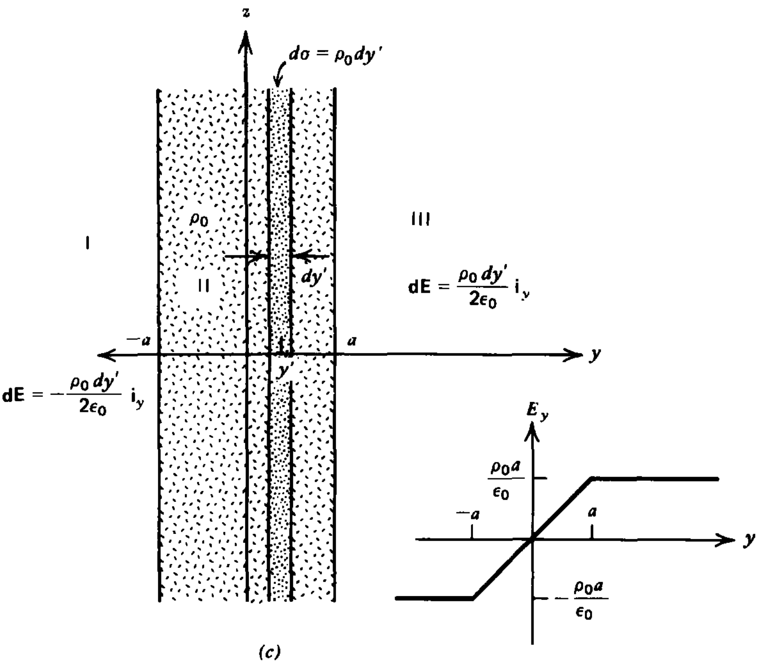



Fig. 2-12(c)

components add:

$$dE_y = \frac{\sigma_0\,dx}{2\pi\varepsilon_0(x^2+y^2)^{1/2}}\cos\theta = \frac{\sigma_0 y\,dx}{2\pi\varepsilon_0(x^2+y^2)} \tag{6}$$

The total field is then obtained by integration over all line charge elements:

$$\begin{aligned} E_y &= \frac{\sigma_0 y}{2\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{dx}{x^2+y^2} \\ &= \frac{\sigma_0 y}{2\pi\varepsilon_0}\frac{1}{y}\tan^{-1}\frac{x}{y}\bigg|_{x=-\infty}^{+\infty} \\ &= \begin{cases} \sigma_0/2\varepsilon_0, & y>0 \\ -\sigma_0/2\varepsilon_0, & y<0 \end{cases} \end{aligned} \tag{7}$$

where we realized that the inverse tangent term takes the sign of the ratio $x/y$ so that the field reverses direction on each side of the sheet. The field strength does not decrease with distance from the infinite sheet.

**(b) Parallel Sheets of Opposite Sign**

A capacitor is formed by two oppositely charged sheets of surface charge a distance $2a$ apart as shown in Figure 2-12$b$.

The fields due to each charged sheet alone are obtained from (7) as

$$\mathbf{E}_1=\begin{cases}\dfrac{\sigma_0}{2\varepsilon_0}\mathbf{i}_y, & y>-a\\[2ex] -\dfrac{\sigma_0}{2\varepsilon_0}\mathbf{i}_y, & y<-a\end{cases}\qquad \mathbf{E}_2=\begin{cases}-\dfrac{\sigma_0}{2\varepsilon_0}\mathbf{i}_y, & y>a\\[2ex] \dfrac{\sigma_0}{2\varepsilon_0}\mathbf{i}_y, & y<a\end{cases}\tag{8}$$

Thus, outside the sheets in regions I and III the fields cancel while they add in the enclosed region II. The nonzero field is confined to the region between the charged sheets and is independent of the spacing:

$$\mathbf{E}=\mathbf{E}_1+\mathbf{E}_2=\begin{cases}(\sigma_0/\varepsilon_0)\mathbf{i}_y, & |y|<a\\ 0 & |y|>a\end{cases}\tag{9}$$

**(c) Uniformly Charged Volume**

A uniformly charged volume with charge density $\rho_0$ of infinite extent in the $x$ and $z$ directions and of width $2a$ is centered about the $y$ axis, as shown in Figure 2-12$c$. We break the volume distribution into incremental sheets of surface charge of width $dy'$ with differential surface charge density $d\sigma=\rho_0\,dy'$. It is necessary to distinguish the position $y'$ of the differential sheet of surface charge from the field point $y$. The total electric field is the sum of all the fields due to each differentially charged sheet. The problem breaks up into three regions. In region I, where $y\le -a$, each surface charge element causes a field in the negative $y$ direction:

$$E_y=\int_{-a}^{a}-\frac{\rho_0}{2\varepsilon_0}\,dy'=-\frac{\rho_0 a}{\varepsilon_0},\quad y\le -a\tag{10}$$

Similarly, in region III, where $y\ge a$, each charged sheet gives rise to a field in the positive $y$ direction:

$$E_y=\int_{-a}^{a}\frac{\rho_0\,dy'}{2\varepsilon_0}=\frac{\rho_0 a}{\varepsilon_0},\quad y\ge a\tag{11}$$

For any position $y$ in region II, where $-a\le y\le a$, the charge to the right of $y$ gives rise to a negatively directed field while the charge to the left of $y$ causes a positively directed field:

$$E_y=\int_{-a}^{y}\frac{\rho_0\,dy'}{2\varepsilon_0}+\int_{y}^{a}(-)\frac{\rho_0}{2\varepsilon_0}\,dy'=\frac{\rho_0 y}{\varepsilon_0},\quad -a\le y\le a\tag{12}$$

The field is thus constant outside of the volume of charge and in opposite directions on either side being the same as for a surface charged sheet with the same total charge per unit area, $\sigma_0=\rho_0 2a$. At the boundaries $y=\pm a$, the field is continuous, changing linearly with position between the boundaries:

$$E_y=\begin{cases}-\dfrac{\rho_0 a}{\varepsilon_0}, & y\leq -a\\ \dfrac{\rho_0 y}{\varepsilon_0}, & -a\leq y\leq a\\ \dfrac{\rho_0 a}{\varepsilon_0}, & y\geq a\end{cases} \tag{13}$$

### 2-3-5 Superposition of Hoops of Line Charge

**(a) Single Hoop**

Using superposition, we can similarly build up solutions starting from a circular hoop of radius $a$ with uniform line charge density $\lambda_0$ centered about the origin in the $z=0$ plane as shown in Figure 2-13$a$. Along the $z$ axis, the distance to the hoop perimeter $(a^2+z^2)^{1/2}$ is the same for all incremental point charge elements $dq=\lambda_0 a\,d\phi$. Each charge element alone contributes $z$- and $r$-directed electric field components. However, along the $z$ axis symmetrically placed elements 180° apart have $z$ components that add but radial components that cancel. The $z$-directed electric field along the $z$ axis is then

$$E_z=\int_0^{2\pi}\frac{\lambda_0 a\,d\phi\cos\theta}{4\pi\varepsilon_0(z^2+a^2)}=\frac{\lambda_0 az}{2\varepsilon_0(a^2+z^2)^{3/2}} \tag{14}$$

The electric field is in the $-z$ direction along the $z$ axis below the hoop.

The total charge on the hoop is $q=2\pi a\lambda_0$ so that (14) can also be written as

$$E_z=\frac{qz}{4\pi\varepsilon_0(a^2+z^2)^{3/2}} \tag{15}$$

When we get far away from the hoop ($|z|\gg a$), the field approaches that of a point charge:

$$\lim_{|z|\gg a}E_z=\pm\frac{q}{4\pi\varepsilon_0 z^2}\begin{cases}z>0\\ z<0\end{cases} \tag{16}$$

**(b) Disk of Surface Charge**

The solution for a circular disk of uniformly distributed surface charge $\sigma_0$ is obtained by breaking the disk into incremental hoops of radius $r$ with line charge $d\lambda=\sigma_0\,dr$ as in

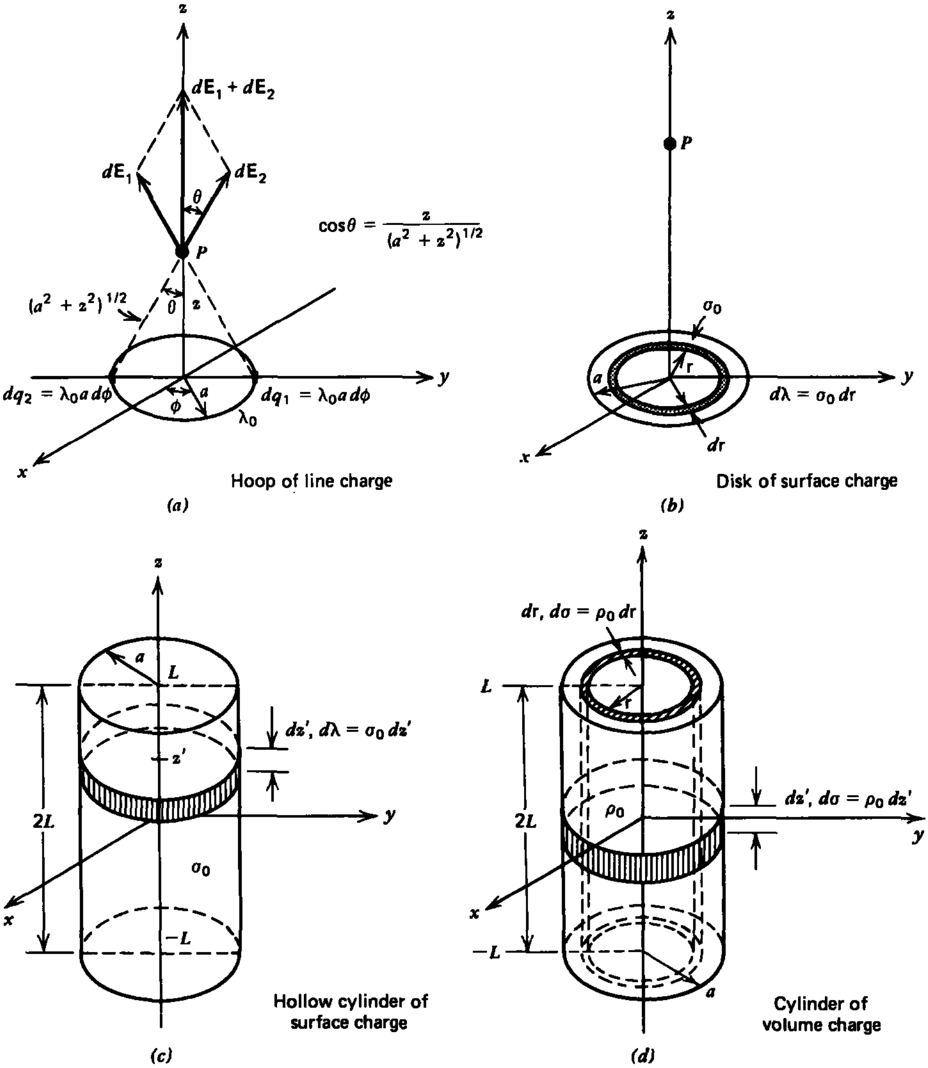



Figure 2-13 (*a*) The electric field along the symmetry $z$ axis of a uniformly distributed hoop of line charge is $z$ directed. (*b*) The axial field from a circular disk of surface charge is obtained by radially summing the contributions of incremental hoops of line charge. (*c*) The axial field from a hollow cylinder of surface charge is obtained by axially summing the contributions of incremental hoops of line charge. (*d*) The axial field from a cylinder of volume charge is found by summing the contributions of axial incremental disks or of radial hollow cylinders of surface charge.

Figure 2-13*b*. Then the incremental $z$-directed electric field along the $z$ axis due to a hoop of radius r is found from (14) as

$$dE_z = \frac{\sigma_0 \mathrm{r} z\, d\mathrm{r}}{2\varepsilon_0(\mathrm{r}^2 + z^2)^{3/2}} \tag{17}$$

where we replace $a$ with r, the radius of the incremental hoop. The total electric field is then

$$\begin{aligned}E_z&=\frac{\sigma_0 z}{2\varepsilon_0}\int_0^a\frac{\mathrm{r}\,d\mathrm{r}}{(\mathrm{r}^2+z^2)^{3/2}}\\&=-\frac{\sigma_0 z}{2\varepsilon_0(\mathrm{r}^2+z^2)^{1/2}}\bigg|_0^a\\&=-\frac{\sigma_0}{2\varepsilon_0}\left(\frac{z}{(a^2+z^2)^{1/2}}-\frac{z}{|z|}\right)\\&=\pm\frac{\sigma_0}{2\varepsilon_0}-\frac{\sigma_0 z}{2\varepsilon_0(a^2+z^2)^{1/2}}\begin{cases}z>0\\z<0\end{cases}\end{aligned}\tag{18}$$

where care was taken at the lower limit ($\mathrm{r}=0$), as the magnitude of the square root must always be used.

As the radius of the disk gets very large, this result approaches that of the uniform field due to an infinite sheet of surface charge:

$$\lim_{a\to\infty}E_z=\pm\frac{\sigma_0}{2\varepsilon_0}\begin{cases}z>0\\z<0\end{cases}\tag{19}$$

**(c) Hollow Cylinder of Surface Charge**

A hollow cylinder of length $2L$ and radius $a$ has its axis along the $z$ direction and is centered about the $z=0$ plane as in Figure 2-13$c$. Its outer surface at $\mathrm{r}=a$ has a uniform distribution of surface charge $\sigma_0$. It is necessary to distinguish between the coordinate of the field point $z$ and the source point at $z'$ $(-L\le z'\le L)$. The hollow cylinder is broken up into incremental hoops of line charge $d\lambda=\sigma_0\,dz'$. Then, the axial distance from the field point at $z$ to any incremental hoop of line charge is $(z-z')$. The contribution to the axial electric field at $z$ due to the incremental hoop at $z'$ is found from (14) as

$$dE_z=\frac{\sigma_0 a(z-z')\,dz'}{2\varepsilon_0[a^2+(z-z')^2]^{3/2}}\tag{20}$$

which when integrated over the length of the cylinder yields

$$\begin{aligned}E_z&=\frac{\sigma_0 a}{2\varepsilon_0}\int_{-L}^{+L}\frac{(z-z')\,dz'}{[a^2+(z-z')^2]^{3/2}}\\&=\frac{\sigma_0 a}{2\varepsilon_0}\frac{1}{[a^2+(z-z')^2]^{1/2}}\bigg|_{z'=-L}^{+L}\\&=\frac{\sigma_0 a}{2\varepsilon_0}\left(\frac{1}{[a^2+(z-L)^2]^{1/2}}-\frac{1}{[a^2+(z+L)^2]^{1/2}}\right)\end{aligned}\tag{21}$$

**(d) Cylinder of Volume Charge**

If this same cylinder is uniformly charged throughout the volume with charge density $\rho_0$, we break the volume into differential-size hollow cylinders of thickness $dr$ with incremental surface charge $d\sigma = \rho_0\, dr$ as in Figure 2-13$d$. Then, the $z$-directed electric field along the $z$ axis is obtained by integration of (21) replacing $a$ by r:

$$\begin{aligned} E_z &= \frac{\rho_0}{2\varepsilon_0}\int_0^a r\left(\frac{1}{[r^2+(z-L)^2]^{1/2}} - \frac{1}{[r^2+(z+L)^2]^{1/2}}\right) dr \\ &= \frac{\rho_0}{2\varepsilon_0}\{[r^2+(z-L)^2]^{1/2} - [r^2+(z+L)^2]^{1/2}\}\Big|_0^a \\ &= \frac{\rho_0}{2\varepsilon_0}\{[a^2+(z-L)^2]^{1/2} - |z-L| - [a^2+(z+L)^2]^{1/2} \\ &\quad + |z+L|\} \end{aligned} \tag{22}$$

where at the lower $r=0$ limit we always take the positive square root.

This problem could have equally well been solved by breaking the volume charge distribution into many differential-sized surface charged disks at position $z'$ $(-L \le z' \le L)$, thickness $dz'$, and effective surface charge density $d\sigma = \rho_0\, dz'$. The field is then obtained by integrating (18).

## 2-4 GAUSS'S LAW

We could continue to build up solutions for given charge distributions using the coulomb superposition integral of Section 2.3.2. However, for geometries with spatial symmetry, there is often a simpler way using some vector properties of the inverse square law dependence of the electric field.

### 2-4-1 Properties of the Vector Distance Between Two Points, $\mathbf{r}_{QP}$

**(a) $\mathbf{r}_{QP}$**

In Cartesian coordinates the vector distance $\mathbf{r}_{QP}$ between a source point at $Q$ and a field point at $P$ directed from $Q$ to $P$ as illustrated in Figure 2-14 is

$$\mathbf{r}_{QP} = (x - x_Q)\mathbf{i}_x + (y - y_Q)\mathbf{i}_y + (z - z_Q)\mathbf{i}_z \tag{1}$$

with magnitude

$$r_{QP} = [(x - x_Q)^2 + (y - y_Q)^2 + (z - z_Q)^2]^{1/2} \tag{2}$$

The unit vector in the direction of $\mathbf{r}_{QP}$ is

$$\mathbf{i}_{QP} = \frac{\mathbf{r}_{QP}}{r_{QP}} \tag{3}$$

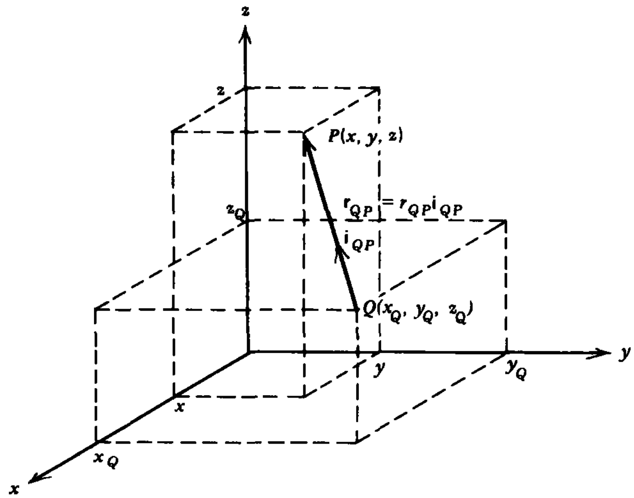



Figure 2-14 The vector distance $\mathbf{r}_{QP}$ between two points $Q$ and $P$.

### (b) Gradient of the Reciprocal Distance, $\nabla(1/r_{QP})$

Taking the gradient of the reciprocal of (2) yields

$$\begin{aligned}\nabla\left(\frac{1}{r_{QP}}\right) &= \mathbf{i}_x\frac{\partial}{\partial x}\left(\frac{1}{r_{QP}}\right)+\mathbf{i}_y\frac{\partial}{\partial y}\left(\frac{1}{r_{QP}}\right)+\mathbf{i}_z\frac{\partial}{\partial z}\left(\frac{1}{r_{QP}}\right)\\ &= -\frac{1}{r_{QP}^3}[(x-x_Q)\mathbf{i}_x+(y-y_Q)\mathbf{i}_y+(z-z_Q)\mathbf{i}_z]\\ &= -\mathbf{i}_{QP}/r_{QP}^2 \end{aligned}\tag{4}$$

which is the negative of the spatially dependent term that we integrate to find the electric field in Section 2.3.2.

### (c) Laplacian of the Reciprocal Distance

Another useful identity is obtained by taking the divergence of the gradient of the reciprocal distance. This operation is called the Laplacian of the reciprocal distance. Taking the divergence of (4) yields

$$\begin{aligned}\nabla^2\left(\frac{1}{r_{QP}}\right) &= \nabla\cdot\left[\nabla\left(\frac{1}{r_{QP}}\right)\right]\\ &= \nabla\cdot\left(\frac{-\mathbf{i}_{QP}}{r_{QP}^2}\right)\\ &= -\frac{\partial}{\partial x}\left(\frac{x-x_Q}{r_{QP}^3}\right)-\frac{\partial}{\partial y}\left(\frac{y-y_Q}{r_{QP}^3}\right)-\frac{\partial}{\partial z}\left(\frac{z-z_Q}{r_{QP}^3}\right)\\ &= -\frac{3}{r_{QP}^3}+\frac{3}{r_{QP}^5}[(x-x_Q)^2+(y-y_Q)^2+(z-z_Q)^2] \end{aligned}\tag{5}$$

Using (2) we see that (5) reduces to

$$\nabla^2\left(\frac{1}{r_{QP}}\right)=\begin{cases}0, & r_{QP}\neq 0\\ \text{undefined} & r_{QP}=0\end{cases} \tag{6}$$

Thus, the Laplacian of the inverse distance is zero for all nonzero distances but is undefined when the field point is coincident with the source point.

## 2-4-2 Gauss's Law In Integral Form

### (a) Point Charge Inside or Outside a Closed Volume

Now consider the two cases illustrated in Figure 2-15 where an arbitrarily shaped closed volume $V$ either surrounds a point charge $q$ or is near a point charge $q$ outside the surface $S$. For either case the electric field emanates radially from the point charge with the spatial inverse square law. We wish to calculate the flux of electric field through the surface $S$ surrounding the volume $V$:

$$\begin{aligned}\Phi&=\oint_S \mathbf{E}\cdot d\mathbf{S}\\ &=\oint_S \frac{q}{4\pi\varepsilon_0 r_{QP}^2}\mathbf{i}_{QP}\cdot d\mathbf{S}\\ &=\oint_S \frac{-q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r_{QP}}\right)\cdot d\mathbf{S}\end{aligned} \tag{7}$$

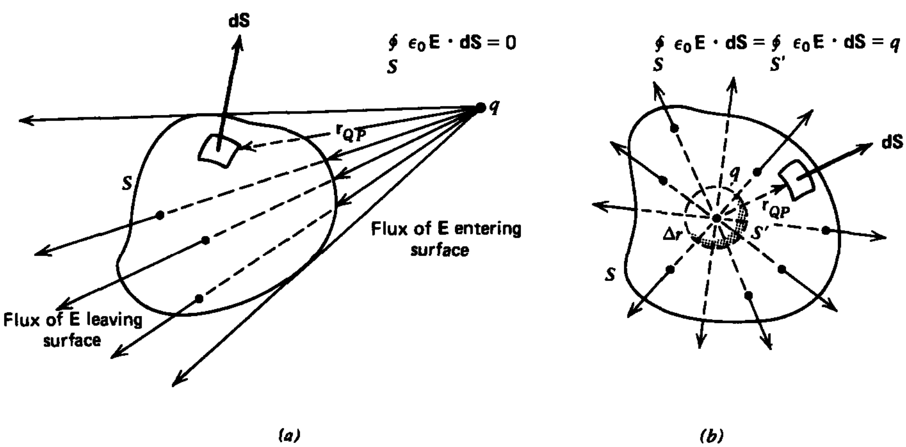



Figure 2-15 ($a$) The net flux of electric field through a closed surface $S$ due to an outside point charge is zero because as much flux enters the near side of the surface as leaves on the far side. ($b$) All the flux of electric field emanating from an enclosed point charge passes through the surface.

where we used (4). We can now use the divergence theorem to convert the surface integral to a volume integral:

$$\oint_S \mathbf{E}\cdot\mathbf{dS} = \frac{-q}{4\pi\varepsilon_0}\int_V \nabla\cdot\left[\nabla\left(\frac{1}{r_{QP}}\right)\right]dV \tag{8}$$

When the point charge $q$ is outside the surface every point in the volume has a nonzero value of $r_{QP}$. Then, using (6) with $r_{QP}\neq 0$, we see that the net flux of $\mathbf{E}$ through the surface is zero.

This result can be understood by examining Figure 2-15*a*. The electric field emanating from $q$ on that part of the surface $S$ nearest $q$ has its normal component oppositely directed to $\mathbf{dS}$ giving a negative contribution to the flux. However, on the opposite side of $S$ the electric field exits with its normal component in the same direction as $\mathbf{dS}$ giving a positive contribution to the flux. We have shown that these flux contributions are equal in magnitude but opposite in sign so that the net flux is zero.

As illustrated in Figure 2-15*b*, assuming $q$ to be positive, we see that when $S$ surrounds the charge the electric field points outwards with normal component in the direction of $\mathbf{dS}$ everywhere on $S$ so that the flux must be positive. If $q$ were negative, $\mathbf{E}$ and $\mathbf{dS}$ would be oppositely directed everywhere so that the flux is also negative. For either polarity with nonzero $q$, the flux cannot be zero. To evaluate the value of this flux we realize that (8) is zero everywhere except where $r_{QP}=0$ so that the surface $S$ in (8) can be shrunk down to a small spherical surface $S'$ of infinitesimal radius $\Delta r$ surrounding the point charge; the rest of the volume has $r_{QP}\neq 0$ so that $\nabla\cdot\nabla(1/r_{QP})=0$. On this incremental surface we know the electric field is purely radial in the same direction as $\mathbf{d}S'$ with the field due to a point charge:

$$\oint_S \mathbf{E}\cdot\mathbf{dS} = \oint_{S'} \mathbf{E}\cdot\mathbf{dS}' = \frac{q}{4\pi\varepsilon_0(\Delta r)^2}4\pi(\Delta r)^2 = \frac{q}{\varepsilon_0} \tag{9}$$

If we had many point charges within the surface $S$, each charge $q_i$ gives rise to a flux $q_i/\varepsilon_0$ so that Gauss's law states that the net flux of $\varepsilon_0\mathbf{E}$ through a closed surface is equal to the net charge enclosed by the surface:

$$\oint_S \varepsilon_0\mathbf{E}\cdot\mathbf{dS} = \sum_{\substack{\text{all } q_i\\ \text{inside } S}} q_i. \tag{10}$$

Any charges outside $S$ do not contribute to the flux.

### (b) Charge Distributions

For continuous charge distributions, the right-hand side of (10) includes the sum of all enclosed incremental charge elements so that the total charge enclosed may be a line, surface, and/or volume integral in addition to the sum of point charges:

$$\oint_S \varepsilon_0 \mathbf{E}\cdot d\mathbf{S} = \sum_{\substack{\text{all } q_i\\ \text{inside } S}} q_i + \int_{\substack{\text{all } q\\ \text{inside } S}} dq$$

$$= \left(\sum q_i + \int_L \lambda\, dl + \int_S \sigma\, dS + \int_V \rho\, dV\right)\Bigg|_{\substack{\text{all charge}\\ \text{inside } S}} \qquad (11)$$

Charges outside the volume give no contribution to the total flux through the enclosing surface.

Gauss's law of (11) can be used to great advantage in simplifying computations for those charges distributed with spatial symmetry. The trick is to find a surface $S$ that has sections tangent to the electric field so that the dot product is zero, or has surfaces perpendicular to the electric field and upon which the field is constant so that the dot product and integration become pure multiplications. If the appropriate surface is found, the surface integral becomes very simple to evaluate.

Coulomb's superposition integral derived in Section 2.3.2 is often used with symmetric charge distributions to determine if any field components are zero. Knowing the direction of the electric field often suggests the appropriate Gaussian surface upon which to integrate (11). This integration is usually much simpler than using Coulomb's law for each charge element.

### 2-4-3 Spherical Symmetry

**(a) Surface Charge**

A sphere of radius $R$ has a uniform distribution of surface charge $\sigma_0$ as in Figure 2-16$a$. Measure the angle $\theta$ from the line joining any point $P$ at radial distance $r$ to the sphere center. Then, the distance from $P$ to any surface charge element on the sphere is independent of the angle $\phi$. Each differential surface charge element at angle $\theta$ contributes field components in the radial and $\theta$ directions, but symmetrically located charge elements at $-\phi$ have equal field magnitude components that add radially but cancel in the $\theta$ direction.

Realizing from the symmetry that the electric field is purely radial and only depends on $r$ and not on $\theta$ or $\phi$, we draw Gaussian spheres of radius $r$ as in Figure 2-16$b$ both inside $(r<R)$ and outside $(r>R)$ the charged sphere. The Gaussian sphere inside encloses no charge while the outside sphere

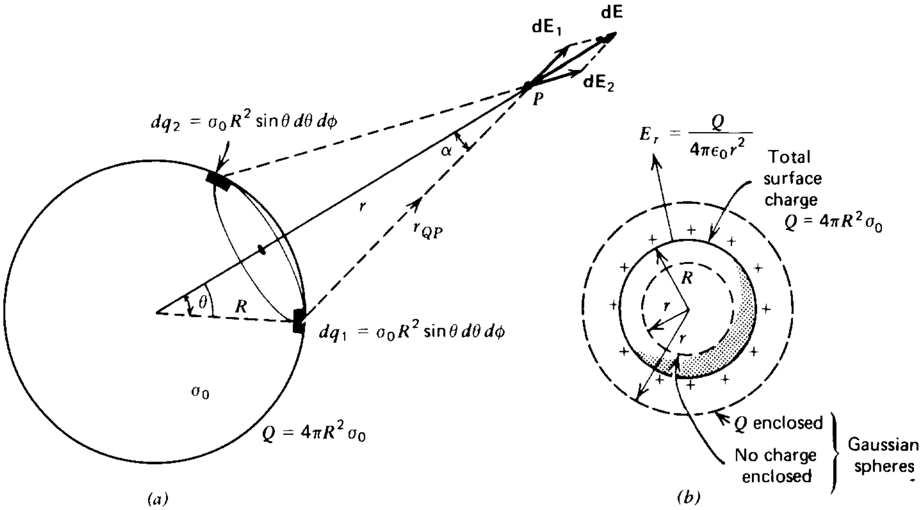



Figure 2-16 A sphere of radius $R$ with uniformly distributed surface charge $\sigma_0$. ($a$) Symmetrically located charge elements show that the electric field is purely radial. ($b$) Gauss's law, applied to concentric spherical surfaces inside ($r<R$) and outside ($r>R$) the charged sphere, easily shows that the electric field within the sphere is zero and outside is the same as if all the charge $Q=4\pi R^2\sigma_0$ were concentrated as a point charge at the origin.

encloses all the charge $Q=\sigma_0 4\pi R^2$:

$$\oint_S \varepsilon_0 \mathbf{E}\cdot\mathbf{dS}=\varepsilon_0 E_r 4\pi r^2=\begin{cases}\sigma_0 4\pi R^2 = Q, & r>R\\ 0, & r<R\end{cases} \tag{12}$$

so that the electric field is

$$E_r=\begin{cases}\dfrac{\sigma_0 R^2}{\varepsilon_0 r^2}=\dfrac{Q}{4\pi\varepsilon_0 r^2}, & r>R\\ 0, & r<R\end{cases} \tag{13}$$

The integration in (12) amounts to just a multiplication of $\varepsilon_0 E_r$ and the surface area of the Gaussian sphere because on the sphere the electric field is constant and in the same direction as the normal $\mathbf{i}_r$. The electric field outside the sphere is the same as if all the surface charge were concentrated as a point charge at the origin.

The zero field solution for $r<R$ is what really proved Coulomb's law. After all, Coulomb's small spheres were not really point charges and his measurements did have small sources of errors. Perhaps the electric force only varied inversely with distance by some power close to two, $r^{-2+\delta}$, where $\delta$ is very small. However, only the inverse square law

gives a zero electric field within a uniformly surface charged sphere. This zero field result is true for any closed conducting body of arbitrary shape charged on its surface with no enclosed charge. Extremely precise measurements were made inside such conducting surface charged bodies and the electric field was always found to be zero. Such a closed conducting body is used for shielding so that a zero field environment can be isolated and is often called a Faraday cage, after Faraday's measurements of actually climbing into a closed hollow conducting body charged on its surface to verify the zero field results.

To appreciate the ease of solution using Gauss's law, let us redo the problem using the superposition integral of Section 2.3.2. From Figure 2-16*a* the incremental radial component of electric field due to a differential charge element is

$$dE_r = \frac{\sigma_0 R^2 \sin\theta\, d\theta\, d\phi}{4\pi\varepsilon_0 r_{QP}^2}\cos\alpha \tag{14}$$

From the law of cosines the angles and distances are related as

$$\begin{aligned} r_{QP}^2 &= r^2 + R^2 - 2rR\cos\theta \\ R^2 &= r^2 + r_{QP}^2 - 2rr_{QP}\cos\alpha \end{aligned} \tag{15}$$

so that $\alpha$ is related to $\theta$ as

$$\cos\alpha = \frac{r - R\cos\theta}{[r^2+R^2-2rR\cos\theta]^{1/2}} \tag{16}$$

Then the superposition integral of Section 2.3.2 requires us to integrate (14) as

$$E_r = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\sigma_0 R^2 \sin\theta (r - R\cos\theta)\, d\theta\, d\phi}{4\pi\varepsilon_0 [r^2+R^2-2rR\cos\theta]^{3/2}} \tag{17}$$

After performing the easy integration over $\phi$ that yields the factor of $2\pi$, we introduce the change of variable:

$$\begin{aligned} u &= r^2 + R^2 - 2rR\cos\theta \\ du &= 2rR\sin\theta\, d\theta \end{aligned} \tag{18}$$

which allows us to rewrite the electric field integral as

$$\begin{aligned} E_r &= \int_{u=(r-R)^2}^{(r+R)^2} \frac{\sigma_0 R[u + r^2 - R^2]\, du}{8\varepsilon_0 r^2 u^{3/2}} \\ &= \frac{\sigma_0 R}{4\varepsilon_0 r^2}\left(u^{1/2} - \frac{(r^2-R^2)}{u^{1/2}}\right)\Bigg|_{(r-R)^2}^{(r+R)^2} \\ &= \frac{\sigma_0 R}{4\varepsilon_0 r^2}\left[(r+R) - |r-R| - (r^2-R^2)\left(\frac{1}{r+R} - \frac{1}{|r-R|}\right)\right] \end{aligned} \tag{19}$$

where we must be very careful to take the positive square root in evaluating the lower limit of the integral for $r<R$. Evaluating (19) for $r$ greater and less than $R$ gives us (13), but with a lot more effort.

**(b) Volume Charge Distribution**

If the sphere is uniformly charged throughout with density $\rho_0$, then the Gaussian surface in Figure 2-17$a$ for $r>R$ still encloses the total charge $Q=\frac{4}{3}\pi R^3\rho_0$. However, now the smaller Gaussian surface with $r<R$ encloses a fraction of the total charge:

$$\oint_S \varepsilon_0\mathbf{E}\cdot\mathbf{dS}=\varepsilon_0E_r4\pi r^2=\begin{cases}\rho_0\frac{4}{3}\pi r^3=Q(r/R)^3, & r<R\\ \\ \rho_0\frac{4}{3}\pi R^3=Q, & r>R\end{cases} \tag{20}$$

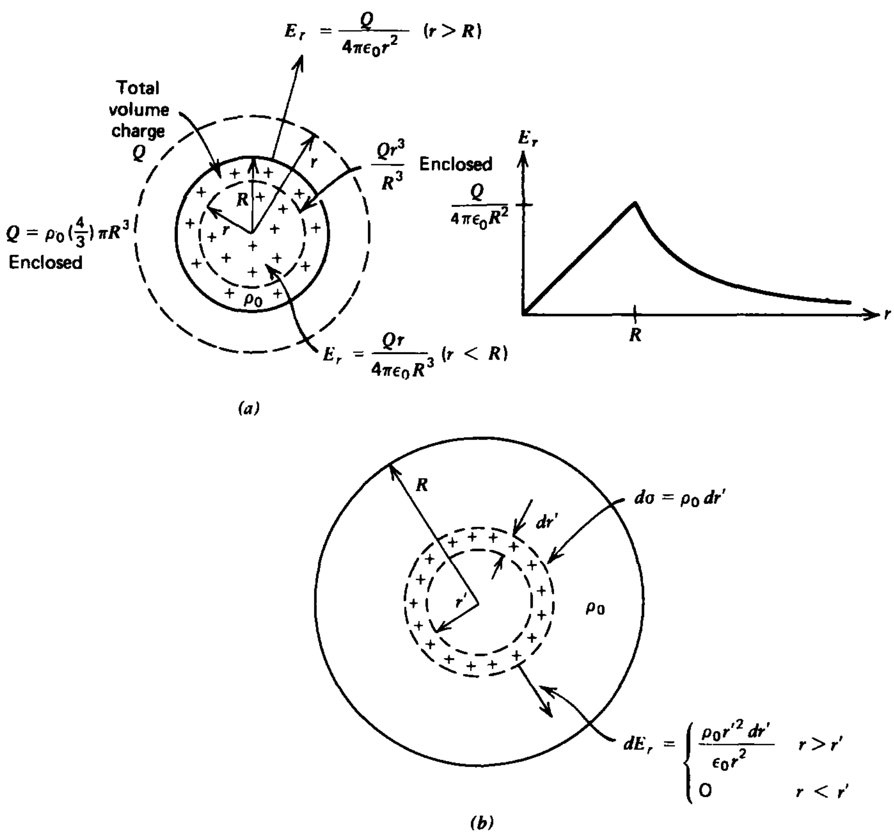



**Figure 2-17** ($a$) Gaussian spheres for a uniformly charged sphere show that the electric field outside the sphere is again the same as if all the charge $Q=\frac{4}{3}\pi R^3\rho_0$ were concentrated as a point charge at $r=0$. ($b$) The solution is also obtained by summing the contributions from incremental spherical shells of surface charge.

so that the electric field is

$$E_r = \begin{cases} \dfrac{\rho_0 r}{3\varepsilon_0} = \dfrac{Qr}{4\pi\varepsilon_0 R^3}, & r<R \\ \dfrac{\rho_0 R^3}{3\varepsilon_0 r^2} = \dfrac{Q}{4\pi\varepsilon_0 r^2}, & r>R \end{cases} \tag{21}$$

This result could also have been obtained using the results of (13) by breaking the spherical volume into incremental shells of radius $r'$, thickness $dr'$, carrying differential surface charge $d\sigma = \rho_0\, dr'$ as in Figure 2-17*b*. Then the contribution to the field is zero inside each shell but nonzero outside:

$$dE_r = \begin{cases} 0, & r<r' \\ \dfrac{\rho_0 r'^2\, dr'}{\varepsilon_0 r^2}, & r>r' \end{cases} \tag{22}$$

The total field outside the sphere is due to all the differential shells, while the field inside is due only to the enclosed shells:

$$E_r = \begin{cases} \displaystyle\int_0^r \frac{r'^2 \rho_0\, dr'}{\varepsilon_0 r^2} = \frac{\rho_0 r}{3\varepsilon_0} = \frac{Qr}{4\pi\varepsilon_0 R^3}, & r<R \\ \displaystyle\int_0^R \frac{r'^2 \rho_0\, dr'}{\varepsilon_0 r^2} = \frac{\rho_0 R^3}{3\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0 r^2}, & r>R \end{cases} \tag{23}$$

which agrees with (21).

## 2-4-4 Cylindrical Symmetry

### (a) Hollow Cylinder of Surface Charge

An infinitely long cylinder of radius $a$ has a uniform distribution of surface charge $\sigma_0$, as shown in Figure 2-18*a*. The angle $\phi$ is measured from the line joining the field point $P$ to the center of the cylinder. Each incremental line charge element $d\lambda = \sigma_0 a\, d\phi$ contributes to the electric field at $P$ as given by the solution for an infinitely long line charge in Section 2.3.3. However, the symmetrically located element at $-\phi$ gives rise to equal magnitude field components that add radially as measured from the cylinder center but cancel in the $\phi$ direction.

Because of the symmetry, the electric field is purely radial so that we use Gauss's law with a concentric cylinder of radius r and height $L$, as in Figure 2-18*b* where $L$ is arbitrary. There is no contribution to Gauss's law from the upper and lower surfaces because the electric field is purely tangential. Along the cylindrical wall at radius r, the electric field is constant and

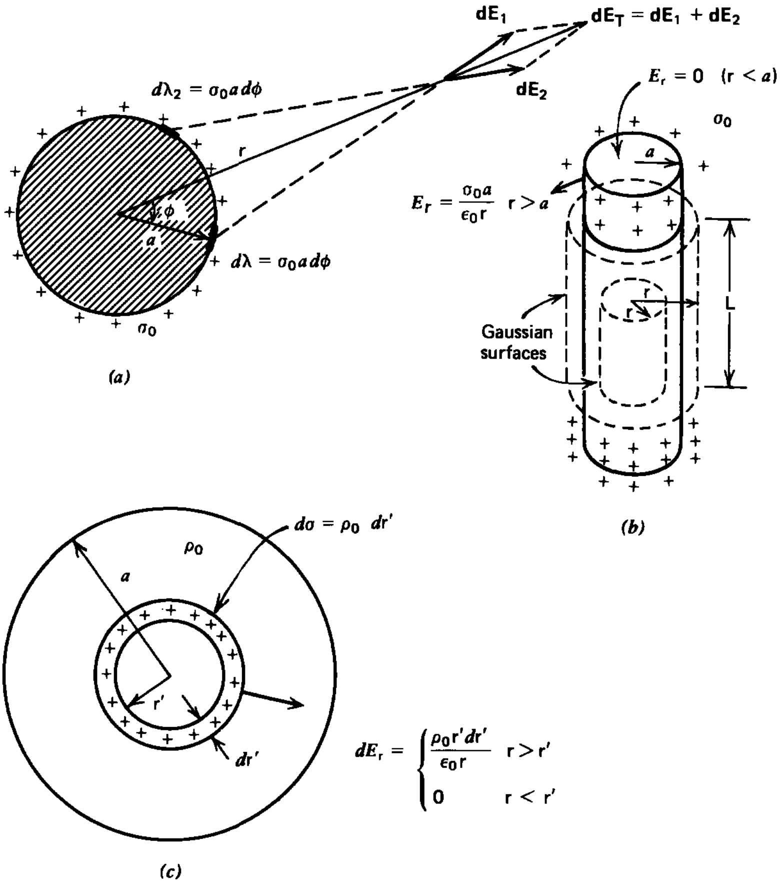



Figure 2-18 (*a*) Symmetrically located line charge elements on a cylinder with uniformly distributed surface charge show that the electric field is purely radial. (*b*) Gauss's law applied to concentric cylindrical surfaces shows that the field inside the surface charged cylinder is zero while outside it is the same as if all the charge per unit length $\sigma_0\, 2\pi a$ were concentrated at the origin as a line charge. (*c*) In addition to using the surfaces of (*b*) with Gauss's law for a cylinder of volume charge, we can also sum the contributions from incremental hollow cylinders of surface charge.

purely normal so that Gauss's law simply yields

$$\oint_S \varepsilon_0 \mathbf{E} \cdot \mathbf{dS} = \varepsilon_0 E_r 2\pi r L = \begin{cases} \sigma_0 2\pi a L, & r > a \\ 0 & r < a \end{cases} \tag{24}$$

where for $r < a$ no charge is enclosed, while for $r > a$ all the charge within a height $L$ is enclosed. The electric field outside the cylinder is then the same as if all the charge per unit length $\lambda = \sigma_0 2\pi a$ were concentrated along the axis of the cylinder:

$$E_r = \begin{cases} \dfrac{\sigma_0 a}{\varepsilon_0 r} = \dfrac{\lambda}{2\pi\varepsilon_0 r} & r > a \\ 0, & r < a \end{cases} \tag{25}$$

Note in (24) that the arbitrary height $L$ canceled out.

**(b) Cylinder of Volume Charge**

If the cylinder is uniformly charged with density $\rho_0$, both Gaussian surfaces in Figure 2-18*b* enclose charge

$$\oint_S \varepsilon_0 \mathbf{E} \cdot \mathbf{dS} = \varepsilon_0 E_r 2\pi r L = \begin{cases} \rho_0 \pi a^2 L, & r > a \\ \rho_0 \pi r^2 L, & r < a \end{cases} \tag{26}$$

so that the electric field is

$$E_r = \begin{cases} \dfrac{\rho_0 a^2}{2\varepsilon_0 r} = \dfrac{\lambda}{2\pi\varepsilon_0 r}, & r > a \\ \dfrac{\rho_0 r}{2\varepsilon_0} = \dfrac{\lambda r}{2\pi\varepsilon_0 a^2} & r < a \end{cases} \tag{27}$$

where $\lambda = \rho_0 \pi a^2$ is the total charge per unit length on the cylinder.

Of course, this result could also have been obtained by integrating (25) for all differential cylindrical shells of radius $r'$ with thickness $dr'$ carrying incremental surface charge $d\sigma = \rho_0\, dr'$, as in Figure 2-18*c*.

$$E_r = \begin{cases} \displaystyle\int_0^a \frac{\rho_0 r'}{\varepsilon_0 r}\, dr' = \frac{\rho_0 a^2}{2\varepsilon_0 r} = \frac{\lambda}{2\pi\varepsilon_0 r}, & r > a \\ \displaystyle\int_0^r \frac{\rho_0 r'}{\varepsilon_0 r}\, dr' = \frac{\rho_0 r}{2\varepsilon_0} = \frac{\lambda r}{2\pi\varepsilon_0 a^2}, & r < a \end{cases} \tag{28}$$

## 2-4-5 Gauss's Law and the Divergence Theorem

If a volume distribution of charge $\rho$ is completely surrounded by a closed Gaussian surface $S$, Gauss's law of (11) is

$$\oint_S \varepsilon_0 \mathbf{E} \cdot \mathbf{dS} = \int_V \rho\, dV \tag{29}$$

The left-hand side of (29) can be changed to a volume integral using the divergence theorem:

$$\oint_S \varepsilon_0 \mathbf{E} \cdot \mathbf{dS} = \int_V \nabla \cdot (\varepsilon_0 \mathbf{E})\, dV = \int_V \rho\, dV \tag{30}$$

Since (30) must hold for any volume, the volume integrands in (30) must be equal, yielding the point form of Gauss's law:

$$\nabla \cdot (\varepsilon_0 \mathbf{E}) = \rho \tag{31}$$

Since the permittivity of free space $\varepsilon_0$ is a constant, it can freely move outside the divergence operator.

### 2-4-6 Electric Field Discontinuity Across a Sheet of Surface Charge

In Section 2.3.4*a* we found that the electric field changes direction discontinuously on either side of a straight sheet of surface charge. We can be more general by applying the surface integral form of Gauss's law in (30) to the differential-sized pill-box surface shown in Figure 2-19 surrounding a small area $dS$ of surface charge:

$$\oint_S \varepsilon_0 \mathbf{E} \cdot \mathbf{dS} = \int_S \sigma\, dS \Rightarrow \varepsilon_0 (E_{2n} - E_{1n})\, dS = \sigma\, dS \tag{32}$$

where $E_{2n}$ and $E_{1n}$ are the perpendicular components of electric field on each side of the interface. Only the upper and lower surfaces of the pill-box contribute in (32) because the surface charge is assumed to have zero thickness so that the short cylindrical surface has zero area. We thus see that the surface charge density is proportional to the discontinuity in the normal component of electric field across the sheet:

$$\varepsilon_0 (E_{2n} - E_{1n}) = \sigma \Rightarrow \mathbf{n} \cdot \varepsilon_0 (\mathbf{E}_2 - \mathbf{E}_1) = \sigma \tag{33}$$

where $\mathbf{n}$ is perpendicular to the interface directed from region 1 to region 2.

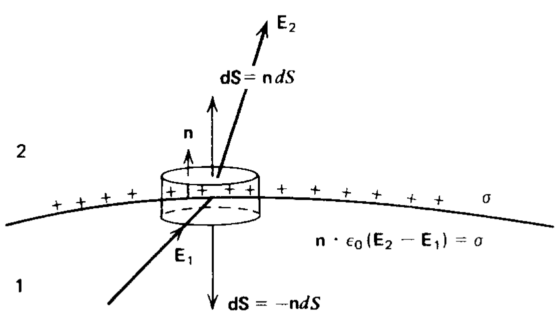



Figure 2-19 Gauss's law applied to a differential sized pill-box surface enclosing some surface charge shows that the normal component of $\varepsilon_0\mathbf{E}$ is discontinuous in the surface charge density.

## 2-5 THE ELECTRIC POTENTIAL

If we have two charges of opposite sign, work must be done to separate them in opposition to the attractive coulomb force. This work can be regained if the charges are allowed to come together. Similarly, if the charges have the same sign, work must be done to push them together; this work can be regained if the charges are allowed to separate. A charge gains energy when moved in a direction opposite to a force. This is called potential energy because the amount of energy depends on the position of the charge in a force field.

### 2-5-1 Work Required to Move a Point Charge

The work $W$ required to move a test charge $q_t$ along any path from the radial distance $r_a$ to the distance $r_b$ with a force that just overcomes the coulombic force from a point charge $q$, as shown in Figure 2-20, is

$$\begin{aligned} W &= -\int_{r_a}^{r_b} \mathbf{F}\cdot d\mathbf{l} \\ &= -\frac{qq_t}{4\pi\varepsilon_0}\int_{r_a}^{r_b} \frac{\mathbf{i}_r \cdot d\mathbf{l}}{r^2} \end{aligned} \tag{1}$$

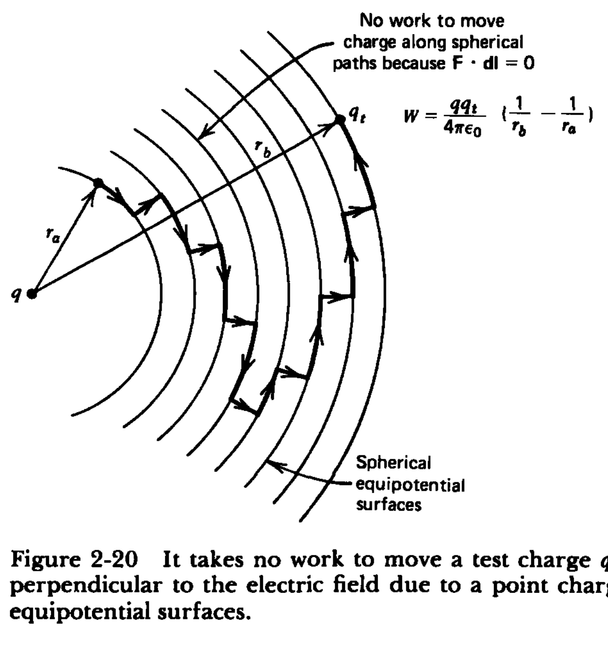



Figure 2-20 It takes no work to move a test charge $q_t$ along the spherical surfaces perpendicular to the electric field due to a point charge $q$. Such surfaces are called equipotential surfaces.

The minus sign in front of the integral is necessary because the quantity $W$ represents the work we must exert on the test charge in opposition to the coulombic force between charges. The dot product in (1) tells us that it takes no work to move the test charge perpendicular to the electric field, which in this case is along spheres of constant radius. Such surfaces are called equipotential surfaces. Nonzero work is necessary to move $q$ to a different radius for which $\mathbf{dl} = dr\, \mathbf{i}_r$. Then, the work of (1) depends only on the starting and ending positions ($r_a$ and $r_b$) of the path and not on the shape of the path itself:

$$\begin{aligned} W &= -\frac{qq_t}{4\pi\varepsilon_0}\int_{r_a}^{r_b}\frac{dr}{r^2} \\ &= \frac{qq_t}{4\pi\varepsilon_0}\left(\frac{1}{r_b}-\frac{1}{r_a}\right) \end{aligned} \tag{2}$$

We can convince ourselves that the sign is correct by examining the case when $r_b$ is bigger than $r_a$ and the charges $q$ and $q_t$ are of opposite sign and so attract each other. To separate the charges further requires us to do work on $q_t$ so that $W$ is positive in (2). If $q$ and $q_t$ are the same sign, the repulsive coulomb force would tend to separate the charges further and perform work on $q_t$. For force equilibrium, we would have to exert a force opposite to the direction of motion so that $W$ is negative.

If the path is closed so that we begin and end at the same point with $r_a = r_b$, the net work required for the motion is zero. If the charges are of the opposite sign, it requires positive work to separate them, but on the return, equal but opposite work is performed on us as the charges attract each other.

If there was a distribution of charges with net field $\mathbf{E}$, the work in moving the test charge against the total field $\mathbf{E}$ is just the sum of the works necessary to move the test charge against the field from each charge alone. Over a closed path this work remains zero:

$$W = \oint_L -q_t\mathbf{E}\cdot\mathbf{dl} = 0 \Rightarrow \oint_L \mathbf{E}\cdot\mathbf{dl} = 0 \tag{3}$$

which requires that the line integral of the electric field around the closed path also be zero.

### 2-5-2 The Electric Field and Stokes' Theorem

Using Stokes' theorem of Section 1.5.3, we can convert the line integral of the electric field to a surface integral of the curl of the electric field:

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \int_S (\nabla\times\mathbf{E})\cdot d\mathbf{S} \tag{4}$$

From Section 1.3.3, we remember that the gradient of a scalar function also has the property that its line integral around a closed path is zero. This means that the electric field can be determined from the gradient of a scalar function $V$ called the potential having units of volts [kg-m$^2$-s$^{-3}$-A$^{-1}$]:

$$\mathbf{E} = -\nabla V \tag{5}$$

The minus sign is introduced by convention so that the electric field points in the direction of decreasing potential. From the properties of the gradient discussed in Section 1.3.1 we see that the electric field is always perpendicular to surfaces of constant potential.

By applying the right-hand side of (4) to an area of differential size or by simply taking the curl of (5) and using the vector identity of Section 1.5.4*a* that the curl of the gradient is zero, we reach the conclusion that the electric field has zero curl:

$$\nabla\times\mathbf{E} = 0 \tag{6}$$

## 2-5-3 The Potential and the Electric Field

The potential difference between the two points at $r_a$ and $r_b$ is the work per unit charge necessary to move from $r_a$ to $r_b$:

$$\begin{aligned} V(r_b) - V(r_a) &= \frac{W}{q_t} \\ &= -\int_{r_a}^{r_b} \mathbf{E}\cdot d\mathbf{l} = +\int_{r_b}^{r_a} \mathbf{E}\cdot d\mathbf{l} \end{aligned} \tag{7}$$

Note that (3), (6), and (7) are the fields version of Kirchoff's circuit voltage law that the algebraic sum of voltage drops around a closed loop is zero.

The advantage to introducing the potential is that it is a scalar from which the electric field can be easily calculated. The electric field must be specified by its three components, while if the single potential function $V$ is known, taking its negative gradient immediately yields the three field components. This is often a simpler task than solving for each field component separately. Note in (5) that adding a constant to the potential does not change the electric field, so the potential is only uniquely defined to within a constant. It is necessary to specify a reference zero potential that is often taken at infinity. In actual practice zero potential is often assigned to the earth's surface so that common usage calls the reference point "ground."

The potential due to a single point charge $q$ is

$$V(r_b)-V(r_a)=-\int_{r_a}^{r_b}\frac{q\,dr}{4\pi\varepsilon_0 r^2}=\frac{q}{4\pi\varepsilon_0 r}\Big|_{r_a}^{r_b}$$

$$=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_b}-\frac{1}{r_a}\right) \tag{8}$$

If we pick our reference zero potential at $r_a=\infty$, $V(r_a)=0$ so that $r_b=r$ is just the radial distance from the point charge. The scalar potential $V$ is then interpreted as the work per unit charge necessary to bring a charge from infinity to some distance $r$ from the point charge $q$:

$$V(r)=\frac{q}{4\pi\varepsilon_0 r} \tag{9}$$

The net potential from many point charges is obtained by the sum of the potentials from each charge alone. If there is a continuous distribution of charge, the summation becomes an integration over all the differential charge elements $dq$:

$$V=\int_{\text{all } q}\frac{dq}{4\pi\varepsilon_0 r_{QP}} \tag{10}$$

where the integration is a line integral for line charges, a surface integral for surface charges, and a volume integral for volume charges.

The electric field formula of Section 2.3.2 obtained by superposition of coulomb's law is easily re-obtained by taking the negative gradient of (10), recognizing that derivatives are to be taken with respect to field positions $(x, y, z)$ while the integration is over source positions $(x_Q, y_Q, z_Q)$. The del operator can thus be brought inside the integral and operates only on the quantity $r_{QP}$:

$$\mathbf{E}=-\nabla V=-\int_{\text{all } q}\frac{dq}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r_{QP}}\right)$$

$$=\int_{\text{all } q}\frac{dq}{4\pi\varepsilon_0 r_{QP}^2}\mathbf{i}_{QP} \tag{11}$$

where we use the results of Section 2.4.1*b* for the gradient of the reciprocal distance.

### 2-5-4 Finite Length Line Charge

To demonstrate the usefulness of the potential function, consider the uniform distribution of line charge $\lambda_0$ of finite length $2L$ centered on the $z$ axis in Figure 2-21. Distinguishing between the position of the charge element $dq = \lambda_0\, dz'$ at $z'$ and the field point at coordinate $z$, the distance between source and field point is

$$r_{QP} = [r^2 + (z - z')^2]^{1/2} \tag{12}$$

Substituting into (10) yields

$$\begin{aligned} V &= \int_{-L}^{L} \frac{\lambda_0\, dz'}{4\pi\varepsilon_0[r^2 + (z - z')^2]^{1/2}} \\ &= -\frac{\lambda_0}{4\pi\varepsilon_0} \ln\left(\frac{z - L + [r^2 + (z - L)^2]^{1/2}}{z + L + [r^2 + (z + L)^2]^{1/2}}\right) \\ &= -\frac{\lambda_0}{4\pi\varepsilon_0}\left(\sinh^{-1}\frac{z - L}{r} - \sinh^{-1}\frac{z + L}{r}\right) \end{aligned} \tag{13}$$

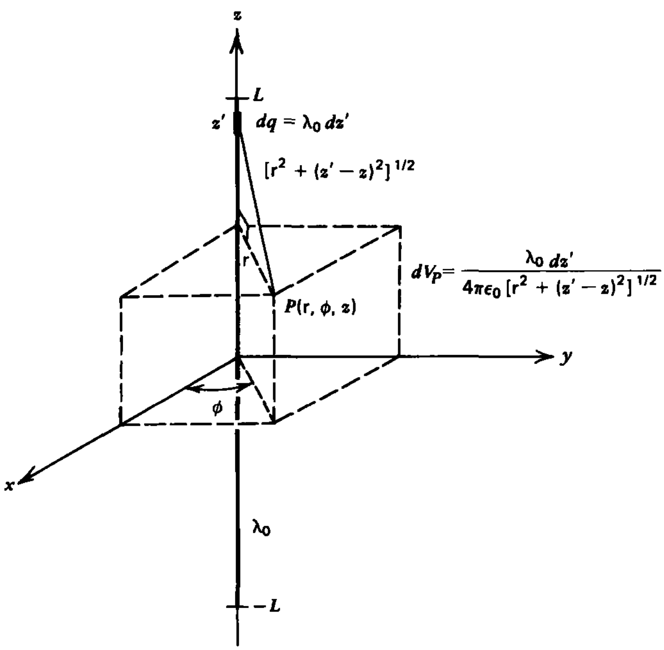



Figure 2-21 The potential from a finite length of line charge is obtained by adding the potentials due to each incremental line charge element.

The field components are obtained from (13) by taking the negative gradient of the potential:

$$E_z = -\frac{\partial V}{\partial z} = \frac{\lambda_0}{4\pi\varepsilon_0}\left(\frac{1}{[r^2+(z-L)^2]^{1/2}} - \frac{1}{[r^2+(z+L)^2]^{1/2}}\right)$$

$$E_r = -\frac{\partial V}{\partial r} = \frac{\lambda_0 r}{4\pi\varepsilon_0}\left(\frac{1}{[r^2+(z-L)^2]^{1/2}[z-L+[r^2+(z-L)^2]^{1/2}]}\right.$$

$$\left. - \frac{1}{[r^2+(z+L)^2]^{1/2}[z+L+[r^2+(z+L)^2]^{1/2}]}\right)$$

$$= -\frac{\lambda_0}{4\pi\varepsilon_0 r}\left(\frac{z-L}{[r^2+(z-L)^2]^{1/2}} - \frac{z+L}{[r^2+(z+L)^2]^{1/2}}\right) \tag{14}$$

As $L$ becomes large, the field and potential approaches that of an infinitely long line charge:

$$\lim_{L\to\infty}\begin{cases} E_z = 0 \\ E_r = \dfrac{\lambda_0}{2\pi\varepsilon_0 r} \\ V = -\dfrac{\lambda_0}{2\pi\varepsilon_0}(\ln r - \ln 2L) \end{cases} \tag{15}$$

The potential has a constant term that becomes infinite when $L$ is infinite. This is because the zero potential reference of (10) is at infinity, but when the line charge is infinitely long the charge at infinity is nonzero. However, this infinite constant is of no concern because it offers no contribution to the electric field.

Far from the line charge the potential of (13) approaches that of a point charge $2\lambda_0 L$:

$$\lim_{r^2=r^2+z^2 \gg L^2} V = \frac{\lambda_0(2L)}{4\pi\varepsilon_0 r} \tag{16}$$

Other interesting limits of (14) are

$$\lim_{z=0}\begin{cases} E_z = 0 \\ E_r = \dfrac{\lambda_0 L}{2\pi\varepsilon_0 r(r^2+L^2)^{1/2}} \end{cases}$$

$$\lim_{r=0}\begin{cases} E_z = \dfrac{\lambda_0}{4\pi\varepsilon_0}\left(\dfrac{1}{|z-L|} - \dfrac{1}{|z+L|}\right) = \begin{cases} \dfrac{\pm\lambda_0 L}{2\pi\varepsilon_0(z^2-L^2)}, & z>L \\ & z<-L \\ \dfrac{\lambda_0 z}{2\pi\varepsilon_0(L^2-z^2)}, & -L\le z\le L \end{cases} \\ E_r = 0 \end{cases} \tag{17}$$

## 2-5-5 Charged Spheres

### (a) Surface Charge

A sphere of radius $R$ supports a uniform distribution of surface charge $\sigma_0$ with total charge $Q=\sigma_0 4\pi R^2$, as shown in Figure 2-22*a*. Each incremental surface charge element contributes to the potential as

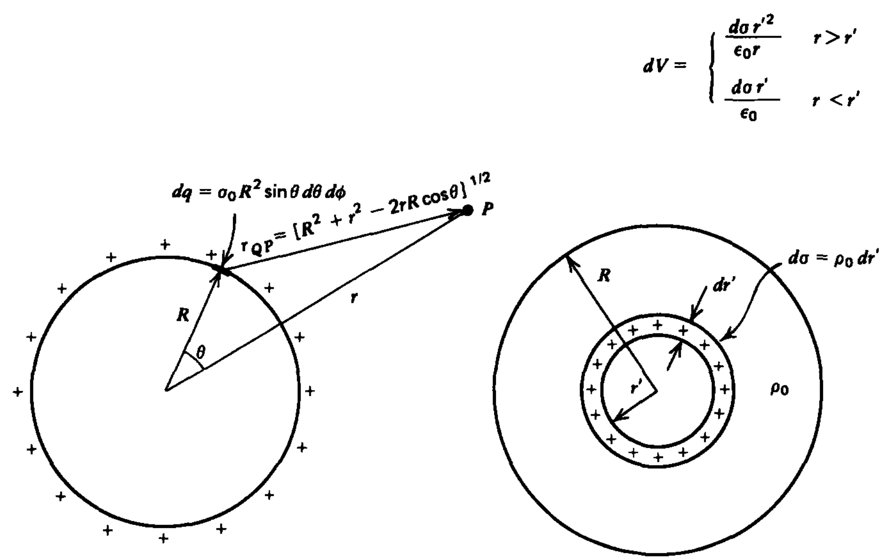
$$dV=\frac{\sigma_0 R^2 \sin\theta\, d\theta\, d\phi}{4\pi\varepsilon_0 r_{QP}} \tag{18}$$

where from the law of cosines

$$r_{QP}^2=R^2+r^2-2rR\cos\theta \tag{19}$$

so that the differential change in $r_{QP}$ about the sphere is

$$2r_{QP}\,dr_{QP}=2rR\sin\theta\, d\theta \tag{20}$$




(*a*) (*b*)

Figure 2-22 (*a*) A sphere of radius $R$ supports a uniform distribution of surface charge $\sigma_0$. (*b*) The potential due to a uniformly volume charged sphere is found by summing the potentials due to differential sized shells.

Therefore, the total potential due to the whole charged sphere is

$$
\begin{aligned}
V &= \int_{r_{QP}=|r-R|}^{r+R} \int_{\phi=0}^{2\pi} \frac{\sigma_0 R}{4\pi\varepsilon_0 r}\, dr_{QP}\, d\phi \\
&= \frac{\sigma_0 R}{2\varepsilon_0 r} r_{QP}\Big|_{|r-R|}^{r+R} \\
&= \begin{cases} \dfrac{\sigma_0 R^2}{\varepsilon_0 r} = \dfrac{Q}{4\pi\varepsilon_0 r}, & r > R \\[2ex] \dfrac{\sigma_0 R}{\varepsilon_0} = \dfrac{Q}{4\pi\varepsilon_0 R}, & r < R \end{cases}
\end{aligned} \tag{21}
$$

Then, as found in Section 2.4.3*a* the electric field is

$$
E_r = -\frac{\partial V}{\partial r} = \begin{cases} \dfrac{\sigma_0 R^2}{\varepsilon_0 r^2} = \dfrac{Q}{4\pi\varepsilon_0 r^2}, & r > R \\[2ex] 0 & r < R \end{cases} \tag{22}
$$

Outside the sphere, the potential of (21) is the same as if all the charge $Q$ were concentrated at the origin as a point charge, while inside the sphere the potential is constant and equal to the surface potential.

**(b) Volume Charge**

If the sphere is uniformly charged with density $\rho_0$ and total charge $Q = \frac{4}{3}\pi R^3 \rho_0$, the potential can be found by breaking the sphere into differential size shells of thickness $dr'$ and incremental surface charge $d\sigma = \rho_0\, dr'$. Then, integrating (21) yields

$$
V = \begin{cases} \displaystyle\int_0^R \frac{\rho_0 r'^2}{\varepsilon_0 r}\, dr' = \frac{\rho_0 R^3}{3\varepsilon_0 r} = \frac{Q}{4\pi\varepsilon_0 r}, & r > R \\[2ex] \displaystyle\int_0^r \frac{\rho_0 r'^2}{\varepsilon_0 r}\, dr' + \int_r^R \frac{\rho_0 r'}{\varepsilon_0}\, dr' = \frac{\rho_0}{2\varepsilon_0}\left(R^2 - \frac{r^2}{3}\right) & \\[2ex] \qquad = \dfrac{3Q}{8\pi\varepsilon_0 R^3}\left(R^2 - \dfrac{r^2}{3}\right) & r < R \end{cases} \tag{23}
$$

where we realized from (21) that for $r < R$ the interior shells have a different potential contribution than exterior shells.

Then, the electric field again agrees with Section 2.4.3*b*:

$$
E_r = -\frac{\partial V}{\partial r} = \begin{cases} \dfrac{\rho_0 R^3}{3\varepsilon_0 r^2} = \dfrac{Q}{4\pi\varepsilon_0 r^2}, & r > R \\[2ex] \dfrac{\rho_0 r}{3\varepsilon_0} = \dfrac{Qr}{4\pi\varepsilon_0 R^3}, & r < R \end{cases} \tag{24}
$$

**(c) Two Spheres**

Two conducting spheres with respective radii $R_1$ and $R_2$ have their centers a long distance $D$ apart as shown in Figure 2-23. Different charges $Q_1$ and $Q_2$ are put on each sphere. Because $D \gg R_1 + R_2$, each sphere can be treated as isolated. The potential on each sphere is then

$$V_1 = \frac{Q_1}{4\pi\varepsilon_0 R_1}, \qquad V_2 = \frac{Q_2}{4\pi\varepsilon_0 R_2} \tag{25}$$

If a wire is connected between the spheres, they are forced to be at the same potential:

$$V_0 = \frac{q_1}{4\pi\varepsilon_0 R_1} = \frac{q_2}{4\pi\varepsilon_0 R_2} \tag{26}$$

causing a redistribution of charge. Since the total charge in the system must be conserved,

$$q_1 + q_2 = Q_1 + Q_2 \tag{27}$$

Eq. (26) requires that the charges on each sphere be

$$q_1 = \frac{R_1(Q_1+Q_2)}{R_1+R_2}, \qquad q_2 = \frac{R_2(Q_1+Q_2)}{R_1+R_2} \tag{28}$$

so that the system potential is

$$V_0 = \frac{Q_1+Q_2}{4\pi\varepsilon_0(R_1+R_2)} \tag{29}$$

Even though the smaller sphere carries less total charge, from (22) at $r=R$, where $E_r(R) = \sigma_0/\varepsilon_0$, we see that the surface electric field is stronger as the surface charge density is larger:

$$\begin{aligned} E_1(r=R_1) &= \frac{q_1}{4\pi\varepsilon_0 R_1^2} = \frac{Q_1+Q_2}{4\pi\varepsilon_0 R_1(R_1+R_2)} = \frac{V_0}{R_1} \\ E_2(r=R_2) &= \frac{q_2}{4\pi\varepsilon_0 R_2^2} = \frac{Q_2+Q_2}{4\pi\varepsilon_0 R_2(R_1+R_2)} = \frac{V_0}{R_2} \end{aligned} \tag{30}$$

For this reason, the electric field is always largest near corners and edges of equipotential surfaces, which is why

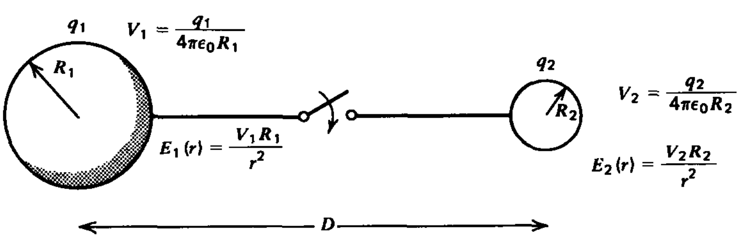



Figure 2-23 The charges on two spheres a long distance apart ($D \gg R_1 + R_2$) must redistribute themselves when connected by a wire so that each sphere is at the same potential. The surface electric field is then larger at the smaller sphere.

sharp points must be avoided in high-voltage equipment. When the electric field exceeds a critical amount $E_b$, called the breakdown strength, spark discharges occur as electrons are pulled out of the surrounding medium. Air has a breakdown strength of $E_b \approx 3 \times 10^6$ volts/m. If the two spheres had the same radius of 1 cm ($10^{-2}$ m), the breakdown strength is reached when $V_0 \approx 30{,}000$ volts. This corresponds to a total system charge of $Q_1 + Q_2 \approx 6.7 \times 10^{-8}$ coul.

### 2-5-6 Poisson's and Laplace's Equations

The general governing equations for the free space electric field in integral and differential form are thus summarized as

$$\oint_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int_V \rho \, dV \Rightarrow \nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \tag{31}$$

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = 0 \Rightarrow \nabla \times \mathbf{E} = 0 \Rightarrow \mathbf{E} = -\nabla V \tag{32}$$

The integral laws are particularly useful for geometries with great symmetry and with one-dimensional fields where the charge distribution is known. Often, the electrical potential of conducting surfaces are constrained by external sources so that the surface charge distributions, themselves sources of electric field are not directly known and are in part due to other charges by induction and conduction. Because of the coulombic force between charges, the charge distribution throughout space itself depends on the electric field and it is necessary to self-consistently solve for the equilibrium between the electric field and the charge distribution. These complications often make the integral laws difficult to use, and it becomes easier to use the differential form of the field equations. Using the last relation of (32) in Gauss's law of (31) yields a single equation relating the Laplacian of the potential to the charge density:

$$\nabla \cdot (\nabla V) = \nabla^2 V = -\rho/\varepsilon_0 \tag{33}$$

which is called Poisson's equation. In regions of zero charge ($\rho = 0$) this equation reduces to Laplace's equation, $\nabla^2 V = 0$.

## 2-6 THE METHOD OF IMAGES WITH LINE CHARGES AND CYLINDERS

### 2-6-1 Two Parallel Line Charges

The potential of an infinitely long line charge $\lambda$ is given in Section 2.5.4 when the length of the line $L$ is made very large. More directly, knowing the electric field of an infinitely long line charge from Section 2.3.3 allows us to obtain the potential by direct integration:

$$E_r = -\frac{\partial V}{\partial r} = \frac{\lambda}{2\pi\varepsilon_0 r} \Rightarrow V = -\frac{\lambda}{2\pi\varepsilon_0} \ln \frac{r}{r_0} \tag{1}$$

where $r_0$ is the arbitrary reference position of zero potential.

If we have two line charges of opposite polarity $\pm\lambda$ a distance $2a$ apart, we choose our origin halfway between, as in Figure 2-24$a$, so that the potential due to both charges is just the superposition of potentials of (1):

$$V = -\frac{\lambda}{2\pi\varepsilon_0} \ln \left(\frac{y^2+(x+a)^2}{y^2+(x-a)^2}\right)^{1/2} \tag{2}$$

where the reference potential point $r_0$ cancels out and we use Cartesian coordinates. Equipotential lines are then

$$\frac{y^2+(x+a)^2}{y^2+(x-a)^2} = e^{-4\pi\varepsilon_0 V/\lambda} = K_1 \tag{3}$$

where $K_1$ is a constant on an equipotential line. This relation is rewritten by completing the squares as

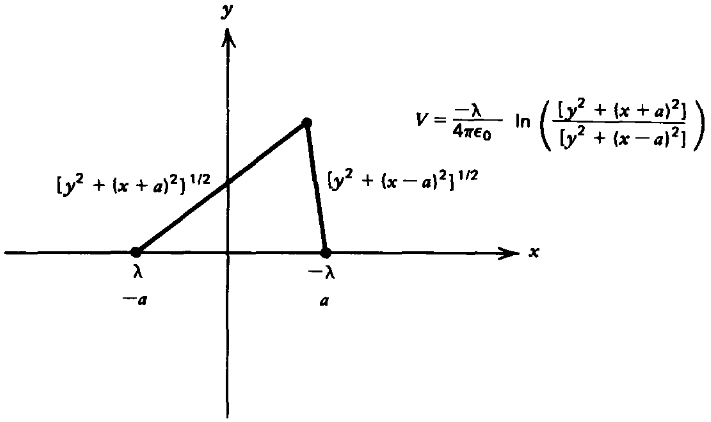
$$\left(x - \frac{a(1+K_1)}{K_1-1}\right)^2 + y^2 = \frac{4K_1 a^2}{(1-K_1)^2} \tag{4}$$

which we recognize as circles of radius $r = 2a\sqrt{K_1}/|1-K_1|$ with centers at $y=0, x = a(1+K_1)/(K_1-1)$, as drawn by dashed lines in Figure 2-24$b$. The value of $K_1 = 1$ is a circle of infinite radius with center at $x = \pm\infty$ and thus represents the $x=0$ plane. For values of $K_1$ in the interval $0 \leq K_1 \leq 1$ the equipotential circles are in the left half-plane, while for $1 \leq K_1 \leq \infty$ the circles are in the right half-plane.

The electric field is found from (2) as

$$\mathbf{E} = -\nabla V = \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{-4axy\mathbf{i}_y + 2a(y^2+a^2-x^2)\mathbf{i}_x}{[y^2+(x+a)^2][y^2+(x-a)^2]}\right) \tag{5}$$

One way to plot the electric field distribution graphically is by drawing lines that are everywhere tangent to the electric field, called field lines or lines of force. These lines are everywhere perpendicular to the equipotential surfaces and tell us the direction of the electric field. The magnitude is proportional to the density of lines. For a single line charge, the field lines emanate radially. The situation is more complicated for the two line charges of opposite polarity in Figure 2-24 with the field lines always starting on the positive charge and terminating on the negative charge.




(a)

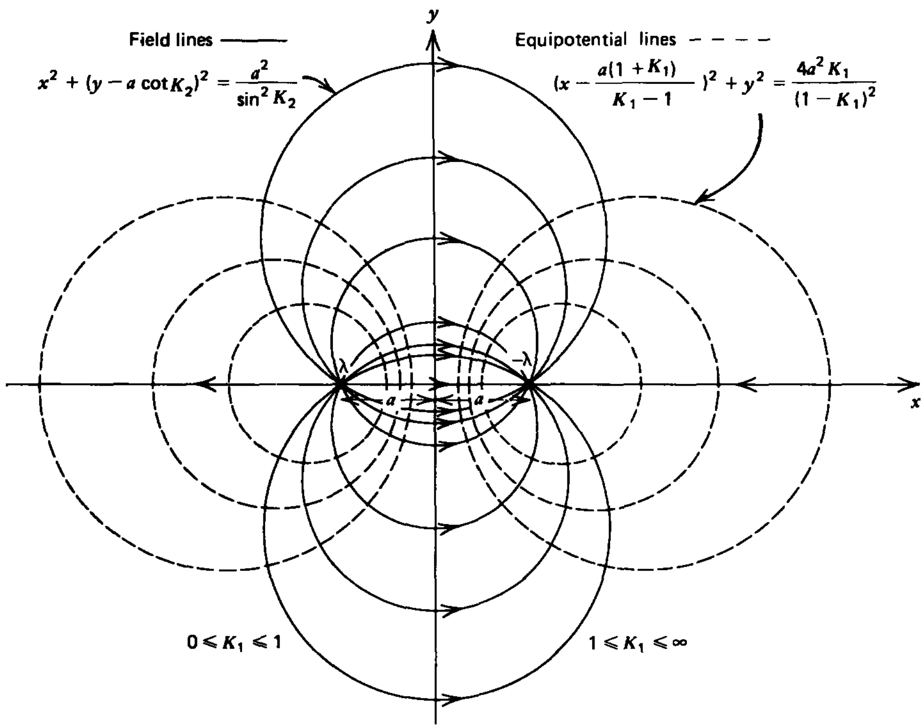



Figure 2-24 (*a*) Two parallel line charges of opposite polarity a distance $2a$ apart. (*b*) The equipotential (dashed) and field (solid) lines form a set of orthogonal circles.

For the field given by (5), the equation for the lines tangent to the electric field is

$$\frac{dy}{dx}=\frac{E_y}{E_x}=-\frac{2xy}{y^2+a^2-x^2}\Rightarrow\frac{d(x^2+y^2)}{a^2-(x^2+y^2)}+d(\ln y)=0 \qquad (6)$$

where the last equality is written this way so the expression can be directly integrated to

$$x^2+(y-a\cot K_2)^2=\frac{a^2}{\sin^2 K_2} \qquad (7)$$

where $K_2$ is a constant determined by specifying a single coordinate $(x_0, y_0)$ along the field line of interest. The field lines are also circles of radius $a/\sin K_2$ with centers at $x=0, y=a\cot K_2$ as drawn by the solid lines in Figure 2-24*b*.

## 2-6-2 The Method of Images

### (a) General properties

When a conductor is in the vicinity of some charge, a surface charge distribution is induced on the conductor in order to terminate the electric field, as the field within the equipotential surface is zero. This induced charge distribution itself then contributes to the external electric field subject to the boundary condition that the conductor is an equipotential surface so that the electric field terminates perpendicularly to the surface. In general, the solution is difficult to obtain because the surface charge distribution cannot be known until the field is known so that we can use the boundary condition of Section 2.4.6. However, the field solution cannot be found until the surface charge distribution is known.

However, for a few simple geometries, the field solution can be found by replacing the conducting surface by equivalent charges within the conducting body, called images, that guarantee that all boundary conditions are satisfied. Once the image charges are known, the problem is solved as if the conductor were not present but with a charge distribution composed of the original charges plus the image charges.

### (b) Line Charge Near a Conducting Plane

The method of images can adapt a known solution to a new problem by replacing conducting bodies with an equivalent charge. For instance, we see in Figure 2-24*b* that the field lines are all perpendicular to the $x=0$ plane. If a conductor were placed along the $x=0$ plane with a single line charge $\lambda$ at $x=-a$, the potential and electric field for $x<0$ is the same as given by (2) and (5).

A surface charge distribution is induced on the conducting plane in order to terminate the incident electric field as the field must be zero inside the conductor. This induced surface charge distribution itself then contributes to the external electric field for $x<0$ in exactly the same way as for a single image line charge $-\lambda$ at $x=+a$.

The force per unit length on the line charge $\lambda$ is due only to the field from the image charge $-\lambda$;

$$\mathbf{f}=\lambda\mathbf{E}(-a,0)=\frac{\lambda^2}{2\pi\varepsilon_0(2a)}\mathbf{i}_x=\frac{\lambda^2}{4\pi\varepsilon_0 a}\mathbf{i}_x \tag{8}$$

From Section 2.4.6 we know that the surface charge distribution on the plane is given by the discontinuity in normal component of electric field:

$$\sigma(x=0)=-\varepsilon_0 E_x(x=0)=\frac{-\lambda a}{\pi(y^2+a^2)} \tag{9}$$

where we recognize that the field within the conductor is zero. The total charge per unit length on the plane is obtained by integrating (9) over the whole plane:

$$\begin{aligned}\lambda_T&=\int_{-\infty}^{+\infty}\sigma(x=0)\,dy\\&=-\frac{\lambda a}{\pi}\int_{-\infty}^{+\infty}\frac{dy}{y^2+a^2}\\&=-\frac{\lambda a}{\pi}\frac{1}{a}\tan^{-1}\frac{y}{a}\bigg|_{-\infty}^{+\infty}\\&=-\lambda\end{aligned} \tag{10}$$

and just equals the image charge.

### 2-6-3 Line Charge and Cylinder

Because the equipotential surfaces of (4) are cylinders, the method of images also works with a line charge $\lambda$ a distance $D$ from the center of a conducting cylinder of radius $R$ as in Figure 2-25. Then the radius $R$ and distance $a$ must fit (4) as

$$R=\frac{2a\sqrt{K_1}}{|1-K_1|},\qquad \pm a+\frac{a(1+K_1)}{K_1-1}=D \tag{11}$$

where the upper positive sign is used when the line charge is outside the cylinder, as in Figure 2-25*a*, while the lower negative sign is used when the line charge is within the cylinder, as in Figure 2-25*b*. Because the cylinder is chosen to be in the right half-plane, $1\le K_1\le\infty$, the unknown parameters $K_1$

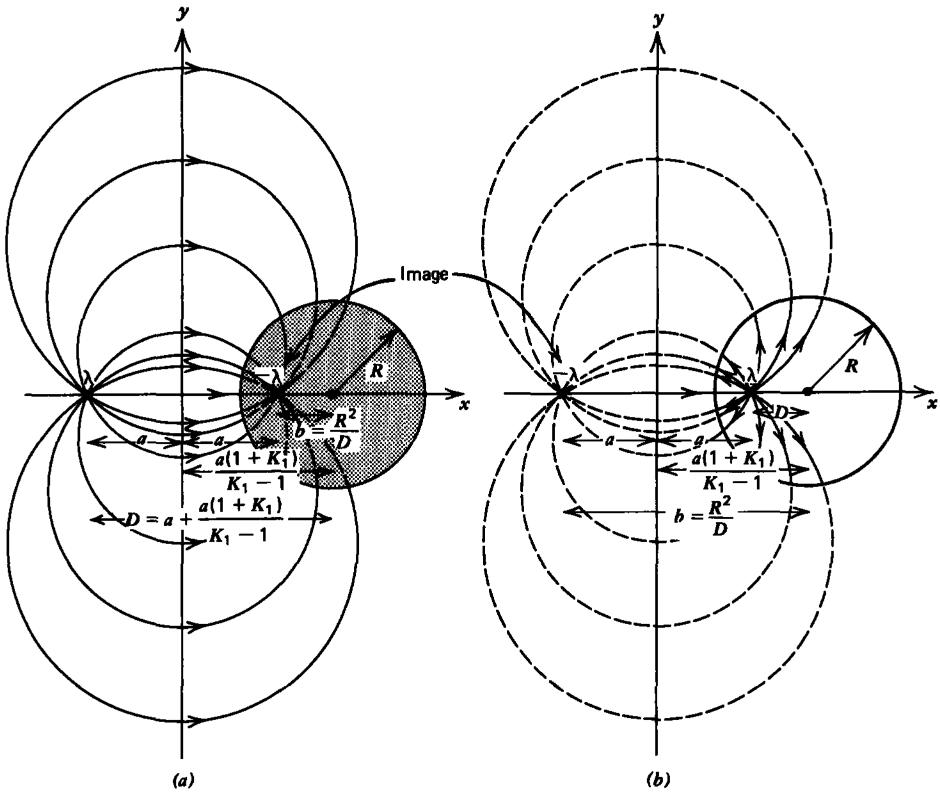



Figure 2-25 The electric field surrounding a line charge $\lambda$ a distance $D$ from the center of a conducting cylinder of radius $R$ is the same as if the cylinder were replaced by an image charge $-\lambda$, a distance $b = R^2/D$ from the center. (*a*) Line charge outside cylinder. (*b*) Line charge inside cylinder.

and $a$ are expressed in terms of the given values $R$ and $D$ from (11) as

$$K_1 = \left(\frac{D^2}{R^2}\right)^{\pm 1}, \qquad a = \pm\frac{D^2 - R^2}{2D} \tag{12}$$

For either case, the image line charge then lies a distance $b$ from the center of the cylinder:

$$b = \frac{a(1+K_1)}{K_1 - 1} \mp a = \frac{R^2}{D} \tag{13}$$

being inside the cylinder when the inducing charge is outside ($R < D$), and vice versa, being outside the cylinder when the inducing charge is inside ($R > D$).

The force per unit length on the cylinder is then just due to the force on the image charge:

$$f_x = -\frac{\lambda^2}{2\pi\varepsilon_0(D-b)} = -\frac{\lambda^2 D}{2\pi\varepsilon_0(D^2-R^2)} \tag{14}$$

## 2-6-4 Two Wire Line

### (a) Image Charges

We can continue to use the method of images for the case of two parallel equipotential cylinders of differing radii $R_1$ and $R_2$ having their centers a distance $D$ apart as in Figure 2-26. We place a line charge $\lambda$ a distance $b_1$ from the center of cylinder 1 and a line charge $-\lambda$ a distance $b_2$ from the center of cylinder 2, both line charges along the line joining the centers of the cylinders. We simultaneously treat the cases where the cylinders are adjacent, as in Figure 2-26*a*, or where the smaller cylinder is inside the larger one, as in Figure 2-26*b*.

The position of the image charges can be found using (13) realizing that the distance from each image charge to the center of the opposite cylinder is $D-b$ so that

$$b_1 = \frac{R_1^2}{D \mp b_2}, \qquad b_2 = \pm\frac{R_2^2}{D-b_1} \tag{15}$$

where the upper signs are used when the cylinders are adjacent and lower signs are used when the smaller cylinder is inside the larger one. We separate the two coupled equations in (15) into two quadratic equations in $b_1$ and $b_2$:

$$\begin{aligned} b_1^2 - \frac{[D^2-R_2^2+R_1^2]}{D} b_1 + R_1^2 &= 0 \\ b_2^2 \mp \frac{[D^2-R_1^2+R_2^2]}{D} b_2 + R_2^2 &= 0 \end{aligned} \tag{16}$$

with resulting solutions

$$\begin{aligned} b_2 &= \pm\frac{[D^2-R_1^2+R_2^2]}{2D} - \left[\left(\frac{D^2-R_1^2+R_2^2}{2D}\right)^2 - R_2^2\right]^{1/2} \\ b_1 &= \frac{[D^2+R_1^2-R_2^2]}{2D} \mp \left[\left(\frac{D^2+R_1^2-R_2^2}{2D}\right)^2 - R_1^2\right]^{1/2} \end{aligned} \tag{17}$$

We were careful to pick the roots that lay outside the region between cylinders. If the equal magnitude but opposite polarity image line charges are located at these positions, the cylindrical surfaces are at a constant potential.

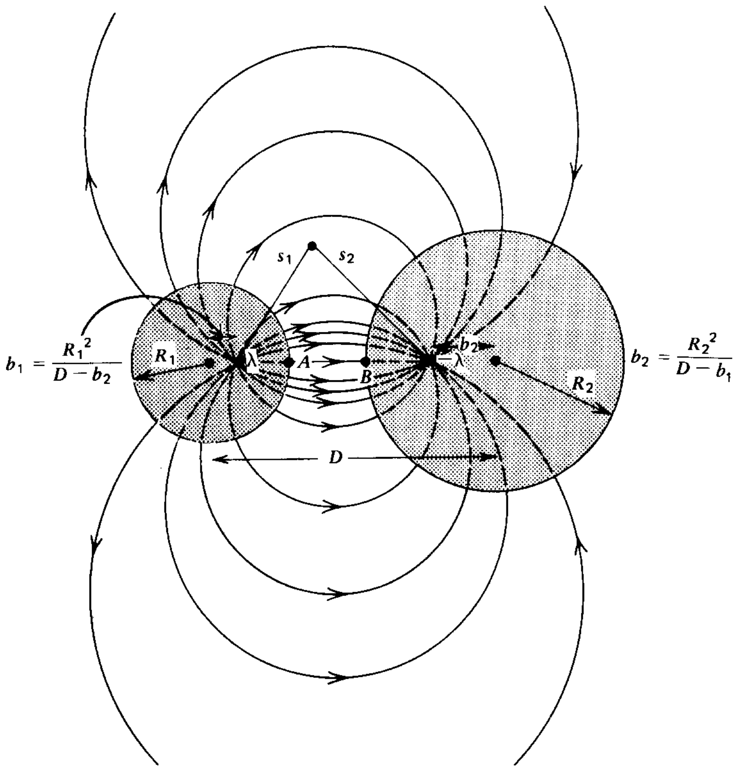



Figure 2-26 The solution for the electric field between two parallel conducting cylinders is found by replacing the cylinders by their image charges. The surface charge density is largest where the cylinder surfaces are closest together. This is called the proximity effect. (*a*) Adjacent cylinders. (*b*) Smaller cylinder inside the larger one.

### (b) Force of Attraction

The attractive force per unit length on cylinder 1 is the force on the image charge $\lambda$ due to the field from the opposite image charge $-\lambda$:

$$f_x = \frac{\lambda^2}{2\pi\varepsilon_0[\pm(D-b_1)-b_2]}$$

$$= \frac{\lambda^2}{4\pi\varepsilon_0\left[\left(\frac{D^2-R_1^2+R_2^2}{2D}\right)^2 - R_2^2\right]^{1/2}}$$

$$= \frac{\lambda^2}{4\pi\varepsilon_0\left[\left(\frac{D^2-R_2^2+R_1^2}{2D}\right)^2 - R_1^2\right]^{1/2}} \qquad (18)$$

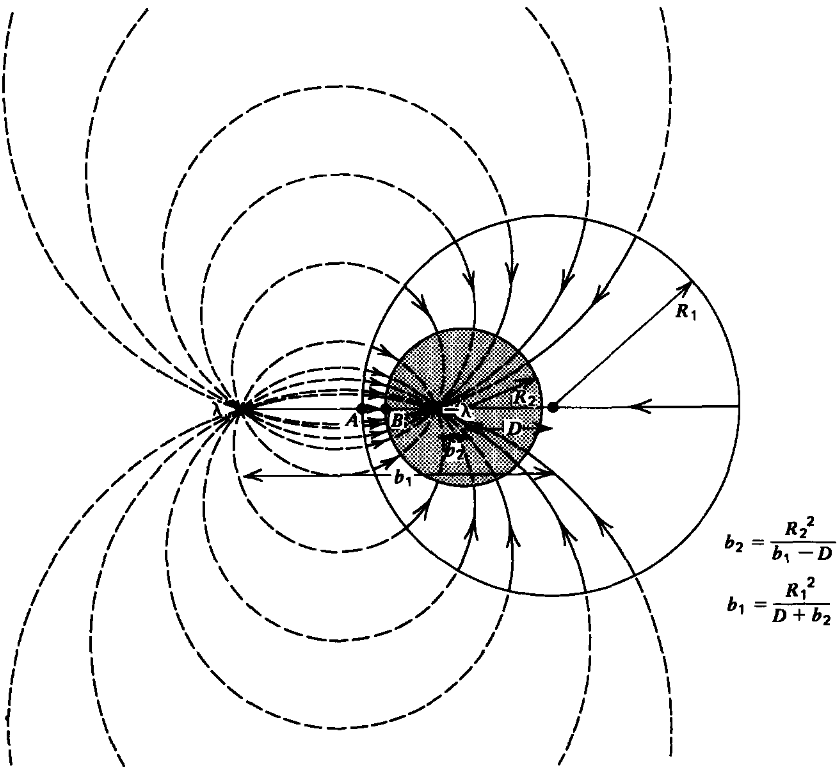



Fig. 2-26(*b*)

### (c) Capacitance Per Unit Length

The potential of (2) in the region between the two cylinders depends on the distances from any point to the line charges:

$$V = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{s_1}{s_2} \tag{19}$$

To find the voltage difference between the cylinders we pick the most convenient points labeled $A$ and $B$ in Figure 2-26:

$$\begin{array}{ll} \quad\quad A & \quad\quad B \\ s_1 = \pm(R_1 - b_1) & s_1 = \pm(D - b_1 \mp R_2) \\ s_2 = \pm(D \mp b_2 - R_1) & s_2 = R_2 - b_2 \end{array} \tag{20}$$

although any two points on the surfaces could have been used. The voltage difference is then

$$V_1 - V_2 = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\pm\frac{(R_1 - b_1)(R_2 - b_2)}{(D \mp b_2 - R_1)(D - b_1 \mp R_2)}\right) \tag{21}$$

This expression can be greatly reduced using the relations

$$D \mp b_2 = \frac{R_1^2}{b_1}, \qquad D - b_1 = \pm \frac{R_2^2}{b_2} \tag{22}$$

to

$$\begin{aligned} V_1 - V_2 &= -\frac{\lambda}{2\pi\varepsilon_0} \ln \frac{b_1 b_2}{R_1 R_2} \\ &= \frac{\lambda}{2\pi\varepsilon_0} \ln \left\{ \pm \frac{[D^2 - R_1^2 - R_2^2]}{2R_1R_2} \right. \\ &\quad \left. + \left[ \left( \frac{D^2 - R_1^2 - R_2^2}{2R_1R_2} \right)^2 - 1 \right]^{1/2} \right\} \end{aligned} \tag{23}$$

The potential difference $V_1 - V_2$ is linearly related to the line charge $\lambda$ through a factor that only depends on the geometry of the conductors. This factor is defined as the capacitance per unit length and is the ratio of charge per unit length to potential difference:

$$\begin{aligned} C = \frac{\lambda}{V_1 - V_2} &= \frac{2\pi\varepsilon_0}{\ln \left\{ \pm \frac{[D^2 - R_1^2 - R_2^2]}{2R_1R_2} + \left[ \left( \frac{D^2 - R_1^2 - R_2^2}{2R_1R_2} \right)^2 - 1 \right]^{1/2} \right\}} \\ &= \frac{2\pi\varepsilon_0}{\cosh^{-1}\left( \pm \frac{D^2 - R_1^2 - R_2^2}{2R_1R_2} \right)} \end{aligned} \tag{24}$$

where we use the identity*

$$\ln [y + (y^2 - 1)^{1/2}] = \cosh^{-1} y \tag{25}$$

We can examine this result in various simple limits. Consider first the case for adjacent cylinders ($D > R_1 + R_2$).

1. If the distance $D$ is much larger than the radii,

$$\lim_{D \gg (R_1 + R_2)} C \approx \frac{2\pi\varepsilon_0}{\ln [D^2/(R_1R_2)]} = \frac{2\pi\varepsilon_0}{\cosh^{-1} [D^2/(2R_1R_2)]} \tag{26}$$

2. The capacitance between a cylinder and an infinite plane can be obtained by letting one cylinder have infinite radius but keeping finite the closest distance $s =$

$$* \; y = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$(e^x)^2 - 2ye^x + 1 = 0$$

$$e^x = y \pm (y^2 - 1)^{1/2}$$

$$x = \cosh^{-1} y = \ln [y \pm (y^2 - 1)^{1/2}]$$

$D-R_1-R_2$ between cylinders. If we let $R_1$ become infinite, the capacitance becomes

$$\lim_{\substack{R_1\to\infty \\ D-R_1-R_2=s \text{ (finite)}}} C=\frac{2\pi\varepsilon_0}{\ln\left\{\dfrac{s+R_2}{R_2}+\left[\left(\dfrac{s+R_2}{R_2}\right)^2-1\right]^{1/2}\right\}}$$

$$=\frac{2\pi\varepsilon_0}{\cosh^{-1}\left(\dfrac{s+R_2}{R_2}\right)} \tag{27}$$

3. If the cylinders are identical so that $R_1=R_2\equiv R$, the capacitance per unit length reduces to

$$\lim_{R_1=R_2\equiv R} C=\frac{\pi\varepsilon_0}{\ln\left\{\dfrac{D}{2R}+\left[\left(\dfrac{D}{2R}\right)^2-1\right]^{1/2}\right\}}=\frac{\pi\varepsilon_0}{\cosh^{-1}\dfrac{D}{2R}} \tag{28}$$

4. When the cylinders are concentric so that $D=0$, the capacitance per unit length is

$$\lim_{D=0} C=\frac{2\pi\varepsilon_0}{\ln(R_1/R_2)}=\frac{2\pi\varepsilon_0}{\cosh^{-1}[(R_1^2+R_2^2)/(2R_1R_2)]} \tag{29}$$

## 2-7 THE METHOD OF IMAGES WITH POINT CHARGES AND SPHERES

### 2-7-1 Point Charge and a Grounded Sphere

A point charge $q$ is a distance $D$ from the center of the conducting sphere of radius $R$ at zero potential as shown in Figure 2-27$a$. We try to use the method of images by placing a single image charge $q'$ a distance $b$ from the sphere center along the line joining the center to the point charge $q$.

We need to find values of $q'$ and $b$ that satisfy the zero potential boundary condition at $r=R$. The potential at any point $P$ outside the sphere is

$$V=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{s}+\frac{q'}{s'}\right) \tag{1}$$

where the distance from $P$ to the point charges are obtained from the law of cosines:

$$\begin{aligned} s&=[r^2+D^2-2rD\cos\theta]^{1/2} \\ s'&=[b^2+r^2-2rb\cos\theta]^{1/2} \end{aligned} \tag{2}$$

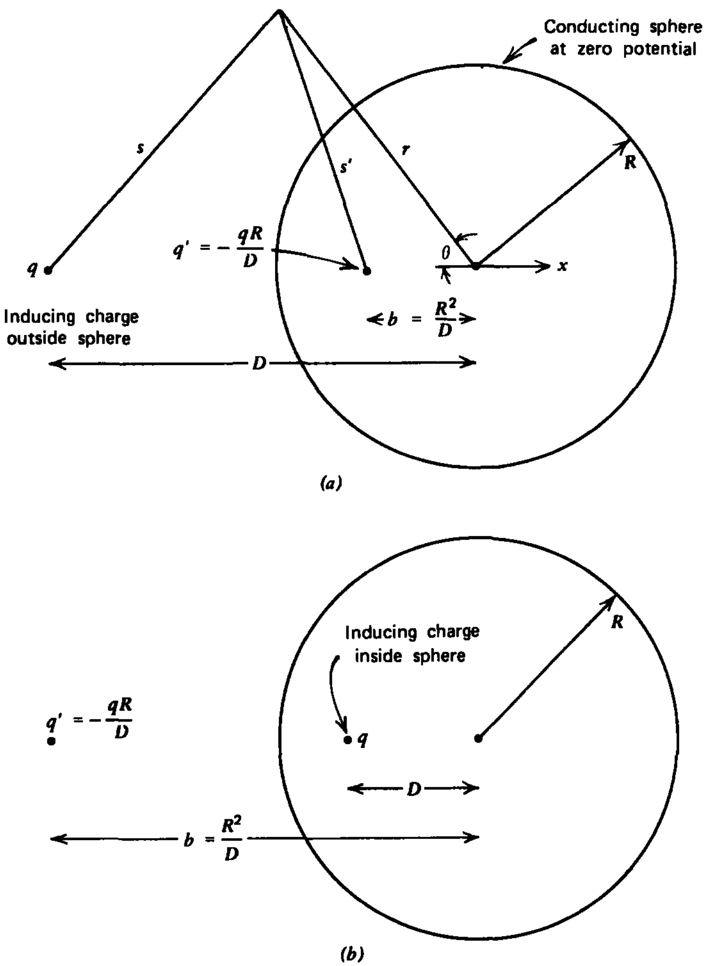



Figure 2-27 (*a*) The field due to a point charge $q$, a distance $D$ outside a conducting sphere of radius $R$, can be found by placing a single image charge $-qR/D$ at a distance $b = R^2/D$ from the center of the sphere. (*b*) The same relations hold true if the charge $q$ is inside the sphere but now the image charge is outside the sphere, since $D < R$.

At $r = R$, the potential in (1) must be zero so that $q$ and $q'$ must be of opposite polarity:

$$\left(\frac{q}{s}+\frac{q'}{s'}\right)\bigg|_{r=R} = 0 \Rightarrow \left(\frac{q}{s}\right)^2 = \left(\frac{q'}{s'}\right)^2\bigg|_{r=R} \tag{3}$$

where we square the equalities in (3) to remove the square roots when substituting (2),

$$q^2[b^2 + R^2 - 2Rb\cos\theta] = q'^2[R^2 + D^2 - 2RD\cos\theta] \tag{4}$$

Since (4) must be true for all values of $\theta$, we obtain the following two equalities:

$$\begin{aligned} q^2(b^2+R^2) &= q'^2(R^2+D^2) \\ q^2 b &= q'^2 D \end{aligned} \tag{5}$$

Eliminating $q$ and $q'$ yields a quadratic equation in $b$:

$$b^2 - bD\left[1+\left(\frac{R}{D}\right)^2\right] + R^2 = 0 \tag{6}$$

with solution

$$\begin{aligned} b &= \frac{D}{2}\left[1+\left(\frac{R}{D}\right)^2\right] \pm \sqrt{\left\{\frac{D}{2}\left[1+\left(\frac{R}{D}\right)^2\right]\right\}^2 - R^2} \\ &= \frac{D}{2}\left[1+\left(\frac{R}{D}\right)^2\right] \pm \sqrt{\left\{\frac{D}{2}\left[1-\left(\frac{R}{D}\right)^2\right]\right\}^2} \\ &= \frac{D}{2}\left\{\left[1+\left(\frac{R}{D}\right)^2\right] \pm \left[1-\left(\frac{R}{D}\right)^2\right]\right\} \end{aligned} \tag{7}$$

We take the lower negative root so that the image charge is inside the sphere with value obtained from using (7) in (5):

$$b = \frac{R^2}{D}, \qquad q' = -q\frac{R}{D} \tag{8}$$

remembering from (3) that $q$ and $q'$ have opposite sign. We ignore the $b = D$ solution with $q' = -q$ since the image charge must always be outside the region of interest. If we allowed this solution, the net charge at the position of the inducing charge is zero, contrary to our statement that the net charge is $q$.

The image charge distance $b$ obeys a similar relation as was found for line charges and cylinders in Section 2.6.3. Now, however, the image charge magnitude does not equal the magnitude of the inducing charge because not all the lines of force terminate on the sphere. Some of the field lines emanating from $q$ go around the sphere and terminate at infinity.

The force on the grounded sphere is then just the force on the image charge $-q'$ due to the field from $q$:

$$f_x = \frac{qq'}{4\pi\varepsilon_0(D-b)^2} = -\frac{q^2 R}{4\pi\varepsilon_0 D(D-b)^2} = -\frac{q^2 RD}{4\pi\varepsilon_0(D^2-R^2)^2} \tag{9}$$

The electric field outside the sphere is found from (1) using (2) as

$$\mathbf{E} = -\nabla V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{s^3}[(r - D\cos\theta)\mathbf{i}_r + D\sin\theta\,\mathbf{i}_\theta] + \frac{q'}{s'^3}[(r - b\cos\theta)\mathbf{i}_r + b\sin\theta\,\mathbf{i}_\theta]\right) \tag{10}$$

On the sphere where $s' = (R/D)s$, the surface charge distribution is found from the discontinuity in normal electric field as given in Section 2.4.6:

$$\sigma(r = R) = \varepsilon_0 E_r(r = R) = -\frac{q(D^2 - R^2)}{4\pi R[R^2 + D^2 - 2RD\cos\theta]^{3/2}} \tag{11}$$

The total charge on the sphere

$$\begin{aligned} q_T &= \int_0^\pi \sigma(r = R) 2\pi R^2 \sin\theta\, d\theta \\ &= -\frac{q}{2}R(D^2 - R^2)\int_0^\pi \frac{\sin\theta\, d\theta}{[R^2 + D^2 - 2RD\cos\theta]^{3/2}} \end{aligned} \tag{12}$$

can be evaluated by introducing the change of variable

$$u = R^2 + D^2 - 2RD\cos\theta, \qquad du = 2RD\sin\theta\, d\theta \tag{13}$$

so that (12) integrates to

$$\begin{aligned} q_T &= -\frac{q(D^2 - R^2)}{4D}\int_{(D-R)^2}^{(D+R)^2} \frac{du}{u^{3/2}} \\ &= -\frac{q(D^2 - R^2)}{4D}\left(-\frac{2}{u^{1/2}}\right)\Bigg|_{(D-R)^2}^{(D+R)^2} = -\frac{qR}{D} \end{aligned} \tag{14}$$

which just equals the image charge $q'$.

If the point charge $q$ is inside the grounded sphere, the image charge and its position are still given by (8), as illustrated in Figure 2-27*b*. Since $D < R$, the image charge is now outside the sphere.

### 2-7-2 Point Charge Near a Grounded Plane

If the point charge is a distance $a$ from a grounded plane, as in Figure 2-28*a*, we consider the plane to be a sphere of infinite radius $R$ so that $D = R + a$. In the limit as $R$ becomes infinite, (8) becomes

$$\lim_{\substack{R\to\infty \\ D=R+a}} q' = -q, \qquad b = \frac{R}{(1 + a/R)} = R - a \tag{15}$$

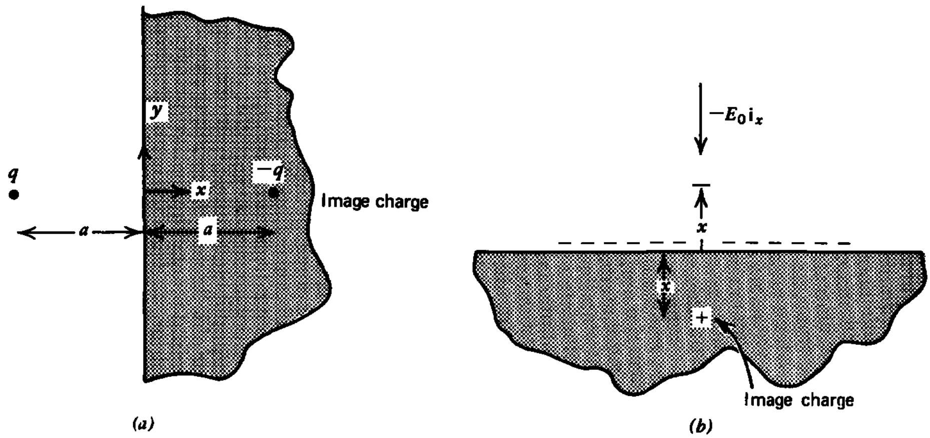



Figure 2-28 ($a$) A point charge $q$ near a conducting plane has its image charge $-q$ symmetrically located behind the plane. ($b$) An applied uniform electric field causes a uniform surface charge distribution on the conducting plane. Any injected charge must overcome the restoring force due to its image in order to leave the electrode.

so that the image charge is of equal magnitude but opposite polarity and symmetrically located on the opposite side of the plane.

The potential at any point $(x, y, z)$ outside the conductor is given in Cartesian coordinates as

$$V=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{[(x+a)^2+y^2+z^2]^{1/2}}-\frac{1}{[(x-a)^2+y^2+z^2]^{1/2}}\right) \tag{16}$$

with associated electric field

$$\mathbf{E}=-\nabla V=\frac{q}{4\pi\varepsilon_0}\left(\frac{(x+a)\mathbf{i}_x+y\mathbf{i}_y+z\mathbf{i}_z}{[(x+a)^2+y^2+z^2]^{3/2}}-\frac{(x-a)\mathbf{i}_x+y\mathbf{i}_y+z\mathbf{i}_z}{[(x-a)^2+y^2+z^2]^{3/2}}\right) \tag{17}$$

Note that as required the field is purely normal to the grounded plane

$$E_y(x=0)=0, \qquad E_z(x=0)=0 \tag{18}$$

The surface charge density on the conductor is given by the discontinuity of normal $\mathbf{E}$:

$$\begin{aligned}\sigma(x=0)&=-\varepsilon_0 E_x(x=0)\\&=-\frac{q}{4\pi}\frac{2a}{[y^2+z^2+a^2]^{3/2}}\\&=-\frac{qa}{2\pi(\mathrm{r}^2+a^2)^{3/2}};\ \mathrm{r}^2=y^2+z^2\end{aligned} \tag{19}$$

where the minus sign arises because the surface normal points in the negative $x$ direction.

The total charge on the conducting surface is obtained by integrating (19) over the whole surface:

$$q_T=\int_0^\infty \sigma(x=0)2\pi r\,dr$$

$$=-qa\int_0^\infty \frac{r\,dr}{(r^2+a^2)^{3/2}}$$

$$=\frac{qa}{(r^2+a^2)^{1/2}}\bigg|_0^\infty=-q \tag{20}$$

As is always the case, the total charge on a conducting surface must equal the image charge.

The force on the conductor is then due only to the field from the image charge:

$$\mathbf{f}=-\frac{q^2}{16\pi\varepsilon_0 a^2}\mathbf{i}_x \tag{21}$$

This attractive force prevents charges from escaping from an electrode surface when an electric field is applied. Assume that an electric field $-E_0\mathbf{i}_x$ is applied perpendicular to the electrode shown in Figure (2-28*b*). A uniform negative surface charge distribution $\sigma=-\varepsilon_0 E_0$ as given in (2.4.6) arises to terminate the electric field as there is no electric field within the conductor. There is then an upwards Coulombic force on the surface charge, so why aren't the electrons pulled out of the electrode? Imagine an ejected charge $-q$ a distance $x$ from the conductor. From (15) we know that an image charge $+q$ then appears at $-x$ which tends to pull the charge $-q$ back to the electrode with a force given by (21) with $a=x$ in opposition to the imposed field that tends to pull the charge away from the electrode. The total force on the charge $-q$ is then

$$f_x=qE_0-\frac{q^2}{4\pi\varepsilon_0(2x)^2} \tag{22}$$

The force is zero at position $x_c$

$$f_x=0\Rightarrow x_c=\left[\frac{q}{16\pi\varepsilon_0 E_0}\right]^{1/2} \tag{23}$$

For an electron ($q=1.6\times10^{-19}$ coulombs) in a field of $E_0=10^6$ v/m, $x_c\approx1.9\times10^{-8}$ m. For smaller values of $x$ the net force is negative tending to pull the charge back to the electrode. If the charge can be propelled past $x_c$ by external forces, the imposed field will then carry the charge away from the electrode. If this external force is due to heating of the electrode, the process is called thermionic emission. High field emission even with a cold electrode occurs when the electric field $E_0$ becomes sufficiently large (on the order of $10^{10}$ v/m) that the coulombic force overcomes the quantum mechanical binding forces holding the electrons within the electrode.

### 2-7-3 Sphere With Constant Charge

If the point charge $q$ is outside a conducting sphere ($D > R$) that now carries a constant total charge $Q_0$, the induced charge is still $q' = -qR/D$. Since the total charge on the sphere is $Q_0$, we must find another image charge that keeps the sphere an equipotential surface and has value $Q_0 + qR/D$. This other image charge must be placed at the center of the sphere, as in Figure 2-29*a*. The original charge $q$ plus the image charge $q' = -qR/D$ puts the sphere at zero potential. The additional image charge at the center of the sphere raises the potential of the sphere to

$$V = \frac{Q_0 + qR/D}{4\pi\varepsilon_0 R} \tag{24}$$

The force on the sphere is now due to the field from the point charge $q$ acting on the two image charges:

$$\begin{aligned} f_x &= \frac{q}{4\pi\varepsilon_0}\left(-\frac{qR}{D(D-b)^2} + \frac{(Q_0 + qR/D)}{D^2}\right) \\ &= \frac{q}{4\pi\varepsilon_0}\left(-\frac{qRD}{(D^2 - R^2)^2} + \frac{(Q_0 + qR/D)}{D^2}\right) \end{aligned} \tag{25}$$

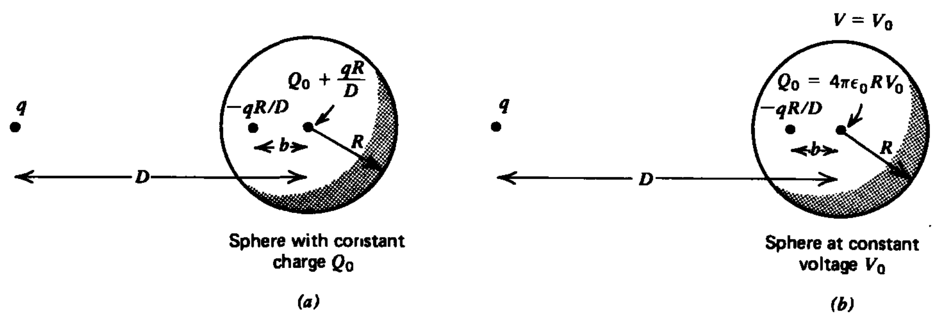



Figure 2-29 (*a*) If a conducting sphere carries a constant charge $Q_0$ or (*b*) is at a constant voltage $V_0$, an additional image charge is needed at the sphere center when a charge $q$ is nearby.

### 2-7-4 Constant Voltage Sphere

If the sphere is kept at constant voltage $V_0$, the image charge $q'=-qR/D$ at distance $b=R^2/D$ from the sphere center still keeps the sphere at zero potential. To raise the potential of the sphere to $V_0$, another image charge,

$$Q_0=4\pi\varepsilon_0 R V_0 \tag{26}$$

must be placed at the sphere center, as in Figure 2-29*b*. The force on the sphere is then

$$f_x=\frac{q}{4\pi\varepsilon_0}\left(-\frac{qR}{D(D-b)^2}+\frac{Q_0}{D^2}\right) \tag{27}$$

## PROBLEMS

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Section 2.1

1. Faraday's "ice-pail" experiment is repeated with the following sequence of steps:

(i) A ball with total charge $Q$ is brought inside an insulated metal ice-pail without touching.
(ii) The outside of the pail is momentarily connected to the ground and then disconnected so that once again the pail is insulated.
(iii) Without touching the pail, the charged ball is removed.

(a) Sketch the charge distribution on the inside and outside of the pail during each step.

(b) What is the net charge on the pail after the charged ball is removed?

2. A sphere initially carrying a total charge $Q$ is brought into momentary contact with an uncharged identical sphere.

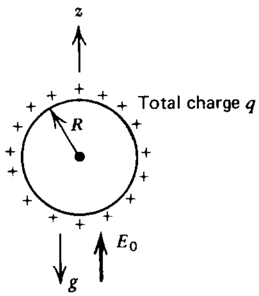
(a) How much charge is on each sphere?

(b) This process is repeated for $N$ identical initially uncharged spheres. How much charge is on each of the spheres including the original charged sphere?

(c) What is the total charge in the system after the $N$ contacts?

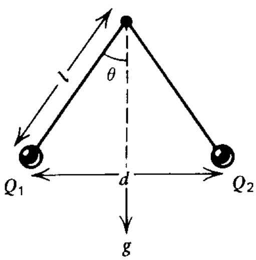
Section 2.2

3. The charge of an electron was first measured by Robert A. Millikan in 1909 by measuring the electric field necessary to levitate a small charged oil drop against its weight. The oil droplets were sprayed and became charged by frictional electrification.



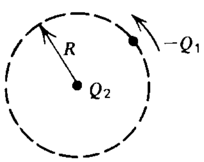

A spherical droplet of radius $R$ and effective mass density $\rho_m$ carries a total charge $q$ in a gravity field $g$. What electric field $E_0\mathbf{i}_z$ will suspend the charged droplet? Millikan found by this method that all droplets carried integer multiples of negative charge $e = -1.6 \times 10^{-19}$ coul.

4. Two small conducting balls, each of mass $m$, are at the end of insulating strings of length $l$ joined at a point. Charges are




placed on the balls so that they are a distance $d$ apart. A charge $Q_1$ is placed on ball 1. What is the charge $Q_2$ on ball 2?

5. A point charge $-Q_1$ of mass $m$ travels in a circular orbit of radius $R$ about a charge of opposite sign $Q_2$.




(a) What is the equilibrium angular speed of the charge $-Q_1$?

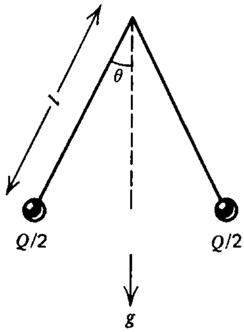
(b) This problem describes Bohr's one electron model of the atom if the charge $-Q_1$ is that of an electron and $Q_2 = Ze$ is the nuclear charge, where $Z$ is the number of protons. According to the postulates of quantum mechanics the angular momentum $L$ of the electron must be quantized,

$$L = mvR = nh/2\pi, \qquad n = 1, 2, 3, \cdots$$

where $h = 6.63 \times 10^{-34}$ joule-sec is Planck's constant. What are the allowed values of $R$?

(c) For the hydrogen atom ($Z = 1$) what is the radius of the smallest allowed orbit and what is the electron's orbital velocity?

6. An electroscope measures charge by the angular deflection of two identical conducting balls suspended by an essentially weightless insulating string of length $l$. Each ball has mass $M$ in the gravity field $g$ and when charged can be considered a point charge.

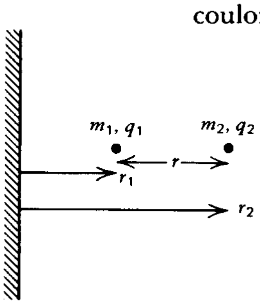



A total charge $Q$ is deposited on the two balls of the electroscope. The angle $\theta$ from the normal obeys a relation of the form

$$\tan\theta \sin^2\theta = \text{const}$$

What is the constant?

7. Two point charges $q_1$ and $q_2$ in vacuum with respective masses $m_1$ and $m_2$ attract (or repel) each other via the coulomb force.




(a) Write a single differential equation for the distance between the charges $r = r_2 - r_1$. What is the effective mass of the charges? (**Hint:** Write Newton's law for each charge and take a mass-weighted difference.)

(b) If the two charges are released from rest at $t = 0$ when a distance $r_0$ from one another, what is their relative velocity $v = dr/dt$ as a function of $r$? **Hint:**

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr} = \frac{d}{dr}\left(\frac{1}{2}v^2\right)$$

(c) What is their position as a function of time? Separately consider the cases when the charges have the same or opposite polarity. **Hint:**

Let $$u=\sqrt{r}$$

$$\int \frac{u^2\,du}{\sqrt{a^2-u^2}}=-\frac{u}{2}\sqrt{a^2-u^2}+\frac{a^2}{2}\sin^{-1}\frac{u}{a}$$

$$\int \frac{u^2\,du}{\sqrt{u^2-a^2}}=\frac{u}{2}\sqrt{u^2-a^2}+\frac{a^2}{2}\ln\left(u+\sqrt{u^2-a^2}\right)$$

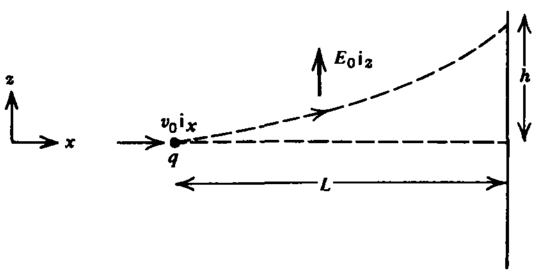
(d) If the charges are of opposite polarity, at what time will they collide? (**Hint:** If you get a negative value of time, check your signs of square roots in (b).)

(e) If the charges are taken out of the vacuum and placed in a viscous medium, the velocity rather than the acceleration is proportional to the force

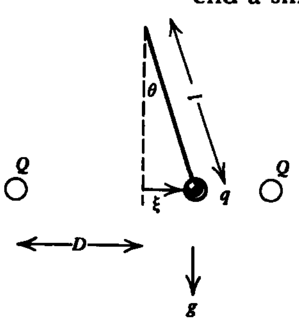
$$\beta_1\mathbf{v}_1=\mathbf{f}_1,\qquad \beta_2\mathbf{v}_2=\mathbf{f}_2$$

where $\beta_1$ and $\beta_2$ are the friction coefficients for each charge. Repeat parts (a)–(d) for this viscous dominated motion.

8. A charge $q$ of mass $m$ with initial velocity $\mathbf{v}=v_0\mathbf{i}_x$ is injected at $x=0$ into a region of uniform electric field $\mathbf{E}=E_0\mathbf{i}_z$. A screen is placed at the position $x=L$. At what height $h$ does the charge hit the screen? Neglect gravity.




9. A pendulum with a weightless string of length $l$ has on its end a small sphere with charge $q$ and mass $m$. A distance $D$



away on either side of the pendulum mass are two fixed spheres each carrying a charge $Q$. The three spheres are of sufficiently small size that they can be considered as point charges and masses.

(a) Assuming the pendulum displacement $\xi$ to be small ($\xi \ll D$), show that Newton's law can be approximately written as

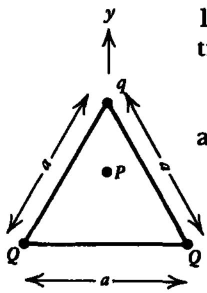
$$\frac{d^2\xi}{dt^2}+\omega_0^2\xi=0$$

What is $\omega_0^2$? **Hint:**

$$\sin\theta\approx\frac{\xi}{l},\quad \frac{1}{(D\pm\xi)^2}\approx\frac{1}{D^2}\mp\frac{2\xi}{D^3}$$

(b) At $t=0$ the pendulum is released from rest with $\xi=\xi_0$. What is the subsequent pendulum motion?

(c) For what values of $qQ$ is the motion unbounded with time?

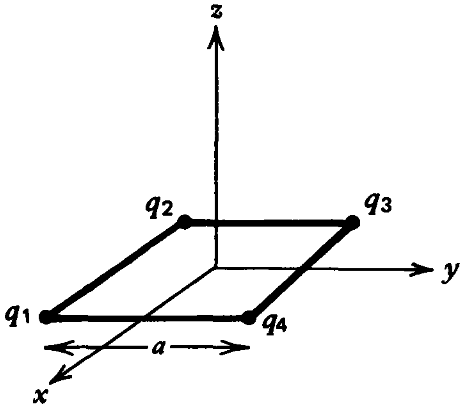



10. Charges $Q$, $Q$, and $q$ lie on the corners of an equilateral triangle with sides of length $a$.

(a) What is the force on the charge $q$?

(b) What must $q$ be for $\mathbf{E}$ to be zero half-way up the altitude at $P$?

11. Find the electric field along the $z$ axis due to four equal magnitude point charges $q$ placed on the vertices of a square with sides of length $a$ in the $xy$ plane centered at the origin



when:

(a) the charges have the same polarity, $q_1 = q_2 = q_3 = q_4 \equiv q$;
(b) the charges alternate in polarity, $q_1 = q_3 \equiv q$, $q_2 = q_4 \equiv -q$;
(c) the charges are $q_1 = q_2 \equiv q$, $q_3 = q_4 \equiv -q$.

Section 2.3

12. Find the total charge in each of the following distributions where $a$ is a constant parameter:

(a) An infinitely long line charge with density

$$\lambda(z) = \lambda_0 e^{-|z|/a}$$

(b) A spherically symmetric volume charge distributed over all space

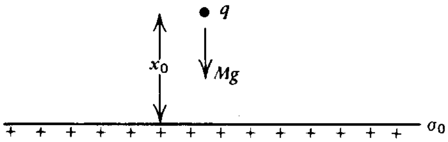
$$\rho(r) = \frac{\rho_0}{[1 + r/a]^4}$$

**(Hint:** Let $u = 1 + r/a$.)

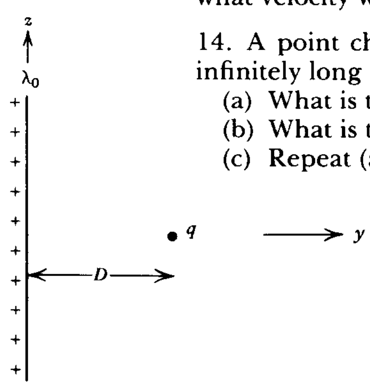
(c) An infinite sheet of surface charge with density

$$\sigma(x, y) = \frac{\sigma_0 e^{-|x|/a}}{[1 + (y/b)^2]}$$

13. A point charge $q$ with mass $M$ in a gravity field $g$ is released from rest a distance $x_0$ above a sheet of surface charge with uniform density $\sigma_0$.



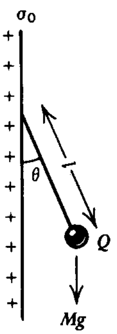

(a) What is the position of the charge as a function of time?
(b) For what value of $\sigma_0$ will the charge remain stationary?
(c) If $\sigma_0$ is less than the value of (b), at what time and with what velocity will the charge reach the sheet?




14. A point charge $q$ at $z = 0$ is a distance $D$ away from an infinitely long line charge with uniform density $\lambda_0$.

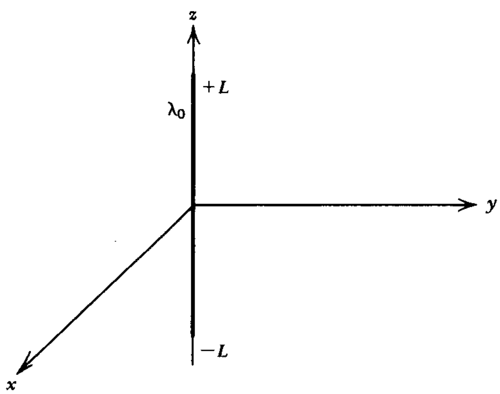
(a) What is the force on the point charge $q$?
(b) What is the force on the line charge?
(c) Repeat (a) and (b) if the line charge has a distribution

$$\lambda(z) = \frac{\lambda_0 |z|}{a}$$

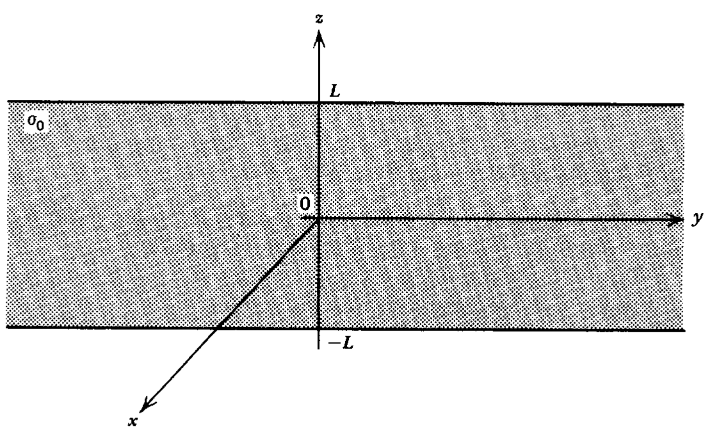



15. A small sphere of mass $M$ in a gravity field $g$ carrying a charge $Q$ is connected by a massless string to a sheet of surface charge of the same polarity with density $\sigma_0$. What is the angle $\theta$ between the sheet and charge?

16. A line charge $\lambda$ along the $z$ axis extends over the interval $-L \leq z \leq L$.




(a)




(b)

(a) Find the electric field in the $z = 0$ plane.

(b) Using the results of (a) find the electric field in the $z = 0$ plane due to an infinite strip ($-\infty \leq y \leq \infty$) of height $2L$ with surface charge density $\sigma_0$. Check your results with the text for $L \to \infty$. **Hint:** Let $u = x^2 + y^2$

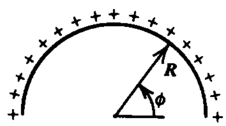
$$\int \frac{du}{u\sqrt{u-x^2}\sqrt{L^2+u}} = \frac{1}{Lx}\sin^{-1}\left(\frac{(L^2-x^2)u - 2L^2x^2}{u(L^2+x^2)}\right)$$

17. An infinitely long hollow semi-cylinder of radius $R$ carries a uniform surface charge distribution $\sigma_0$.

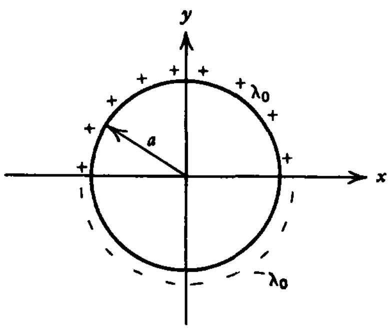
(a) What is the electric field along the axis of the cylinder?

(b) Use the results of (a) to find the electric field along the axis due to a semi-cylinder of volume charge $\rho_0$.

(c) Repeat (a) and (b) to find the electric field at the center of a uniformly surface or volume charged hemisphere.



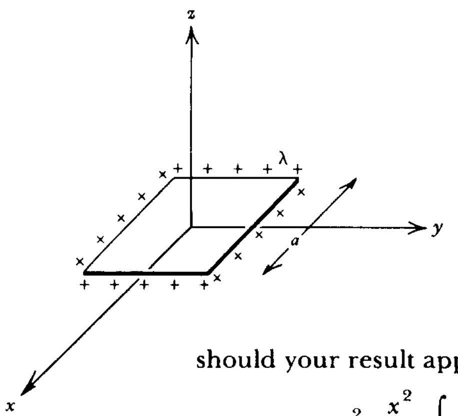

18. (a) Find the electric field along the $z$ axis of a circular loop centered in the $xy$ plane of radius $a$ carrying a uniform line charge $\lambda_0$ for $y > 0$ and $-\lambda_0$ for $y < 0$.




(b) Use the results of (a) to find the electric field along the $z$ axis of a circular disk of radius $a$ carrying a uniform surface charge $\sigma_0$ for $y > 0$ and $-\sigma_0$ for $y < 0$.

19. (a) Find the electric field along the $z$ axis due to a square loop with sides of length $a$ centered about the $z$ axis in the $xy$ plane carrying a uniform line charge $\lambda$. What should your result approach for $z \gg a$?

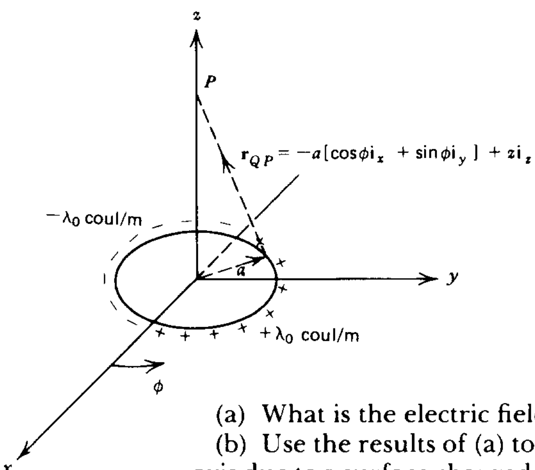
(b) Use the results of (a) to find the electric field along the $z$ axis due to a square of uniform surface charge $\sigma_0$. What




should your result approach as $a \to \infty$? **Hint:** Let

$$u = z^2 + \frac{x^2}{4}, \quad \int \frac{du}{u\sqrt{2u - z^2}} = \frac{2}{|z|} \tan^{-1} \sqrt{\frac{2u - z^2}{z^2}}$$

20. A circular loop of radius $a$ in the $xy$ plane has a uniform line charge distribution $\lambda_0$ for $y > 0$ and $-\lambda_0$ for $y < 0$.




(a) What is the electric field along the $z$ axis?

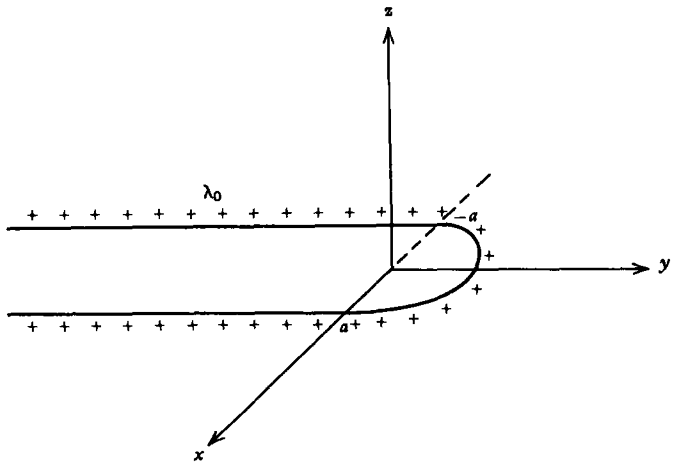
(b) Use the results of (a) to find the electric field along the $z$ axis due to a surface charged disk, whose density is $\sigma_0$ for $y > 0$ and $-\sigma_0$ for $y < 0$. **Hint:**

$$\int \frac{r^2\,dr}{(r^2 + z^2)^{3/2}} = -\frac{r}{\sqrt{r^2 + z^2}} + \ln\,(r + \sqrt{r^2 + z^2})$$

(c) Repeat (a) if the line charge has distribution $\lambda = \lambda_0 \sin\phi$.

(*d*) Repeat (b) if the surface charge has distribution $\sigma = \sigma_0 \sin\phi$.

21. An infinitely long line charge with density $\lambda_0$ is folded in half with both halves joined by a half-circle of radius $a$. What is the electric field along the $z$ axis passing through the center




of the circle. **Hint:**

$$\int \frac{x\,dx}{[x^2+a^2]^{3/2}} = \frac{-1}{[x^2+a^2]^{1/2}}$$

$$\int \frac{dx}{[x^2+a^2]^{3/2}} = \frac{x}{a^2[x^2+a^2]^{1/2}}$$

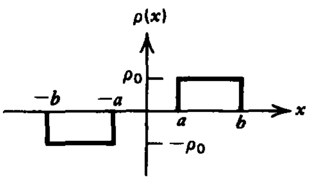
$$\mathbf{i}_r = \cos\phi\ \mathbf{i}_x + \sin\phi\ \mathbf{i}_y$$

Section 2.4

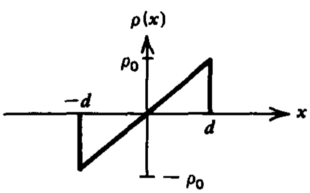
22. Find the total charge enclosed within each of the following volumes for the given electric fields:
 (a) $\mathbf{E} = Ar^2\mathbf{i}_r$ for a sphere of radius $R$;
 (b) $\mathbf{E} = Ar^2\mathbf{i}_r$ for a cylinder of radius $a$ and length $L$;
 (c) $\mathbf{E} = A(x\mathbf{i}_x + y\mathbf{i}_y)$ for a cube with sides of length $a$ having a corner at the origin.

23. Find the electric field everywhere for the following planar volume charge distributions:

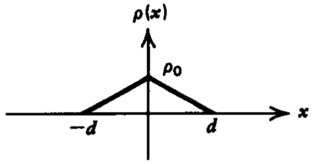
(a) $\rho(x) = \rho_0 e^{-|x|/a}, \quad -\infty \le x \le \infty$




(b) $\rho(x) = \begin{cases} -\rho_0, & -b \le x \le -a \\ \rho_0, & a \le x \le b \end{cases}$




(c) $\rho(x) = \dfrac{\rho_0 x}{d}, \quad -d \le x \le d$




(d) $\rho(x)=\begin{cases}\rho_0(1+x/d), & -d\le x\le 0\\ \rho_0(1-x/d), & 0\le x\le d\end{cases}$

24. Find the electric field everywhere for the following spherically symmetric volume charge distributions:

(a) $\rho(r)=\rho_0 e^{-r/a}, \quad 0\le r\le\infty$

$\left(\textbf{Hint:} \quad \int r^2 e^{-r/a}\,dr=-a\,e^{-r/a}[r^2+2a^2(r/a+1)].\right)$

(b) $\rho(r)=\begin{cases}\rho_1, & 0\le r<R_1\\ \rho_2, & R_1<r<R_2\end{cases}$

(c) $\rho(r)=\rho_0 r/R, \quad 0<r<R$

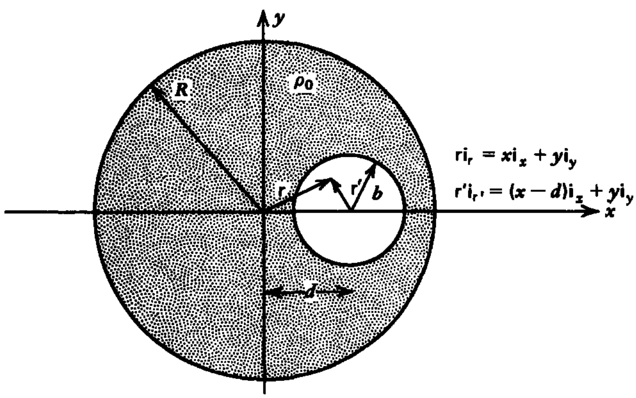
25. Find the electric field everywhere for the following cylindrically symmetric volume charge distributions:

(a) $\rho(\mathrm{r})=\rho_0\,e^{-\mathrm{r}/a}, \quad 0<\mathrm{r}<\infty$

$\left[\textbf{Hint:} \quad \int \mathrm{r}e^{-\mathrm{r}/a}\,d\mathrm{r}=-a^2\,e^{-\mathrm{r}/a}\,(\mathrm{r}/a+1).\right]$

(b) $\rho(\mathrm{r})=\begin{cases}\rho_1, & 0<\mathrm{r}<a\\ \rho_2, & a<\mathrm{r}<b\end{cases}$

(c) $\rho(\mathrm{r})=\rho_0\mathrm{r}/a, \quad 0<\mathrm{r}<a$

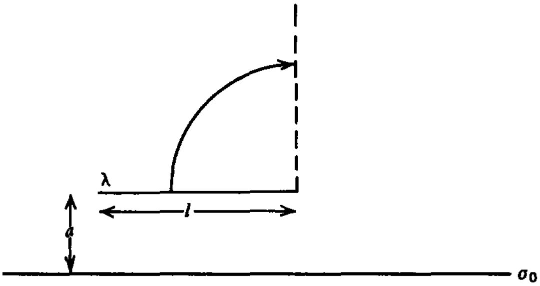



26. An infinitely long cylinder of radius $R$ with uniform volume charge density $\rho_0$ has an off-axis hole of radius $b$ with center a distance $d$ away from the center of the cylinder.

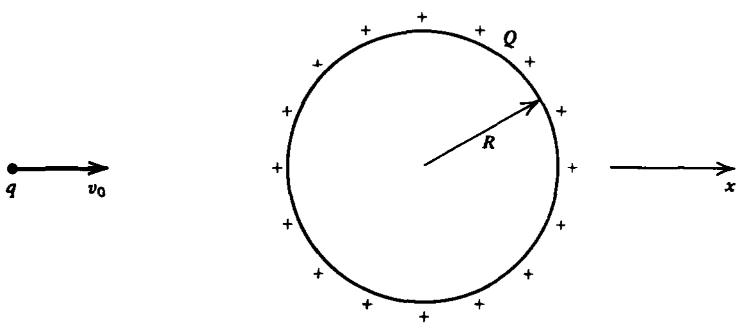
What is the electric field within the hole? (**Hint:** Replace the hole by the superposition of volume charge distributions of density $\rho_0$ and $-\rho_0$ and use the results of (27). Convert the cylindrical coordinates to Cartesian coordinates for ease of vector addition.)

Section 2.5

27. A line charge $\lambda$ of length $l$ lies parallel to an infinite sheet of surface charge $\sigma_0$. How much work is required to rotate the line charge so that it is vertical?




28. A point charge $q$ of mass $m$ is injected at infinity with initial velocity $v_0\mathbf{i}_x$ towards the center of a uniformly charged sphere of radius $R$. The total charge on the sphere $Q$ is the same sign as $q$.




(a) What is the minimum initial velocity necessary for the point charge to collide with the sphere?

(b) If the initial velocity is half of the result in (a), how close does the charge get to the sphere?

29. Find the electric field and volume charge distributions for the following potential distributions:

(a) $V = Ax^2$

(b) $V = Axyz$

(c) $V = Ar^2 \sin\phi + Brz$

(d) $V = Ar^2 \sin\theta \cos\phi$

30. Which of the following vectors can be an electric field? If so, what is the volume charge density?

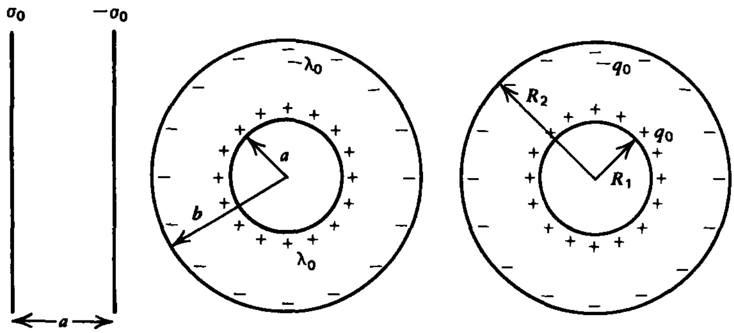
(a) $\mathbf{E} = ax^2y^2\mathbf{i}_x$

(b) $\mathbf{E} = a(\mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta)$

(c) $\mathbf{E} = a(y\mathbf{i}_x - x\mathbf{i}_y)$

(d) $\mathbf{E} = (a/r^2)[\mathbf{i}_r(1+\cos\phi) + \mathbf{i}_\phi \sin\phi]$

31. Find the potential difference $V$ between the following surface charge distributions:



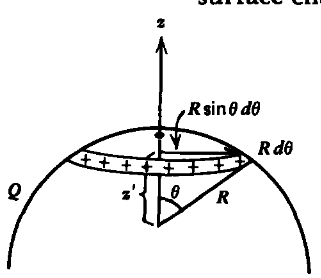

(a) (b) (c)

(a) Two parallel sheets of surface charge of opposite polarity $\pm\sigma_0$ and spacing $a$.

(b) Two coaxial cylinders of surface charge having infinite length and respective radii $a$ and $b$. The total charge per unit length on the inner cylinder is $\lambda_0$ while on the outer cylinder is $-\lambda_0$.

(c) Two concentric spheres of surface charge with respective radii $R_1$ and $R_2$. The inner sphere carries a uniformly distributed surface charge with total charge $q_0$. The outer sphere has total charge $-q_0$.

32. A hemisphere of radius $R$ has a uniformly distributed surface charge with total charge $Q$.




(a) Break the spherical surface into hoops of line charge of thickness $R\,d\theta$. What is the radius of the hoop, its height $z'$, and its total incremental charge $dq$?

(b) What is the potential along the $z$ axis due to this incremental charged hoop? Eliminate the dependence on $\theta$ and express all variables in terms of $z'$, the height of the differential hoop of line charge.

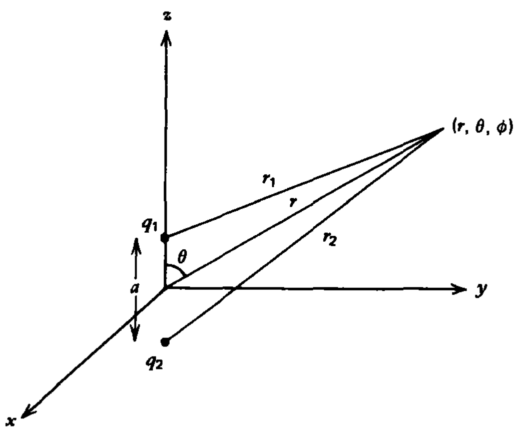
(c) What is the potential at any position along the $z$ axis due to the entire hemisphere of surface charge? **Hint:**

$$\int \frac{dz'}{[a+bz']^{1/2}} = \frac{2\sqrt{a+bz'}}{b}$$

(d) What is the electric field along the $z$ axis?

(e) If the hemisphere is uniformly charged throughout its volume with total charge $Q$, find the potential and electric field at all points along the $z$ axis. (**Hint:** $\int r\sqrt{z^2+r^2}\,dr = \frac{1}{3}(z^2+r^2)^{3/2}$.)

33. Two point charges $q_1$ and $q_2$ lie along the $z$ axis a distance $a$ apart.




(a) Find the potential at the coordinate $(r, \theta, \phi)$. (**Hint:** $r_1^2 = r^2 + (a/2)^2 - ar\cos\theta$.)

(b) What is the electric field?

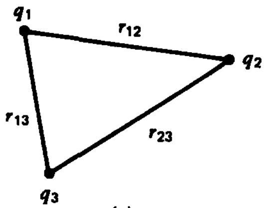
(c) An electric dipole is formed if $q_2 = -q_1$. Find an approximate expression for the potential and electric field for points far from the dipole, $r \gg a$.

(d) What is the equation of the field lines in this far field limit that is everywhere tangent to the electric field

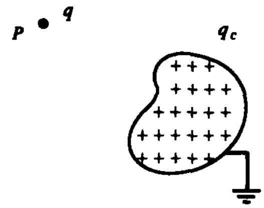
$$\frac{dr}{r\,d\theta} = \frac{E_r}{E_\theta}$$

Find the equation of the field line that passes through the point $(r = r_0, \theta = \pi/2)$. (**Hint:** $\int \cot\theta\,d\theta = \ln\sin\theta$.)

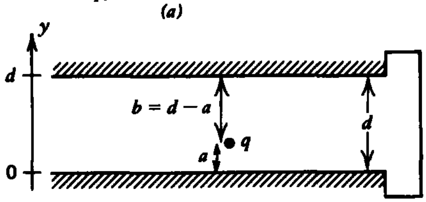
34. (a) Find the potentials $V_1$, $V_2$, and $V_3$ at the location of each of the three-point charges shown.




(a)




(d)




(g)

(b) Now consider another set of point charges $q_1'$, $q_2'$, and $q_3'$ at the same positions and calculate the potentials $V_1'$, $V_2'$, and $V_3'$. Verify by direct substitution that

$$q_1' V_1 + q_2' V_2 + q_3' V_3 = q_1 V_1' + q_2 V_2' + q_3 V_3'$$

The generalized result for any number of charges is called Green's reciprocity theorem,

$$\sum_{i=1}^{N} (q_i V_i' - q_i' V_i) = 0$$

(c) Show that Green's reciprocity theorem remains unchanged for perfect conductors as the potential on the conductor is constant. The $q_i$ is then the total charge on the conductor.

(d) A charge $q$ at the point $P$ is in the vicinity of a zero potential conductor. It is known that if the conductor is charged to a voltage $V_c$, the potential at the point $P$ in the absence of the point charge is $V_p$. Find the total charge $q_c$ induced on the grounded conductor. (**Hint:** Let $q_1 = q$, $q_2 = q_c$, $V_2 = 0$, $q_1' = 0$, $V_1' = V_p$, $V_2' = V_c$.)

(e) If the conductor is a sphere of radius $R$ and the point $P$ is a distance $D$ from the center of the sphere, what is $q_c$? Is this result related to the method of images?

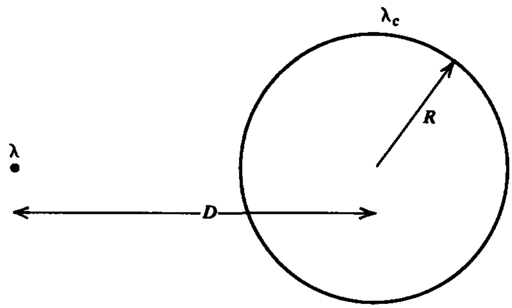
(f) A line charge $\lambda$ is a distance $D$ from the center of a grounded cylinder of radius $a$. What is the total charge per unit length induced on the cylinder?

(g) A point charge $q$ is between two zero potential perfect conductors. What is the total charge induced on each conducting surface? (**Hint:** Try $q_1 = q$, $q_2 = q(y = 0)$, $q_3 = q(y = d)$, $V_2 = 0$, $V_3 = 0$, $q_1' = 0$, $V_2' = V_0$, $V_3' = 0$.)

(h) A point charge $q$ travels at constant velocity $v_0$ between shorted parallel plate electrodes of spacing $d$. What is the short circuit current as a function of time?

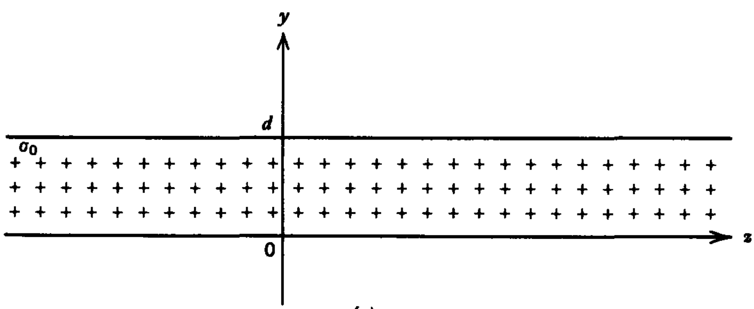
Section 2.6

35. An infinitely long line charge $\lambda$ is a distance $D$ from the center of a conducting cylinder of radius $R$ that carries a total charge per unit length $\lambda_c$. What is the force per unit length on

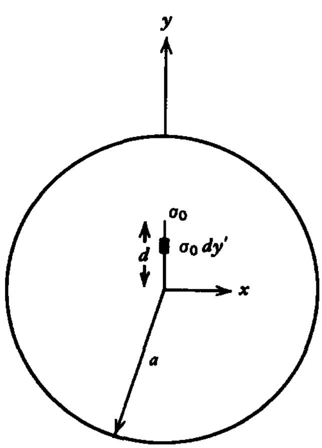



the cylinder? (**Hint:** Where can another image charge be placed with the cylinder remaining an equipotential surface?)

36. An infinitely long sheet of surface charge of width $d$ and uniform charge density $\sigma_0$ is placed in the $yz$ plane.




(a)




(b)

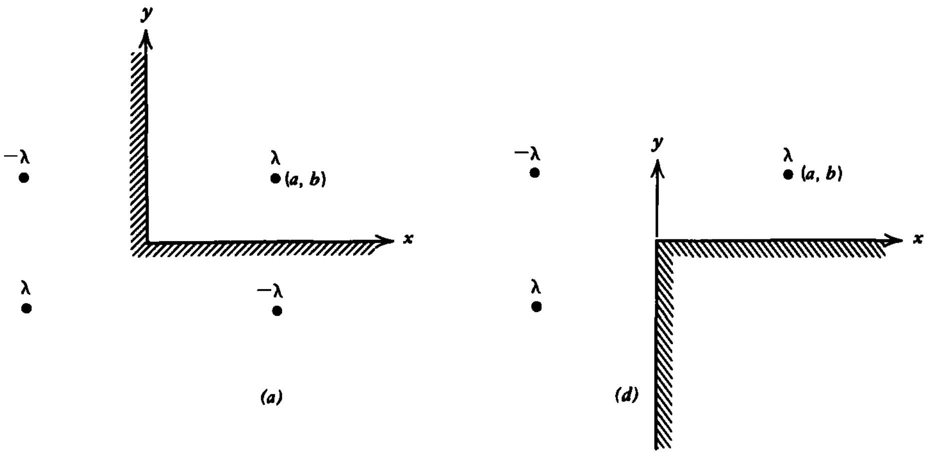
(a) Find the electric field everywhere in the $yz$ plane. (**Hint:** Break the sheet into differential line charge elements $d\lambda = \sigma_0\, dy'$.)

(b) An infinitely long conducting cylinder of radius $a$ surrounds the charged sheet that has one side along the axis of the cylinder. Find the image charge and its location due to an incremental line charge element $\sigma_0\, dy'$ at distance $y'$.

(c) What is the force per unit length on the cylinder? **Hint:**

$$\int \ln(1-cy')\, dy' = -\left(\frac{1-cy'}{c}\right)[\ln(1-cy')-1]$$

37. A line charge $\lambda$ is located at coordinate $(a, b)$ near a right-angled conducting corner.




(a) Verify that the use of the three image line charges shown satisfy all boundary conditions.

(b) What is the force per unit length on $\lambda$?

(c) What charge per unit length is induced on the surfaces $x = 0$ and $y = 0$?

(d) Now consider the inverse case when three line charges of alternating polarity $\pm\lambda$ are outside a conducting corner. What is the force on the conductor?

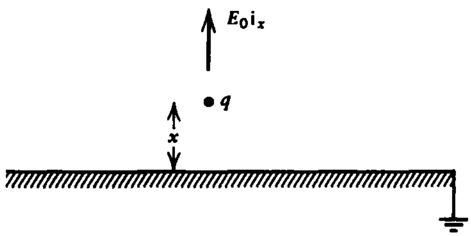
(e) Repeat (a)–(d) with point charges.

Section 2.7

38. A positive point charge $q$ within a uniform electric field $E_0\mathbf{i}_x$ is a distance $x$ from a grounded conducting plane.

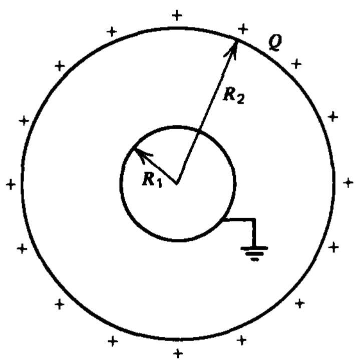
(a) At what value of $x$ is the force on the charge equal to zero?

(b) If the charge is initially at a position equal to half the value found in (a), what minimum initial velocity is necessary for the charge to continue on to $x = +\infty$? (**Hint:** $E_x = -dV/dx$.)




(c) If $E_0 = 0$, how much work is necessary to move the point charge from $x = d$ to $x = +\infty$?

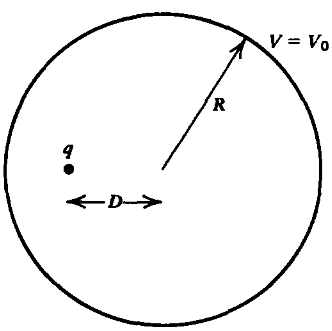
39. A sphere of radius $R_2$ having a uniformly distributed surface charge $Q$ surrounds a grounded sphere of radius $R_1$.



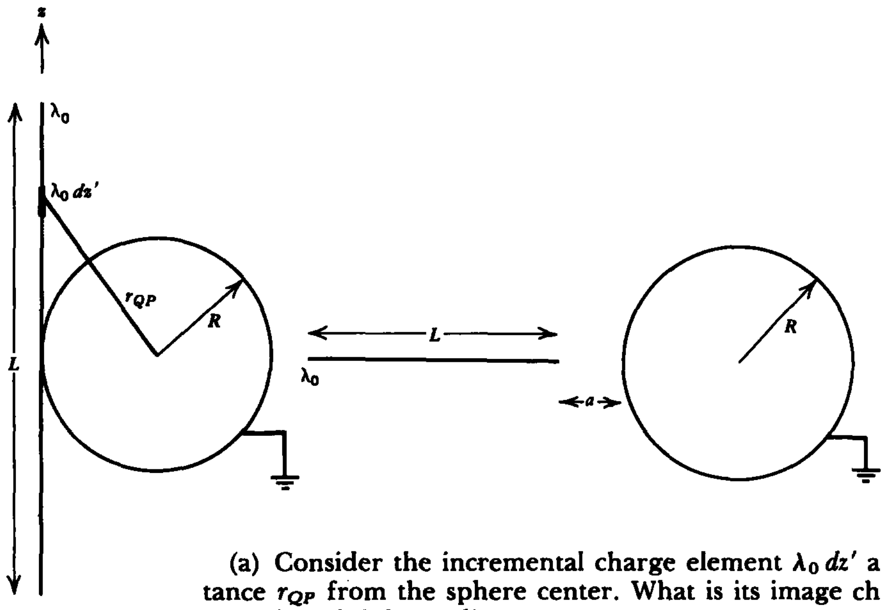

(a) What is the total charge induced on the grounded sphere? (**Hint:** Consider the image charge due to an incremental charge $dq = (Q/4\pi) \sin\theta \, d\theta \, d\phi$ at $r = R_2$.)

(b) What are the potential and electric field distributions everywhere?

40. A point charge $q$ located a distance $D$ $(D < R)$ from the center is within a conducting sphere of radius $R$ that is at constant potential $V_0$. What is the force on $q$?




41. A line charge of length $L$ with uniform density $\lambda_0$ is orientated the two ways shown with respect to a grounded sphere of radius $R$. For both cases:

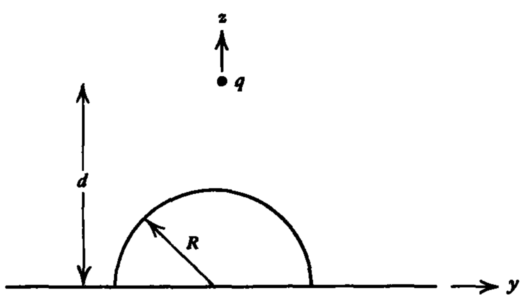



(a) Consider the incremental charge element $\lambda_0\, dz'$ a distance $r_{QP}$ from the sphere center. What is its image charge and where is it located?

(b) What is the total charge induced on the sphere? **Hint:**

$$\int \frac{dz'}{\sqrt{R^2+z'^2}} = \ln\left(z'+\sqrt{R^2+z'^2}\right)$$

42. A conducting hemispherical projection of radius $R$ is placed upon a ground plane of infinite extent. A point charge $q$ is placed a distance $d$ $(d>R)$ above the center of the hemisphere.

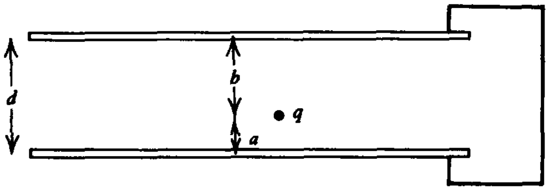



(a) What is the force on $q$? (**Hint:** Try placing three image charges along the $z$ axis to make the plane and hemisphere have zero potential.)

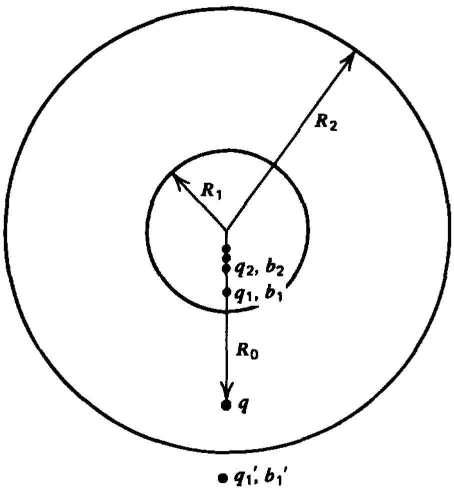
(b) What is the total charge induced on the hemisphere at $r=R$ and on the ground plane $|y|>R$? **Hint:**

$$\int \frac{r\, dr}{[r^2+d^2]^{3/2}} = \frac{-1}{\sqrt{r^2+d^2}}$$

43. A point charge $q$ is placed between two parallel grounded conducting planes a distance $d$ apart.




(a)




(c)

(a) The point charge $q$ a distance $a$ above the lower plane and a distance $b$ below the upper conductor has symmetrically located image charges. However, each image charge itself has an image in the opposite conductor. Show that an infinite number of image charges are necessary. What are the locations of these image charges?

(b) Show that the total charge on each conductor cannot be found by this method as the resulting series is divergent.

(c) Now consider a point charge $q$, a radial distance $R_0$ from the center of two concentric grounded conducting spheres of radii $R_1$ and $R_2$. Show that an infinite number of image charges in each sphere are necessary where, if we denote the $n$th image charge in the smaller sphere as $q_n$ a distance $b_n$ from the center and the $n$th image charge in the outer sphere as $q_n'$ a distance $b_n'$ from the center, then

$$q_{n+1} = -\frac{R_1}{b_n'} q_n', \qquad q_{n+1}' = -\frac{R_2}{b_n} q_n$$

$$b_{n+1} = \frac{R_1^2}{b_n'}, \qquad b_{n+1}' = \frac{R_2^2}{b_n}$$

(d) Show that the equations in (c) can be simplified to

$$q_{n+1} - q_{n-1}\left(\frac{R_1}{R_2}\right) = 0$$

$$b_{n+1} - b_{n-1}\left(\frac{R_1}{R_2}\right)^2 = 0$$

(e) Try power-law solutions

$$q_n = A\lambda^n, \qquad b_n = B\alpha^n$$

and find the characteristic values of $\lambda$ and $\alpha$ that satisfy the equations in (d).

(f) Taking a linear combination of the solutions in (e), evaluate the unknown amplitude coefficients by substituting in values for $n = 1$ and $n = 2$. What are all the $q_n$ and $b_n$?

(g) What is the total charge induced on the inner sphere?

(**Hint:** $\sum_{n=1}^{\infty} a^n = a/(1-a)$ for $a < 1$)

(h) Using the solutions of ($f$) with the difference relations of (c), find $q'_n$ and $b'_n$.

(i) Show that $\sum_{n=1}^{\infty} q'_n$ is not a convergent series so that the total charge on the outer sphere cannot be found by this method.

(j) Why must the total induced charge on both spheres be $-q$? What then is the total induced charge on the outer sphere?

(k) Returning to our original problem in (a) and (b) of a point charge between parallel planes, let the radii of the spheres approach infinity such that the distances

$$d = R_2 - R_1, \qquad a = R_2 - R_0, \qquad b = R_0 - R_1$$

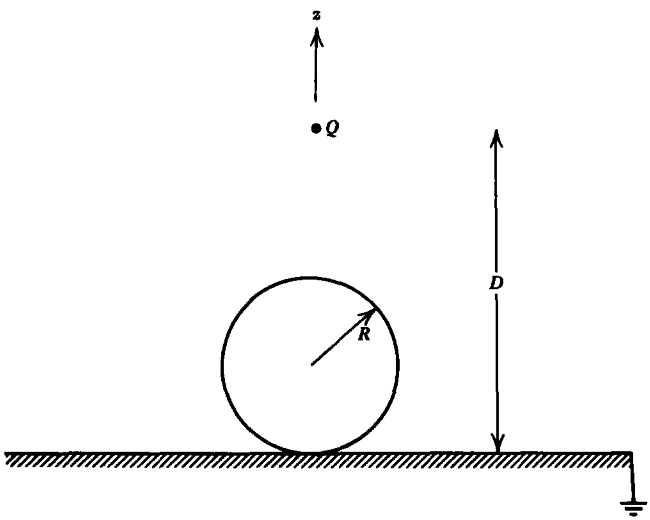
remains finite. What is the total charge induced on each plane conductor?

44. A point charge $Q$ is a distance $D$ above a ground plane. Directly below is the center of a small conducting sphere of radius $R$ that rests on the plane.

(a) Find the first image charges and their positions in the sphere and in the plane.

(b) Now find the next image of each induced in the other. Show that two sets of image charges are induced on the sphere where each obey the difference equations

$$q_{n+1} = \frac{q_n R}{2R - b_n}, \qquad b_{n+1} = \frac{R^2}{2R - b_n}$$




(c) Eliminating the $b_n$, show that the governing difference equation is

$$\frac{1}{q_{n+1}}-\frac{2}{q_n}+\frac{1}{q_{n-1}}=0$$

Guess solutions of the form

$$P_n=1/q_n=A\lambda^n$$

and find the allowed values of $\lambda$ that satisfy the difference equation. (**Hint:** For double roots of $\lambda$ the total solution is of the form $P_n=(A_1+A_2n)\lambda^n$.)

(d) Find all the image charges and their positions in the sphere and in the plane.

(e) Write the total charge induced on the sphere in the form

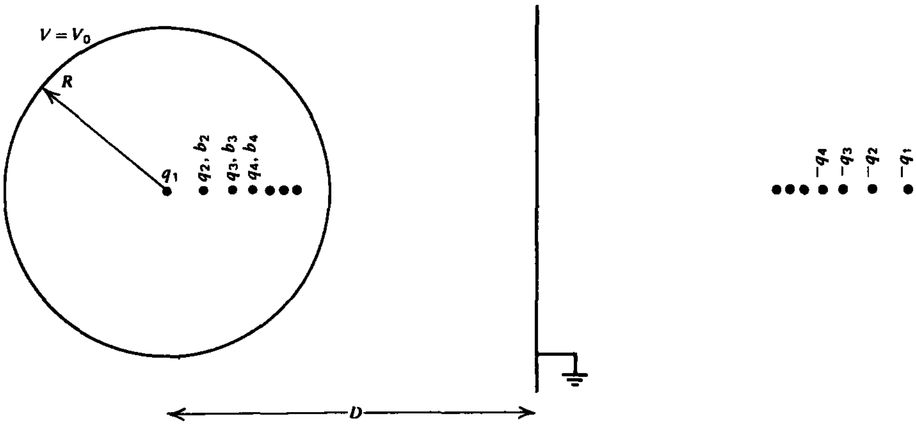
$$q_T=\sum_{n=1}^{\infty}\frac{A}{[1-an^2]}$$

What are $A$ and $a$?

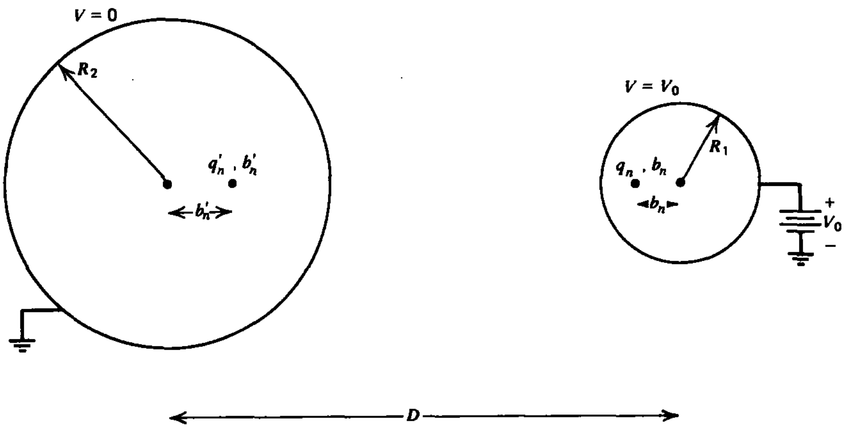
(f) We wish to generalize this problem to that of a sphere resting on the ground plane with an applied field $\mathbf{E}=-E_0\mathbf{i}_z$ at infinity. What must the ratio $Q/D^2$ be, such that as $Q$ and $D$ become infinite the field far from the sphere in the $\theta=\pi/2$ plane is $-E_0\mathbf{i}_z$?

(g) In this limit what is the total charge induced on the sphere? (**Hint:** $\sum_{n=1}^{\infty}\frac{1}{n^2}=\pi^2/6$.)

45. A conducting sphere of radius $R$ at potential $V_0$ has its center a distance $D$ from an infinite grounded plane.




(a)




(f)

(a) Show that an infinite number of image charges in the plane and in the sphere are necessary to satsify the boundary conditions

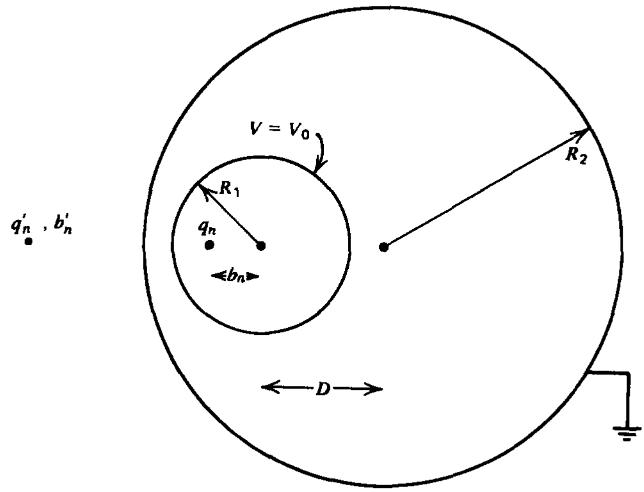
$$q_n = \frac{q_{n-1}R}{2D-b_{n-1}}, \qquad b_n = \frac{R^2}{2D-b_{n-1}}$$

What are $q_1$ and $q_2$?

(b) Show that the governing difference equation is

$$\frac{1}{q_{n-1}} - \frac{c}{q_n} + \frac{1}{q_{n+1}} = 0$$

What is $c$?




(f)

(c) Solve the difference equation in (b) assuming solutions of the form

$$P_n = 1/q_n = A\lambda^n$$

What values of $\lambda$ satisfy (b)? **Hint:**

$$c/2 + \sqrt{(c/2)^2 - 1} = \frac{1}{c/2 - \sqrt{(c/2)^2 - 1}}$$

(d) What is the position of each image charge? What is the limiting position of the image charges as $n \to \infty$?

(e) Show that the capacitance (the ratio of the total charge on the sphere to the voltage $V_0$) can be written as an infinite series

$$C = C_0(\lambda^2 - 1)\left(\frac{1}{\lambda^2 - 1} + \frac{\lambda}{\lambda^4 - 1} + \frac{\lambda^2}{\lambda^6 - 1} + \frac{\lambda^3}{\lambda^8 - 1} + \cdots\right)$$

What are $C_0$ and $\lambda$?

(f) Show that the image charges and their positions for two spheres obey the difference equations

$$q_{n+1} = \mp\frac{q'_n R_1}{D - b'_n}, \qquad b_{n+1} = \pm\frac{R_1^2}{D - b'_n}$$

$$q'_n = -\frac{R_2 q_n}{D \mp b_n}, \qquad b'_n = \frac{R_2^2}{D \mp b_n}$$

where we use the upper signs for adjacent spheres and the lower signs when the smaller sphere of radius $R_1$ is inside the larger one.

(g) Show that the governing difference equation is of the form

$$P_{n+1} \mp cP_n + P_{n-1} = 0$$

What are $P_n$ and $c$?

(h) Solve (g) assuming solutions of the form

$$P_n = A\lambda^n$$

(i) Show that the capacitance is of the form

$$C = C_0(1-\xi^2)\left(\frac{1}{1-\xi^2} + \frac{\lambda}{1-\xi^2\lambda^2} + \frac{\lambda^2}{1-\xi^4\lambda^4} + \cdots\right)$$

What are $C_0$, $\xi$, and $\lambda$?

(j) What is the capacitance when the two spheres are concentric so that $D = 0$. (**Hint:** $\sum_{n=0}^{\infty} a^n = 1/(1-a)$ for $a < 1$.)

# chapter 3

## *polarization and conduction*

The presence of matter modifies the electric field because even though the material is usually charge neutral, the field within the material can cause charge motion, called conduction, or small charge displacements, called polarization. Because of the large number of atoms present, $6.02 \times 10^{23}$ per gram molecular weight (Avogadro's number), slight imbalances in the distribution have large effects on the fields inside and outside the materials. We must then self-consistently solve for the electric field with its effect on charge motion and redistribution in materials, with the charges. resultant effect back as another source of electric field.

## 3-1 POLARIZATION

In many electrically insulating materials, called dielectrics, electrons are tightly bound to the nucleus. They are not mobile, but if an electric field is applied, the negative cloud of electrons can be slightly displaced from the positive nucleus, as illustrated in Figure 3-1*a*. The material is then said to have an electronic polarization. Orientational polarizability as in Figure 3-1*b* occurs in polar molecules that do not share their

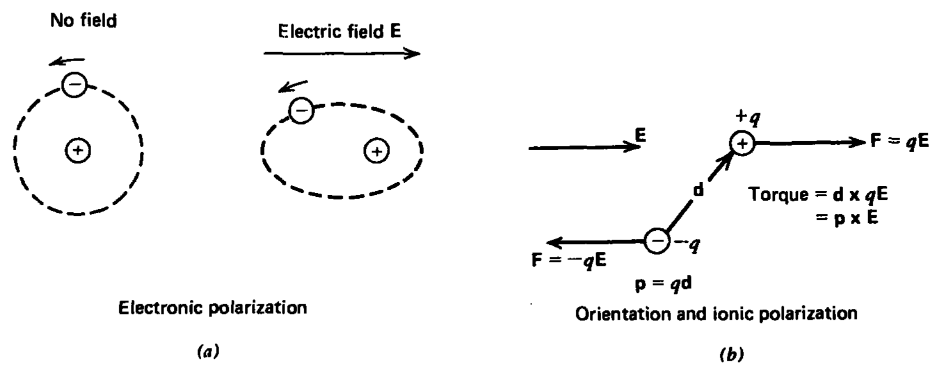



Figure 3-1 An electric dipole consists of two charges of equal magnitude but opposite sign, separated by a small vector distance **d**. (*a*) Electronic polarization arises when the average motion of the electron cloud about its nucleus is slightly displaced. (*b*) Orientation polarization arises when an asymmetric polar molecule tends to line up with an applied electric field. If the spacing **d** also changes, the molecule has ionic polarization.

electrons symmetrically so that the net positive and negative charges are separated. An applied electric field then exerts a torque on the molecule that tends to align it with the field. The ions in a molecule can also undergo slight relative displacements that gives rise to ionic polarizability.

The slightly separated charges for these cases form electric dipoles. Dielectric materials have a distribution of such dipoles. Even though these materials are charge neutral because each dipole contains an equal amount of positive and negative charges, a net charge can accumulate in a region if there is a local imbalance of positive or negative dipole ends. The net polarization charge in such a region is also a source of the electric field in addition to any other free charges.

### 3-1-1 The Electric Dipole

The simplest model of an electric dipole, shown in Figure 3-2*a*, has a positive and negative charge of equal magnitude $q$ separated by a small vector displacement **d** directed from the negative to positive charge along the $z$ axis. The electric potential is easily found at any point $P$ as the superposition of potentials from each point charge alone:

$$V=\frac{q}{4\pi\varepsilon_0 r_+}-\frac{q}{4\pi\varepsilon_0 r_-} \tag{1}$$

The general potential and electric field distribution for any displacement **d** can be easily obtained from the geometry relating the distances $r_+$ and $r_-$ to the spherical coordinates $r$ and $\theta$. By symmetry, these distances are independent of the angle $\phi$. However, in dielectric materials the separation between charges are of atomic dimensions and so are very small compared to distances of interest far from the dipole. So, with $r_+$ and $r_-$ much greater than the dipole spacing $d$, we approximate them as

$$\lim_{r\gg d}\begin{array}{l} r_+\approx r-\dfrac{d}{2}\cos\theta \\ r_-\approx r+\dfrac{d}{2}\cos\theta \end{array} \tag{2}$$

Then the potential of (1) is approximately

$$V\approx\frac{qd\cos\theta}{4\pi\varepsilon_0 r^2}=\frac{\mathbf{p}\cdot\mathbf{i}_r}{4\pi\varepsilon_0 r^2} \tag{3}$$

where the vector **p** is called the dipole moment and is defined as

$$\mathbf{p}=q\mathbf{d}\ (\text{coul-m}) \tag{4}$$

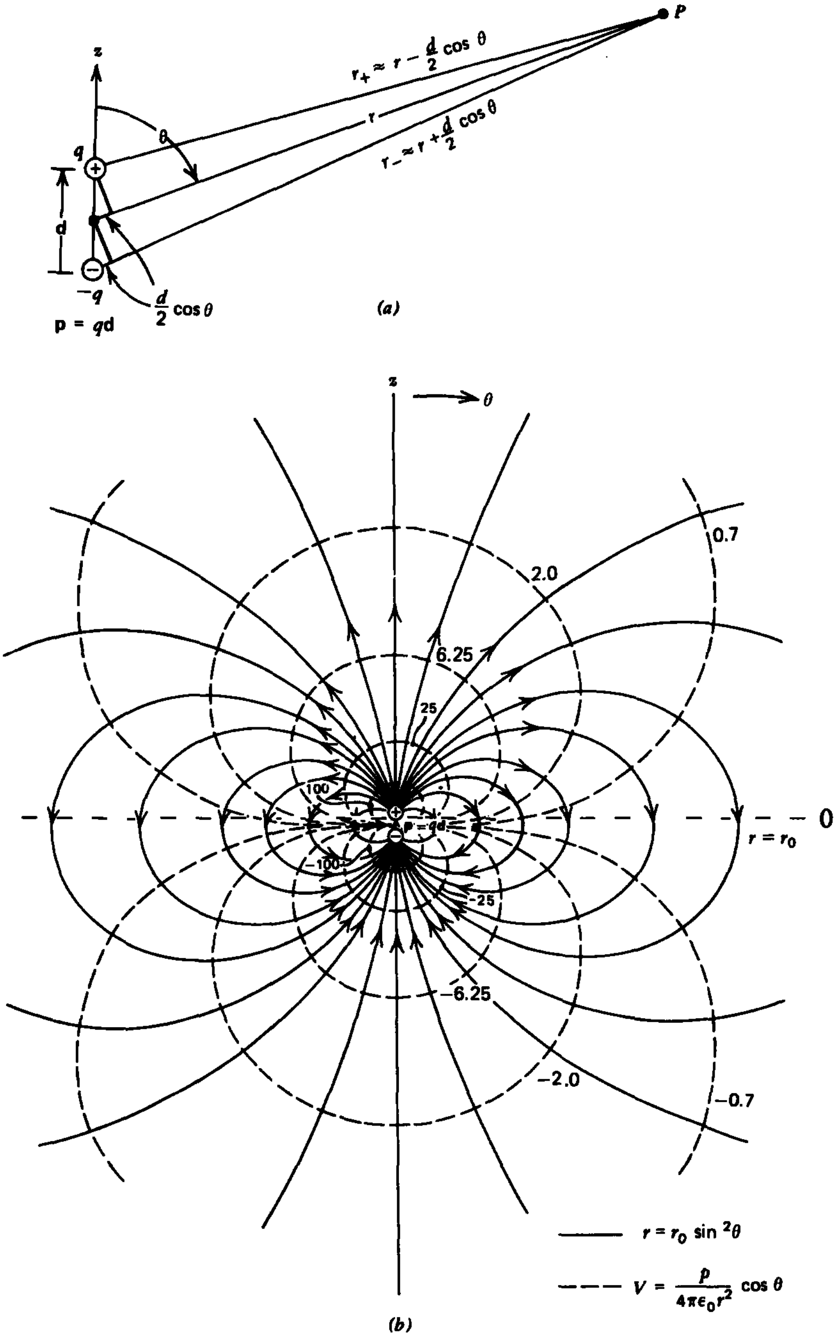



Figure 3-2 (*a*) The potential at any point $P$ due to the electric dipole is equal to the sum of potentials of each charge alone. (*b*) The equi-potential (dashed) and field lines (solid) for a point electric dipole calibrated for $4\pi\varepsilon_0/p = 100$.

Because the separation of atomic charges is on the order of 1 Å($10^{-10}$ m) with a charge magnitude equal to an integer multiple of the electron charge ($q = 1.6\times 10^{-19}$ coul), it is convenient to express dipole moments in units of debyes defined as 1 debye $= 3.33\times 10^{-30}$ coul-m so that dipole moments are of order $p = 1.6\times 10^{-29}$ coul-m $\approx 4.8$ debyes. The electric field for the point dipole is then

$$\mathbf{E} = -\nabla V = \frac{p}{4\pi\varepsilon_0 r^3}[2\cos\theta\,\mathbf{i}_r + \sin\theta\,\mathbf{i}_\theta] = \frac{3(\mathbf{p}\cdot\mathbf{i}_r)\mathbf{i}_r - \mathbf{p}}{4\pi\varepsilon_0 r^3} \tag{5}$$

the last expressions in (3) and (5) being coordinate independent. The potential and electric field drop off as a single higher power in $r$ over that of a point charge because the net charge of the dipole is zero. As one gets far away from the dipole, the fields due to each charge tend to cancel. The point dipole equipotential and field lines are sketched in Figure 3-2$b$. The lines tangent to the electric field are

$$\frac{dr}{r\,d\theta} = \frac{E_r}{E_\theta} = 2\cot\theta \Rightarrow r = r_0\sin^2\theta \tag{6}$$

where $r_0$ is the position of the field line when $\theta = \pi/2$. All field lines start on the positive charge and terminate on the negative charge.

If there is more than one pair of charges, the definition of dipole moment in (4) is generalized to a sum over all charges,

$$\mathbf{p} = \sum_{\text{all charges}} q_i\mathbf{r}_i \tag{7}$$

where $\mathbf{r}_i$ is the vector distance from an origin to the charge $q_i$ as in Figure 3-3. When the net charge in the system is zero ($\sum q_i = 0$), the dipole moment is independent of the choice of origins for if we replace $\mathbf{r}_i$ in (7) by $\mathbf{r}_i + \mathbf{r}_0$, where $\mathbf{r}_0$ is the constant vector distance between two origins:

$$\begin{aligned}\mathbf{p} &= \sum q_i(\mathbf{r}_i + \mathbf{r}_0)\\ &= \sum q_i\mathbf{r}_i + \mathbf{r}_0 \cancelto{0}{\sum q_i}\\ &= \sum q_i\mathbf{r}_i\end{aligned} \tag{8}$$

The result is unchanged from (7) as the constant $\mathbf{r}_0$ could be taken outside the summation.

If we have a continuous distribution of charge (7) is further generalized to

$$\mathbf{p} = \int_{\text{all } q} \mathbf{r}\,dq \tag{9}$$

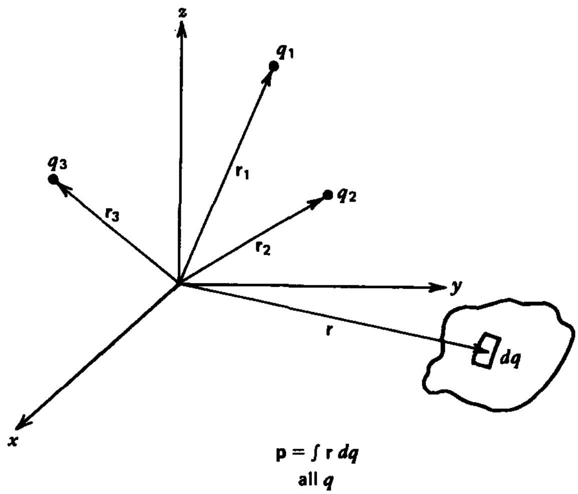



Figure 3-3 The dipole moment can be defined for any distribution of charge. If the net charge in the system is zero, the dipole moment is independent of the location of the origin.

Then the potential and electric field far away from any dipole distribution is given by the coordinate independent expressions in (3) and (5) where the dipole moment **p** is given by (7) and (9).

### 3-1-2 Polarization Charge

We enclose a large number of dipoles within a dielectric medium with the differential-sized rectangular volume $\Delta x\, \Delta y\, \Delta z$ shown in Figure 3-4*a*. All totally enclosed dipoles, being charge neutral, contribute no net charge within the volume. Only those dipoles within a distance $\mathbf{d}\cdot\mathbf{n}$ of each surface are cut by the volume and thus contribute a net charge where $\mathbf{n}$ is the unit normal to the surface at each face, as in Figure 3-4*b*. If the number of dipoles per unit volume is $N$, it is convenient to define the number density of dipoles as the polarization vector **P**:

$$\mathbf{P}=N\mathbf{p}=Nq\mathbf{d} \tag{10}$$

The net charge enclosed near surface 1 is

$$dq_1=(Nqd_x)|_x\, \Delta y\, \Delta z=P_x(x)\, \Delta y\, \Delta z \tag{11}$$

while near the opposite surface 2

$$dq_2=-(Nqd_x)|_{x+\Delta x}\, \Delta y\, \Delta z=-P_x(x+\Delta x)\, \Delta y\, \Delta z \tag{12}$$

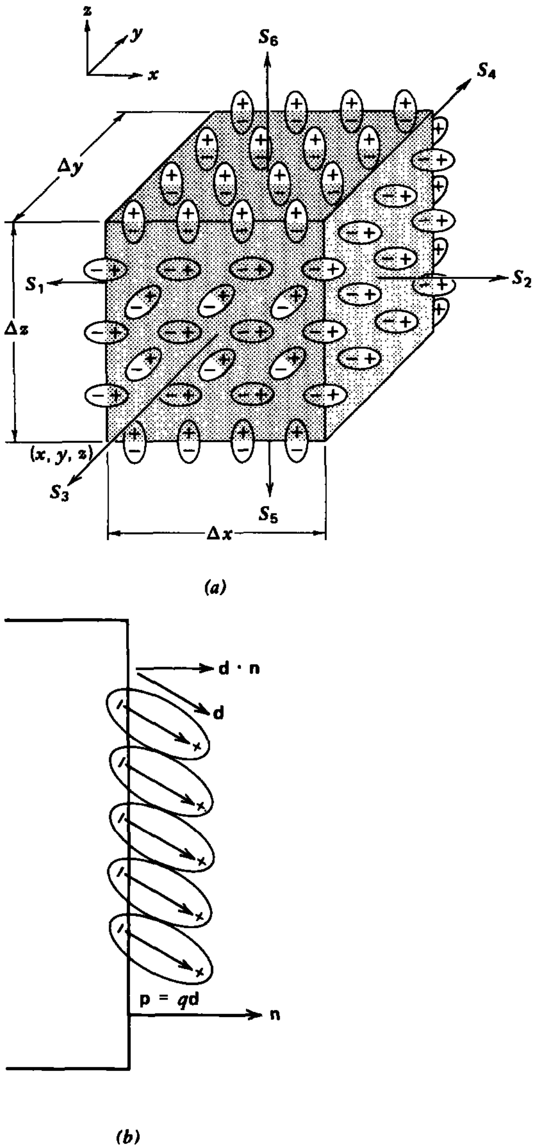



Figure 3-4 (*a*) The net charge enclosed within a differential-sized volume of dipoles has contributions only from the dipoles that are cut by the surfaces. All totally enclosed dipoles contribute no net charge. (*b*) Only those dipoles within a distance $\mathbf{d} \cdot \mathbf{n}$ of the surface are cut by the volume.

where we assume that $\Delta y$ and $\Delta z$ are small enough that the polarization $\mathbf{P}$ is essentially constant over the surface. The polarization can differ at surface 1 at coordinate $x$ from that at surface 2 at coordinate $x + \Delta x$ if either the number density

$N$, the charge $q$, or the displacement $\mathbf{d}$ is a function of $x$. The difference in sign between (11) and (12) is because near $S_1$ the positive charge is within the volume, while near $S_2$ negative charge remains in the volume. Note also that only the component of $\mathbf{d}$ normal to the surface contributes to the volume of net charge.

Similarly, near the surfaces $S_3$ and $S_4$ the net charge enclosed is

$$\begin{aligned} dq_3 &= (Nqd_y)|_y\, \Delta x\, \Delta z = P_y(y)\, \Delta x\, \Delta z \\ dq_4 &= -(Nqd_y)|_{y+\Delta y}\, \Delta x\, \Delta z = -P_y(y+\Delta y)\, \Delta x\, \Delta z \end{aligned} \tag{13}$$

while near the surfaces $S_5$ and $S_6$ with normal in the $z$ direction the net charge enclosed is

$$\begin{aligned} dq_5 &= (Nqd_z)|_z\, \Delta x\, \Delta y = P_z(z)\, \Delta x\, \Delta y \\ dq_6 &= -(Nqd_z)|_{z+\Delta z}\, \Delta x\, \Delta y = -P_z(z+\Delta z)\, \Delta x\, \Delta y \end{aligned} \tag{14}$$

The total charge enclosed within the volume is the sum of (11)–(14):

$$\begin{aligned} dq_T &= dq_1 + dq_2 + dq_3 + dq_4 + dq_5 + dq_6 \\ &= \left(\frac{P_x(x) - P_x(x+\Delta x)}{\Delta x} + \frac{P_y(y) - P_y(y+\Delta y)}{\Delta y} \right. \\ &\quad \left. + \frac{P_z(z) - P_z(z+\Delta z)}{\Delta z}\right) \Delta x\, \Delta y\, \Delta z \end{aligned} \tag{15}$$

As the volume shrinks to zero size, the polarization terms in (15) define partial derivatives so that the polarization volume charge density is

$$\rho_p = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \frac{dq_T}{\Delta x\, \Delta y\, \Delta z} = -\left(\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y} + \frac{\partial P_z}{\partial z}\right) = -\nabla \cdot \mathbf{P} \tag{16}$$

This volume charge is also a source of the electric field and needs to be included in Gauss's law

$$\nabla \cdot (\varepsilon_0 \mathbf{E}) = \rho_f + \rho_p = \rho_f - \nabla \cdot \mathbf{P} \tag{17}$$

where we subscript the free charge $\rho_f$ with the letter $f$ to distinguish it from the polarization charge $\rho_p$. The total polarization charge within a region is obtained by integrating (16) over the volume,

$$q_p = \int_V \rho_p\, dV = -\int_V \nabla \cdot \mathbf{P}\, dV = -\oint_S \mathbf{P} \cdot \mathbf{dS} \tag{18}$$

where we used the divergence theorem to relate the polarization charge to a surface integral of the polarization vector.

### 3-1-3 The Displacement Field

Since we have no direct way of controlling the polarization charge, it is convenient to cast Gauss's law only in terms of free charge by defining a new vector **D** as

$$\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P} \tag{19}$$

This vector **D** is called the displacement field because it differs from $\varepsilon_0\mathbf{E}$ due to the slight charge displacements in electric dipoles. Using (19), (17) can be rewritten as

$$\nabla\cdot(\varepsilon_0\mathbf{E}+\mathbf{P})=\nabla\cdot\mathbf{D}=\rho_f \tag{20}$$

where $\rho_f$ only includes the free charge and not the bound polarization charge. By integrating both sides of (20) over a volume and using the divergence theorem, the new integral form of Gauss's law is

$$\int_V \nabla\cdot\mathbf{D}\,dV=\oint_S \mathbf{D}\cdot d\mathbf{S}=\int_V \rho_f\,dV \tag{21}$$

In free space, the polarization **P** is zero so that $\mathbf{D}=\varepsilon_0\mathbf{E}$ and (20)–(21) reduce to the free space laws used in Chapter 2.

### 3-1-4 Linear Dielectrics

It is now necessary to find the constitutive law relating the polarization **P** to the applied electric field **E**. An accurate discussion would require the use of quantum mechanics, which is beyond the scope of this text. However, a simplified classical model can be used to help us qualitatively understand the most interesting case of a linear dielectric.

**(a) Polarizability**

We model the atom as a fixed positive nucleus with a surrounding uniform spherical negative electron cloud, as shown in Figure 3-5*a*. In the absence of an applied electric field, the dipole moment is zero because the center of charge for the electron cloud is coincident with the nucleus. More formally, we can show this using (9), picking our origin at the position of the nucleus:

$$\mathbf{p}=\cancelto{0}{Q(0)}-\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{R_0}\mathbf{i}_r\rho_0 r^3\sin\theta\,dr\,d\theta\,d\phi \tag{22}$$

Since the radial unit vector $\mathbf{i}_r$ changes direction in space, it is necessary to use Table 1-2 to write $\mathbf{i}_r$ in terms of the constant Cartesian unit vectors:

$$\mathbf{i}_r=\sin\theta\cos\phi\,\mathbf{i}_x+\sin\theta\sin\phi\,\mathbf{i}_y+\cos\theta\,\mathbf{i}_z \tag{23}$$

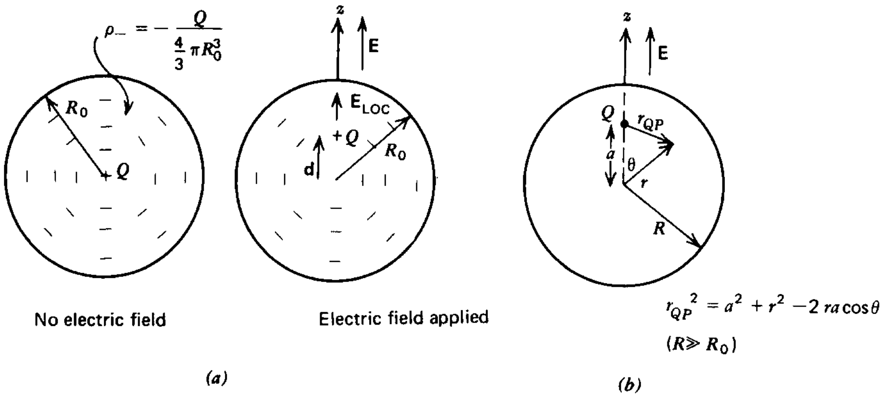



Figure 3-5 ($a$) A simple atomic classical model has a negative spherical electron cloud of small radius $R_0$ centered about a positive nucleus when no external electric field is present. An applied electric field tends to move the positive charge in the direction of the field and the negative charge in the opposite direction creating an electric dipole. ($b$) The average electric field within a large sphere of radius $R$ ($R \gg R_0$) enclosing many point dipoles is found by superposing the average fields due to each point charge.

When (23) is used in (22) the $x$ and $y$ components integrate to zero when integrated over $\phi$, while the $z$ component is zero when integrated over $\theta$ so that $\mathbf{p} = 0$.

An applied electric field tends to push the positive charge in the direction of the field and the negative charge in the opposite direction causing a slight shift $\mathbf{d}$ between the center of the spherical cloud and the positive nucleus, as in Figure 3-5$a$. Opposing this movement is the attractive coulombic force. Considering the center of the spherical cloud as our origin, the self-electric field within the cloud is found from Section 2.4.3$b$ as

$$E_r = -\frac{Qr}{4\pi\varepsilon_0 R_0^3} \tag{24}$$

In equilibrium the net force $\mathbf{F}$ on the positive charge is zero,

$$\mathbf{F} = Q\left(\mathbf{E}_{\text{Loc}} - \frac{Q\mathbf{d}}{4\pi\varepsilon_0 R_0^3}\right) = 0 \tag{25}$$

where we evaluate (24) at $r = d$ and $\mathbf{E}_{\text{Loc}}$ is the local polarizing electric field acting on the dipole. From (25) the equilibrium dipole spacing is

$$\mathbf{d} = \frac{4\pi\varepsilon_0 R_0^3}{Q}\mathbf{E}_{\text{Loc}} \tag{26}$$

so that the dipole moment is written as

$$\mathbf{p} = Q\mathbf{d} = \alpha\mathbf{E}_{\text{Loc}}, \qquad \alpha = 4\pi\varepsilon_0 R_0^3 \tag{27}$$

where $\alpha$ is called the polarizability.

**(b) The Local Electric Field**

If this dipole were isolated, the local electric field would equal the applied macroscopic field. However, a large number density $N$ of neighboring dipoles also contributes to the polarizing electric field. The electric field changes drastically from point to point within a small volume containing many dipoles, being equal to the superposition of fields due to each dipole given by (5). The macroscopic field is then the average field over this small volume.

We calculate this average field by first finding the average field due to a single point charge $Q$ a distance $a$ along the $z$ axis from the center of a spherical volume with radius $R$ much larger than the radius of the electron cloud ($R \gg R_0$) as in Figure 3-5$b$. The average field due to this charge over the spherical volume is

$$<\mathbf{E}> = \frac{1}{\frac{4}{3}\pi R^3}\int_{r=0}^{R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\frac{Q(r\mathbf{i}_r - a\mathbf{i}_z)r^2\sin\theta\,dr\,d\theta\,d\phi}{4\pi\varepsilon_0[a^2+r^2-2ra\cos\theta]^{3/2}} \tag{28}$$

where we used the relationships

$$r_{QP}^2 = a^2 + r^2 - 2ra\cos\theta, \qquad \mathbf{r}_{QP} = r\mathbf{i}_r - a\mathbf{i}_z \tag{29}$$

Using (23) in (28) again results in the $x$ and $y$ components being zero when integrated over $\phi$. Only the $z$ component is now nonzero:

$$<E_z> = \frac{Q}{\frac{4}{3}\pi R^3}\frac{2\pi}{(4\pi\varepsilon_0)}\int_{\theta=0}^{\pi}\int_{r=0}^{R}\frac{r^3(\cos\theta - a/r)\sin\theta\,dr\,d\theta}{[a^2+r^2-2ra\cos\theta]^{3/2}} \tag{30}$$

We introduce the change of variable from $\theta$ to $u$

$$u = r^2 + a^2 - 2ar\cos\theta, \qquad du = 2ar\sin\theta\,d\theta \tag{31}$$

so that (30) can be integrated over $u$ and $r$. Performing the $u$ integration first we have

$$\begin{aligned}<E_z> &= \frac{3Q}{8\pi R^3\varepsilon_0}\int_{r=0}^{R}\int_{(r-a)^2}^{(r+a)^2}\frac{r}{4a^2}\frac{(r^2-a^2-u)}{u^{3/2}}\,dr\,du\\ &= \frac{3Q}{8\pi R^3\varepsilon_0}\int_{r=0}^{R}\frac{r}{4a^2}\left(-\frac{2(r^2-a^2+u)}{u^{1/2}}\right)\Bigg|_{u=(r-a)^2}^{(r+a)^2}dr\\ &= -\frac{3Q}{8\pi R^3\varepsilon_0 a^2}\int_{r=0}^{R}r^2\left(1-\frac{r-a}{|r-a|}\right)dr\end{aligned} \tag{32}$$

We were careful to be sure to take the positive square root in the lower limit of $u$. Then for $r > a$, the integral is zero so that the integral limits over $r$ range from 0 to $a$:

$$<E_z> = -\frac{3Q}{8\pi R^3\varepsilon_0 a^2}\int_{r=0}^{a} 2r^2\,dr = \frac{-Qa}{4\pi\varepsilon_0 R^3} \tag{33}$$

To form a dipole we add a negative charge $-Q$, a small distance $d$ below the original charge. The average electric field due to the dipole is then the superposition of (33) for both charges:

$$<E_z> = -\frac{Q}{4\pi\varepsilon_0 R^3}[a-(a-d)] = -\frac{Qd}{4\pi\varepsilon_0 R^3} = -\frac{p}{4\pi\varepsilon_0 R^3} \tag{34}$$

If we have a number density $N$ of such dipoles within the sphere, the total number of dipoles enclosed is $\frac{4}{3}\pi R^3 N$ so that superposition of (34) gives us the average electric field due to all the dipoles in terms of the polarization vector $\mathbf{P} = N\mathbf{p}$:

$$<\mathbf{E}> = -\frac{\frac{4}{3}\pi R^3 N\mathbf{p}}{4\pi\varepsilon_0 R^3} = -\frac{\mathbf{P}}{3\varepsilon_0} \tag{35}$$

The total macroscopic field is then the sum of the local field seen by each dipole and the average resulting field due to all the dipoles

$$\mathbf{E} = <\mathbf{E}> + \mathbf{E}_{\mathrm{Loc}} = -\frac{\mathbf{P}}{3\varepsilon_0} + \mathbf{E}_{\mathrm{Loc}} \tag{36}$$

so that the polarization $\mathbf{P}$ is related to the macroscopic electric field from (27) as

$$\mathbf{P} = N\mathbf{p} = N\alpha\mathbf{E}_{\mathrm{Loc}} = N\alpha\left(\mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}\right) \tag{37}$$

which can be solved for $\mathbf{P}$ as

$$\mathbf{P} = \frac{N\alpha}{1 - N\alpha/3\varepsilon_0}\mathbf{E} = \chi_e\varepsilon_0\mathbf{E}, \qquad \chi_e = \frac{N\alpha/\varepsilon_0}{1 - N\alpha/3\varepsilon_0} \tag{38}$$

where we introduce the electric susceptibility $\chi_e$ as the proportionality constant between $\mathbf{P}$ and $\varepsilon_0\mathbf{E}$. Then, use of (38) in (19) relates the displacement field $\mathbf{D}$ linearly to the electric field:

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0(1+\chi_e)\mathbf{E} = \varepsilon_0\varepsilon_r\mathbf{E} = \varepsilon\mathbf{E} \tag{39}$$

where $\varepsilon_r = 1 + \chi_e$ is called the relative dielectric constant and $\varepsilon = \varepsilon_r\varepsilon_0$ is the permittivity of the dielectric, also simply called the dielectric constant. In free space the susceptibility is zero ($\chi_e = 0$) so that $\varepsilon_r = 1$ and the permittivity is that of free space, $\varepsilon = \varepsilon_0$. The last relation in (39) is usually the most convenient to use as all the results of Chapter 2 are also correct within linear dielectrics if we replace $\varepsilon_0$ by $\varepsilon$. Typical values of relative permittivity are listed in Table 3-1 for various common substances. High dielectric constant materials are usually composed of highly polar molecules.

**Table 3-1 The relative permittivity for various common substances at room temperature**

| | $\varepsilon_r = \varepsilon/\varepsilon_0$ |
|---|---|
| Carbon Tetrachloride[a] | 2.2 |
| Ethanol[a] | 24 |
| Methanol[a] | 33 |
| *n*-Hexane[a] | 1.9 |
| Nitrobenzene[a] | 35 |
| Pure Water[a] | 80 |
| Barium Titanate[b] (with 20% Strontium Titanate) | >2100 |
| Borosilicate Glass[b] | 4.0 |
| Ruby Mica (Muscovite)[b] | 5.4 |
| Polyethylene[b] | 2.2 |
| Polyvinyl Chloride[b] | 6.1 |
| Teflon[b] (Polytetrafluorethylene) | 2.1 |
| Plexiglas[b] | 3.4 |
| Paraffin Wax[b] | 2.2 |

[a] *From* Lange's Handbook of Chemistry, *10th ed., McGraw-Hill, New York, 1961, pp. 1234–37.*

[b] *From A. R. von Hippel (Ed.) Dielectric Materials and Applications, M.I.T., Cambridge, Mass., 1966, pp. 301–370*

The polarizability and local electric field were only introduced so that we could relate microscopic and macroscopic fields. For most future problems we will describe materials by their permittivity $\varepsilon$ because this constant is most easily measured. The polarizability is then easily found as

$$\varepsilon-\varepsilon_0=\frac{N\alpha}{1-N\alpha/3\varepsilon_0}\Rightarrow\frac{N\alpha}{3\varepsilon_0}=\frac{\varepsilon-\varepsilon_0}{\varepsilon+2\varepsilon_0} \tag{40}$$

It then becomes simplest to work with the field vectors $\mathbf{D}$ and $\mathbf{E}$. The polarization can always be obtained if needed from the definition

$$\mathbf{P}=\mathbf{D}-\varepsilon_0\mathbf{E}=(\varepsilon-\varepsilon_0)\mathbf{E} \tag{41}$$

## EXAMPLE 3-1 POINT CHARGE WITHIN A DIELECTRIC SPHERE

Find all the fields and charges due to a point charge $q$ within a linear dielectric sphere of radius $R$ and permittivity $\varepsilon$ surrounded by free space, as in Figure 3-6.

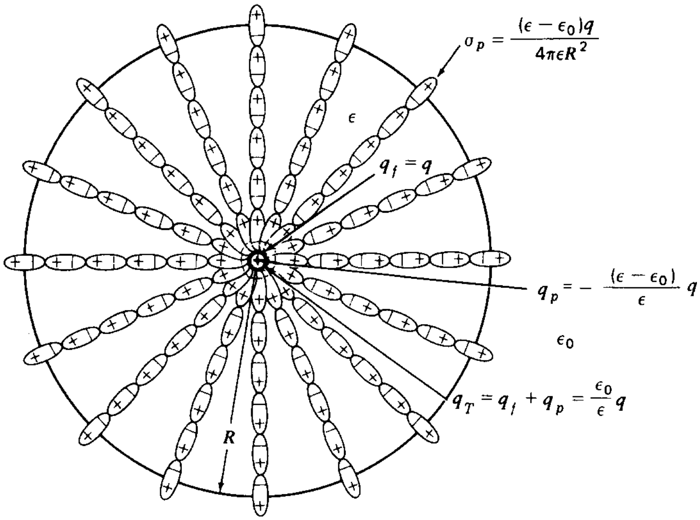



Figure 3-6 The electric field due to a point charge within a dielectric sphere is less than the free space field because of the partial neutralization of the point charge by the accumulation of dipole ends of opposite charge. The total polarization charge on the sphere remains zero as an equal magnitude but opposite sign polarization charge appears at the spherical interface.

## SOLUTION

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Applying Gauss's law of (21) to a sphere of any radius $r$ whether inside or outside the sphere, the enclosed free charge is always $q$:

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = D_r 4\pi r^2 = q \Rightarrow D_r = \frac{q}{4\pi r^2} \quad \text{all } r$$

The electric field is then discontinuous at $r = R$,

$$E_r = \begin{cases} \dfrac{D_r}{\varepsilon} = \dfrac{q}{4\pi\varepsilon r^2} & r < R \\[2ex] \dfrac{D_r}{\varepsilon_0} = \dfrac{q}{4\pi\varepsilon_0 r^2}, & r > R \end{cases}$$

due to the abrupt change of permittivities. The polarization field is

$$P_r = D_r - \varepsilon_0 E_r = \begin{cases} \dfrac{(\varepsilon - \varepsilon_0)q}{4\pi\varepsilon r^2}, & r < R \\[2ex] 0, & r > R \end{cases}$$

The volume polarization charge $\rho_P$ is zero everywhere,

$$\rho_p = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r) = 0, \quad 0 < r < R$$

except at $r = 0$ where a point polarization charge is present, and at $r = R$ where we have a surface polarization charge found by using (18) for concentric Gaussian spheres of radius $r$ inside and outside the dielectric sphere:

$$q_p = -\oint_S \mathbf{P} \cdot d\mathbf{S} = -P_r 4\pi r^2 = \begin{cases} -(\varepsilon - \varepsilon_0)q/\varepsilon, & r < R \\ 0, & r > R \end{cases}$$

We know that for $r < R$ this polarization charge must be a point charge at the origin as there is no volume charge contribution yielding a total point charge at the origin:

$$q_T = q_p + q = \frac{\varepsilon_0}{\varepsilon} q$$

This reduction of net charge is why the electric field within the sphere is less than the free space value. The opposite polarity ends of the dipoles are attracted to the point charge, partially neutralizing it. The total polarization charge enclosed by the sphere with $r > R$ is zero as there is an opposite polarity surface polarization charge at $r = R$ with density,

$$\sigma_p = \frac{(\varepsilon - \varepsilon_0)q}{4\pi\varepsilon R^2}$$

The total surface charge $\sigma_p 4\pi R^2 = (\varepsilon - \varepsilon_0)q/\varepsilon$ is equal in magnitude but opposite in sign to the polarization point charge at the origin. The total polarization charge always sums to zero.

## 3-1-5 Spontaneous Polarization

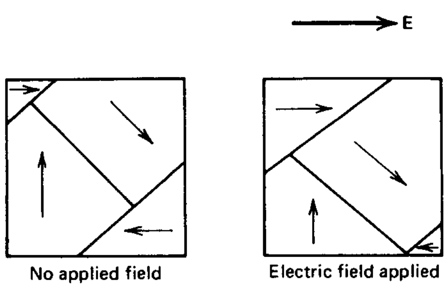
**(a) Ferro-electrics**

Examining (38) we see that when $N\alpha/3\varepsilon_0 = 1$ the polarization can be nonzero even if the electric field is zero. We can just meet this condition using the value of polarizability in (27) for electronic polarization if the whole volume is filled with contacting dipole spheres of the type in Figure 3-5$a$ so that we have one dipole for every volume of $\frac{4}{3}\pi R_0^3$. Then any slight fluctuation in the local electric field increases the polarization, which in turn increases the local field resulting in spontaneous polarization so that all the dipoles over a region are aligned. In a real material dipoles are not so densely packed. Furthermore, more realistic statistical models including thermally induced motions have shown that most molecules cannot meet the conditions for spontaneous polarization.

However, some materials have sufficient additional contributions to the polarizabilities due to ionic and orientational polarization that the condition for spontaneous polarization is met. Such materials are called ferro-electrics, even though they are not iron compounds, because of their similarity in behavior to iron compound ferro-magnetic materials, which we discuss in Section 5.5.3*c*. Ferro-electrics are composed of permanently polarized regions, called domains, as illustrated in Figure 3-7*a*. In the absence of an electric field, these domains are randomly distributed so that the net macroscopic polarization field is zero. When an electric field is applied, the dipoles tend to align with the field so that domains with a polarization component along the field grow at the expense of nonaligned domains. Ferro-electrics typically have very high permittivities such as barium titanate listed in Table 3-1.

The domains do not respond directly with the electric field as domain orientation and growth is not a reversible process but involves losses. The polarization $\mathbf{P}$ is then nonlinearly related to the electric field $\mathbf{E}$ by the hysteresis curve shown in Figure 3-8. The polarization of an initially unpolarized sample increases with electric field in a nonlinear way until the saturation value $\mathbf{P}_{sat}$ is reached when all the domains are completely aligned with the field. A further increase in $\mathbf{E}$ does not increase $\mathbf{P}$ as all the dipoles are completely aligned.

As the field decreases, the polarization does not retrace its path but follows a new path as the dipoles tend to stick to their previous positions. Even when the electric field is zero, a




(*a*) (*b*)

Figure 3-7 (*a*) In the absence of an applied electric field, a ferro-electric material generally has randomly distributed permanently polarized domains. Over a macroscopic volume, the net polarization due to all the domains is zero. (*b*) When an electric field is applied, domains with a polarization component in the direction of the field grow at the expense of nonaligned domains so that a net polarization can result.

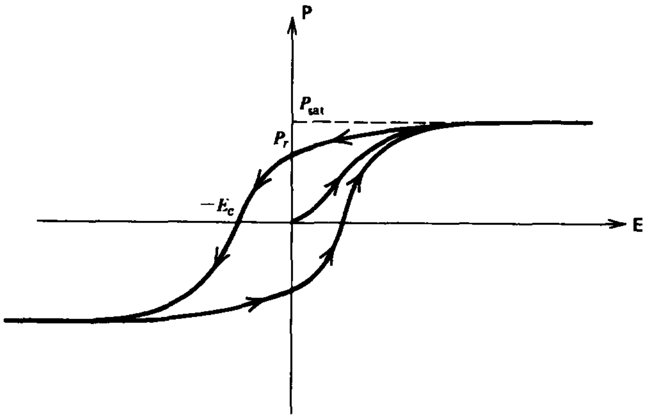



Figure 3-8 A typical ferro-electric hysteresis curve shows a saturation value $\mathbf{P}_{sat}$ when all the domains align with the field, a remanent polarization $\mathbf{P}_r$ when the electric field is removed, and a negative coercive electric field $-\mathbf{E}_c$, necessary to bring the polarization back to zero.

remanent polarization $\mathbf{P}_r$ remains. To bring the polarization to zero requires a negative coercive field $-E_c$. Further magnitude increases in negative electric field continues the symmetric hysteresis loop until a negative saturation is reached where all the dipoles have flipped over. If the field is now brought to zero and continued to positive field values, the whole hysteresis curve is traversed.

**(b) Electrets**

There are a class of materials called electrets that also exhibit a permanent polarization even in the absence of an applied electric field. Electrets are typically made using certain waxes or plastics that are heated until they become soft. They are placed within an electric field, tending to align the dipoles in the same direction as the electric field, and then allowed to harden. The dipoles are then frozen in place so that even when the electric field is removed a permanent polarization remains.

Other interesting polarization phenomena are:

*Electrostriction*—slight change in size of a dielectric due to the electrical force on the dipoles.

*Piezo-electricity*—when the electrostrictive effect is reversible so that a mechanical strain also creates a field.

*Pyro-electricity*—induced polarization due to heating or cooling.

## 3-2 CONDUCTION

### 3-2-1 Conservation of Charge

In contrast to dielectrics, most metals have their outermost band of electrons only weakly bound to the nucleus and are free to move in an applied electric field. In electrolytic solutions, ions of both sign are free to move. The flow of charge, called a current, is defined as the total charge flowing through a surface per unit time. In Figure 3-9*a* a single species of free charge with density $\rho_{fi}$ and velocity $\mathbf{v}_i$ flows through a small differential sized surface $\mathbf{dS}$. The total charge flowing through this surface in a time $\Delta t$ depends only on the velocity component perpendicular to the surface:

$$\Delta Q_i = \rho_{fi}\Delta t \mathbf{v}_i \cdot \mathbf{dS} \tag{1}$$

The tangential component of the velocity parallel to the surface $\mathbf{dS}$ only results in charge flow along the surface but not through it. The total differential current through $\mathbf{dS}$ is then defined as

$$dI_i = \frac{\Delta Q_i}{\Delta t} = \rho_{fi}\mathbf{v}_i \cdot \mathbf{dS} = \mathbf{J}_{fi} \cdot \mathbf{dS} \text{ ampere} \tag{2}$$

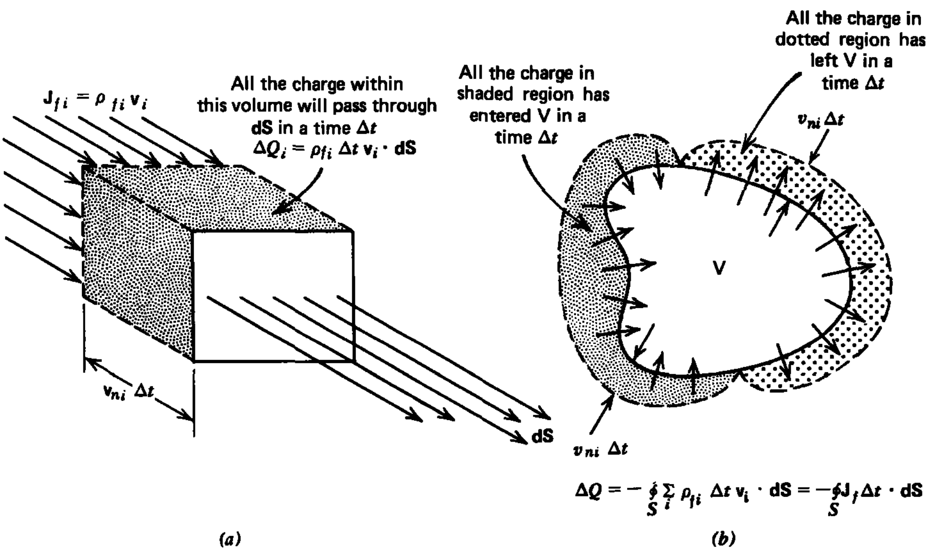



Figure 3-9 The current through a surface is defined as the number of charges per second passing through the surface. (*a*) The current is proportional to the component of charge velocity perpendicular to the surface. (*b*) The net change of total charge within a volume is equal to the difference of the charge entering to that leaving in a small time $\Delta t$.

where the free current density of these charges $\mathbf{J}_{fi}$ is a vector and is defined as

$$\mathbf{J}_{fi} = \rho_{fi}\mathbf{v}_i \text{ amp/m}^2 \tag{3}$$

If there is more than one type of charge carrier, the net charge density is equal to the algebraic sum of all the charge densities, while the net current density equals the vector sum of the current densities due to each carrier:

$$\rho_f = \sum \rho_{fi}, \quad \mathbf{J}_f = \sum \rho_{fi}\mathbf{v}_i \tag{4}$$

Thus, even if we have charge neutrality so that $\rho_f = 0$, a net current can flow if the charges move with different velocities. For example, two oppositely charged carriers with densities $\rho_1 = -\rho_2 \equiv \rho_0$ moving with respective velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ have

$$\rho_f = \rho_1 + \rho_2 = 0, \qquad \mathbf{J}_f = \rho_1\mathbf{v}_1 + \rho_2\mathbf{v}_2 = \rho_0(\mathbf{v}_1 - \mathbf{v}_2) \tag{5}$$

With $\mathbf{v}_1 \neq \mathbf{v}_2$ a net current flows with zero net charge. This is typical in metals where the electrons are free to flow while the oppositely charged nuclei remain stationary.

The total current $I$, a scalar, flowing through a macroscopic surface $S$, is then just the sum of the total differential currents of all the charge carriers through each incremental size surface element:

$$I = \int_S \mathbf{J}_f \cdot \mathbf{dS} \tag{6}$$

Now consider the charge flow through the closed volume V with surface $S$ shown in Figure 3-9*b*. A time $\Delta t$ later, that charge within the volume near the surface with the velocity component outward will leave the volume, while that charge just outside the volume with a velocity component inward will just enter the volume. The difference in total charge is transported by the current:

$$\begin{aligned}\Delta Q &= \int_V [\rho_f(t+\Delta t) - \rho_f(t)]\, dV \\ &= -\oint_S \sum \rho_{fi}\mathbf{v}_i\Delta t \cdot \mathbf{dS} = -\oint_S \mathbf{J}_f\Delta t \cdot \mathbf{dS}\end{aligned} \tag{7}$$

The minus sign on the right is necessary because when $\mathbf{v}_i$ is in the direction of $\mathbf{dS}$, charge has left the volume so that the enclosed charge decreases. Dividing (7) through by $\Delta t$ and taking the limit as $\Delta t \to 0$, we use (3) to derive the integral conservation of charge equation:

$$\oint_S \mathbf{J}_f \cdot \mathbf{dS} + \int_V \frac{\partial \rho_f}{\partial t}\, dV = 0 \tag{8}$$

Using the divergence theorem, the surface integral can be converted to a volume integral:

$$\int_V \left[\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t}\right] dV = 0 \Rightarrow \nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0 \tag{9}$$

where the differential point form is obtained since the integral must be true for any volume so that the bracketed term must be zero at each point. From Gauss's law ($\nabla \cdot \mathbf{D} = \rho_f$) (8) and (9) can also be written as

$$\oint_S \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot \mathbf{dS} = 0, \qquad \nabla \cdot \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}\right) = 0 \tag{10}$$

where $\mathbf{J}_f$ is termed the conduction current density and $\partial \mathbf{D}/\partial t$ is called the displacement current density.

This is the field form of Kirchoff's circuit current law that the algebraic sum of currents at a node sum to zero. Equation (10) equivalently tells us that the net flux of total current, conduction plus displacement, is zero so that all the current that enters a surface must leave it. The displacement current does not involve any charge transport so that time-varying current can be transmitted through space without charge carriers. Under static conditions, the displacement current is zero.

## 3-2-2 Charged Gas Conduction Models

**(a) Governing Equations.**

In many materials, including good conductors like metals, ionized gases, and electrolytic solutions as well as poorer conductors like lossy insulators and semiconductors, the charge carriers can be classically modeled as an ideal gas within the medium, called a plasma. We assume that we have two carriers of equal magnitude but opposite sign $\pm q$ with respective masses $m_\pm$ and number densities $n_\pm$. These charges may be holes and electrons in a semiconductor, oppositely charged ions in an electrolytic solution, or electrons and nuclei in a metal. When an electric field is applied, the positive charges move in the direction of the field while the negative charges move in the opposite direction. These charges collide with the host medium at respective frequencies $\nu_+$ and $\nu_-$, which then act as a viscous or frictional dissipation opposing the motion. In addition to electrical and frictional forces, the particles exert a force on themselves through a pressure term due to thermal agitation that would be present even if the particles were uncharged. For an ideal gas the partial pressure $p$ is

$$p = nkT \text{ Pascals } [\text{kg-s}^{-2}\text{-m}^{-1}] \tag{11}$$

where $n$ is the number density of charges, $T$ is the absolute temperature, and $k=1.38\times10^{-23}$ joule/°K is called Boltzmann's constant.

The net pressure force on the small rectangular volume shown in Figure 3-10 is

$$\mathbf{f}_p=\left(\frac{p(x-\Delta x)-p(x)}{\Delta x}\mathbf{i}_x+\frac{p(y)-p(y+\Delta y)}{\Delta y}\mathbf{i}_y\right.$$
$$\left.+\frac{p(z)-p(z+\Delta z)}{\Delta z}\mathbf{i}_z\right)\Delta x\,\Delta y\,\Delta z \tag{12}$$

where we see that the pressure only exerts a net force on the volume if it is different on each opposite surface. As the volume shrinks to infinitesimal size, the pressure terms in (12) define partial derivatives so that the volume force density becomes

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}}\frac{\mathbf{f}_p}{\Delta x\,\Delta y\,\Delta z}=-\left(\frac{\partial p}{\partial x}\mathbf{i}_x+\frac{\partial p}{\partial y}\mathbf{i}_y+\frac{\partial p}{\partial z}\mathbf{i}_z\right)=-\nabla p \tag{13}$$

Then using (11)–(13), Newton's force law for each charge carrier within the small volume is

$$m_\pm\frac{\partial \mathbf{v}_\pm}{\partial t}=\pm q\mathbf{E}-m_\pm\nu_\pm\mathbf{v}_\pm-\frac{1}{n_\pm}\nabla(n_\pm kT) \tag{14}$$

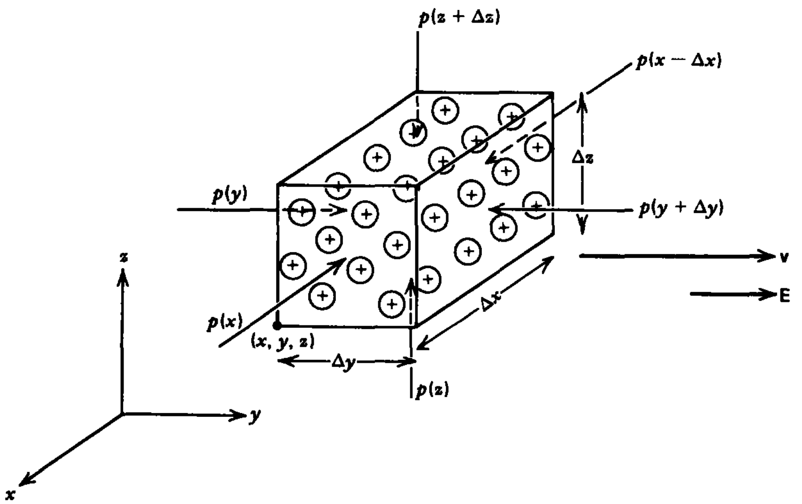



Figure 3-10 Newton's force law, applied to a small rectangular volume $\Delta x\,\Delta y\,\Delta z$ moving with velocity $\mathbf{v}$, enclosing positive charges with number density $n$. The pressure is the force per unit area acting normally inward on each surface and only contributes to the net force if it is different on opposite faces.

where the electric field $\mathbf{E}$ is due to the imposed field plus the field generated by the charges, as given by Gauss's law.

**(b) Drift-Diffusion Conduction**

Because in many materials the collision frequencies are typically on the order of $\nu \approx 10^{13}$ Hz, the inertia terms in (14) are often negligible. In this limit we can easily solve (14) for the velocity of each carrier as

$$\lim_{\partial \mathbf{v}_\pm/\partial t \ll \nu_\pm \mathbf{v}_\pm} \mathbf{v}_\pm = \frac{1}{m_\pm \nu_\pm}\left(\pm q\mathbf{E} - \frac{1}{n_\pm}\nabla(n_\pm kT)\right) \tag{15}$$

The charge and current density for each carrier are simply given as

$$\rho_\pm = \pm q n_\pm, \qquad \mathbf{J}_\pm = \rho_\pm \mathbf{v}_\pm = \pm q n_\pm \mathbf{v}_\pm \tag{16}$$

Multiplying (15) by the charge densities then gives us the constitutive law for each current as

$$\mathbf{J}_\pm = \pm q n_\pm \mathbf{v}_\pm = \pm \rho_\pm \mu_\pm \mathbf{E} - D_\pm \nabla \rho_\pm \tag{17}$$

where $\mu_\pm$ are called the particle mobilities and $D_\pm$ are their diffusion coefficients

$$\mu_\pm = \frac{q}{m_\pm \nu_\pm}\,[\text{A-kg}^{-1}\text{-s}^{-2}], \quad D_\pm = \frac{kT}{m_\pm \nu_\pm}\,[\text{m}^2\text{-s}^{-1}] \tag{18}$$

assuming that the system is at constant temperature. We see that the ratio $D_\pm/\mu_\pm$ for each carrier is the same having units of voltage, thus called the thermal voltage:

$$\frac{D_\pm}{\mu_\pm} = \frac{kT}{q}\ \text{volts}\ [\text{kg-m}^2\text{-A}^{-1}\text{-s}^{-3}] \tag{19}$$

This equality is known as Einstein's relation.

In equilibrium when the net current of each carrier is zero, (17) can be written in terms of the potential as ($\mathbf{E} = -\nabla V$)

$$\mathbf{J}_+ = \mathbf{J}_- = 0 = -\rho_\pm \mu_\pm \nabla V \mp D_\pm \nabla \rho_\pm \tag{20}$$

which can be rewritten as

$$\nabla\left[\pm \frac{\mu_\pm}{D_\pm} V + \ln \rho_\pm\right] = 0 \tag{21}$$

The bracketed term can then only be a constant, so the charge density is related to the potential by the Boltzmann distribution:

$$\rho_\pm = \pm \rho_0\, e^{\mp qV/kT} \tag{22}$$

where we use the Einstein relation of (19) and $\pm\rho_0$ is the equilibrium charge density of each carrier when $V = 0$ and are of equal magnitude because the system is initially neutral.

To find the spatial dependence of $\rho$ and $V$ we use (22) in Poisson's equation derived in Section 2.5.6:

$$\nabla^2 V = -\frac{(\rho_+ + \rho_-)}{\varepsilon} = -\frac{\rho_0}{\varepsilon}(e^{-qV/kT} - e^{qV/kT}) = \frac{2\rho_0}{\varepsilon}\sinh\frac{qV}{kT} \tag{23}$$

This equation is known as the Poisson–Boltzmann equation because the charge densities obey Boltzmann distributions.

Consider an electrode placed at $x = 0$ raised to the potential $V_0$ with respect to a zero potential at $x = \pm\infty$, as in Figure 3-11*a*. Because the electrode is long, the potential only varies

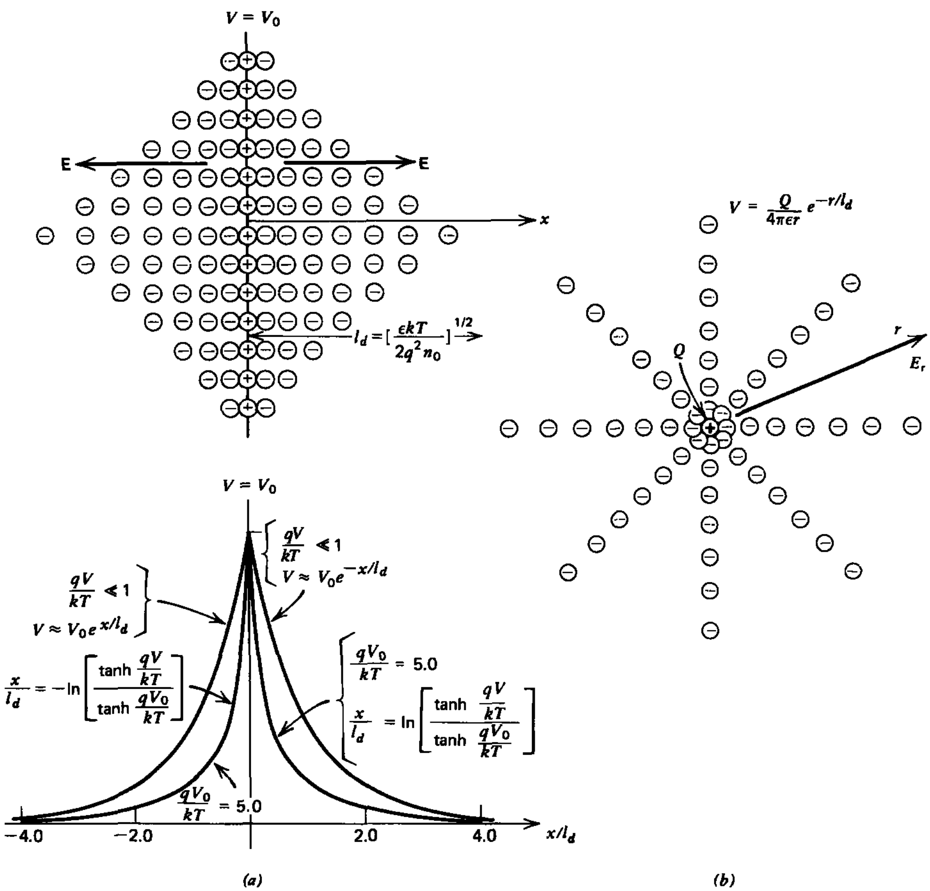



Figure 3-11 Opposite polarity mobile charges accumulate around any net charge inserted into a conductor described by the drift-diffusion equations, and tend to shield out its field for distances larger than the Debye length. (*a*) Electrode at potential $V_0$ with respect to a zero potential at $x = \pm\infty$. The spatial decay is faster for larger values of $V_0$. (*b*) Point charge.

with the $x$ coordinate so that (23) becomes

$$\frac{d^2\tilde{V}}{dx^2}-\frac{1}{l_d^2}\sinh\tilde{V}=0,\qquad \tilde{V}=\frac{qV}{kT},\qquad l_d^2=\frac{\varepsilon kT}{2\rho_0 q}\tag{24}$$

where we normalize the voltage to the thermal voltage $kT/q$ and $l_d$ is called the Debye length.

If (24) is multiplied by $d\tilde{V}/dx$, it can be written as an exact differential:

$$\frac{d}{dx}\left[\frac{1}{2}\left(\frac{d\tilde{V}}{dx}\right)^2-\frac{\cosh\tilde{V}}{l_d^2}\right]=0\tag{25}$$

The bracketed term must then be a constant that is evaluated far from the electrode where the potential and electric field $\tilde{E}_x=-d\tilde{V}/dx$ are zero:

$$\frac{d\tilde{V}}{dx}=-\tilde{E}_x=\left[\frac{2}{l_d^2}(\cosh\tilde{V}-1)\right]^{1/2}=\mp\frac{2}{l_d}\sinh\frac{\tilde{V}}{2}\begin{cases}x>0\\x<0\end{cases}\tag{26}$$

The different signs taken with the square root are necessary because the electric field points in opposite directions on each side of the electrode. The potential is then implicitly found by direct integration as

$$\frac{\tanh(\tilde{V}/4)}{\tanh(\tilde{V}_0/4)}=e^{\mp x/l_d}\begin{cases}x>0\\x<0\end{cases}\tag{27}$$

The Debye length thus describes the characteristic length over which the applied potential exerts influence. In many materials the number density of carriers is easily of the order of $n_0\approx 10^{20}/\mathrm{m}^3$, so that at room temperature ($T\approx 293°\mathrm{K}$), $l_d$ is typically $10^{-7}$ m.

Often the potentials are very small so that $qV/kT\ll 1$. Then, the hyperbolic terms in (27), as well as in the governing equation of (23), can be approximated by their arguments:

$$\nabla^2 V-\frac{V}{l_d^2}=0\tag{28}$$

This approximation is only valid when the potentials are much less than the thermal voltage $kT/q$, which at room temperature is about 25 mv. In this limit the solution of (27) shows that the voltage distribution decreases exponentially. At higher values of $V_0$, the decay is faster, as shown in Figure 3-11*a*.

If a point charge $Q$ is inserted into the plasma medium, as in Figure 3-11*b*, the potential only depends on the radial distance $r$. In the small potential limit, (28) in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)-\frac{V}{l_d^2}=0\tag{29}$$

Realizing that this equation can be rewritten as

$$\frac{\partial^2}{\partial r^2}(rV)-\frac{(rV)}{l_d^2}=0 \tag{30}$$

we have a linear constant coefficient differential equation in the variable $(rV)$ for which solutions are

$$rV=A_1\,e^{-r/l_d}+A_2\,e^{+r/l_d} \tag{31}$$

Because the potential must decay and not grow far from the charge, $A_2=0$ and the solution is

$$V=\frac{Q}{4\pi\varepsilon r}\,e^{-r/l_d} \tag{32}$$

where we evaluated $A_1$ by realizing that as $r\to 0$ the potential must approach that of an isolated point charge. Note that for small $r$ the potential becomes very large and the small potential approximation is violated.

**(c) Ohm's Law**

We have seen that the mobile charges in a system described by the drift-diffusion equations accumulate near opposite polarity charge and tend to shield out its effect for distances larger than the Debye length. Because this distance is usually so much smaller than the characteristic system dimensions, most regions of space outside the Debye sheath are charge neutral with equal amounts of positive and negative charge density $\pm\rho_0$. In this region, the diffusion term in (17) is negligible because there are no charge density gradients. Then the total current density is proportional to the electric field:

$$\mathbf{J}=\mathbf{J}_+ + \mathbf{J}_- = \rho_0(\mathbf{v}_+ - \mathbf{v}_-) = qn_0(\mu_+ + \mu_-)\mathbf{E}=\sigma\mathbf{E} \tag{33}$$

where $\sigma$ [siemans/m (m$^{-3}$-kg$^{-1}$-s$^3$-A$^2$)] is called the Ohmic conductivity and (33) is the point form of Ohm's law. Sometimes it is more convenient to work with the reciprocal conductivity $\rho_r=(1/\sigma)$ (ohm-m) called the resistivity. We will predominantly use Ohm's law to describe most media in this text, but it is important to remember that it is often not valid within the small Debye distances near charges. When Ohm's law is valid, the net charge is zero, thus giving no contribution to Gauss's law. Table 3-2 lists the Ohmic conductivities for various materials. We see that different materials vary over wide ranges in their ability to conduct charges.

The Ohmic conductivity of "perfect conductors" is large and is idealized to be infinite. Since all physical currents in (33) must remain finite, the electric field within the conductor is zero so that it imposes an equipotential surface:

$$\lim_{\sigma\to\infty} \mathbf{J} = \sigma\mathbf{E} \Rightarrow \begin{cases} \mathbf{E} = 0 \\ V = \text{const} \\ \mathbf{J} = \text{finite} \end{cases} \tag{34}$$

**Table 3-2 The Ohmic conductivity for various common substances at room temperature**

| | $\sigma$ [siemen/m] |
|---|---|
| Silver[a] | $6.3 \times 10^7$ |
| Copper[a] | $5.9 \times 10^7$ |
| Gold[a] | $4.2 \times 10^7$ |
| Lead[a] | $0.5 \times 10^7$ |
| Tin[a] | $0.9 \times 10^7$ |
| Zinc[a] | $1.7 \times 10^7$ |
| Carbon[a] | $7.3 \times 10^{-4}$ |
| Mercury[b] | $1.06 \times 10^6$ |
| Pure Water[b] | $4 \times 10^{-6}$ |
| Nitrobenzene[b] | $5 \times 10^{-7}$ |
| Methanol[b] | $4 \times 10^{-5}$ |
| Ethanol[b] | $1.3 \times 10^{-7}$ |
| Hexane[b] | $< 1 \times 10^{-18}$ |

[a] *From* Handbook of Chemistry and Physics, *49th ed., The Chemical Rubber Co., 1968, p. E80.*
[b] *From* Lange's Handbook of Chemistry, *10th ed., McGraw-Hill, New York, 1961, pp. 1220–21.*

Throughout this text electrodes are generally assumed to be perfectly conducting and thus are at a constant potential. The external electric field must then be incident perpendicularly to the surface.

### (d) Superconductors

One notable exception to Ohm's law is for superconducting materials at cryogenic temperatures. Then, with collisions negligible ($\nu_\pm = 0$) and the absolute temperature low ($T \approx 0$), the electrical force on the charges is only balanced by their inertia so that (14) becomes simply

$$\frac{\partial \mathbf{v}_\pm}{\partial t} = \pm \frac{q}{m_\pm} \mathbf{E} \tag{35}$$

We multiply (35) by the charge densities that we assume to be constant so that the constitutive law relating the current density to the electric field is

$$\frac{\partial(\pm q n_\pm \mathbf{v}_\pm)}{\partial t}=\frac{\partial \mathbf{J}_\pm}{\partial t}=\frac{q^2 n_\pm}{m_\pm}\mathbf{E}=\omega_{p\pm}^2\,\varepsilon\mathbf{E},\qquad \omega_{p\pm}^2=\frac{q^2 n_\pm}{m_\pm \varepsilon} \tag{36}$$

where $\omega_{p\pm}$ is called the plasma frequency for each carrier.

For electrons ($q=-1.6\times 10^{-19}$ coul, $m_-=9.1\times 10^{-31}$ kg) of density $n_-\approx 10^{20}/\text{m}^3$ within a material with the permittivity of free space, $\varepsilon=\varepsilon_0\approx 8.854\times 10^{-12}$ farad/m, the plasma frequency is

$$\begin{aligned}\omega_{p-}&=\sqrt{q^2 n_-/m_-\varepsilon}\approx 5.6\times 10^{11}\ \text{radian/sec}\\ &\Rightarrow f_{p-}=\omega_{p-}/2\pi\approx 9\times 10^{10}\ \text{Hz}\end{aligned} \tag{37}$$

If such a material is placed between parallel plate electrodes that are open circuited, the electric field and current density $\mathbf{J}=\mathbf{J}_+ + \mathbf{J}_-$ must be perpendicular to the electrodes, which we take as the $x$ direction. If the electrode spacing is small compared to the width, the interelectrode fields far from the ends must then be $x$ directed and be only a function of $x$. Then the time derivative of the charge conservation equation in (10) is

$$\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}(J_+ + J_-)+\varepsilon\frac{\partial^2 E}{\partial t^2}\right]=0 \tag{38}$$

The bracketed term is just the time derivative of the total current density, which is zero because the electrodes are open circuited so that using (36) in (38) yields

$$\frac{\partial^2 E}{\partial t^2}+\omega_p^2 E=0,\qquad \omega_p^2=\omega_{p+}^2+\omega_{p-}^2 \tag{39}$$

which has solutions

$$E=A_1\sin\omega_p t+A_2\cos\omega_p t \tag{40}$$

Any initial perturbation causes an oscillatory electric field at the composite plasma frequency $\omega_p$. The charges then execute simple harmonic motion about their equilibrium position.

## 3-3 FIELD BOUNDARY CONDITIONS

In many problems there is a surface of discontinuity separating dissimilar materials, such as between a conductor and a dielectric, or between different dielectrics. We must determine how the fields change as we cross the interface where the material properties change abruptly.

### 3-3-1 Tangential Component of E

We apply the line integral of the electric field around a contour of differential size enclosing the interface between dissimilar materials, as shown in Figure 3-12*a*. The long sections *a* and *c* of length **dl** are tangential to the surface and the short joining sections *b* and *d* are of zero length as the interface is assumed to have zero thickness. Applying the line integral of the electric field around this contour, from Section 2.5.6 we obtain

$$\oint_L \mathbf{E}\cdot d\mathbf{l}=(E_{1t}-E_{2t})\,dl=0 \tag{1}$$

where $E_{1t}$ and $E_{2t}$ are the components of the electric field tangential to the interface. We get no contribution from the normal components of field along sections *b* and *d* because the contour lengths are zero. The minus sign arises along *c* because the electric field is in the opposite direction of the contour traversal. We thus have that the tangential

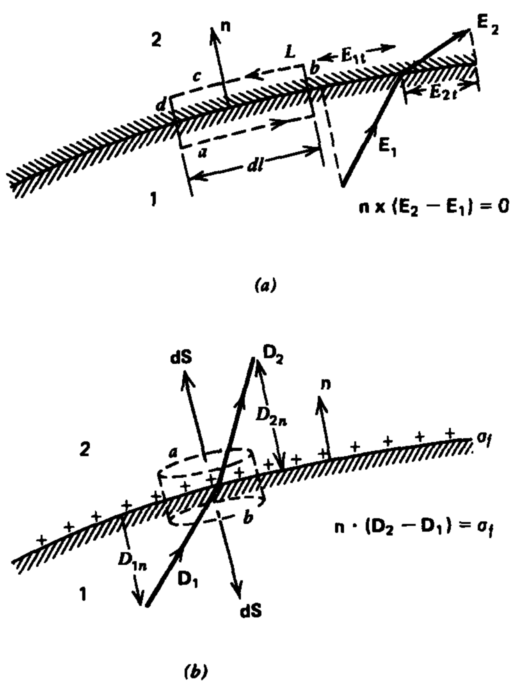



Figure 3-12 (*a*) Stokes' law applied to a line integral about an interface of discontinuity shows that the tangential component of electric field is continuous across the boundary. (b) Gauss's law applied to a pill-box volume straddling the interface shows that the normal component of displacement vector is discontinuous in the free surface charge density $\sigma_f$.

components of the electric field are continuous across the interface

$$E_{1t} = E_{2t} \Rightarrow \mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \tag{2}$$

where $\mathbf{n}$ is the interfacial normal shown in Figure 3-12*a*.

Within a perfect conductor the electric field is zero. Therefore, from (2) we know that the tangential component of $\mathbf{E}$ outside the conductor is also zero. Thus the electric field must always terminate perpendicularly to a perfect conductor.

### 3-3-2 Normal Component of D

We generalize the results of Section 2.4.6 to include dielectric media by again choosing a small Gaussian surface whose upper and lower surfaces of area $dS$ are parallel to a surface charged interface and are joined by an infinitely thin cylindrical surface with zero area, as shown in Figure 3-12*b*. Then only faces $a$ and $b$ contribute to Gauss's law:

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = (D_{2n} - D_{1n})\, dS = \sigma_f\, dS \tag{3}$$

where the interface supports a free surface charge density $\sigma_f$ and $D_{2n}$ and $D_{1n}$ are the components of the displacement vector on either side of the interface in the direction of the normal $\mathbf{n}$ shown, pointing from region 1 to region 2. Reducing (3) and using more compact notation we have

$$D_{2n} - D_{1n} = \sigma_f, \quad \mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_f \tag{4}$$

where the minus sign in front of $\mathbf{D}_1$ arises because the normal on the lower surface $b$ is $-\mathbf{n}$. The normal components of the displacement vector are discontinuous if the interface has a surface charge density. If there is no surface charge ($\sigma_f = 0$), the normal components of $\mathbf{D}$ are continuous. If each medium has no polarization, (4) reduces to the free space results of Section 2.4.6.

At the interface between two different lossless dielectrics, there is usually no surface charge ($\sigma_f = 0$), unless it was deliberately placed, because with no conductivity there is no current to transport charge. Then, even though the normal component of the $\mathbf{D}$ field is continuous, the normal component of the electric field is discontinuous because the dielectric constant in each region is different.

At the interface between different conducting materials, free surface charge may exist as the current may transport charge to the surface discontinuity. Generally for such cases, the surface charge density is nonzero. In particular, if one region is a perfect conductor with zero internal electric field, the surface charge density on the surface is just equal to the normal component of **D** field at the conductor's surface,

$$\sigma_f = \mathbf{n} \cdot \mathbf{D} \tag{5}$$

where **n** is the outgoing normal from the perfect conductor.

### 3-3-3 Point Charge Above a Dielectric Boundary

If a point charge $q$ within a region of permittivity $\varepsilon_1$ is a distance $d$ above a planar boundary separating region I from region II with permittivity $\varepsilon_2$, as in Figure 3-13, the tangential component of **E** and in the absence of free surface charge the normal component of **D**, must be continuous across the interface. Let us try to use the method of images by placing an image charge $q'$ at $y=-d$ so that the solution in region I is due to this image charge plus the original point charge $q$. The solution for the field in region II will be due to an image charge $q''$ at $y=d$, the position of the original point charge. Note that the appropriate image charge is always outside the region where the solution is desired. At this point we do not know if it is possible to satisfy the boundary conditions with these image charges, but we will try to find values of $q'$ and $q''$ to do so.

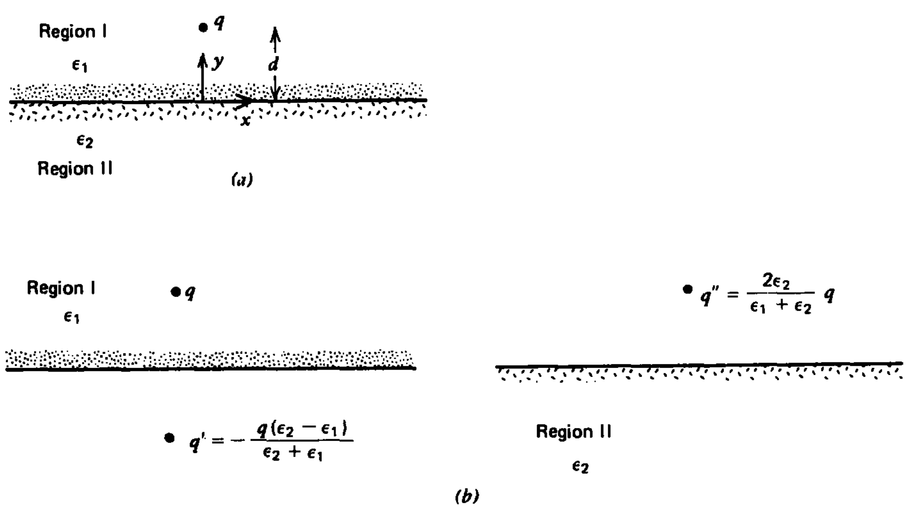



Figure 3-13 (*a*) A point charge $q$ above a flat dielectric boundary requires different sets of image charges to solve for the fields in each region. (*b*) The field in region *I* is due to the original charge and the image charge $q'$ while the field in region II is due only to image charge $q''$.

The potential in each region is

$$\begin{aligned} V_{\mathrm{I}} &= \frac{1}{4\pi\varepsilon_1}\left(\frac{q}{[x^2+(y-d)^2+z^2]^{1/2}}+\frac{q'}{[x^2+(y+d)^2+z^2]^{1/2}}\right), \quad y \geq 0 \\ V_{\mathrm{II}} &= \frac{1}{4\pi\varepsilon_2}\frac{q''}{[x^2+(y-d)^2+z^2]^{1/2}}, \quad y \leq 0 \end{aligned} \tag{6}$$

with resultant electric field

$$\begin{aligned} \mathbf{E}_{\mathrm{I}} &= -\nabla V_{\mathrm{I}} \\ &= \frac{1}{4\pi\varepsilon_1}\left(\frac{q[x\mathbf{i}_x+(y-d)\mathbf{i}_y+z\mathbf{i}_z]}{[x^2+(y-d)^2+z^2]^{3/2}}+\frac{q'[x\mathbf{i}_x+(y+d)\mathbf{i}_y+z\mathbf{i}_z]}{[x^2+(y+d)^2+z^2]^{3/2}}\right) \end{aligned} \tag{7}$$

$$\mathbf{E}_{\mathrm{II}} = -\nabla V_{\mathrm{II}} = \frac{q''}{4\pi\varepsilon_2}\left(\frac{x\mathbf{i}_x+(y-d)\mathbf{i}_y+z\mathbf{i}_z}{[x^2+(y-d)^2+z^2]^{3/2}}\right)$$

To satisfy the continuity of tangential electric field at $y=0$ we have

$$\left.\begin{aligned} E_{x\mathrm{I}} &= E_{x\mathrm{II}} \\ E_{z\mathrm{I}} &= E_{z\mathrm{II}} \end{aligned}\right\} \Rightarrow \frac{q+q'}{\varepsilon_1} = \frac{q''}{\varepsilon_2} \tag{8}$$

With no surface charge, the normal component of $\mathbf{D}$ must be continuous at $y=0$,

$$\varepsilon_1 E_{y\mathrm{I}} = \varepsilon_2 E_{y\mathrm{II}} \Rightarrow -q+q' = -q'' \tag{9}$$

Solving (8) and (9) for the unknown charges we find

$$\begin{aligned} q' &= -\frac{(\varepsilon_2-\varepsilon_1)}{\varepsilon_1+\varepsilon_2}q \\ q'' &= \frac{2\varepsilon_2}{(\varepsilon_1+\varepsilon_2)}q \end{aligned} \tag{10}$$

The force on the point charge $q$ is due only to the field from image charge $q'$:

$$\mathbf{f} = \frac{qq'}{4\pi\varepsilon_1(2d)^2}\mathbf{i}_y = -\frac{q^2(\varepsilon_2-\varepsilon_1)}{16\pi\varepsilon_1(\varepsilon_1+\varepsilon_2)d^2}\mathbf{i}_y \tag{11}$$

### 3-3-4 Normal Component of P and $\varepsilon_0$E

By integrating the flux of polarization over the same Gaussian pillbox surface, shown in Figure 3-12$b$, we relate the discontinuity in normal component of polarization to the surface polarization charge density $\sigma_p$ using the relations from Section 3.1.2:

$$\oint_S \mathbf{P}\cdot\mathbf{dS} = -\int_S \sigma_p\, dS \Rightarrow P_{2n} - P_{1n} = -\sigma_p \Rightarrow \mathbf{n}\cdot(\mathbf{P}_2 - \mathbf{P}_1) = -\sigma_p \tag{12}$$

The minus sign in front of $\sigma_p$ results because of the minus sign relating the volume polarization charge density to the divergence of **P**.

To summarize, polarization charge is the source of **P**, free charge is the source of **D**, and the total charge is the source of $\varepsilon_0\mathbf{E}$. Using (4) and (12), the electric field interfacial discontinuity is

$$\mathbf{n}\cdot(\mathbf{E}_2 - \mathbf{E}_1) = \frac{\mathbf{n}\cdot[(\mathbf{D}_2 - \mathbf{D}_1) - (\mathbf{P}_2 - \mathbf{P}_1)]}{\varepsilon_0} = \frac{\sigma_f + \sigma_p}{\varepsilon_0} \tag{13}$$

For linear dielectrics it is often convenient to lump polarization effects into the permittivity $\varepsilon$ and never use the vector **P**, only **D** and **E**.

For permanently polarized materials, it is usually convenient to replace the polarization **P** by the equivalent polarization volume charge density and surface charge density of (12) and solve for **E** using the coulombic superposition integral of Section 2.3.2. In many dielectric problems, there is no volume polarization charge, but at surfaces of discontinuity a surface polarization charge is present as given by (12).

## EXAMPLE 3-2 CYLINDER PERMANENTLY POLARIZED ALONG ITS AXIS

A cylinder of radius $a$ and height $L$ is centered about the $z$ axis and has a uniform polarization along its axis, $\mathbf{P} = P_0\mathbf{i}_z$, as shown in Figure 3-14. Find the electric field **E** and displacement vector **D** everywhere on its axis.

### SOLUTION

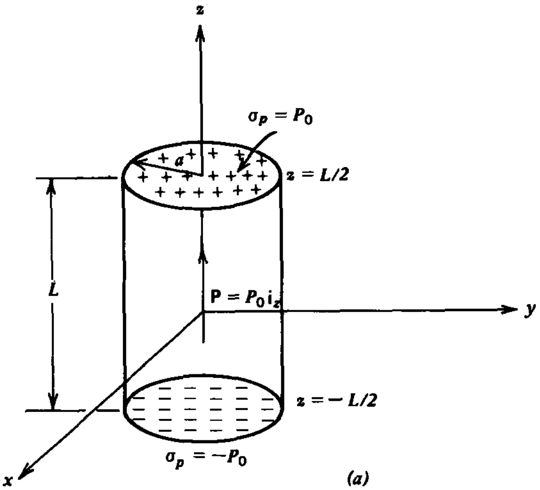
---

With a constant polarization **P**, the volume polarization charge density is zero:

$$\rho_p = -\nabla\cdot\mathbf{P} = 0$$

Since $\mathbf{P} = 0$ outside the cylinder, the normal component of **P** is discontinuous at the upper and lower surfaces yielding uniform surface polarization charges:

$$\sigma_p(z = L/2) = P_0, \quad \sigma_P(z = -L/2) = -P_0$$



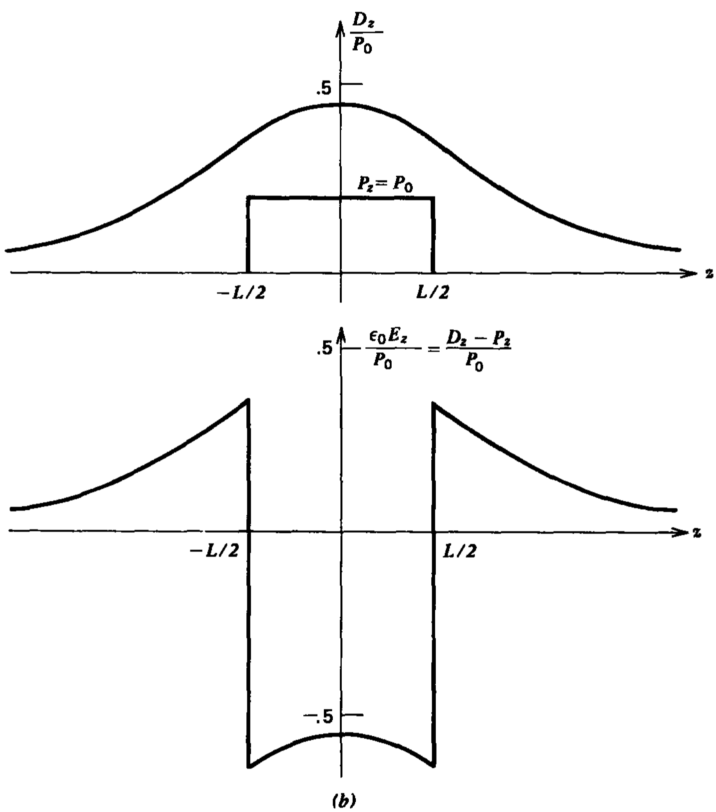




Figure 3-14 (*a*) The electric field due to a uniformly polarized cylinder of length $L$ is the same as for two disks of surface charge of opposite polarity $\pm P_0$ at $z = L/2$. (*b*) The perpendicular displacement field $D_z$ is continuous across the interfaces at $z = \pm L/2$ while the electric field $E_z$ is discontinuous.

The solution for a single disk of surface charge was obtained in Section 2.3.5*b*. We superpose the results for the two disks taking care to shift the axial distance appropriately by $\pm L/2$ yielding the concise solution for the displacement field:

$$D_z = \frac{P_0}{2}\left(\frac{(z+L/2)}{[a^2+(z+L/2)^2]^{1/2}} - \frac{(z-L/2)}{[a^2+(z-L/2)^2]^{1/2}}\right)$$

The electric field is then

$$E_z = \begin{cases} D_z/\varepsilon_0, & |z| > L/2 \\ (D_z - P_0)/\varepsilon_0 & |z| < L/2 \end{cases}$$

These results can be examined in various limits. If the radius $a$ becomes very large, the electric field should approach that of two parallel sheets of surface charge $\pm P_0$, as in Section 2.3.4*b*:

$$\lim_{a\to\infty} E_z = \begin{cases} 0, & |z| > L/2 \\ -P_0/\varepsilon_0, & |z| < L/2 \end{cases}$$

with a zero displacement field everywhere.

In the opposite limit, for large $z$ ($z \gg a$, $z \gg L$) far from the cylinder, the axial electric field dies off as the dipole field with $\theta = 0$

$$\lim_{z\to\infty} E_z = \frac{p}{2\pi\varepsilon_0 z^3}, \quad p = P_0 \pi a^2 L$$

with effective dipole moment $p$ given by the product of the total polarization charge at $z = L/2$, $(P_0\pi a^2)$, and the length $L$.

## 3-3-5 Normal Component of J

Applying the conservation of total current equation in Section 3.2.1 to the same Gaussian pillbox surface in Figure 3-12*b* results in contributions again only from the upper and lower surfaces labeled "*a*" and "*b*":

$$\mathbf{n}\cdot\left(\mathbf{J}_2 - \mathbf{J}_1 + \frac{\partial}{\partial t}(\mathbf{D}_2 - \mathbf{D}_1)\right) = 0 \tag{14}$$

where we assume that no surface currents flow along the interface. From (4), relating the surface charge density to the discontinuity in normal $\mathbf{D}$, this boundary condition can also be written as

$$\mathbf{n}\cdot(\mathbf{J}_2 - \mathbf{J}_1) + \frac{\partial \sigma_f}{\partial t} = 0 \tag{15}$$

which tells us that if the current entering a surface is different from the current leaving, charge has accumulated at the interface. In the dc steady state the normal component of **J** is continuous across a boundary.

## 3-4 RESISTANCE

### 3-4-1 Resistance Between Two Electrodes

Two conductors maintained at a potential difference $V$ within a conducting medium will each pass a total current $I$, as shown in Figure 3-15. By applying the surface integral form of charge conservation in Section 3.2.1 to a surface $S'$ which surrounds both electrodes but is far enough away so that **J** and **D** are negligibly small, we see that the only nonzero current contributions are from the terminal wires that pass through the surface. These must sum to zero so that the

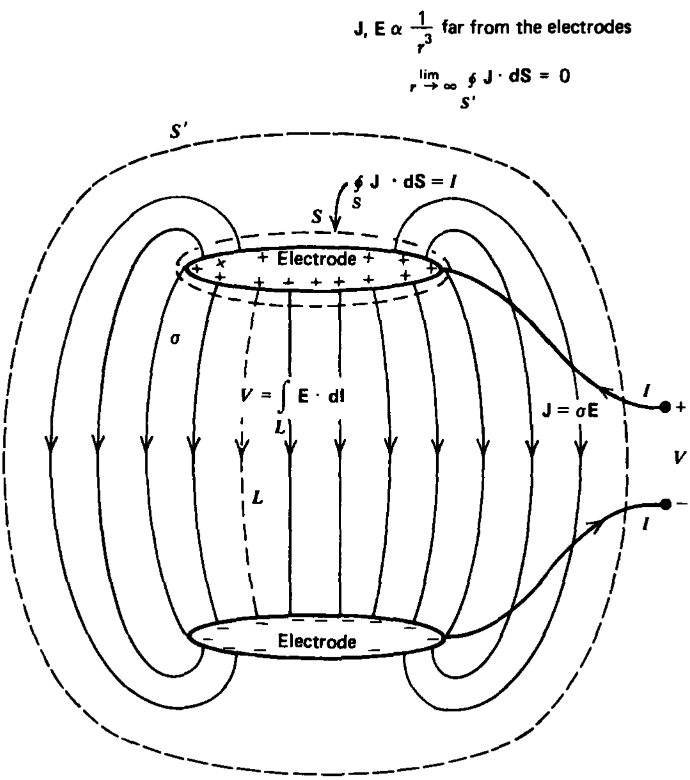



Figure 3-15 A voltage applied across two electrodes within an ohmic medium causes a current to flow into one electrode and out the other. The electrodes have equal magnitude but opposite polarity charges so that far away the fields die off as a dipole $\propto(1/r^3)$. Then, even though the surface $S'$ is increasing as $r^2$, the flux of current goes to zero as $1/r$.

currents have equal magnitudes but flow in opposite directions. Similarly, applying charge conservation to a surface $S$ just enclosing the upper electrode shows that the current $I$ entering the electrode via the wire must just equal the total current (conduction plus displacement) leaving the electrode. This total current travels to the opposite electrode and leaves via the connecting wire.

The dc steady-state ratio of voltage to current between the two electrodes in Figure 3-15 is defined as the resistance:

$$\mathrm{R}=\frac{V}{I}\ \text{ohm}\ [\text{kg-m}^2\text{-s}^{-3}\text{-A}^{-2}] \tag{1}$$

For an arbitrary geometry, (1) can be expressed in terms of the fields as

$$\mathrm{R}=\frac{\int_L \mathbf{E}\cdot d\mathbf{l}}{\oint_S \mathbf{J}\cdot d\mathbf{S}}=\frac{\int_L \mathbf{E}\cdot d\mathbf{l}}{\oint_S \sigma\mathbf{E}\cdot d\mathbf{S}} \tag{2}$$

where $S$ is a surface completely surrounding an electrode and $L$ is any path joining the two electrodes. Note that the field line integral is taken along the line from the high to low potential electrode so that the voltage difference $V$ is equal to the positive line integral. From (2), we see that the resistance only depends on the geometry and conductivity $\sigma$ and not on the magnitude of the electric field itself. If we were to increase the voltage by any factor, the field would also increase by this same factor everywhere so that this factor would cancel out in the ratio of (2). The conductivity $\sigma$ may itself be a function of position.

### 3-4-2 Parallel Plate Resistor

Two perfectly conducting parallel plate electrodes of arbitrarily shaped area $A$ and spacing $l$ enclose a cylinder of material with Ohmic conductivity $\sigma$, as in Figure 3-16*a*. The current must flow tangential to the outer surface as the outside medium being free space has zero conductivity so that no current can pass through the interface. Because the tangential component of electric field is continuous, a field does exist in the free space region that decreases with increasing distance from the resistor. This three-dimensional field is difficult to calculate because it depends on three coordinates.

The electric field within the resistor is much simpler to calculate because it is perpendicular to the electrodes in the $x$ direction. Gauss's law with no volume charge then tells us that

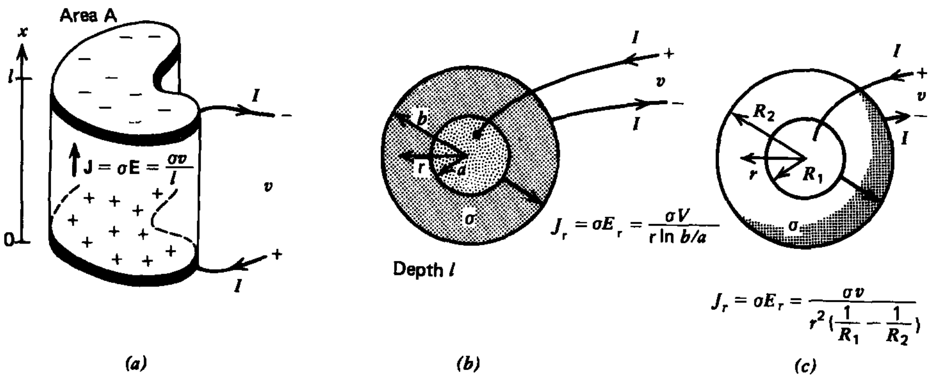



Figure 3-16 Simple resistor electrode geometries. (*a*) Parallel plates. (*b*) Coaxial cylinders. (*c*) Concentric spheres.

this field is constant:

$$\nabla \cdot (\varepsilon \mathbf{E}) = 0 \Rightarrow \frac{dE_x}{dx} = 0 \Rightarrow E_x = E_0 \tag{3}$$

However, the line integral of $\mathbf{E}$ between the electrodes must be the applied voltage $v$:

$$\int_0^l E_x\, dx = v \Rightarrow E_0 = v/l \tag{4}$$

The current density is then

$$\mathbf{J} = \sigma E_0 \mathbf{i}_x = (\sigma v/l)\mathbf{i}_x \tag{5}$$

so that the total current through the electrodes is

$$I = \oint_S \mathbf{J} \cdot \mathbf{dS} = (\sigma v/l)A \tag{6}$$

where the surface integral is reduced to a pure product because the constant current density is incident perpendicularly on the electrodes. The resistance is then

$$\mathrm{R} = \frac{v}{I} = \frac{l}{\sigma A} = \frac{\text{spacing}}{\text{(conductivity) (electrode area)}} \tag{7}$$

Typical resistance values can vary over many orders of magnitude. If the electrodes have an area $A = 1\ \text{cm}^2\ (10^{-4}\ \text{m}^2)$ with spacing $l = 1$ mm $(10^{-3}$ m) a material like copper has a resistance $R \approx 0.17 \times 10^{-6}$ ohm while carbon would have a resistance $R \approx 1.4 \times 10^4$ ohm. Because of this large range of resistance values sub-units often used are micro-ohms $(1\ \mu\Omega = 10^{-6}\ \Omega)$, milli-ohms $(1\ \text{m}\Omega = 10^{-3}\ \Omega)$, kilohm $(1\ k\Omega = 10^3\ \Omega)$, and megohms $(1\ \text{M}\Omega = 10^6\ \Omega)$, where the symbol $\Omega$ is used to represent the unit of ohms.

Although the field outside the resistor is difficult to find, we do know that for distances far from the resistor the field approaches that of a point dipole due to the oppositely charged electrodes with charge density

$$\sigma_f(x=0) = -\sigma_f(x=l) = \varepsilon E_0 = \varepsilon v/l \tag{8}$$

and thus dipole moment

$$\mathbf{p} = -\sigma_f(x=0)Al\mathbf{i}_x = -\varepsilon Av\mathbf{i}_x \tag{9}$$

The minus sign arises because the dipole moment points from negative to positive charge. Note that (8) is only approximate because all of the external field lines in the free space region must terminate on the side and back of the electrodes giving further contributions to the surface charge density. Generally, if the electrode spacing $l$ is much less than any of the electrode dimensions, this extra contribution is very small.

### 3-4-3 Coaxial Resistor

Two perfectly conducting coaxial cylinders of length $l$, inner radius $a$, and outer radius $b$ are maintained at a potential difference $v$ and enclose a material with Ohmic conductivity $\sigma$, as in Figure 3-16$b$. The electric field must then be perpendicular to the electrodes so that with no free charge Gauss's law requires

$$\nabla \cdot (\varepsilon \mathbf{E}) = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rE_r) = 0 \Rightarrow E_r = \frac{c}{r} \tag{10}$$

where $c$ is an integration constant found from the voltage condition

$$\int_a^b E_r\,dr = c \ln r \Big|_a^b = v \Rightarrow c = \frac{v}{\ln(b/a)} \tag{11}$$

The current density is then

$$J_r = \sigma E_r = \frac{\sigma v}{r \ln(b/a)} \tag{12}$$

with the total current at any radius r being a constant

$$I = \int_{z=0}^{l} \int_{\phi=0}^{2\pi} J_r r\,d\phi\,dz = \frac{\sigma v 2\pi l}{\ln(b/a)} \tag{13}$$

so that the resistance is

$$R = \frac{v}{I} = \frac{\ln(b/a)}{2\pi\sigma l} \tag{14}$$

### 3-4-4 Spherical Resistor

We proceed in the same way for two perfectly conducting concentric spheres at a potential difference $v$ with inner radius $R_1$ and outer radius $R_2$, as in Figure 3-16c. With no free charge, symmetry requires the electric field to be purely radial so that Gauss's law yields

$$\nabla \cdot (\varepsilon \mathbf{E}) = 0 \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = 0 \Rightarrow E_r = \frac{c}{r^2} \tag{15}$$

where $c$ is a constant found from the voltage condition as

$$\int_{R_1}^{R_2} E_r\, dr = -\frac{c}{r}\bigg|_{R_1}^{R_2} = v \Rightarrow c = \frac{v}{(1/R_1 - 1/R_2)} \tag{16}$$

The electric field and current density are inversely proportional to the square of the radius

$$J_r = \sigma E_r = \frac{\sigma v}{r^2(1/R_1 - 1/R_2)} \tag{17}$$

so that the current density is constant at any radius $r$

$$I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} J_r r^2 \sin\theta\, d\theta\, d\phi = \frac{4\pi\sigma v}{(1/R_1 - 1/R_2)} \tag{18}$$

with resistance

$$R = \frac{v}{I} = \frac{(1/R_1 - 1/R_2)}{4\pi\sigma} \tag{19}$$

## 3-5 CAPACITANCE

### 3-5-1 Parallel Plate Electrodes

Parallel plate electrodes of finite size constrained to potential difference $v$ enclose a dielectric medium with permittivity $\varepsilon$. The surface charge density does not distribute itself uniformly, as illustrated by the fringing field lines for infinitely thin parallel plate electrodes in Figure 3-17a. Near the edges the electric field is highly nonuniform decreasing in magnitude on the back side of the electrodes. Between the electrodes, far from the edges the electric field is uniform, being the same as if the electrodes were infinitely long. Fringing field effects can be made negligible if the electrode spacing $l$ is much less than the depth $d$ or width $w$. For more accurate work, end effects can be made even more negligible by using a guard ring encircling the upper electrode, as in Figure 3-17b. The guard ring is maintained at the same potential as the electrode, thus except for the very tiny gap, the field between

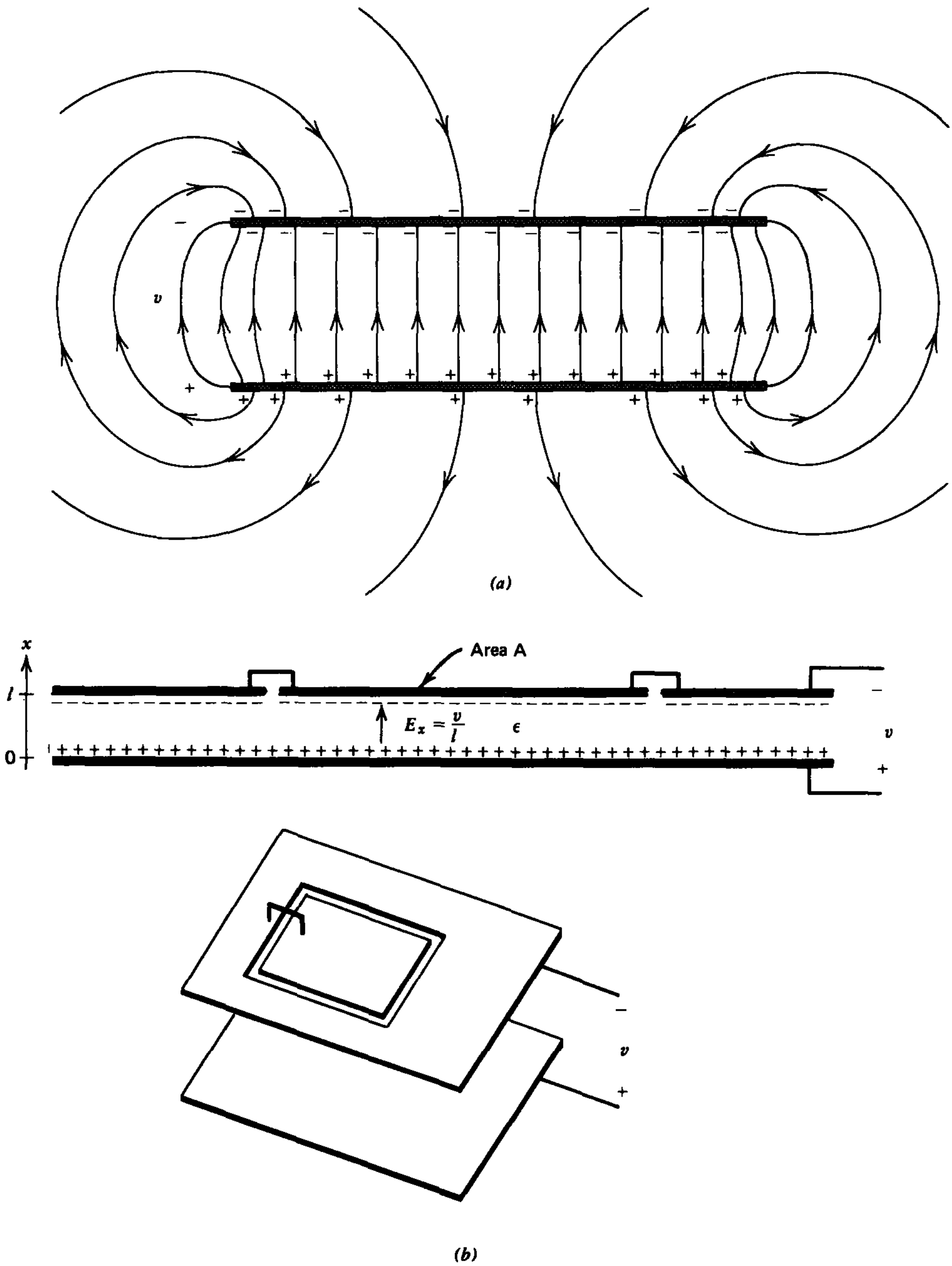



Figure 3-17 (*a*) Two infinitely thin parallel plate electrodes of finite area at potential difference $v$ have highly nonuniform fields outside the interelectrode region. (*b*) A guard ring around one electrode removes end effects so that the field between the electrodes is uniform. The end effects now arise at the edge of the guard ring, which is far from the region of interest.

the electrodes is as if the end effects were very far away and not just near the electrode edges.

We often use the phrase "neglect fringing" to mean that the nonuniform field effects near corners and edges are negligible.

With the neglect of fringing field effects near the electrode ends, the electric field is perpendicular to the electrodes and related to the voltage as

$$\int_0^l E_x\,dx = v \Rightarrow E_x = v/l \tag{1}$$

The displacement vector is then proportional to the electric field terminating on each electrode with an equal magnitude but opposite polarity surface charge density given by

$$D_x = \varepsilon E_x = \sigma_f(x=0) = -\sigma_f(x=l) = \varepsilon v/l \tag{2}$$

The charge is positive where the voltage polarity is positive, and vice versa, with the electric field directed from the positive to negative electrode. The magnitude of total free charge on each electrode is

$$q_f = \sigma_f(x=0)A = \frac{\varepsilon A}{l} v \tag{3}$$

The capacitance $C$ is defined as the magnitude of the ratio of total free charge on either electrode to the voltage difference between electrodes:

$$\begin{aligned} C &= \frac{q_f}{v} = \frac{\varepsilon A}{l} \\ &= \frac{\text{(permittivity) (electrode area)}}{\text{spacing}} \text{ farad } [\mathrm{A^2\text{-}s^4\text{-}kg^{-1}\text{-}m^{-2}}] \end{aligned} \tag{4}$$

Even though the system remains neutral, mobile electrons on the lower electrode are transported through the voltage source to the upper electrode in order to terminate the displacement field at the electrode surfaces, thus keeping the fields zero inside the conductors. Note that no charge is transported through free space. The charge transport between electrodes is due to work by the voltage source and results in energy stored in the electric field.

In SI units, typical capacitance values are very small. If the electrodes have an area of $A = 1\ \mathrm{cm}^2$ ($10^{-4}\ \mathrm{m}^2$) with spacing of $l = 1$ mm ($10^{-3}$ m), the free space capacitance is $C \approx 0.9 \times 10^{-12}$ farad. For this reason usual capacitance values are expressed in microfarads ($1\ \mu\mathrm{f} = 10^{-6}$ farad), nanofarads ($1\ \mathrm{nf} = 10^{-9}$ farad), and picofarads ($1\ \mathrm{pf} = 10^{-12}$ farad).

With a linear dielectric of permittivity $\varepsilon$ as in Figure 3-18*a*, the field of (1) remains unchanged for a given voltage but the charge on the electrodes and thus the capacitance increases with the permittivity, as given by (3). However, if the total free charge on each electrode were constrained, the voltage difference would decrease by the same factor.

These results arise because of the presence of polarization charges on the electrodes that partially cancel the free charge. The polarization vector within the dielectric-filled parallel plate capacitor is a constant

$$P_x = D_x - \varepsilon_0 E_x = (\varepsilon - \varepsilon_0)E_x = (\varepsilon - \varepsilon_0)v/l \tag{5}$$

so that the volume polarization charge density is zero. However, with zero polarization in the electrodes, there is a discontinuity in the normal component of polarization at the electrode surfaces. The boundary condition of Section 3.3.4 results in an equal magnitude but opposite polarity surface polarization charge density on each electrode, as illustrated in

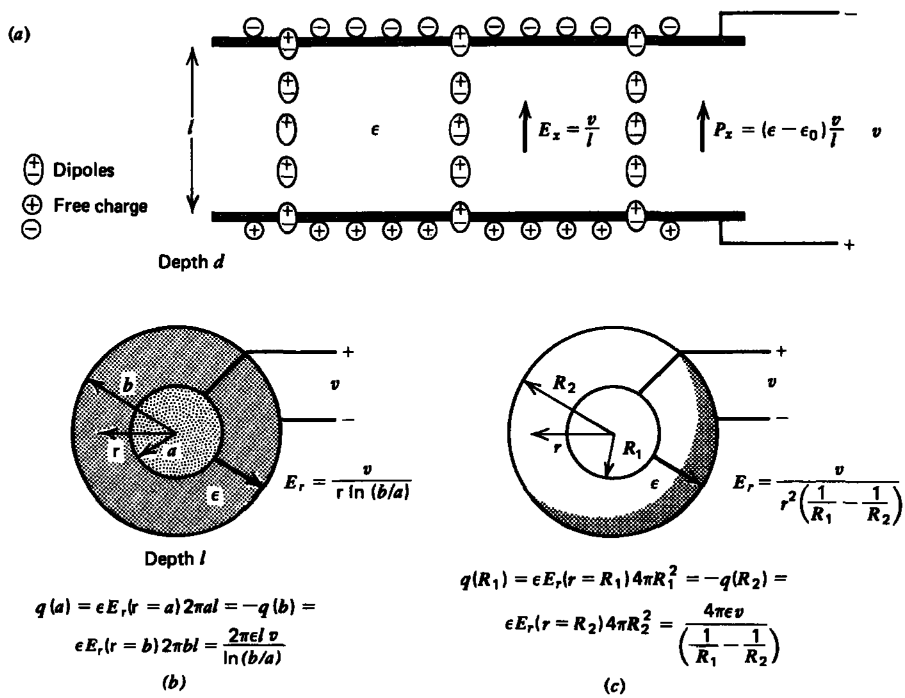



Figure 3-18 The presence of a dielectric between the electrodes increases the capacitance because for a given voltage additional free charge is needed on each electrode to overcome the partial neutralization of the attracted opposite polarity dipole ends. (*a*) Parallel plate electrodes. (*b*) Coaxial cylinders. (*c*) Concentric spheres.

Figure 3-18$a$:

$$\sigma_p(x=0) = -\sigma_p(x=l) = -P_x = -(\varepsilon-\varepsilon_0)v/l \tag{6}$$

Note that negative polarization charge appears on the positive polarity electrode and vice versa. This is because opposite charges attract so that the oppositely charged ends of the dipoles line up along the electrode surface partially neutralizing the free charge.

### 3-5-2 Capacitance for any Geometry

We have based our discussion around a parallel plate capacitor. Similar results hold for any shape electrodes in a dielectric medium with the capacitance defined as the magnitude of the ratio of total free charge on an electrode to potential difference. The capacitance is always positive by definition and for linear dielectrics is only a function of the geometry and dielectric permittivity and not on the voltage levels,

$$C = \frac{q_f}{v} = \frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\int_L \mathbf{E}\cdot d\mathbf{l}} = \frac{\oint_S \varepsilon\mathbf{E}\cdot d\mathbf{S}}{\int_L \mathbf{E}\cdot d\mathbf{l}} \tag{7}$$

as multiplying the voltage by a constant factor also increases the electric field by the same factor so that the ratio remains unchanged.

The integrals in (7) are similar to those in Section 3.4.1 for an Ohmic conductor. For the same geometry filled with a homogenous Ohmic conductor or a linear dielectric, the resistance-capacitance product is a constant independent of the geometry:

$$\mathrm{R}C = \frac{\int_L \mathbf{E}\cdot d\mathbf{l}}{\sigma\oint_S \mathbf{E}\cdot d\mathbf{S}}\,\frac{\varepsilon\oint_S \mathbf{E}\cdot d\mathbf{S}}{\int_L \mathbf{E}\cdot d\mathbf{l}} = \frac{\varepsilon}{\sigma} \tag{8}$$

Thus, for a given geometry, if either the resistance or capacitance is known, the other quantity is known immediately from (8). We can thus immediately write down the capacitance of the geometries shown in Figure 3-18 assuming the medium between electrodes is a linear dielectric with permittivity $\varepsilon$ using the results of Sections 3.4.2–3.4.4:

$$\begin{aligned}
&\text{Parallel Plate}\quad \mathrm{R} = \frac{l}{\sigma A} \Rightarrow C = \frac{\varepsilon A}{l}\\
&\text{Coaxial}\quad \mathrm{R} = \frac{\ln(b/a)}{2\pi\sigma l} \Rightarrow C = \frac{2\pi\varepsilon l}{\ln(b/a)}\\
&\text{Spherical}\quad \mathrm{R} = \frac{1/R_1 - 1/R_2}{4\pi\sigma} \Rightarrow C = \frac{4\pi\varepsilon}{(1/R_1 - 1/R_2)}
\end{aligned} \tag{9}$$

### 3-5-3 Current Flow Through a Capacitor

From the definition of capacitance in (7), the current to an electrode is

$$i=\frac{dq_f}{dt}=\frac{d}{dt}(Cv)=C\frac{dv}{dt}+v\frac{dC}{dt} \tag{10}$$

where the last term only arises if the geometry or dielectric permittivity changes with time. For most circuit applications, the capacitance is independent of time and (10) reduces to the usual voltage-current circuit relation.

In the capacitor of arbitrary geometry, shown in Figure 3-19, a conduction current $i$ flows through the wires into the upper electrode and out of the lower electrode changing the amount of charge on each electrode, as given by (10). There is no conduction current flowing in the dielectric between the electrodes. As discussed in Section 3.2.1 the total current, displacement plus conduction, is continuous. Between the electrodes in a lossless capacitor, this current is entirely displacement current. The displacement field is itself related to the time-varying surface charge distribution on each electrode as given by the boundary condition of Section 3.3.2.

### 3-5-4 Capacitance of Two Contacting Spheres

If the outer radius $R_2$ of the spherical capacitor in (9) is put at infinity, we have the capacitance of an isolated sphere of radius $R$ as

$$C=4\pi\varepsilon R \tag{11}$$

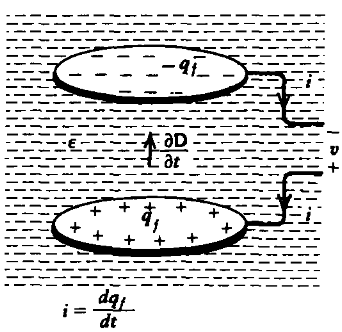



Figure 3-19 The conduction current $i$ that travels through the connecting wire to an electrode in a lossless capacitor is transmitted through the dielectric medium to the opposite electrode via displacement current. No charge carriers travel through the lossless dielectric.

If the surrounding medium is free space ($\varepsilon = \varepsilon_0$) for $R = 1$ m, we have that $C \approx \frac{1}{9} \times 10^{-9}$ farad $\approx 111$ pf.

We wish to find the self-capacitance of two such contacting spheres raised to a potential $V_0$, as shown in Figure 3-20. The capacitance is found by first finding the total charge on the two spheres. We can use the method of images by first placing an image charge $q_1 = Q = 4\pi\varepsilon R V_0$ at the center of each sphere to bring each surface to potential $V_0$. However, each of these charges will induce an image charge $q_2$ in the other sphere at distance $b_2$ from the center,

$$q_2 = -\frac{Q}{2}, \qquad b_2 = \frac{R^2}{D} = \frac{R}{2} \tag{12}$$

where we realize that the distance from inducing charge to the opposite sphere center is $D = 2R$. This image charge does not raise the potential of either sphere. Similarly, each of these image charges induces another image charge $q_3$ in the other sphere at disance $b_3$,

$$q_3 = -\frac{q_2 R}{D - b_2} = \frac{Q}{3}, \qquad b_3 = \frac{R^2}{D - b_2} = \frac{2}{3}R \tag{13}$$

which will induce a further image charge $q_4$, ad infinitum. An infinite number of image charges will be necessary, but with the use of difference equations we will be able to add all the image charges to find the total charge and thus the capacitance.

The $n$th image charge $q_n$ and its distance from the center $b_n$ are related to the $(n-1)$th images as

$$q_n = -\frac{q_{n-1} R}{D - b_{n-1}}, \qquad b_n = \frac{R^2}{D - b_{n-1}} \tag{14}$$

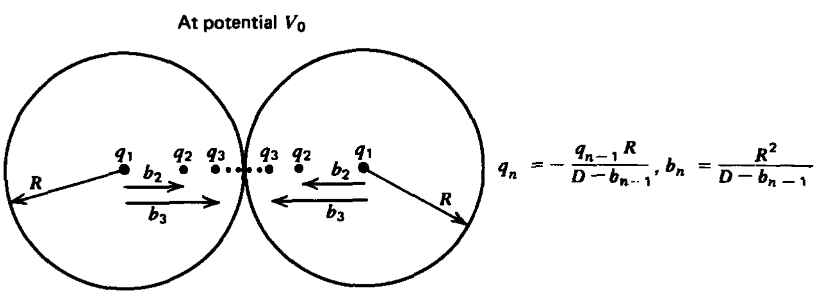



Figure 3-20 Two identical contacting spheres raised to a potential $V_0$ with respect to infinity are each described by an infinite number of image charges $q_n$ each a distance $b_n$ from the sphere center.

where $D=2R$. We solve the first relation for $b_{n-1}$ as

$$D-b_{n-1}=-\frac{q_{n-1}}{q_n}R$$
$$b_n=\frac{q_n}{q_{n+1}}R+D \tag{15}$$

where the second relation is found by incrementing $n$ in the first relation by 1. Substituting (15) into the second relation of (14) gives us a single equation in the $q_n$'s:

$$\frac{q_nR}{q_{n+1}}+D=-\frac{Rq_n}{q_{n-1}}\Rightarrow\frac{1}{q_{n+1}}+\frac{2}{q_n}+\frac{1}{q_{n-1}}=0 \tag{16}$$

If we define the reciprocal charges as

$$p_n=1/q_n \tag{17}$$

then (16) becomes a homogeneous linear constant coefficient difference equation

$$p_{n+1}+2p_n+p_{n-1}=0 \tag{18}$$

Just as linear constant coefficient differential equations have exponential solutions, (18) has power law solutions of the form

$$p_n=A\lambda^n \tag{19}$$

where the characteristic roots $\lambda$, analogous to characteristic frequencies, are found by substitution back into (18),

$$\lambda^{n+1}+2\lambda^n+\lambda^{n-1}=0\Rightarrow\lambda^2+2\lambda+1=(\lambda+1)^2=0 \tag{20}$$

to yield a double root with $\lambda=-1$. Because of the double root, the superposition of both solutions is of the form

$$p_n=A_1(-1)^n+A_2n(-1)^n \tag{21}$$

similar to the behavior found in differential equations with double characteristic frequencies. The correctness of (21) can be verified by direct substitution back into (18). The constants $A_1$ and $A_2$ are determined from $q_1$ and $q_2$ as

$$\left.\begin{aligned}p_1&=1/Q=-A_1-A_2\\ p_2&=\frac{1}{q_2}=-\frac{2}{Q}=+A_1+2A_2\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}A_1&=0\\ A_2&=-\frac{1}{Q}\end{aligned}\right. \tag{22}$$

so that the $n$th image charge is

$$q_n=\frac{1}{p_n}=\frac{1}{-(-1)^n n/Q}=\frac{-(-1)^nQ}{n} \tag{23}$$

The capacitance is then given as the ratio of the total charge on the two spheres to the voltage,

$$C=\frac{2\sum_{n=0}^{\infty} q_n}{V_0}=-\frac{2Q}{V_0}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}=+\frac{2Q}{V_0}[1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots]$$
$$=8\pi\varepsilon R\ln 2 \tag{24}$$

where we recognize the infinite series to be the Taylor series expansion of $\ln(1+x)$ with $x=1$. The capacitance of two contacting spheres is thus $2\ln 2\approx 1.39$ times the capacitance of a single sphere given by (11).

The distance from the center to each image charge is obtained from (23) substituted into (15) as

$$b_n=\left(\frac{(-1)^n\,(n+1)}{n(-1)^{n+1}}+2\right)R=\frac{(n-1)}{n}R \tag{25}$$

We find the force of attraction between the spheres by taking the sum of the forces on each image charge on one of the spheres due to all the image charges on the other sphere. The force on the $n$th image charge on one sphere due to the $m$th image charge in the other sphere is

$$f_{nm}=\frac{-q_nq_m}{4\pi\varepsilon[2R-b_n-b_m]^2}=\frac{-Q_0^2(-1)^{n+m}}{4\pi\varepsilon R^2}\frac{nm}{(m+n)^2} \tag{26}$$

where we used (23) and (25). The total force on the left sphere is then found by summing over all values of $m$ and $n$,

$$f=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}f_{nm}=\frac{-Q_0^2}{4\pi\varepsilon R^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+m}nm}{(n+m)^2}$$
$$=\frac{-Q_0^2}{4\pi\varepsilon R^2}\frac{1}{6}[\ln 2-\tfrac{1}{4}] \tag{27}$$

where the double series can be explicitly expressed.* The force is negative because the like charge spheres repel each other. If $Q_0=1$ coul with $R=1$ m, in free space this force is $f\approx 6.6\times 10^8$ nt, which can lift a mass in the earth's gravity field of $6.8\times 10^7$ kg ($\approx 3\times 10^7$ lb).

## 3-6 LOSSY MEDIA

Many materials are described by both a constant permittivity $\varepsilon$ and constant Ohmic conductivity $\sigma$. When such a material is placed between electrodes do we have a capacitor

* *See Albert D. Wheelon,* Tables of Summable Series and Integrals Involving Bessel Functions, *Holden Day, (1968) pp. 55, 56.*

or a resistor? We write the governing equations of charge conservation and Gauss's law with linear constitutive laws:

$$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0, \quad \mathbf{J}_f = \sigma \mathbf{E} + \rho_f \mathbf{U} \tag{1}$$

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \mathbf{D} = \varepsilon \mathbf{E} \tag{2}$$

We have generalized Ohm's law in (1) to include convection currents if the material moves with velocity $\mathbf{U}$. In addition to the conduction charges, any free charges riding with the material also contribute to the current. Using (2) in (1) yields a single partial differential equation in $\rho_f$:

$$\sigma \underbrace{(\nabla \cdot \mathbf{E})}_{\rho_f/\varepsilon} + \nabla \cdot (\rho_f \mathbf{U}) + \frac{\partial \rho_f}{\partial t} = 0 \Rightarrow \frac{\partial \rho_f}{\partial t} + \nabla \cdot (\rho_f \mathbf{U}) + \frac{\sigma}{\varepsilon} \rho_f = 0 \tag{3}$$

## 3-6-1 Transient Charge Relaxation

Let us first assume that the medium is stationary so that $\mathbf{U} = 0$. Then the solution to (3) for any initial possibly spatially varying charge distribution $\rho_0(x, y, z, t = 0)$ is

$$\rho_f = \rho_0(x, y, z, t = 0)\, e^{-t/\tau}, \quad \tau = \varepsilon/\sigma \tag{4}$$

where $\tau$ is the relaxation time. This solution is the continuum version of the resistance-capacitance (RC) decay time in circuits.

The solution of (4) tells us that at all positions within a conductor, any initial charge density dies off exponentially with time. It does not spread out in space. This is our justification of not considering any net volume charge in conducting media. If a system has no volume charge at $t = 0$ ($\rho_0 = 0$), it remains uncharged for all further time. Charge is transported through the region by the Ohmic current, but the net charge remains zero. Even if there happens to be an initial volume charge distribution, for times much longer than the relaxation time the volume charge density becomes negligibly small. In metals, $\tau$ is on the order of $10^{-19}$ sec, which is the justification of assuming the fields are zero within an electrode. Even though their large conductivity is not infinite, for times longer than the relaxation time $\tau$, the field solutions are the same as if a conductor were perfectly conducting.

The question remains as to where the relaxed charge goes. The answer is that it is carried by the conduction current to surfaces of discontinuity where the conductivity abruptly changes.

### 3-6-2 Uniformly Charged Sphere

A sphere of radius $R_2$ with constant permittivity $\varepsilon$ and Ohmic conductivity $\sigma$ is uniformly charged up to the radius $R_1$ with charge density $\rho_0$ at time $t=0$, as in Figure 3-21. From $R_1$ to $R_2$ the sphere is initially uncharged so that it remains uncharged for all time. The sphere is surrounded by free space with permittivity $\varepsilon_0$ and zero conductivity.

From (4) we can immediately write down the volume charge distribution for all time,

$$\rho_f = \begin{cases} \rho_0 e^{-t/\tau}, & r < R_1 \\ 0, & r > R_1 \end{cases} \tag{5}$$

where $\tau = \varepsilon/\sigma$. The total charge on the sphere remains constant, $Q = \frac{4}{3}\pi R_1^3 \rho_0$, but the volume charge is transported by the Ohmic current to the interface at $r = R_2$ where it becomes a surface charge. Enclosing the system by a Gaussian surface with $r > R_2$ shows that the external electric field is time independent,

$$E_r = \frac{Q}{4\pi\varepsilon_0 r^2}, \quad r > R_2 \tag{6}$$

Similarly, applying Gaussian surfaces for $r < R_1$ and $R_1 < r < R_2$ yields

$$E_r = \begin{cases} \dfrac{\rho_0 r e^{-t/\tau}}{3\varepsilon} = \dfrac{Q r e^{-t/\tau}}{4\pi\varepsilon R_1^3}, & 0 < r < R_1 \\ \dfrac{Q e^{-t/\tau}}{4\pi\varepsilon r^2}, & R_1 < r < R_2 \end{cases} \tag{7}$$

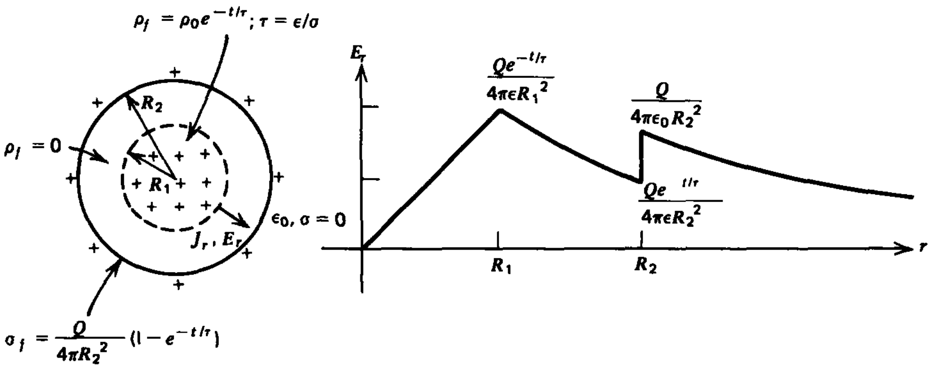



Figure 3-21 An initial volume charge distribution within an Ohmic conductor decays exponentially towards zero with relaxation time $\tau = \varepsilon/\sigma$ and appears as a surface charge at an interface of discontinuity. Initially uncharged regions are always uncharged with the charge transported through by the current.

The surface charge density at $r=R_2$ builds up exponentially with time:

$$\sigma_f(r=R_2)=\varepsilon_0 E_r(r=R_{2+})-\varepsilon E_r(r=R_{2-})$$
$$=\frac{Q}{4\pi R_2^2}(1-e^{-t/\tau}) \tag{8}$$

The charge is carried from the charged region ($r<R_1$) to the surface at $r=R_2$ via the conduction current with the charge density inbetween ($R_1<r<R_2$) remaining zero:

$$J_c=\sigma E_r=\begin{cases}\dfrac{\sigma Q r}{4\pi\varepsilon R_1^3}e^{-t/\tau}, & 0<r<R_1\\[2mm] \dfrac{\sigma Q\, e^{-t/\tau}}{4\pi\varepsilon r^2}, & R_1<r<R_2\\[2mm] 0, & r>R_2\end{cases} \tag{9}$$

Note that the total current, conduction plus displacement, is zero everywhere:

$$-J_c=J_d=\varepsilon\frac{\partial E_r}{\partial t}=\begin{cases}-\dfrac{Qr\sigma\, e^{-t/\tau}}{4\pi\varepsilon R_1^3}, & 0<r<R_1\\[2mm] -\dfrac{\sigma Q\, e^{-t/\tau}}{4\pi\varepsilon r^2}, & R_1<r<R_2\\[2mm] 0, & r>R_2\end{cases} \tag{10}$$

### 3-6-3 Series Lossy Capacitor

**(a) Charging transient**

To exemplify the difference between resistive and capacitive behavior we examine the case of two different materials in series stressed by a step voltage first turned on at $t=0$, as shown in Figure 3-22$a$. Since it takes time to charge up the interface, the interfacial surface charge cannot instantaneously change at $t=0$ so that it remains zero at $t=0_+$. With no surface charge density, the displacement field is continuous across the interface so that the solution at $t=0_+$ is the same as for two lossless series capacitors independent of the conductivities:

$$D_x=\varepsilon_1E_1=\varepsilon_2E_2 \tag{11}$$

The voltage constraint requires that

$$\int_0^{a+b}E_x\,dx=E_1a+E_2b=V \tag{12}$$

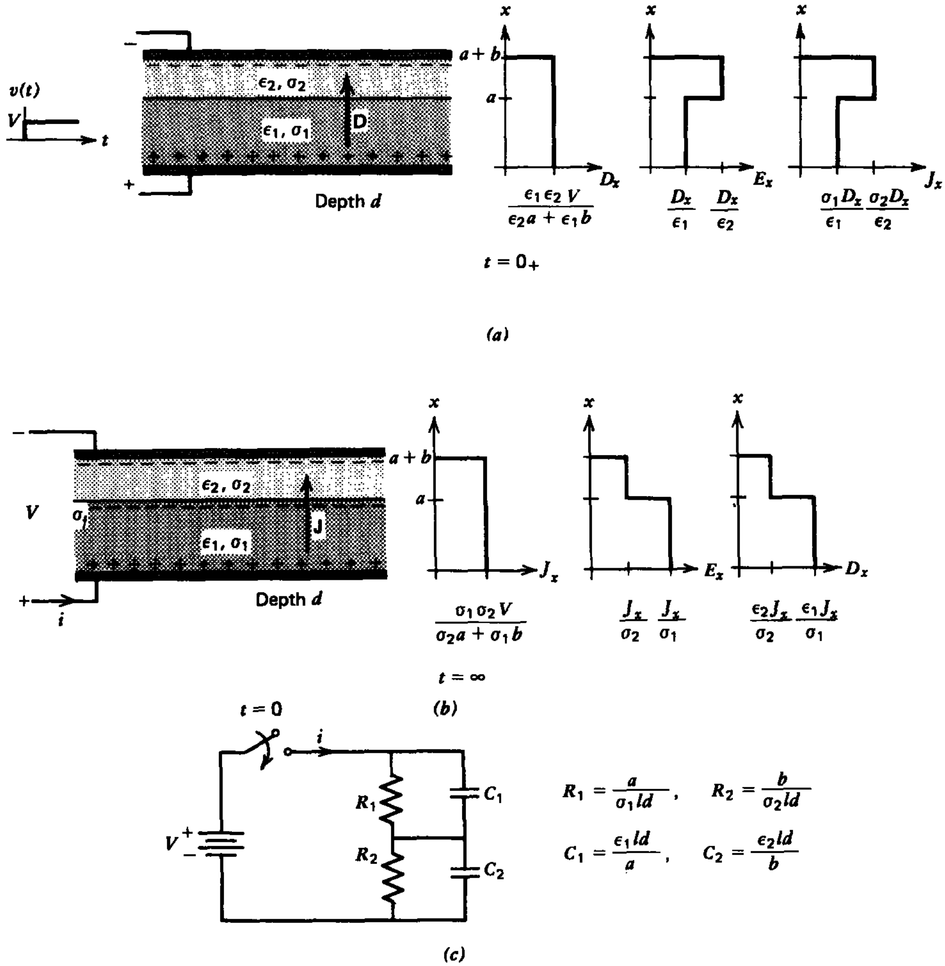



Figure 3-22 Two different lossy dielectric materials in series between parallel plate electrodes have permittivities and Ohmic conductivities that change abruptly across the interface. (*a*) At $t = 0_+$, right after a step voltage is applied, the interface is uncharged so that the displacement field is continuous with the solution the same as for two lossless dielectrics in series. (b) Since the current is discontinuous across the boundary between the materials, the interface will charge up. In the dc steady state the current is continuous. (c) Each region is equivalent to a resistor and capacitor in parallel.

so that the displacement field is

$$D_x(t = 0_+) = \frac{\varepsilon_1\varepsilon_2 V}{\varepsilon_2 a + \varepsilon_1 b} \tag{13}$$

The total current from the battery is due to both conduction and displacement currents. At $t = 0$, the displacement current is infinite (an impulse) as the displacement field instantaneously changes from zero to (13) to produce the surface charge on each electrode:

$$\sigma_f(x=0) = -\sigma_f(x=a+b) = D_x \tag{14}$$

After the voltage has been on a long time, the fields approach their steady-state values, as in Figure 3-22*b*. Because there are no more time variations, the current density must be continuous across the interface just the same as for two series resistors independent of the permittivities,

$$J_x(t\to\infty) = \sigma_1 E_1 = \sigma_2 E_2 = \frac{\sigma_1\sigma_2 V}{\sigma_2 a + \sigma_1 b} \tag{15}$$

where we again used (12). The interfacial surface charge is now

$$\sigma_f(x=a) = \varepsilon_2 E_2 - \varepsilon_1 E_1 = \frac{(\varepsilon_2\sigma_1 - \varepsilon_1\sigma_2)V}{\sigma_2 a + \sigma_1 b} \tag{16}$$

What we have shown is that for early times the system is purely capacitive, while for long times the system is purely resistive. The inbetween transient interval is found by using (12) with charge conservation applied at the interface:

$$\mathbf{n}\cdot\left(\mathbf{J}_2 - \mathbf{J}_1 + \frac{d}{dt}(\mathbf{D}_2 - \mathbf{D}_1)\right) = 0$$

$$\Rightarrow \sigma_2 E_2 - \sigma_1 E_1 + \frac{d}{dt}[\varepsilon_2 E_2 - \varepsilon_1 E_1] = 0 \tag{17}$$

With (12) to relate $E_2$ to $E_1$ we obtain a single ordinary differential equation in $E_1$,

$$\frac{dE_1}{dt} + \frac{E_1}{\tau} = \frac{\sigma_2 V}{\varepsilon_2 a + \varepsilon_1 b} \tag{18}$$

where the relaxation time is a weighted average of relaxation times of each material:

$$\tau = \frac{\varepsilon_1 b + \varepsilon_2 a}{\sigma_1 b + \sigma_2 a} \tag{19}$$

Using the initial condition of (13) the solutions for the fields are

$$\begin{aligned} E_1 &= \frac{\sigma_2 V}{\sigma_2 a + \sigma_1 b}(1 - e^{-t/\tau}) + \frac{\varepsilon_2 V}{\varepsilon_2 a + \varepsilon_1 b} e^{-t/\tau} \\ E_2 &= \frac{\sigma_1 V}{\sigma_2 a + \sigma_1 b}(1 - e^{-t/\tau}) + \frac{\varepsilon_1 V}{\varepsilon_2 a + \varepsilon_1 b} e^{-t/\tau} \end{aligned} \tag{20}$$

Note that as $t\to\infty$ the solutions approach those of (15). The interfacial surface charge is

$$\sigma_f(x=a)=\varepsilon_2E_2-\varepsilon_1E_1=\frac{(\varepsilon_2\sigma_1-\varepsilon_1\sigma_2)}{\sigma_2a+\sigma_1b}(1-e^{-t/\tau})V \quad (21)$$

which is zero at $t=0$ and agrees with (16) for $t\to\infty$.

The total current delivered by the voltage source is

$$\begin{aligned} i&=\left(\sigma_1E_1+\varepsilon_1\frac{dE_1}{dt}\right)ld=\left(\sigma_2E_2+\varepsilon_2\frac{dE_2}{dt}\right)ld\\ &=\left[\frac{\sigma_1\sigma_2}{\sigma_2a+\sigma_1b}+\left(\sigma_1-\frac{\varepsilon_1}{\tau}\right)\left(\frac{\varepsilon_2}{\varepsilon_2a+\varepsilon_1b}-\frac{\sigma_2}{\sigma_2a+\sigma_1b}\right)e^{-t/\tau}\right.\\ &\left.\quad+\frac{\varepsilon_1\varepsilon_2}{\varepsilon_2a+\varepsilon_1b}\delta(t)\right]ldV \end{aligned} \quad (22)$$

where the last term is the impulse current that instantaneously puts charge on each electrode in zero time at $t=0$:

$$\delta(t)=\begin{cases}0, & t\neq 0\\ \infty, & t=0\end{cases}\Rightarrow\int_{0_-}^{0_+}\delta(t)\,dt=1$$

To reiterate, we see that for early times the capacitances dominate and that in the steady state the resistances dominate with the transition time depending on the relaxation times and geometry of each region. The equivalent circuit for the system is shown in Figure 3-22*c* as a series combination of a parallel resistor-capacitor for each region.

**(b) Open Circuit**

Once the system is in the dc steady state, we instantaneously open the circuit so that the terminal current is zero. Then, using (22) with $i=0$, we see that the fields decay independently in each region with the relaxation time of each region:

$$\begin{aligned} E_1&=\frac{\sigma_2V}{\sigma_2a+\sigma_1b}e^{-t/\tau_1},\quad \tau_1=\frac{\varepsilon_1}{\sigma_1}\\ E_2&=\frac{\sigma_1V}{\sigma_2a+\sigma_1b}e^{-t/\tau_2},\quad \tau_2=\frac{\varepsilon_2}{\sigma_2} \end{aligned} \quad (23)$$

The open circuit voltage and interfacial charge then decay as

$$\begin{aligned} V_{oc}&=E_1a+E_2b=\frac{V}{\sigma_2a+\sigma_1b}[\sigma_2a\,e^{-t/\tau_1}+\sigma_1b\,e^{-t/\tau_2}]\\ \sigma_f&=\varepsilon_2E_2-\varepsilon_1E_1=\frac{V}{\sigma_2a+\sigma_1b}[\varepsilon_2\sigma_1\,e^{-t/\tau_2}-\varepsilon_1\sigma_2\,e^{-t/\tau_1}] \end{aligned} \quad (24)$$

**(c) Short Circuit**

If the dc steady-state system is instead short circuited, we set $V=0$ in (12) and (18),

$$E_1 a + E_2 b = 0$$
$$\frac{dE_1}{dt}+\frac{E_1}{\tau}=0 \tag{25}$$

where $\tau$ is still given by (19). Since at $t=0$ the interfacial surface charge cannot instantaneously change, the initial fields must obey the relation

$$\lim_{t=0}(\varepsilon_2 E_2 - \varepsilon_1 E_1) = -\left(\frac{\varepsilon_2 a}{b}+\varepsilon_1\right)E_1 = \frac{(\varepsilon_2\sigma_1-\varepsilon_1\sigma_2)V}{\sigma_2 a+\sigma_1 b} \tag{26}$$

to yield the solutions

$$E_1 = -\frac{E_2 b}{a} = -\frac{(\varepsilon_2\sigma_1-\varepsilon_1\sigma_2)bV}{(\varepsilon_2 a+\varepsilon_1 b)(\sigma_2 a+\sigma_1 b)}e^{-t/\tau} \tag{27}$$

The short circuit current and surface charge are then

$$i = \left[\left(\frac{\sigma_1\varepsilon_2-\varepsilon_1\sigma_2}{\varepsilon_1 b+\varepsilon_2 a}\right)^2\frac{abV}{(\sigma_2 a+\sigma_1 b)}e^{-t/\tau}-\frac{V\varepsilon_1\varepsilon_2}{\varepsilon_2 a+\varepsilon_1 b}\delta(t)\right]ld$$
$$\sigma_f = \varepsilon_2 E_2 - \varepsilon_1 E_1 = \frac{(\varepsilon_2\sigma_1-\varepsilon_1\sigma_2)}{\sigma_2 a+\sigma_1 b}Ve^{-t/\tau} \tag{28}$$

The impulse term in the current is due to the instantaneous change in displacement field from the steady-state values found from (15) to the initial values of (26).

**(d) Sinusoidal Steady State**

Now rather than a step voltage, we assume that the applied voltage is sinusoidal,

$$v(t) = V_0\cos\omega t \tag{29}$$

and has been on a long time.

The fields in each region are still only functions of time and not position. It is convenient to use complex notation so that all quantities are written in the form

$$v(t) = \mathrm{Re}\,(V_0 e^{j\omega t})$$
$$E_1(t) = \mathrm{Re}\,(\hat{E}_1 e^{j\omega t}),\quad E_2(t) = \mathrm{Re}\,(\hat{E}_2 e^{j\omega t}) \tag{30}$$

Using carets above a term to designate a complex amplitude, the applied voltage condition of (12) requires

$$\hat{E}_1 a + \hat{E}_2 b = V_0 \tag{31}$$

while the interfacial charge conservation equation of (17) becomes

$$\sigma_2\hat{E}_2-\sigma_1\hat{E}_1+j\omega\,(\varepsilon_2\hat{E}_2-\varepsilon_1\hat{E}_1)=[\sigma_2+j\omega\varepsilon_2]\hat{E}_2 -[\sigma_1+j\omega\varepsilon_1]\hat{E}_1=0 \tag{32}$$

The solutions are

$$\frac{\hat{E}_1}{(j\omega\varepsilon_2+\sigma_2)}=\frac{\hat{E}_2}{(j\omega\varepsilon_1+\sigma_1)}=\frac{V_0}{[b(\sigma_1+j\omega\varepsilon_1)+a(\sigma_2+j\omega\varepsilon_2)]} \tag{33}$$

which gives the interfacial surface charge amplitude as

$$\hat{\sigma}_f=\varepsilon_2\hat{E}_2-\varepsilon_1\hat{E}_1=\frac{(\varepsilon_2\sigma_1-\varepsilon_1\sigma_2)V_0}{[b(\sigma_1+j\omega\varepsilon_1)+a(\sigma_2+j\omega\varepsilon_2)]} \tag{34}$$

As the frequency becomes much larger than the reciprocal relaxation times,

$$\omega \gg \frac{\sigma_1}{\varepsilon_1},\quad \omega \gg \frac{\sigma_2}{\varepsilon_2} \tag{35}$$

the surface charge density goes to zero. This is because the surface charge cannot keep pace with the high-frequency alternations, and thus the capacitive component dominates. Thus, in experimental work charge accumulations can be prevented if the excitation frequencies are much faster than the reciprocal charge relaxation times.

The total current through the electrodes is

$$\begin{aligned}\hat{I}&=(\sigma_1+j\omega\varepsilon_1)\hat{E}_1ld=(\sigma_2+j\omega\varepsilon_2)\hat{E}_2ld\\&=\frac{ld(\sigma_1+j\omega\varepsilon_1)(\sigma_2+j\omega\varepsilon_2)V_0}{[b(\sigma_1+j\omega\varepsilon_1)+a(\sigma_2+j\omega\varepsilon_2)]}\\&=\frac{V_0}{\dfrac{R_2}{R_2C_2j\omega+1}+\dfrac{R_1}{R_1C_1j\omega+1}}\end{aligned} \tag{36}$$

with the last result easily obtained from the equivalent circuit in Figure 3-22$c$.

### 3-6-4 Distributed Systems

#### (a) Governing Equations

In all our discussions we have assumed that the electrodes are perfectly conducting so that they have no resistance and the electric field terminates perpendicularly. Consider now the parallel plate geometry shown in Figure 3-23$a$, where the electrodes have a large but finite conductivity $\sigma_c$. The electrodes are no longer equi-potential surfaces since as the current passes along the conductor an Ohmic $i$R drop results.

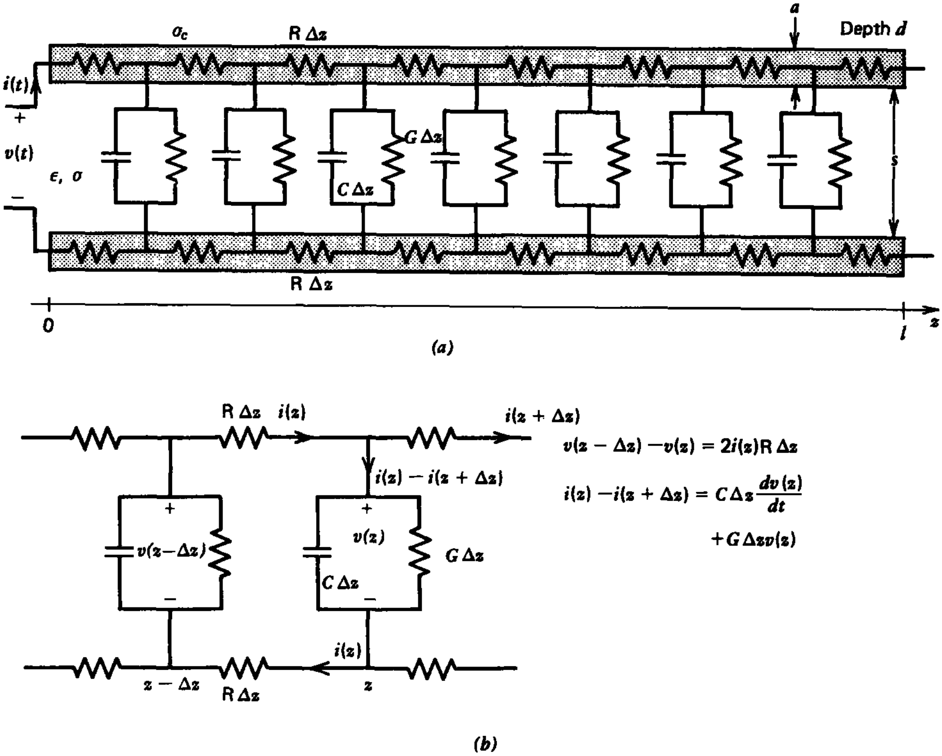



Figure 3-23 Lossy parallel plate electrodes with finite Ohmic conductivity $\sigma_c$ enclose a lossy dielectric with permittivity $\varepsilon$ and conductivity $\sigma$. (*a*) This system can be modeled by a distributed resistor-capacitor network. (*b*) Kirchoff's voltage and current laws applied to a section of length $\Delta z$ allow us to describe the system by partial differential equations.

The current is also shunted through the lossy dielectric so that less current flows at the far end of the conductor than near the source. We can find approximate solutions by breaking the continuous system into many small segments of length $\Delta z$. The electrode resistance of this small section is

$$\mathrm{R}\,\Delta z = \frac{\Delta z}{\sigma_c a d} \tag{37}$$

where $\mathrm{R} = 1/(\sigma_c a d)$ is just the resistance per unit length. We have shown in the previous section that the dielectric can be modeled as a parallel resistor-capacitor combination,

$$C\Delta z = \frac{\varepsilon d \Delta z}{s}, \quad \frac{1}{G\Delta z} = \frac{s}{\sigma d \Delta z} \tag{38}$$

$C$ is the capacitance per unit length and $G$ is the conductance per unit length where the conductance is the reciprocal of the resistance. It is more convenient to work with the conductance because it is in parallel with the capacitance.

We apply Kirchoff's voltage and current laws for the section of equivalent circuit shown in Figure 3-23*b*:

$$\begin{aligned} v(z-\Delta z)-v(z)&=2i(z)\mathrm{R}\,\Delta z\\ i(z)-i(z+\Delta z)&=C\Delta z\frac{dv(z)}{dt}+G\Delta z v(z)\end{aligned} \tag{39}$$

The factor of 2 in the upper equation arises from the equal series resistances of the upper and lower conductors. Dividing through by $\Delta z$ and taking the limit as $\Delta z$ becomes infinitesimally small yields the partial differential equations

$$\begin{aligned} -\frac{\partial v}{\partial z}&=2i\mathrm{R}\\ -\frac{\partial i}{\partial z}&=C\frac{\partial v}{\partial t}+Gv\end{aligned} \tag{40}$$

Taking $\partial/\partial z$ of the upper equation allows us to substitute in the lower equation to eliminate $i$,

$$\frac{\partial^2 v}{\partial z^2}=2\mathrm{R}C\frac{\partial v}{\partial t}+2\mathrm{R}Gv \tag{41}$$

which is called a transient diffusion equation. Equations (40) and (41) are also valid for any geometry whose cross sectional area remains constant over its length. The 2R represents the series resistance per unit length of both electrodes, while $C$ and $G$ are the capacitance and conductance per unit length of the dielectric medium.

### (b) Steady State

If a dc voltage $V_0$ is applied, the steady-state voltage is

$$\frac{d^2 v}{dz^2}-2\mathrm{R}Gv=0\Rightarrow v=A_1\sinh\sqrt{2\mathrm{R}G}z+A_2\cosh\sqrt{2\mathrm{R}G}z \tag{42}$$

where the constants are found by the boundary conditions at $z=0$ and $z=l$,

$$v(z=0)=V_0,\qquad i(z=l)=0 \tag{43}$$

We take the $z=l$ end to be open circuited. Solutions are

$$\begin{aligned} v(z)&=V_0\frac{\cosh\sqrt{2\mathrm{R}G}\,(z-l)}{\cosh\sqrt{2\mathrm{R}G}\,l}\\ i(z)&=-\frac{1}{2\mathrm{R}}\frac{dv}{dz}=V_0\sqrt{\frac{G}{2\mathrm{R}}}\frac{\sinh\sqrt{2\mathrm{R}G}(z-l)}{\cosh\sqrt{2\mathrm{R}G}\,l}\end{aligned} \tag{44}$$

**(c) Transient Solution**

If this dc voltage is applied as a step at $t=0$, it takes time for the voltage and current to reach these steady-state distributions. Because (41) is linear, we can use superposition and guess a solution for the voltage that is the sum of the steady-state solution in (44) and a transient solution that dies off with time:

$$v(z,t)=\frac{V_0\cosh\sqrt{2RG}\,(z-l)}{\cosh\sqrt{2RG}\,l}+\hat{v}(z)\,e^{-\alpha t} \tag{45}$$

At this point we do not know the function $\hat{v}(z)$ or $\alpha$. Substituting the assumed solution of (45) back into (41) yields the ordinary differential equation

$$\frac{d^2\hat{v}}{dz^2}+p^2\hat{v}=0,\qquad p^2=2RC\alpha-2RG \tag{46}$$

which has the trigonometric solutions

$$\hat{v}(z)=a_1\sin pz+a_2\cos pz \tag{47}$$

Since the time-independent part of (45) already satisfies the boundary conditions at $z=0$, the transient part must be zero there so that $a_2=0$. The transient contribution to the current $i$, found from (40),

$$\begin{aligned}i(z,t)&=V_0\sqrt{\frac{G}{2R}}\frac{\sinh\sqrt{2RG}\,(z-l)}{\cosh\sqrt{2RG}\,l}+\hat{\imath}(z)\,e^{-\alpha t}\\ \hat{\imath}(z)&=-\frac{1}{2R}\frac{d\hat{v}(z)}{dz}=-\frac{pa_1}{2R}\cos pz\end{aligned} \tag{48}$$

must still be zero at $z=l$, which means that $pl$ must be an odd integer multiple of $\pi/2$,

$$pl=(2n+1)\frac{\pi}{2}\Rightarrow\alpha_n=\frac{1}{2RC}\left((2n+1)\frac{\pi}{2l}\right)^2+\frac{G}{C},\quad n=0,1,2,\cdots \tag{49}$$

Since the boundary conditions allow an infinite number of values of $\alpha$, the most general solution is the superposition of all allowed solutions:

$$v(z,t)=V_0\frac{\cosh\sqrt{2RG}\,(z-l)}{\cosh\sqrt{2RG}\,l}+\sum_{n=0}^{\infty}A_n\sin(2n+1)\frac{\pi z}{2l}e^{-\alpha_n t} \tag{50}$$

This solution satisfies the boundary conditions but not the initial conditions at $t=0$ when the voltage is first turned on. Before the voltage source is applied, the voltage distribution throughout the system is zero. It must remain zero right after being turned on otherwise the time derivative in (40) would be infinite, which requires nonphysical infinite currents. Thus we impose the initial condition

$$v(z, t=0)=0=V_0\frac{\cosh\sqrt{2RG}(z-l)}{\cosh\sqrt{2RG}\,l}+\sum_{n=0}^{\infty}A_n\sin(2n+1)\frac{\pi z}{2l} \tag{51}$$

We can solve for the amplitudes $A_n$ by multiplying (51) through by $\sin(2m+1)\pi z/2l$ and then integrating over $z$ from 0 to $l$:

$$\begin{aligned}0=&\frac{V_0}{\cosh\sqrt{2RG}\,l}\int_0^l\cosh\sqrt{2RG}(z-l)\sin(2m+1)\frac{\pi z}{2l}\,dz\\&+\int_0^l\sum_{n=0}^{\infty}A_n\sin(2n+1)\frac{\pi z}{2l}\sin(2m+1)\frac{\pi z}{2l}\,dz\end{aligned} \tag{52}$$

The first term is easily integrated by writing the hyperbolic cosine in terms of exponentials,* while the last term integrates to zero for all values of $m$ not equal to $n$ so that the amplitudes are

$$A_n=-\frac{1}{l^2}\frac{\pi V_0(2n+1)}{2RG+[(2n+1)\pi/2l]^2} \tag{53}$$

The total solutions are then

$$\begin{aligned}v(z,t)=&\frac{V_0\cosh\sqrt{2RG}(z-l)}{\cosh\sqrt{2RG}\,l}\\&-\frac{\pi V_0}{l^2}\sum_{n=0}^{\infty}\frac{(2n+1)\sin[(2n+1)(\pi z/2l)]e^{-\alpha_n t}}{2RG+[(2n+1)(\pi/2l)]^2}\end{aligned}$$

$$\begin{aligned}i(z,t)=&-\frac{1}{2R}\frac{\partial v}{\partial z}\\=&-\frac{V_0\sqrt{G/2R}\sinh\sqrt{2RG}(z-l)}{\cosh\sqrt{2RG}\,l}\\&+\frac{\pi^2V_0}{4l^3R}\sum_{n=0}^{\infty}\frac{(2n+1)^2\cos[(2n+1)(\pi z/2l)]e^{-\alpha_n t}}{2RG+[(2n+1)(\pi/2l)]^2}\end{aligned} \tag{54}$$

* $\int\cosh a(z-l)\sin bz\,dz$

$$=\frac{1}{a^2+b^2}[a\sin bz\sinh a(z-l)-b\cos bz\cosh a(z-l)]$$

$$\int_0^l\sin(2n+1)bz\sin(2m+1)bz\,dz=\begin{cases}0 & m\neq n\\ l/2 & m=n\end{cases}$$

The fundamental time constant corresponds to the smallest value of $\alpha$, which is when $n=0$:

$$\tau_0=\frac{1}{\alpha_0}=\frac{C}{G+\dfrac{1}{2R}\left(\dfrac{\pi}{2l}\right)^2} \tag{55}$$

For times long compared to $\tau_0$ the system is approximately in the steady state. Because of the fast exponential decrease for times greater than zero, the infinite series in (54) can often be approximated by the first term. These solutions are plotted in Figure 3-24 for the special case where $G=0$. Then the voltage distribution builds up from zero to a constant value diffusing in from the left. The current near $z=0$ is initially very large. As time increases, with $G=0$, the current everywhere decreases towards a zero steady state.

### 3-6-5 Effects of Convection

We have seen that in a stationary medium any initial charge density decays away to a surface of discontinuity. We now wish to focus attention on a dc steady-state system of a conducting medium moving at constant velocity $U\mathbf{i}_x$, as in Figure 3-25. A source at $x=0$ maintains a constant charge density $\rho_0$. Then (3) in the dc steady state with constant

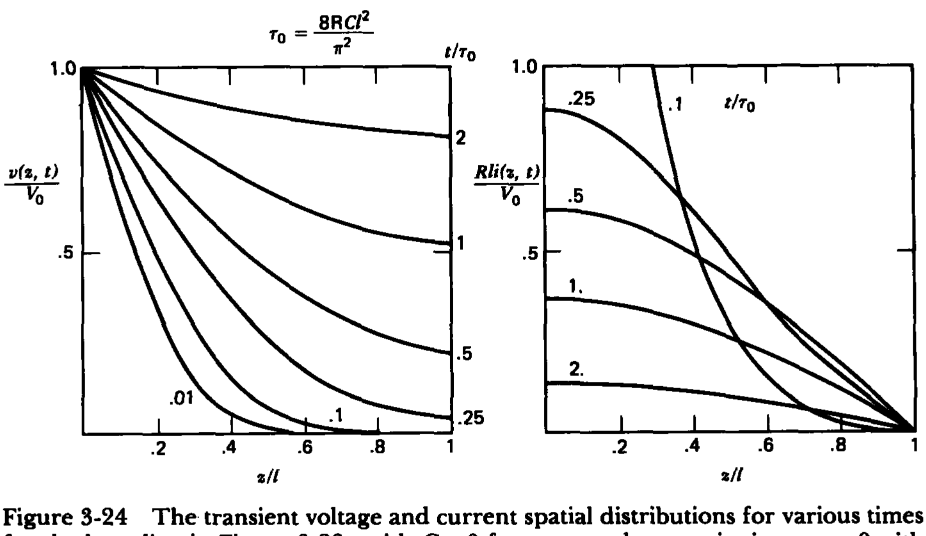



Figure 3-24 The transient voltage and current spatial distributions for various times for the lossy line in Figure 3-23*a* with $G=0$ for a step voltage excitation at $z=0$ with the $z=l$ end open circuited. The diffusion effects arise because of the lossy electrodes where the longest time constant is $\tau_0=8RCl^2/\pi^2$.

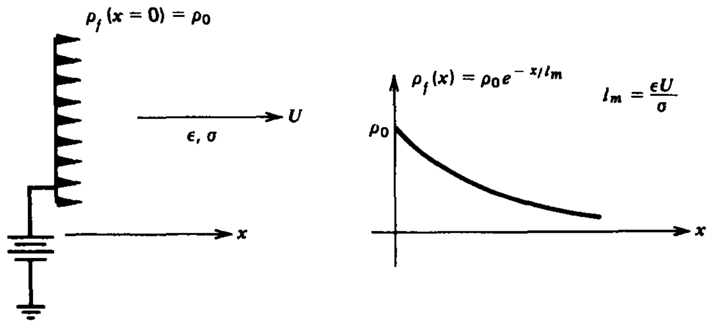



Figure 3-25 A moving conducting material with velocity $U\mathbf{i}_x$ tends to take charge injected at $x=0$ with it. The steady-state charge density decreases exponentially from the source.

velocity becomes

$$\frac{d\rho_f}{dx}+\frac{\sigma}{\varepsilon U}\rho_f=0 \tag{56}$$

which has exponentially decaying solutions

$$\rho_f=\rho_0\, e^{-x/l_m}, \qquad l_m=\frac{\varepsilon U}{\sigma} \tag{57}$$

where $l_m$ represents a characteristic spatial decay length. If the system has cross-sectional area $A$, the total charge $q$ in the system is

$$q=\int_0^\infty \rho_f A\, dx=\rho_0 l_m A \tag{58}$$

### 3-6-6 The Earth and its Atmosphere as a Leaky Spherical Capacitor*

In fair weather, at the earth's surface exists a dc electric field with approximate strength of 100 V/m directed radially toward the earth's center. The magnitude of the electric field decreases with height above the earth's surface because of the nonuniform electrical conductivity $\sigma(r)$ of the atmosphere approximated as

$$\sigma(r)=\sigma_0+a(r-R)^2 \text{ siemen/m} \tag{59}$$

where measurements have shown that

$$\begin{aligned}\sigma_0 &\approx 3\times 10^{-14}\\ a &\approx .5\times 10^{-20}\end{aligned} \tag{60}$$

* *M. A. Uman, "The Earth and Its Atmosphere as a Leaky Spherical Capacitor,"* Am. J. Phys. *V. 42, Nov. 1974, pp. 1033–1035.*

and $R\approx 6\times 10^6$ meter is the earth's radius. The conductivity increases with height because of cosmic radiation in the lower atmosphere. Because of solar radiation the atmosphere acts as a perfect conductor above 50 km.

In the dc steady state, charge conservation of Section 3-2-1 with spherical symmetry requires

$$\nabla\cdot\mathbf{J}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2J_r)=0\Rightarrow J_r=\sigma(r)E_r=\frac{C}{r^2} \tag{61}$$

where the constant of integration $C$ is found by specifying the surface electric field $E_r(R)\approx -100$ V/m

$$J_r(r)=\frac{\sigma(R)E_r(R)R^2}{r^2} \tag{62}$$

At the earth's surface the current density is then

$$J_r(R)=\sigma(R)E_r(R)=\sigma_0E_r(R)\approx -3\times 10^{-12}\ \text{amp/m}^2 \tag{63}$$

The total current directed radially inwards over the whole earth is then

$$I=|J_r(R)4\pi R^2|\approx 1350\ \text{amp} \tag{64}$$

The electric field distribution throughout the atmosphere is found from (62) as

$$E_r(r)=\frac{J_r(r)}{\sigma(r)}=\frac{\sigma(R)E_r(R)R^2}{r^2\sigma(r)} \tag{65}$$

The surface charge density on the earth's surface is

$$\sigma_f(r=R)=\varepsilon_0E_r(R)\approx -8.85\times 10^{-10}\ \text{Coul/m}^2 \tag{66}$$

This negative surface charge distribution (remember: $E_r(r)<0$) is balanced by positive volume charge distribution throughout the atmosphere

$$\begin{aligned}\rho_f(r)=\varepsilon_0\nabla\cdot\mathbf{E}&=\frac{\varepsilon_0}{r^2}\frac{\partial}{\partial r}(r^2E_r)=\frac{\varepsilon_0\sigma(R)E_r(R)R^2}{r^2}\frac{d}{dr}\left(\frac{1}{\sigma(r)}\right)\\&=\frac{-\varepsilon_0\sigma(R)E_r(R)R^2}{r^2(\sigma(r))^2}2a(r-R)\end{aligned} \tag{67}$$

The potential difference between the upper atmosphere and the earth's surface is

$$\begin{aligned}V&=-\int_R^\infty E_r(r)dr\\&=-\sigma(R)E_r(R)R^2\int_R^\infty\frac{dr}{r^2[\sigma_0+a(r-R)^2]}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\sigma(R)E_r(R)R^2}{a}\left\{-\frac{R}{\left(R^2+\frac{\sigma_0}{a}\right)^2}\ln\left[\frac{(r-R)^2+\frac{\sigma_0}{a}}{r^2}\right]\right.\\
&\qquad\left.\left.-\frac{1}{r\left(R^2+\frac{\sigma_0}{a}\right)}+\frac{\left(R^2-\frac{\sigma_0}{a}\right)}{\sqrt{\frac{\sigma_0}{a}}\left(R^2+\frac{\sigma_0}{a}\right)^2}\tan^{-1}\frac{(r-R)}{\sqrt{\frac{\sigma_0}{a}}}\right\}\right|_{r=R}^{\infty}\\
&= -\frac{\sigma(R)E_r(R)}{a\left(R^2+\frac{\sigma_0}{a}\right)^2}R^2\left\{R\ln\frac{\sigma_0}{aR^2}+\frac{\left(R^2+\frac{\sigma_0}{a}\right)}{R}+\frac{\frac{\pi}{2}\left(R^2-\frac{\sigma_0}{a}\right)}{\sqrt{\frac{\sigma_0}{a}}}\right\}
\end{aligned}
\tag{68}
$$

Using the parameters of (60), we see that $\sigma_0/a \ll R^2$ so that (68) approximately reduces to

$$
\begin{aligned}
\mathrm{V} &\approx -\frac{\sigma_0 E_r(R)}{aR^2}\left\{R\left(\ln\frac{\sigma_0}{aR^2}+1\right)+\frac{\pi R^2}{2\sqrt{\frac{\sigma_0}{a}}}\right\}\\
&\approx 384{,}000 \text{ volts}
\end{aligned}
\tag{69}
$$

If the earth's charge were not replenished, the current flow would neutralize the charge at the earth's surface with a time constant of order

$$
\tau=\frac{\varepsilon_0}{\sigma_0}\approx 300 \text{ seconds} \tag{70}
$$

It is thought that localized stormy regions simultaneously active all over the world serve as "batteries" to keep the earth charged via negatively charged lightning to ground and corona at ground level, producing charge that moves from ground to cloud. This thunderstorm current must be upwards and balances the downwards fair weather current of (64).

## 3.7 FIELD-DEPENDENT SPACE CHARGE DISTRIBUTIONS

A stationary Ohmic conductor with constant conductivity was shown in Section 3-6-1 to not support a steady-state volume charge distribution. This occurs because in our classical Ohmic model in Section 3-2-2$c$ one species of charge (e.g., electrons in metals) move relative to a stationary background species of charge with opposite polarity so that charge neutrality is maintained. However, if only one species of charge is injected into a medium, a net steady-state volume charge distribution can result.

Because of the electric force, this distribution of volume charge $\rho_f$ contributes to and also in turn depends on the electric field. It now becomes necessary to simultaneously satisfy the coupled electrical and mechanical equations.

### 3-7-1 Space Charge Limited Vacuum Tube Diode

In vacuum tube diodes, electrons with charge $-e$ and mass $m$ are boiled off the heated cathode, which we take as our zero potential reference. This process is called thermionic emission. A positive potential $V_0$ applied to the anode at $x=l$ accelerates the electrons, as in Figure 3-26. Newton's law for a particular electron is

$$m\frac{dv}{dt}=-eE=e\frac{dV}{dx} \tag{1}$$

In the dc steady state the velocity of the electron depends only on its position $x$ so that

$$m\frac{dv}{dt}=m\frac{dv}{dx}\frac{dx}{dt}=mv\frac{dv}{dx}\Rightarrow\frac{d}{dx}\left(\tfrac{1}{2}mv^2\right)=\frac{d}{dx}(eV) \tag{2}$$

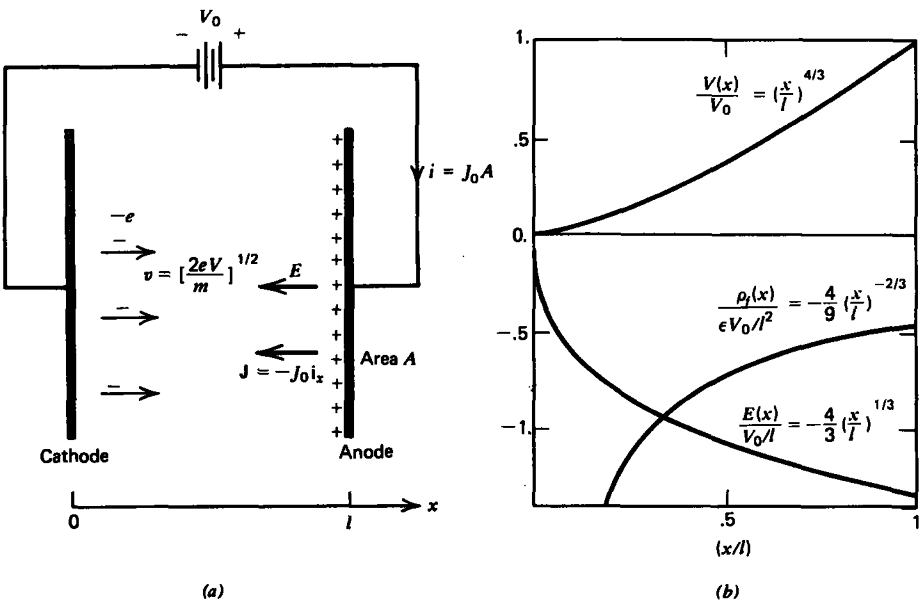



Figure 3-26 Space charge limited vacuum tube diode. (*a*) Thermionic injection of electrons from the heated cathode into vacuum with zero initial velocity. The positive anode potential attracts the electrons whose acceleration is proportional to the local electric field. (*b*) Steady-state potential, electric field, and volume charge distributions.

With this last equality, we have derived the energy conservation theorem

$$\frac{d}{dx}[\tfrac{1}{2}mv^2 - eV] = 0 \Rightarrow \tfrac{1}{2}mv^2 - eV = \text{const} \tag{3}$$

where we say that the kinetic energy $\frac{1}{2}mv^2$ plus the potential energy $-eV$ is the constant total energy. We limit ourselves here to the simplest case where the injected charge at the cathode starts out with zero velocity. Since the potential is also chosen to be zero at the cathode, the constant in (3) is zero. The velocity is then related to the electric potential as

$$v = \left(\frac{2e}{m}V\right)^{1/2} \tag{4}$$

In the time-independent steady state the current density is constant,

$$\nabla \cdot \mathbf{J} = 0 \Rightarrow \frac{dJ_x}{dx} = 0 \Rightarrow \mathbf{J} = -J_0\mathbf{i}_x \tag{5}$$

and is related to the charge density and velocity as

$$J_0 = -\rho_f v \Rightarrow \rho_f = -J_0\left(\frac{m}{2e}\right)^{1/2} V^{-1/2} \tag{6}$$

Note that the current flows from anode to cathode, and thus is in the negative $x$ direction. This minus sign is incorporated in (5) and (6) so that $J_0$ is positive. Poisson's equation then requires that

$$\nabla^2 V = \frac{-\rho_f}{\varepsilon} \Rightarrow \frac{d^2V}{dx^2} = \frac{J_0}{\varepsilon}\left(\frac{m}{2e}\right)^{1/2} V^{-1/2} \tag{7}$$

Power law solutions to this nonlinear differential equation are guessed of the form

$$V = Bx^p \tag{8}$$

which when substituted into (7) yields

$$Bp(p-1)x^{p-2} = \frac{J_0}{\varepsilon}\left(\frac{m}{2e}\right)^{1/2} B^{-1/2}x^{-p/2} \tag{9}$$

For this assumed solution to hold for all $x$ we require that

$$p - 2 = -\frac{p}{2} \Rightarrow p = \frac{4}{3} \tag{10}$$

which then gives us the amplitude $B$ as

$$B = \left[\frac{9}{4}\frac{J_0}{\varepsilon}\left(\frac{m}{2e}\right)^{1/2}\right]^{2/3} \tag{11}$$

so that the potential is

$$V(x)=\left[\frac{9}{4}\frac{J_0}{\varepsilon}\left(\frac{m}{2e}\right)^{1/2}\right]^{2/3}x^{4/3} \tag{12}$$

The potential is zero at the cathode, as required, while the anode potential $V_0$ requires the current density to be

$$V(x=l)=V_0=\left[\frac{9}{4}\frac{J_0}{\varepsilon}\left(\frac{m}{2e}\right)^{1/2}\right]^{2/3}l^{4/3}$$

$$\Rightarrow J_0=\frac{4}{9}\frac{\varepsilon}{l^2}\left(\frac{2e}{m}\right)^{1/2}V_0^{3/2} \tag{13}$$

which is called the Langmuir–Child law.

The potential, electric field, and charge distributions are then concisely written as

$$\begin{aligned} V(x)&=V_0\left(\frac{x}{l}\right)^{4/3}\\ E(x)&=-\frac{dV(x)}{dx}=-\frac{4}{3}\frac{V_0}{l}\left(\frac{x}{l}\right)^{1/3}\\ \rho_f(x)&=\varepsilon\frac{dE(x)}{dx}=-\frac{4}{9}\varepsilon\frac{V_0}{l^2}\left(\frac{x}{l}\right)^{-2/3}\end{aligned} \tag{14}$$

and are plotted in Figure 3-26*b*. We see that the charge density at the cathode is infinite but that the total charge between the electrodes is finite,

$$q_T=\int_{x=0}^{l}\rho_f(x)A\,dx=-\frac{4}{3}\varepsilon\frac{V_0}{l}A \tag{15}$$

being equal in magnitude but opposite in sign to the total surface charge on the anode:

$$q_A=\sigma_f(x=l)A=-\varepsilon E(x=l)A=+\frac{4}{3}\varepsilon\frac{V_0}{l}A \tag{16}$$

There is no surface charge on the cathode because the electric field is zero there.

This displacement $x$ of each electron can be found by substituting the potential distribution of (14) into (4),

$$v=\frac{dx}{dt}=\left(\frac{2eV_0}{m}\right)^{1/2}\left(\frac{x}{l}\right)^{2/3}\Rightarrow\frac{dx}{x^{2/3}}=\left(\frac{2eV_0}{ml^{4/3}}\right)^{1/2}dt \tag{17}$$

which integrates to

$$x=\frac{1}{27}\left(\frac{2eV_0}{ml^{4/3}}\right)^{3/2}t^3 \tag{18}$$

The charge transit time $\tau$ between electrodes is found by solving (18) with $x = l$:

$$\tau = 3l\left(\frac{m}{2eV_0}\right)^{1/2} \tag{19}$$

For an electron ($m = 9.1\times 10^{-31}$ kg, $e = 1.6\times 10^{-19}$ coul) with 100 volts applied across $l = 1$ cm ($10^{-2}$ m) this time is $\tau \approx 5\times 10^{-9}$ sec. The peak electron velocity when it reaches the anode is $v(x = l) \approx 6\times 10^6$ m/sec, which is approximately 50 times less than the vacuum speed of light.

Because of these fast response times vacuum tube diodes are used in alternating voltage applications for rectification as current only flows when the anode is positive and as nonlinear circuit elements because of the three-halves power law of (13) relating current and voltage.

### 3-7-2 Space Charge Limited Conduction in Dielectrics

Conduction properties of dielectrics are often examined by injecting charge. In Figure 3-27, an electron beam with current density $\mathbf{J} = -J_0\mathbf{i}_x$ is suddenly turned on at $t = 0$.* In media, the acceleration of the charge is no longer proportional to the electric field. Rather, collisions with the medium introduce a frictional drag so that the velocity is proportional to the electric field through the electron mobility $\mu$:

$$\mathbf{v} = -\mu\mathbf{E} \tag{20}$$

As the electrons penetrate the dielectric, the space charge front is a distance $s$ from the interface where (20) gives us

$$ds/dt = -\mu E(s) \tag{21}$$

Although the charge density is nonuniformly distributed behind the wavefront, the total charge $Q$ within the dielectric behind the wave front at time $t$ is related to the current density as

$$J_0 A = \rho_f \mu E_x A = -Q/t \Rightarrow Q = -J_0 A t \tag{22}$$

Gauss's law applied to the rectangular surface enclosing all the charge within the dielectric then relates the fields at the interface and the charge front to this charge as

$$\oint_S \varepsilon\mathbf{E}\cdot d\mathbf{S} = [\varepsilon E(s) - \varepsilon_0 E(0)]A = Q = -J_0 A t \tag{23}$$

* *See P. K. Watson, J. M. Schneider, and H. R. Till, Electrohydrodynamic Stability of Space Charge Limited Currents In Dielectric Liquids. II. Experimental Study,* Phys. Fluids ***13*** *(1970), p. 1955.*

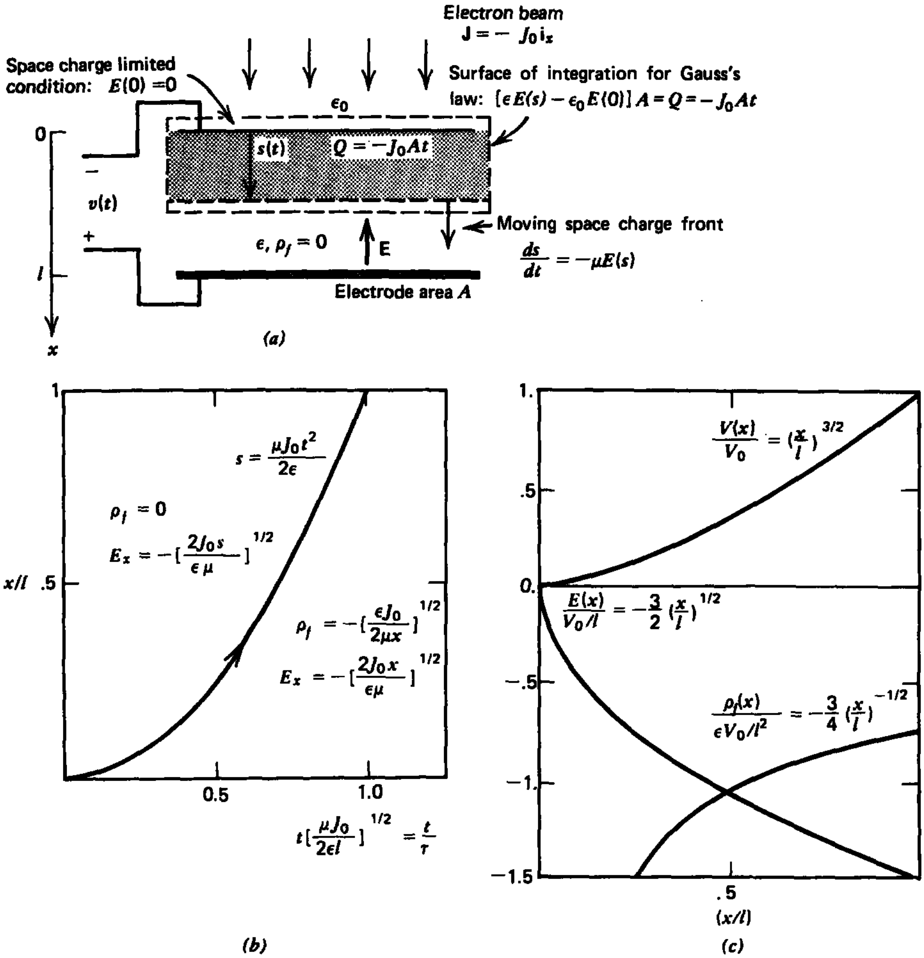



Figure 3-27 (*a*) An electron beam carrying a current $-J_0\mathbf{i}_x$ is turned on at $t=0$. The electrons travel through the dielectric with mobility $\mu$. (*b*) The space charge front, at a distance $s$ in front of the space charge limited interface at $x=0$, travels towards the opposite electrode. (*c*) After the transit time $t_c=[2\varepsilon l/\mu J_0]^{1/2}$ the steady-state potential, electric field, and space charge distributions.

The maximum current flows when $E(0)=0$, which is called space charge limited conduction. Then using (23) in (21) gives us the time dependence of the space charge front:

$$\frac{ds}{dt}=\frac{\mu J_0 t}{\varepsilon}\Rightarrow s(t)=\frac{\mu J_0 t^2}{2\varepsilon} \tag{24}$$

Behind the front Gauss's law requires

$$\frac{dE_x}{dx}=\frac{\rho_f}{\varepsilon}=\frac{J_0}{\varepsilon\mu E_x}\Rightarrow E_x\frac{dE_x}{dx}=\frac{J_0}{\varepsilon\mu} \tag{25}$$

while ahead of the moving space charge the charge density is zero so that the current is carried entirely by displacement current and the electric field is constant in space. The spatial distribution of electric field is then obtained by integrating (25) to

$$E_x=\begin{cases}-\sqrt{2J_0x/\varepsilon\mu}, & 0\le x\le s(t)\\ -\sqrt{2J_0s/\varepsilon\mu}, & s(t)\le x\le l\end{cases} \tag{26}$$

while the charge distribution is

$$\rho_f=\varepsilon\frac{dE_x}{dx}=\begin{cases}-\sqrt{\varepsilon J_0/(2\mu x)}, & 0\le x\le s(t)\\ 0, & s(t)\le x\le l\end{cases} \tag{27}$$

as indicated in Figure 3-27*b*.

The time dependence of the voltage across the dielectric is then

$$\begin{aligned}v(t)&=\int_0^l E_x\,dx=\int_0^{s(t)}\sqrt{\frac{2J_0x}{\varepsilon\mu}}\,dx+\int_{s(t)}^l\sqrt{\frac{2J_0s}{\varepsilon\mu}}\,dx\\&=\frac{J_0lt}{\varepsilon}-\frac{\mu J_0^2t^3}{6\varepsilon^2},\qquad s(t)\le l\end{aligned} \tag{28}$$

These transient solutions are valid until the space charge front $s$, given by (24), reaches the opposite electrode with $s=l$ at time

$$\tau=\sqrt{2\varepsilon l/\mu J_0} \tag{29}$$

Thereafter, the system is in the dc steady state with the terminal voltage $V_0$ related to the current density as

$$J_0=\frac{9}{8}\frac{\varepsilon\mu V_0^2}{l^3} \tag{30}$$

which is the analogous Langmuir–Child's law for collision dominated media. The steady-state electric field and space charge density are then concisely written as

$$E_x=-\frac{3}{2}\frac{V_0}{l}\left(\frac{x}{l}\right)^{1/2},\quad \rho_f=\varepsilon\frac{dE}{dx}=-\frac{3}{4}\frac{\varepsilon V_0}{l^2}\left(\frac{x}{l}\right)^{-1/2} \tag{31}$$

and are plotted in Figure 3-27*c*.

In liquids a typical ion mobility is of the order of $10^{-7}\ \mathrm{m^2/(volt\text{-}sec)}$ with a permittivity of $\varepsilon=2\varepsilon_0\approx 1.77\times10^{-11}$ farad/m. For a spacing of $l=10^{-2}$ m with a potential difference of $V_0=10^4$ V the current density of (30) is $J_0\approx2\times10^{-4}\ \mathrm{amp/m^2}$ with the transit time given by (29) $\tau\approx0.133$ sec. Charge transport times in collison dominated media are much larger than in vacuum.

## 3-8 ENERGY STORED IN A DIELECTRIC MEDIUM

The work needed to assemble a charge distribution is stored as potential energy in the electric field because if the charges are allowed to move this work can be regained as kinetic energy or mechanical work.

### 3-8-1 Work Necessary to Assemble a Distribution of Point Charges

**(a) Assembling the Charges**

Let us compute the work necessary to bring three already existing free charges $q_1$, $q_2$, and $q_3$ from infinity to any position, as in Figure 3-28. It takes no work to bring in the first charge as there is no electric field present. The work necessary to bring in the second charge must overcome the field due to the first charge, while the work needed to bring in the third charge must overcome the fields due to both other charges. Since the electric potential developed in Section 2-5-3 is defined as the work per unit charge necessary to bring a point charge in from infinity, the total work necessary to bring in the three charges is

$$W=q_1(0)+q_2\left(\frac{q_1}{4\pi\varepsilon r_{12}}\right)+q_3\left(\frac{q_1}{4\pi\varepsilon r_{13}}+\frac{q_2}{4\pi\varepsilon r_{23}}\right) \tag{1}$$

where the final distances between the charges are defined in Figure 3-28 and we use the permittivity $\varepsilon$ of the medium. We can rewrite (1) in the more convenient form

$$W=\frac{1}{2}\left\{q_1\left[\frac{q_2}{4\pi\varepsilon r_{12}}+\frac{q_3}{4\pi\varepsilon r_{13}}\right]+q_2\left[\frac{q_1}{4\pi\varepsilon r_{12}}+\frac{q_3}{4\pi\varepsilon r_{23}}\right]\right.$$
$$\left.+q_3\left[\frac{q_1}{4\pi\varepsilon r_{13}}+\frac{q_2}{4\pi\varepsilon r_{23}}\right]\right\} \tag{2}$$

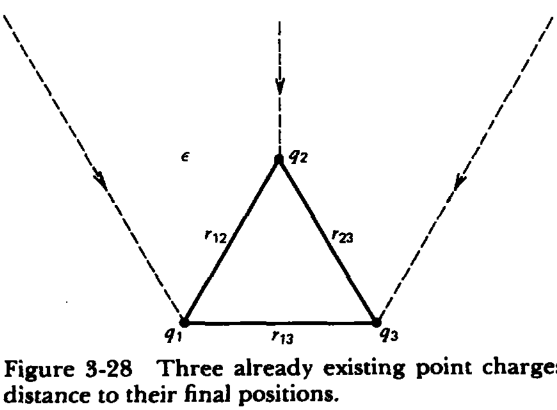



Figure 3-28 Three already existing point charges are brought in from an infinite distance to their final positions.

where we recognize that each bracketed term is just the potential at the final position of each charge and includes contributions from all the other charges, except the one located at the position where the potential is being evaluated:

$$W=\tfrac{1}{2}[q_1V_1+q_2V_2+q_3V_3] \tag{3}$$

Extending this result for any number $N$ of already existing free point charges yields

$$W=\frac{1}{2}\sum_{n=1}^{N} q_nV_n \tag{4}$$

The factor of $\frac{1}{2}$ arises because the potential of a point charge at the time it is brought in from infinity is less than the final potential when all the charges are assembled.

**(b) Binding Energy of a Crystal**

One major application of (4) is in computing the largest contribution to the binding energy of ionic crystals, such as salt (NaCl), which is known as the Madelung electrostatic energy. We take a simple one-dimensional model of a crystal consisting of an infinitely long string of alternating polarity point charges $\pm q$ a distance $a$ apart, as in Figure 3-29. The average work necessary to bring a positive charge as shown in Figure 3-29 from infinity to its position on the line is obtained from (4) as

$$W=\frac{1}{2}\frac{2q^2}{4\pi\varepsilon a}\left[-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}-\cdots\right] \tag{5}$$

The extra factor of 2 in the numerator is necessary because the string extends to infinity on each side. The infinite series is recognized as the Taylor series expansion of the logarithm

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots \tag{6}$$

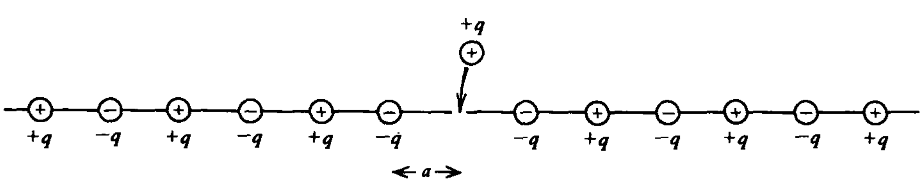



Figure 3-29 A one-dimensional crystal with alternating polarity charges $\pm q$ a distance $a$ apart.

where $x=1$ so that*

$$W=\frac{-q^2}{4\pi\varepsilon a}\ln 2 \tag{7}$$

This work is negative because the crystal pulls on the charge as it is brought in from infinity. This means that it would take positive work to remove the charge as it is bound to the crystal. A typical ion spacing is about 3 Å ($3\times10^{-10}$ m) so that if $q$ is a single proton ($q=1.6\times10^{-19}$ coul), the binding energy is $W\approx5.3\times10^{-19}$ joule. Since this number is so small it is usually more convenient to work with units of energy per unit electronic charge called electron volts (ev), which are obtained by dividing $W$ by the charge on an electron so that, in this case, $W\approx3.3$ ev.

If the crystal was placed in a medium with higher permittivity, we see from (7) that the binding energy decreases. This is why many crystals are soluble in water, which has a relative dielectric constant of about 80.

### 3-8-2 Work Necessary to Form a Continuous Charge Distribution

Not included in (4) is the self-energy of each charge itself or, equivalently, the work necessary to assemble each point charge. Since the potential $V$ from a point charge $q$ is proportional to $q$, the self-energy is proportional $q^2$. However, evaluating the self-energy of a point charge is difficult because the potential is infinite at the point charge.

To understand the self-energy concept better it helps to model a point charge as a small uniformly charged spherical volume of radius $R$ with total charge $Q=\frac{4}{3}\pi R^3\rho_0$. We assemble the sphere of charge from spherical shells, as shown in Figure 3-30, each of thickness $dr_n$ and incremental charge $dq_n=4\pi r_n^2\,dr_n\rho_0$. As we bring in the $n$th shell to be placed at radius $r_n$ the total charge already present and the potential there are

$$q_n=\frac{4}{3}\pi r_n^3\rho_0,\qquad V_n=\frac{q_n}{4\pi\varepsilon r_n}=\frac{r_n^2\rho_0}{3\varepsilon} \tag{8}$$

* *Strictly speaking, this series is only conditionally convergent for $x=1$ and its sum depends on the grouping of individual terms. If the series in* (6) *for $x=1$ is rewritten as*

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\cdots+\frac{1}{2k-1}-\frac{1}{4k-2}-\frac{1}{4k}\cdots,\qquad k\geq1$$

*then its sum is $\frac{1}{2}\ln 2$. [See J. Pleines and S. Mahajan, On Conditionally Divergent Series and a Point Charge Between Two Parallel Grounded Planes,* Am. J. Phys. ***45*** *(1977) p. 868.]*

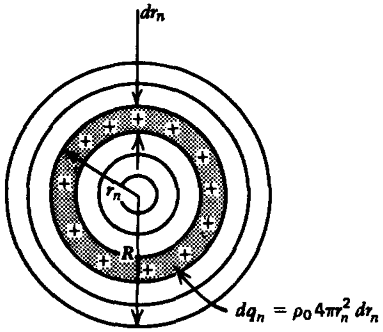



Figure 3-30 A point charge is modelled as a small uniformly charged sphere. It is assembled by bringing in spherical shells of differential sized surface charge elements from infinity.

so that the work required to bring in the $n$th shell is

$$dW_n = V_n\, dq_n = \frac{\rho_0^2 4\pi r_n^4}{3\varepsilon}\, dr_n \tag{9}$$

The total work necessary to assemble the sphere is obtained by adding the work needed for each shell:

$$W = \int dW_n = \int_0^R \frac{4\pi\rho_0^2 r^4}{3\varepsilon}\, dr = \frac{4\pi\rho_0^2 R^5}{15\varepsilon} = \frac{3Q^2}{20\pi\varepsilon R} \tag{10}$$

For a finite charge $Q$ of zero radius the work becomes infinite. However, Einstein's theory of relativity tells us that this work necessary to assemble the charge is stored as energy that is related to the mass as

$$W = mc^2 = \frac{3Q^2}{20\pi\varepsilon R} \Rightarrow R = \frac{3Q^2}{20\pi\varepsilon mc^2} \tag{11}$$

which then determines the radius of the charge. For the case of an electron ($Q = 1.6\times 10^{-19}$ coul, $m = 9.1\times 10^{-31}$ kg) in free space ($\varepsilon = \varepsilon_0 = 8.854\times 10^{-12}$ farad/m), this radius is

$$\begin{aligned} R_{\text{electron}} &= \frac{3(1.6\times 10^{-19})^2}{20\pi(8.854\times 10^{-12})(9.1\times 10^{-31})(3\times 10^8)^2} \\ &\approx 1.69\times 10^{-15}\text{ m} \end{aligned} \tag{12}$$

We can also obtain the result of (10) by using (4) where each charge becomes a differential element $dq$, so that the summation becomes an integration over the continuous free charge distribution:

$$W = \tfrac{1}{2}\int_{\text{all } q_f} V\, dq_f \tag{13}$$

For the case of the uniformly charged sphere, $dq_f=\rho_0\,d\mathrm{V}$, the final potential within the sphere is given by the results of Section 2-5-5*b*:

$$V=\frac{\rho_0}{2\varepsilon}\left(R^2-\frac{r^2}{3}\right) \tag{14}$$

Then (13) agrees with (10):

$$W=\frac{1}{2}\int \rho_0 V\,d\mathrm{V}=\frac{4\pi\rho_0^2}{4\varepsilon}\int_0^R\left(R^2-\frac{r^2}{3}\right)r^2\,dr=\frac{4\pi\rho_0^2R^5}{15\varepsilon}=\frac{3Q^2}{20\pi\varepsilon R} \tag{15}$$

Thus, in general, we define (13) as the energy stored in the electric field, including the self-energy term. It differs from (4), which only includes interaction terms between different charges and not the infinite work necessary to assemble each point charge. Equation (13) is valid for line, surface, and volume charge distributions with the differential charge elements given in Section 2-3-1. Remember when using (4) and (13) that the zero reference for the potential is assumed to be at infinity. Adding a constant $V_0$ to the potential will change the energy unless the total charge in the system is zero

$$\begin{aligned} W&=\tfrac{1}{2}\int (V+V_0)\,dq_f\\ &=\tfrac{1}{2}\int V\,dq_f+\tfrac{1}{2}V_0\int \cancelto{0}{dq_f}\\ &=\tfrac{1}{2}\int V\,dq_f \end{aligned} \tag{16}$$

## 3-8-3 Energy Density of the Electric Field

It is also convenient to express the energy $W$ stored in a system in terms of the electric field. We assume that we have a volume charge distribution with density $\rho_f$. Then, $dq_f=\rho_f\,d\mathrm{V}$, where $\rho_f$ is related to the displacement field from Gauss's law:

$$W=\tfrac{1}{2}\int_{\mathrm{V}}\rho_f V\,d\mathrm{V}=\tfrac{1}{2}\int_{\mathrm{V}} V(\nabla\cdot\mathbf{D})\,d\mathrm{V} \tag{17}$$

Let us examine the vector expansion

$$\nabla\cdot(V\mathbf{D})=(\mathbf{D}\cdot\nabla)V+V(\nabla\cdot\mathbf{D})\Rightarrow V(\nabla\cdot\mathbf{D})=\nabla\cdot(V\mathbf{D})+\mathbf{D}\cdot\mathbf{E} \tag{18}$$

where $\mathbf{E}=-\nabla V$. Then (17) becomes

$$W=\tfrac{1}{2}\int_{\mathrm{V}}\mathbf{D}\cdot\mathbf{E}\,d\mathrm{V}+\tfrac{1}{2}\int_{\mathrm{V}}\nabla\cdot(V\mathbf{D})\,d\mathrm{V} \tag{19}$$

The last term on the right-hand side can be converted to a surface integral using the divergence theorem:

$$\int_V \nabla \cdot (V\mathbf{D})\,dV = \oint_S V\,\mathbf{D}\cdot d\mathbf{S} \tag{20}$$

If we let the volume V be of infinite extent so that the enclosing surface $S$ is at infinity, the charge distribution that only extends over a finite volume looks like a point charge for which the potential decreases as $1/r$ and the displacement vector dies off as $1/r^2$. Thus the term, $V\mathbf{D}$ at best dies off as $1/r^3$. Then, even though the surface area of $S$ increases as $r^2$, the surface integral tends to zero as $r$ becomes infinite as $1/r$. Thus, the second volume integral in (19) approaches zero:

$$\lim_{r\to\infty}\int_V \nabla \cdot (V\mathbf{D})\,dV = \oint_S V\mathbf{D}\cdot d\mathbf{S} = 0 \tag{21}$$

This conclusion is not true if the charge distribution is of infinite extent, since for the case of an infinitely long line or surface charge, the potential itself becomes infinite at infinity because the total charge on the line or surface is infinite. However, for finite size charge distributions, which is always the case in reality, (19) becomes

$$\begin{aligned} W &= \tfrac{1}{2}\int_{\text{all space}} \mathbf{D}\cdot\mathbf{E}\,dV \\ &= \int_{\text{all space}} \tfrac{1}{2}\varepsilon E^2\,dV \end{aligned} \tag{22}$$

where the integration extends over all space. This result is true even if the permittivity $\varepsilon$ is a function of position. It is convenient to define the energy density as the positive-definite quantity:

$$w = \tfrac{1}{2}\varepsilon E^2 \text{ joule/m}^3\ [\text{kg-m}^{-1}\text{-s}^{-2}] \tag{23}$$

where the total energy is

$$W = \int_{\text{all space}} w\,dV \tag{24}$$

Note that although (22) is numerically equal to (13), (22) implies that electric energy exists in those regions where a nonzero electric field exists even if no charge is present in that region, while (13) implies that electric energy exists only where the charge is nonzero. The answer as to where the energy is stored—in the charge distribution or in the electric field—is a matter of convenience since you cannot have one without the other. Numerically both equations yield the same answers but with contributions from different regions of space.

### 3-8-4 Energy Stored in Charged Spheres

**(a) Volume Charge**

We can also find the energy stored in a uniformly charged sphere using (22) since we know the electric field in each region from Section 2-4-3*b*. The energy density is then

$$w = \frac{\varepsilon}{2}E_r^2 = \begin{cases} \dfrac{Q^2}{32\pi^2\varepsilon r^4}, & r > R \\ \dfrac{Q^2 r^2}{32\pi^2\varepsilon R^6}, & r < R \end{cases} \tag{25}$$

with total stored energy

$$\begin{aligned} W &= \int_{\text{all space}} w\, dV \\ &= \frac{Q^2}{8\pi\varepsilon}\left(\int_0^R \frac{r^4}{R^6}\,dr + \int_R^\infty \frac{dr}{r^2}\right) = \frac{3}{20}\frac{Q^2}{\pi\varepsilon R} \end{aligned} \tag{26}$$

which agrees with (10) and (15).

**(b) Surface Charge**

If the sphere is uniformly charged on its surface $Q = 4\pi R^2\sigma_0$, the potential and electric field distributions are

$$V(r) = \begin{cases} \dfrac{Q}{4\pi\varepsilon R} \\ \dfrac{Q}{4\pi\varepsilon r} \end{cases}; \qquad E_r = \begin{cases} 0, & r < R \\ \dfrac{Q}{4\pi\varepsilon r^2}, & r > R \end{cases} \tag{27}$$

Using (22) the energy stored is

$$W = \frac{\varepsilon}{2}\left(\frac{Q}{4\pi\varepsilon}\right)^2 4\pi \int_R^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon R} \tag{28}$$

This result is equally as easy obtained using (13):

$$\begin{aligned} W &= \tfrac{1}{2}\int_S \sigma_0 V(r = R)\, dS \\ &= \tfrac{1}{2}\sigma_0 V(r = R) 4\pi R^2 = \frac{Q^2}{8\pi\varepsilon R} \end{aligned} \tag{29}$$

The energy stored in a uniformly charged sphere is 20% larger than the surface charged sphere for the same total charge $Q$. This is because of the additional energy stored throughout the sphere's volume. Outside the sphere $(r > R)$ the fields are the same as is the stored energy.

**(c) Binding Energy of an Atom**

In Section 3-1-4 we modeled an atom as a fixed positive point charge nucleus $Q$ with a surrounding uniform spherical cloud of negative charge with total charge $-Q$, as in Figure 3-31. Potentials due to the positive point and negative volume charges are found from Section 2-5-5*b* as

$$V_+(r)=\frac{Q}{4\pi\varepsilon_0 r}$$

$$V_-(r)=\begin{cases}-\dfrac{3Q}{8\pi\varepsilon_0 R^3}\left(R^2-\dfrac{r^2}{3}\right), & r<R\\[2ex] -\dfrac{Q}{4\pi\varepsilon_0 r}, & r>R\end{cases}\tag{30}$$

The binding energy of the atom is easily found by superposition considering first the uniformly charged negative sphere with self-energy given in (10), (15), and (26) and then adding the energy of the positive point charge:

$$W=\frac{3Q^2}{20\pi\varepsilon_0 R}+Q[V_-(r=0)]=-\frac{9Q^2}{40\pi\varepsilon_0 R}\tag{31}$$

This is the work necessary to assemble the atom from charges at infinity. Once the positive nucleus is in place, it attracts the following negative charges so that the field does work on the charges and the work of assembly in (31) is negative. Equivalently, the magnitude of (31) is the work necessary for us to disassemble the atom by overcoming the attractive coulombic forces between the opposite polarity charges.

When alternatively using (4) and (13), we only include the potential of the negative volume charge at $r=0$ acting on the positive charge, while we include the total potential due to both in evaluating the energy of the volume charge. We do

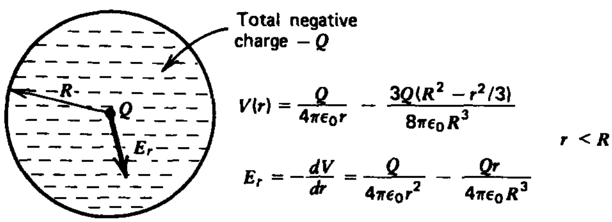



Figure 3-31 An atom can be modelled as a point charge $Q$ representing the nucleus, surrounded by a cloud of uniformly distributed electrons with total charge $-Q$ within a sphere of radius $R$.

not consider the infinite self-energy of the point charge that would be included if we used (22):

$$
\begin{aligned}
W &= \tfrac{1}{2}QV_-(r=0) - \tfrac{1}{2}\int_{r=0}^{R} [V_+(r)+V_-(r)]\frac{3Qr^2}{R^3}\,dr \\
&= -\frac{3Q^2}{16\pi\varepsilon_0 R} - \frac{3Q^2}{8\pi\varepsilon_0 R^3}\int_0^R \left(r - \frac{3}{2}\frac{r^2}{R} + \frac{r^4}{2R^3}\right) dr \\
&= -\frac{9Q^2}{40\pi\varepsilon_0 R}
\end{aligned}
\tag{32}
$$

### 3-8-5 Energy Stored in a Capacitor

In a capacitor all the charge resides on the electrodes as a surface charge. Consider two electrodes at voltage $V_1$ and $V_2$ with respect to infinity, and thus at voltage difference $V = V_2 - V_1$, as shown in Figure 3-32. Each electrode carries opposite polarity charge with magnitude $Q$. Then (13) can be used to compute the total energy stored as

$$
W = \tfrac{1}{2}\left[\int_{S_1} V_1\sigma_1\,dS_1 + \int_{S_2} V_2\sigma_2\,dS_2\right] \tag{33}
$$

Since each surface is an equipotential, the voltages $V_1$ and $V_2$ may be taken outside the integrals. The integral then reduces to the total charge $\pm Q$ on each electrode:

$$
\begin{aligned}
W &= \tfrac{1}{2}\left[V_1\int_{S_1}\underbrace{\sigma_1\,dS_1}_{-Q} + V_2\int_{S_2}\underbrace{\sigma_2\,dS_2}_{Q}\right] \\
&= \tfrac{1}{2}(V_2 - V_1)Q = \tfrac{1}{2}QV
\end{aligned}
\tag{34}
$$

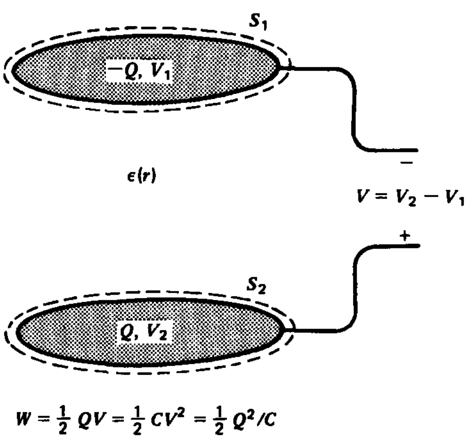



Figure 3-32 A capacitor stores energy in the electric field.

Since in a capacitor the charge and voltage are linearly related through the capacitance

$$Q = CV \tag{35}$$

the energy stored in the capacitor can also be written as

$$W = \tfrac{1}{2}QV = \tfrac{1}{2}CV^2 = \tfrac{1}{2}\frac{Q^2}{C} \tag{36}$$

This energy is equivalent to (22) in terms of the electric field and gives us an alternate method to computing the capacitance if we know the electric field distribution.

### EXAMPLE 3-3 CAPACITANCE OF AN ISOLATED SPHERE

A sphere of radius $R$ carries a uniformly distributed surface charge $Q$. What is its capacitance?

**SOLUTION**

---

The stored energy is given by (28) or (29) so that (36) gives us the capacitance:

$$C = Q^2/2W = 4\pi\varepsilon R$$

## 3.9 FIELDS AND THEIR FORCES

### 3-9-1 Force Per Unit Area on a Sheet of Surface Charge

A confusion arises in applying Coulomb's law to find the perpendicular force on a sheet of surface charge as the normal electric field is different on each side of the sheet. Using the over-simplified argument that half the surface charge resides on each side of the sheet yields the correct force

$$\mathbf{f} = \tfrac{1}{2}\int_S \sigma_f(\mathbf{E}_1 + \mathbf{E}_2)\, d\mathbf{S} \tag{1}$$

where, as shown in Figure 3-33a, $\mathbf{E}_1$ and $\mathbf{E}_2$ are the electric fields on each side of the sheet. Thus, the correct field to use is the average electric field $\frac{1}{2}(\mathbf{E}_1 + \mathbf{E}_2)$ across the sheet.

For the tangential force, the tangential components of $\mathbf{E}$ are continuous across the sheet ($E_{1t} = E_{2t} \equiv E_t$) so that

$$f_t = \tfrac{1}{2}\int_S \sigma_f(E_{1t} + E_{2t})\, dS = \int_S \sigma_f E_t\, dS \tag{2}$$

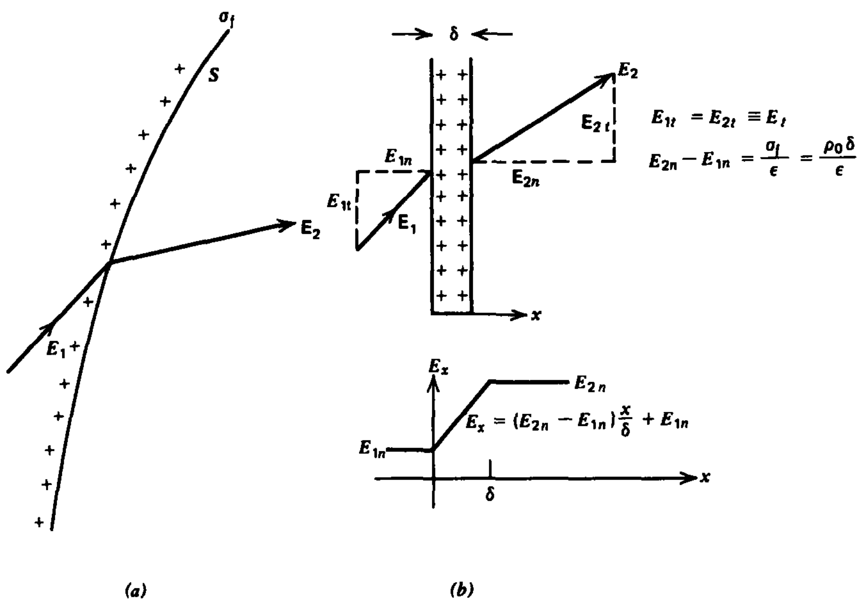



Figure 3-33 (*a*) The normal component of electric field is discontinuous across the sheet of surface charge. (*b*) The sheet of surface charge can be modeled as a thin layer of volume charge. The electric field then varies linearly across the volume.

The normal fields are discontinuous across the sheet so that the perpendicular force is

$$\sigma_f = \varepsilon(E_{2n} - E_{1n}) \Rightarrow f_n = \tfrac{1}{2}\int_S \varepsilon(E_{2n} - E_{1n})(E_{1n} + E_{2n})\, dS$$

$$= \tfrac{1}{2}\int_S \varepsilon(E_{2n}^2 - E_{1n}^2)\, dS \tag{3}$$

To be mathematically rigorous we can examine the field transition through the sheet more closely by assuming the surface charge is really a uniform volume charge distribution $\rho_0$ of very narrow thickness $\delta$, as shown in Figure 3-33*b*. Over the small surface element $dS$, the surface appears straight so that the electric field due to the volume charge can then only vary with the coordinate $x$ perpendicular to the surface. Then the point form of Gauss's law within the volume yields

$$\frac{dE_x}{dx} = \frac{\rho_0}{\varepsilon} \Rightarrow E_x = \frac{\rho_0 x}{\varepsilon} + \text{const} \tag{4}$$

The constant in (4) is evaluated by the boundary conditions on the normal components of electric field on each side of the sheet

$$E_x(x=0)=E_{1n}, \qquad E_x(x=\delta)=E_{2n} \tag{5}$$

so that the electric field is

$$E_x=(E_{2n}-E_{1n})\frac{x}{\delta}+E_{1n} \tag{6}$$

As the slab thickness $\delta$ becomes very small, we approach a sheet charge relating the surface charge density to the discontinuity in electric fields as

$$\lim_{\substack{\rho_0\to\infty\\ \delta\to 0}} \rho_0\delta=\sigma_f=\varepsilon(E_{2n}-E_{1n}) \tag{7}$$

Similarly the force per unit area on the slab of volume charge is

$$\begin{aligned} F_x &= \int_0^\delta \rho_0 E_x\,dx \\ &= \int_0^\delta \rho_0\left[(E_{2n}-E_{1n})\frac{x}{\delta}+E_{1n}\right]dx \\ &= \left[\rho_0(E_{2n}-E_{1n})\frac{x^2}{2\delta}+E_{1n}x\right]\Bigg|_0^\delta \\ &= \frac{\rho_0\delta}{2}(E_{1n}+E_{2n}) \end{aligned} \tag{8}$$

In the limit of (7), the force per unit area on the sheet of surface charge agrees with (3):

$$\lim_{\rho_0\delta=\sigma_f} F_x=\frac{\sigma_f}{2}(E_{1n}+E_{2n})=\frac{\varepsilon}{2}(E_{2n}^2-E_{1n}^2) \tag{9}$$

### 3-9-2 Forces on a Polarized Medium

**(a) Force Density**

In a uniform electric field there is no force on a dipole because the force on each charge is equal in magnitude but opposite in direction, as in Figure 3-34*a*. However, if the dipole moment is not aligned with the field there is an aligning torque given by $\mathbf{t}=\mathbf{p}\times\mathbf{E}$. The torque per unit volume $\mathbf{T}$ on a polarized medium with $N$ dipoles per unit volume is then

$$\mathbf{T}=N\mathbf{t}=N\mathbf{p}\times\mathbf{E}=\mathbf{P}\times\mathbf{E} \tag{10}$$

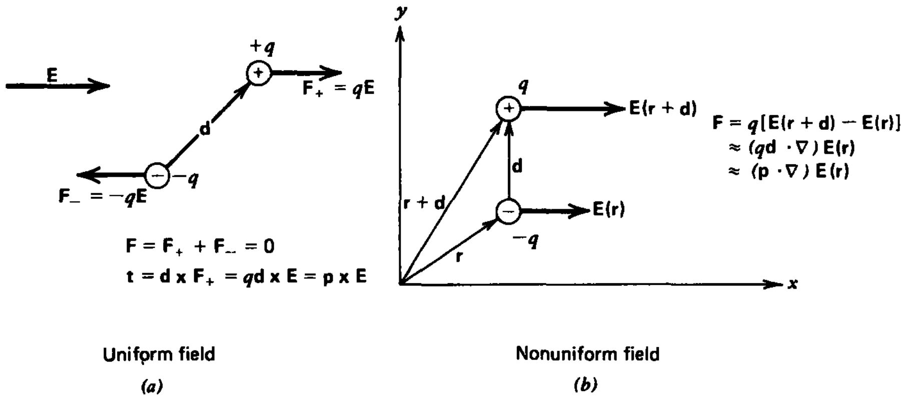



Figure 3-34 (*a*) A torque is felt by a dipole if its moment is not aligned with the electric field. In a uniform electric field there is no net force on a dipole because the force on each charge is equal in magnitude but opposite in direction. (*b*) There is a net force on a dipole only in a nonuniform field.

For a linear dielectric, this torque is zero because the polarization is induced by the field so that **P** and **E** are in the same direction.

A net force can be applied to a dipole if the electric field is different on each end, as in Figure 3-34*b*:

$$\mathbf{f} = -q[\mathbf{E}(\mathbf{r}) - \mathbf{E}(\mathbf{r}+\mathbf{d})] \tag{11}$$

For point dipoles, the dipole spacing **d** is very small so that the electric field at $\mathbf{r}+\mathbf{d}$ can be expanded in a Taylor series as

$$\begin{aligned}\mathbf{E}(\mathbf{r}+\mathbf{d}) &\approx \mathbf{E}(\mathbf{r}) + d_x \frac{\partial}{\partial x}\mathbf{E}(\mathbf{r}) + d_y \frac{\partial}{\partial y}\mathbf{E}(\mathbf{r}) + d_z \frac{\partial}{\partial z}\mathbf{E}(\mathbf{r}) \\ &= \mathbf{E}(\mathbf{r}) + (\mathbf{d}\cdot\nabla)\mathbf{E}(\mathbf{r})\end{aligned} \tag{12}$$

Then the force on a point dipole is

$$\mathbf{f} = (q\mathbf{d}\cdot\nabla)\mathbf{E}(\mathbf{r}) = (\mathbf{p}\cdot\nabla)\mathbf{E}(\mathbf{r}) \tag{13}$$

If we have a distribution of such dipoles with number density $N$, the polarization force density is

$$\mathbf{F} = N\mathbf{f} = (N\mathbf{p}\cdot\nabla)\mathbf{E} = (\mathbf{P}\cdot\nabla)\mathbf{E} \tag{14}$$

Of course, if there is any free charge present we must also add the coulombic force density $\rho_f\mathbf{E}$.

**(b) Permanently Polarized Medium**

A permanently polarized material with polarization $P_0\mathbf{i}_y$ is free to slide between parallel plate electrodes, as is shown in Figure 3-35.

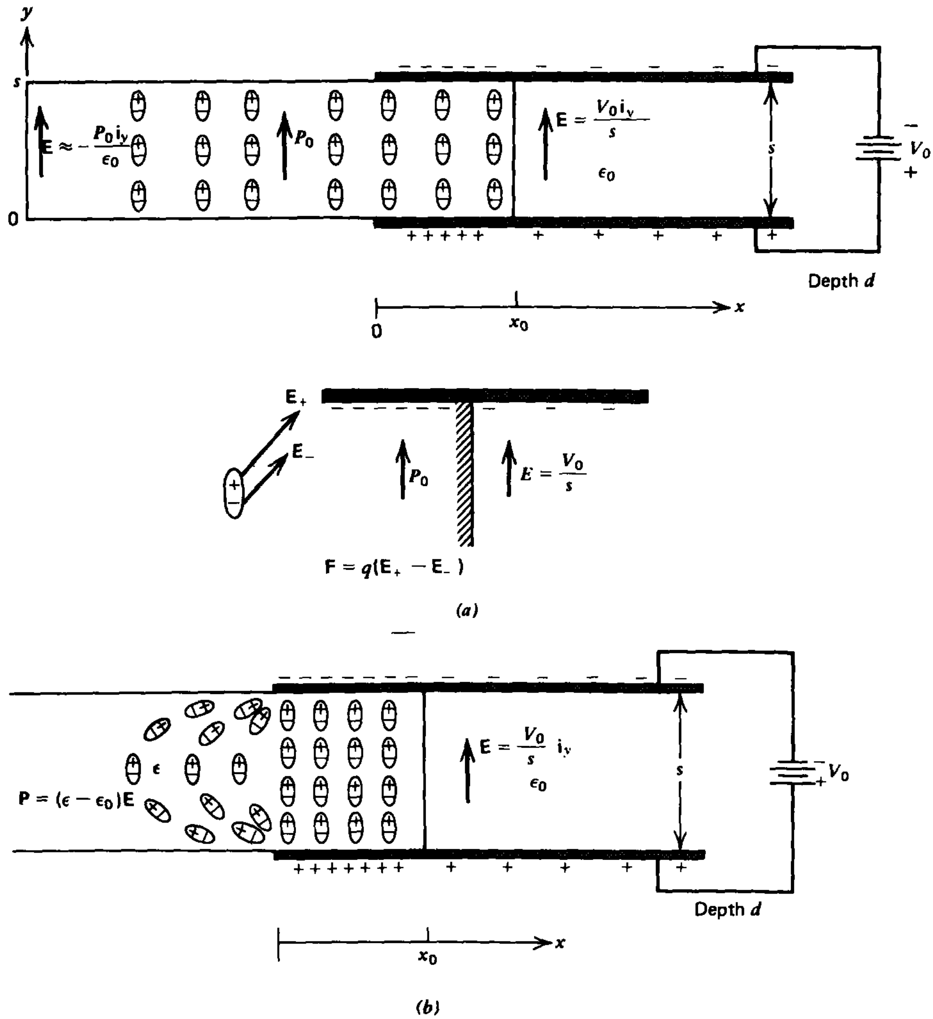



Figure 3-35 (*a*) A permanently polarized electret partially inserted into a capacitor has a force on it due to the Coulombic attraction between the dipole charges and the surface charge on the electrodes. The net force arises in the fringing field region as the end of the dipole further from the electrode edge feels a smaller electric field. Depending on the voltage magnitude and polarity, the electret can be pulled in or pushed out of the capacitor. (*b*) A linear dielectric is always attracted into a free space capacitor because of the net force on dipoles in the nonuniform field. The dipoles are now aligned with the electric field, no matter the voltage polarity.

We only know the electric field in the interelectrode region and from Example 3-2 far away from the electrodes:

$$E_y(x = x_0) = \frac{V_0}{s}, \qquad E_y(x = -\infty) = -\frac{P_0}{\varepsilon_0} \tag{15}$$

Unfortunately, neither of these regions contribute to the force because the electric field is uniform and (14) requires a field gradient for a force. The force arises in the fringing fields near the electrode edges where the field is nonuniform and, thus, exerts less of a force on the dipole end further from the electrode edges. At first glance it looks like we have a difficult problem because we do not know the fields where the force acts. However, because the electric field has zero curl,

$$\nabla\times\mathbf{E}=0\Rightarrow\frac{\partial E_x}{\partial y}=\frac{\partial E_y}{\partial x}\tag{16}$$

the $x$ component of the force density can be written as

$$\begin{aligned}F_x&=P_y\frac{\partial E_x}{\partial y}\\&=P_y\frac{\partial E_y}{\partial x}\\&=\frac{\partial}{\partial x}(P_yE_y)-E_y\cancelto{0}{\frac{\partial P_y}{\partial x}}\end{aligned}\tag{17}$$

The last term in (17) is zero because $P_y=P_0$ is a constant. The total $x$ directed force is then

$$\begin{aligned}f_x&=\int F_x\,dx\,dy\,dz\\&=\int_{x=-\infty}^{x_0}\int_{y=0}^{s}\int_{z=0}^{d}\frac{\partial}{\partial x}(P_yE_y)\,dx\,dy\,dz\end{aligned}\tag{18}$$

We do the $x$ integration first so that the $y$ and $z$ integrations are simple multiplications as the fields at the limits of the $x$ integration are independent of $y$ and $z$:

$$f_x=P_0E_ysd\big|_{x=-\infty}^{x_0}=P_0V_0d+\frac{P_0^2sd}{\varepsilon_0}\tag{19}$$

There is a force pulling the electret between the electrodes even if the voltage were zero due to the field generated by the surface charge on the electrodes induced by the electret. This force is increased if the imposed electric field and polarization are in the same direction. If the voltage polarity is reversed, the force is negative and the electret is pushed out if the magnitude of the voltage exceeds $P_0s/\varepsilon_0$.

**(c) Linearly Polarized Medium**

The problem is different if the slab is polarized by the electric field, as the polarization will then be in the direction of the electric field and thus have $x$ and $y$ components in the fringing fields near the electrode edges where the force arises, as in Figure 3-35*b*. The dipoles tend to line up as shown with the positive ends attracted towards the negative electrode and the negative dipole ends towards the positive electrode. Because the farther ends of the dipoles are in a slightly weaker field, there is a net force to the right tending to draw the dielectric into the capacitor.

The force density of (14) is

$$F_x = P_x \frac{\partial E_x}{\partial x} + P_y \frac{\partial E_x}{\partial y} = (\varepsilon - \varepsilon_0)\left(E_x \frac{\partial E_x}{\partial x} + E_y \frac{\partial E_x}{\partial y}\right) \tag{20}$$

Because the electric field is curl free, as given in (16), the force density is further simplified to

$$F_x = \frac{(\varepsilon - \varepsilon_0)}{2} \frac{\partial}{\partial x}(E_x^2 + E_y^2) \tag{21}$$

The total force is obtained by integrating (21) over the volume of the dielectric:

$$\begin{aligned} f_x &= \int_{x=-\infty}^{x_0} \int_{y=0}^{s} \int_{z=0}^{d} \frac{(\varepsilon - \varepsilon_0)}{2} \frac{\partial}{\partial x}(E_x^2 + E_y^2)\, dx\, dy\, dz \\ &= \frac{(\varepsilon - \varepsilon_0) sd}{2} (E_x^2 + E_y^2)\Big|_{x=-\infty}^{x_0} = \frac{(\varepsilon - \varepsilon_0)}{2} \frac{V_0^2 d}{s} \end{aligned} \tag{22}$$

where we knew that the fields were zero at $x = -\infty$ and uniform at $x = x_0$:

$$E_y(x_0) = V_0/s, \qquad E_x(x_0) = 0 \tag{23}$$

The force is now independent of voltage polarity and always acts in the direction to pull the dielectric into the capacitor if $\varepsilon > \varepsilon_0$.

### 3-9-3 Forces on a Capacitor

Consider a capacitor that has one part that can move in the $x$ direction so that the capacitance depends on the coordinate $x$:

$$q = C(x)v \tag{24}$$

The current is obtained by differentiating the charge with respect to time:

$$\begin{aligned} i &= \frac{dq}{dt} = \frac{d}{dt}[C(x)v] = C(x)\frac{dv}{dt} + v\frac{dC(x)}{dt} \\ &= C(x)\frac{dv}{dt} + v\frac{dC(x)}{dx}\frac{dx}{dt} \end{aligned} \tag{25}$$

Note that this relation has an extra term over the usual circuit formula, proportional to the speed of the moveable member, where we expanded the time derivative of the capacitance by the chain rule of differentiation. Of course, if the geometry is fixed and does not change with time ($dx/dt = 0$), then (25) reduces to the usual circuit expression. The last term is due to the electro-mechanical coupling.

The power delivered to a time-dependent capacitance is

$$p = vi = v\frac{d}{dt}[C(x)v] \tag{26}$$

which can be expanded to the form

$$\begin{aligned} p &= \frac{d}{dt}[\tfrac{1}{2}C(x)v^2] + \tfrac{1}{2}v^2\frac{dC(x)}{dt} \\ &= \frac{d}{dt}[\tfrac{1}{2}C(x)v^2] + \tfrac{1}{2}v^2\frac{dC(x)}{dx}\frac{dx}{dt} \end{aligned} \tag{27}$$

where the last term is again obtained using the chain rule of differentiation. This expression can be put in the form

$$p = \frac{dW}{dt} + f_x\frac{dx}{dt} \tag{28}$$

where we identify the power $p$ delivered to the capacitor as going into increasing the energy storage $W$ and mechanical power $f_x\,dx/dt$ in moving a part of the capacitor:

$$W = \tfrac{1}{2}C(x)v^2, \qquad f_x = \tfrac{1}{2}v^2\frac{dC(x)}{dx} \tag{29}$$

Using (24), the stored energy and force can also be expressed in terms of the charge as

$$W = \frac{1}{2}\frac{q^2}{C(x)}, \qquad f_x = \frac{1}{2}\frac{q^2}{C^2(x)}\frac{dC(x)}{dx} = -\tfrac{1}{2}q^2\frac{d[1/C(x)]}{dx} \tag{30}$$

To illustrate the ease in using (29) or (30) to find the force, consider again the partially inserted dielectric in Figure 3-35*b*. The capacitance when the dielectric extends a distance $x$ into the electrodes is

$$C(x) = \frac{\varepsilon x d}{s} + \varepsilon_0\frac{(l-x)d}{s} \tag{31}$$

so that the force on the dielectric given by (29) agrees with (22):

$$f_x = \tfrac{1}{2}V_0^2\frac{dC(x)}{dx} = \tfrac{1}{2}(\varepsilon - \varepsilon_0)\frac{V_0^2 d}{s} \tag{32}$$

Note that we neglected the fringing field contributions to the capacitance in (31) even though they are the physical origin of the force. The results agree because this extra capacitance does not depend on the position $x$ of the dielectric when $x$ is far from the electrode edges.

This method can only be used for linear dielectric systems described by (24). It is not valid for the electret problem treated in Section 3-9-2*b* because the electrode charge is not linearly related to the voltage, being in part induced by the electret.

## EXAMPLE 3-4 FORCE ON A PARALLEL PLATE CAPACITOR

Two parallel, perfectly conducting electrodes of area $A$ and a distance $x$ apart are shown in Figure 3-36. For each of the following two configurations, find the force on the upper electrode in the $x$ direction when the system is constrained to constant voltage $V_0$ or constant charge $Q_0$.

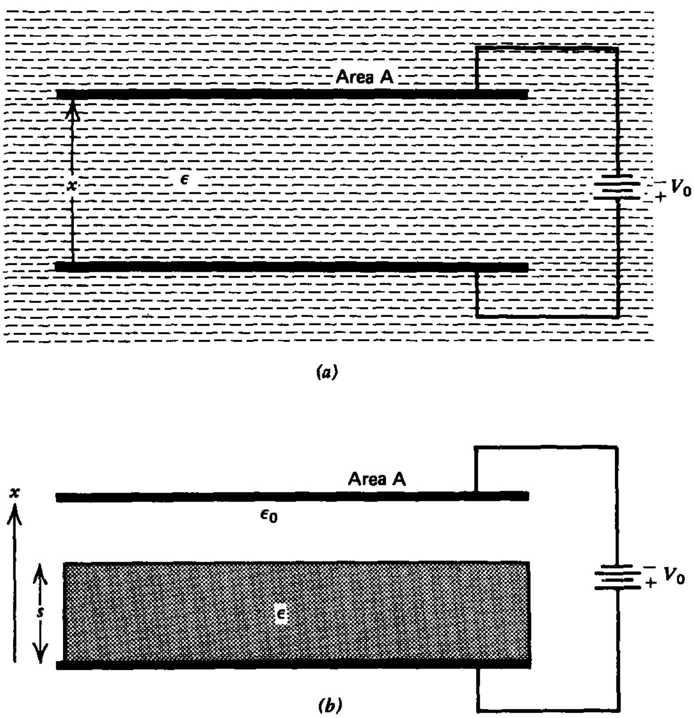



Figure 3-36 A parallel plate capacitor (*a*) immersed within a dielectric fluid or with (*b*) a free space region in series with a solid dielectric.

**(a) Liquid Dielectric**
The electrodes are immersed within a liquid dielectric with permittivity $\varepsilon$, as shown in Figure 3-36*a*.

**SOLUTION**

---

The capacitance of the system is

$$C(x)=\varepsilon A/x$$

so that the force from (29) for constant voltage is

$$f_x=\frac{1}{2}V_0^2\frac{dC(x)}{dx}=-\frac{1}{2}\frac{\varepsilon A V_0^2}{x^2}$$

The force being negative means that it is in the direction opposite to increasing $x$, in this case downward. The capacitor plates attract each other because they are oppositely charged and opposite charges attract. The force is independent of voltage polarity and gets infinitely large as the plate spacing approaches zero. The result is also valid for free space with $\varepsilon=\varepsilon_0$. The presence of the dielectric increases the attractive force.

If the electrodes are constrained to a constant charge $Q_0$ the force is then attractive but independent of $x$:

$$f_x=-\frac{1}{2}Q_0^2\frac{d}{dx}\frac{1}{C(x)}=-\frac{1}{2}\frac{Q_0^2}{\varepsilon A}$$

For both these cases, the numerical value of the force is the same because $Q_0$ and $V_0$ are related by the capacitance, but the functional dependence on $x$ is different. The presence of a dielectric now decreases the force over that of free space.

**(b) Solid Dielectric**
A solid dielectric with permittivity $\varepsilon$ of thickness $s$ is inserted between the electrodes with the remainder of space having permittivity $\varepsilon_0$, as shown in Figure 3-36*b*.

**SOLUTION**

---

The total capacitance for this configuration is given by the series combination of capacitance due to the dielectric block and the free space region:

$$C(x)=\frac{\varepsilon\varepsilon_0 A}{\varepsilon_0 s+\varepsilon(x-s)}$$

The force on the upper electrode for constant voltage is

$$f_x = \frac{1}{2} V_0^2 \frac{d}{dx} C(x) = -\frac{\varepsilon^2 \varepsilon_0 A V_0^2}{2[\varepsilon_0 s + \varepsilon (x - s)]^2}$$

If the electrode just rests on the dielectric so that $x = s$, the force is

$$f_x = -\frac{\varepsilon^2 A V_0^2}{2\varepsilon_0 s^2}.$$

This result differs from that of part (*a*) when $x = s$ by the factor $\varepsilon_r = \varepsilon/\varepsilon_0$ because in this case moving the electrode even slightly off the dielectric leaves a free space region in between. In part (*a*) no free space gap develops as the liquid dielectric fills in the region, so that the dielectric is always in contact with the electrode. The total force on the electrode-dielectric interface is due to both free and polarization charge.

With the electrodes constrained to constant charge, the force on the upper electrode is independent of position and also independent of the permittivity of the dielectric block:

$$f_x = -\frac{1}{2} Q_0^2 \frac{d}{dx} \frac{1}{C(x)} = -\frac{1}{2} \frac{Q_0^2}{\varepsilon_0 A}$$

## 3-10 ELECTROSTATIC GENERATORS

### 3-10-1 Van de Graaff Generator

In the 1930s, reliable means of generating high voltages were necessary to accelerate charged particles in atomic studies. In 1931, Van de Graaff developed an electrostatic generator where charge is sprayed onto an insulating moving belt that transports this charge onto a conducting dome, as illustrated in Figure 3-37a. If the dome was considered an isolated sphere of radius $R$, the capacitance is given as $C = 4\pi\varepsilon_0 R$. The transported charge acts as a current source feeding this capacitance, as in Figure 3-37*b*, so that the dome voltage builds up linearly with time:

$$i = C\frac{dv}{dt} \Rightarrow v = \frac{i}{C} t \qquad (1)$$

This voltage increases until the breakdown strength of the surrounding atmosphere is reached, whereupon a spark discharge occurs. In air, the electric field breakdown strength $E_b$ is $3 \times 10^6$ V/m. The field near the dome varies as $E_r = VR/r^2$, which is maximum at $r = R$, which implies a maximum voltage of $V_{max} = E_b R$. For $V_{max} = 10^6$ V, the radius of the sphere

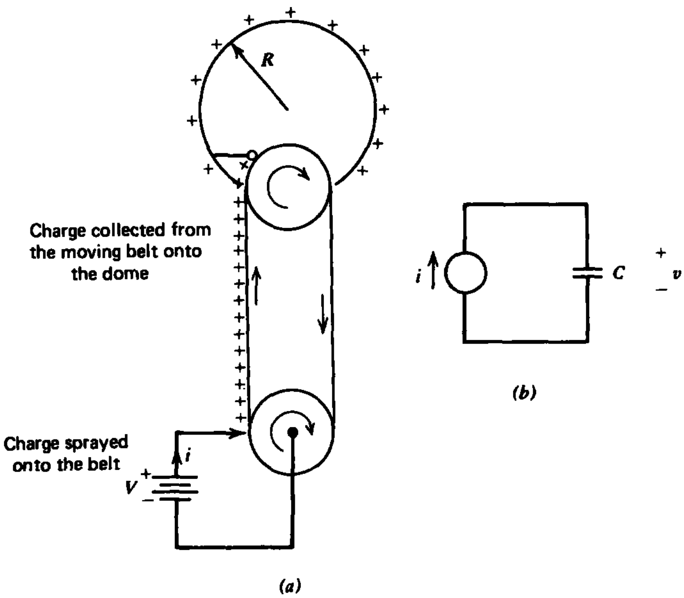



Figure 3-37 (*a*) A Van de Graaff generator consists of a moving insulating belt that transports injected charge onto a conducting dome which can thus rise to very high voltages, easily in excess of a million volts. (*b*) A simple equivalent circuit consists of the convecting charge modeled as a current source charging the capacitance of the dome.

must be $R \approx \frac{1}{3}$ m so that the capacitance is $C \approx 37$ pf. With a charging current of one microampere, it takes $t \approx 37$ sec to reach this maximum voltage.

### 3-10-2 Self-Excited Electrostatic Induction Machines

In the Van de Graaff generator, an external voltage source is necessary to deposit charge on the belt. In the late 1700s, self-excited electrostatic induction machines were developed that did not require any external electrical source. To understand how these devices work, we modify the Van de Graaff generator configuration, as in Figure 3-38a, by putting conducting segments on the insulating belt. Rather than spraying charge, we place an electrode at voltage $V$ with respect to the lower conducting pulley so that opposite polarity charge is induced on the moving segments. As the segments move off the pulley, they carry their charge with them. So far, this device is similar to the Van de Graaff generator using induced charge rather than sprayed charge and is described by the same equivalent circuit where the current source now depends on the capacitance $C_i$ between the inducing electrode and the segmented electrodes, as in Figure 3-38*b*.

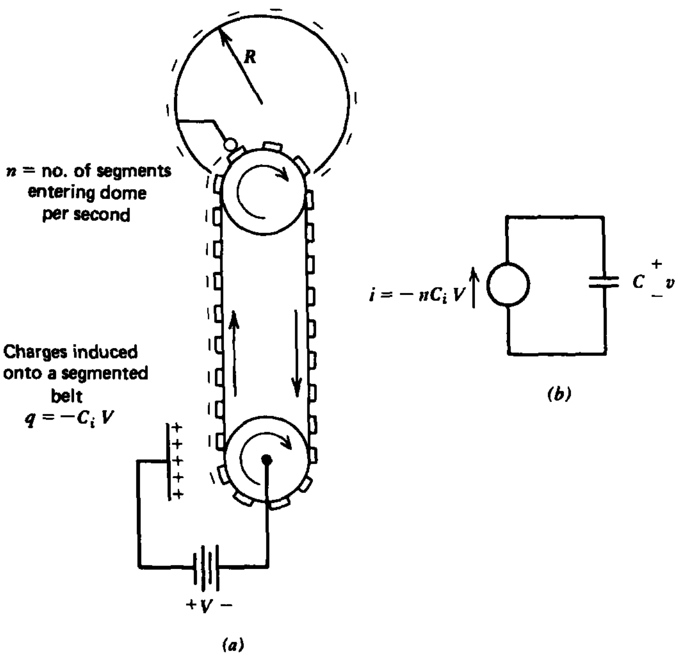



Figure 3-38 A modified Van de Graaff generator as an electrostatic induction machine. (*a*) Here charges are induced onto a segmented belt carrying insulated conductors as the belt passes near an electrode at voltage $V$. (*b*) Now the current source feeding the capacitor equivalent circuit depends on the capacitance $C_i$ between the electrode and the belt.

Now the early researchers cleverly placed another induction machine nearby as in Figure 3-39*a*. Rather than applying a voltage source, because one had not been invented yet, they electrically connected the dome of each machine to the inducer electrode of the other. The induced charge on one machine was proportional to the voltage on the other dome. Although no voltage is applied, any charge imbalance on an inducer electrode due to random noise or stray charge will induce an opposite charge on the moving segmented belt that carries this charge to the dome of which some appears on the other inducer electrode where the process is repeated with opposite polarity charge. The net effect is that the charge on the original inducer has been increased.

More quantitatively, we use the pair of equivalent circuits in Figure 3-39*b* to obtain the coupled equations

$$-nC_iv_1 = C\frac{dv_2}{dt}, \qquad -nC_iv_2 = C\frac{dv_1}{dt} \tag{2}$$

where $n$ is the number of segments per second passing through the dome. All voltages are referenced to the lower pulleys that are electrically connected together. Because these

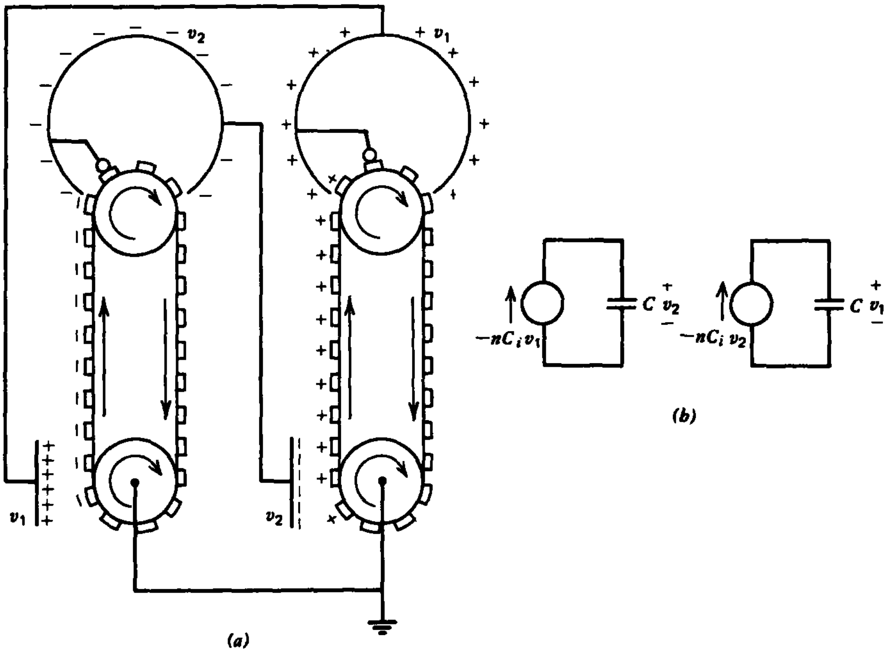



Figure 3-39 (*a*) A pair of coupled self-excited electrostatic induction machines generate their own inducing voltage. (b) The system is described by two simple coupled circuits.

are linear constant coefficient differential equations, the solutions must be exponentials:

$$v_1 = \hat{V}_1 e^{st}, \qquad v_2 = \hat{V}_2 e^{st} \tag{3}$$

Substituting these assumed solutions into (2) yields the following characteristic roots:

$$s^2 = \left(\frac{nC_i}{C}\right)^2 \Rightarrow s = \pm\frac{nC_i}{C} \tag{4}$$

so that the general solution is

$$\begin{aligned} v_1 &= A_1\, e^{(nC_i/C)t} + A_2\, e^{-(nC_i/C)t} \\ v_2 &= -A_1\, e^{(nC_i/C)t} + A_2\, e^{-(nC_i/C)t} \end{aligned} \tag{5}$$

where $A_1$ and $A_2$ are determined from initial conditions.

The negative root of (4) represents the uninteresting decaying solutions while the positive root has solutions that grow unbounded with time. This is why the machine is self-excited. Any initial voltage perturbation, no matter how small, increases without bound until electrical breakdown is reached. Using representative values of $n = 10$, $C_i = 2$ pf, and $C = 10$ pf, we have that $s = \pm 2$ so that the time constant for voltage build-up is about one-half second.

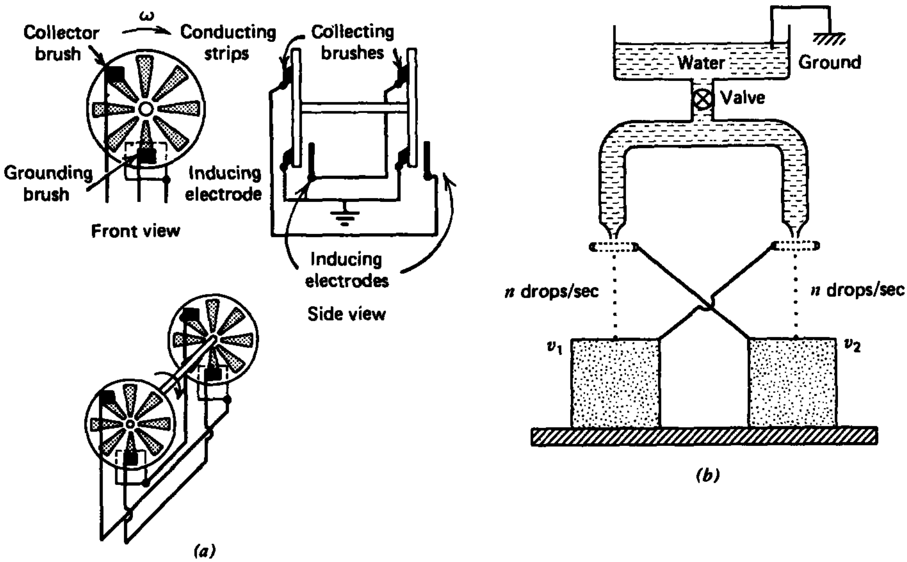



Figure 3-40 Other versions of self-excited electrostatic induction machines use (*a*) rotating conducting strips (Wimshurst machine) or (*b*) falling water droplets (Lord Kelvin's water dynamo). These devices are also described by the coupled equivalent circuits in Figure 3-39*b*.

The early electrical scientists did not use a segmented belt but rather conducting disks embedded in an insulating wheel that could be turned by hand, as shown for the Wimshurst machine in Figure 3-40*a*. They used the exponentially growing voltage to charge up a capacitor called a Leyden jar (credited to scientists from Leyden, Holland), which was a glass bottle silvered on the inside and outside to form two electrodes with the glass as the dielectric.

An analogous water drop dynamo was invented by Lord Kelvin (then Sir W. Thomson) in 1861, which replaced the rotating disks by falling water drops, as in Figure 3-40*b*. All these devices are described by the coupled equivalent circuits in Figure 3-39*b*.

## 3-10-3 Self-Excited Three-Phase Alternating Voltages

In 1967, Euerle* modified Kelvin's original dynamo by adding a third stream of water droplets so that three-phase

* *W. C. Euerle, "A Novel Method of Generating Polyphase Power," M.S. Thesis, Massachusetts Institute of Technology, 1967. See also J. R. Melcher, Electric Fields and Moving Media,* IEEE Trans. Education *E-17 (1974), pp. 100–110, and the film by the same title produced for the National Committee on Electrical Engineering Films by the Educational Development Center, 39 Chapel St., Newton, Mass. 02160.*

alternating voltages were generated. The analogous three-phase Wimshurst machine is drawn in Figure 3-41*a* with equivalent circuits in Figure 3-41*b*. Proceeding as we did in (2) and (3),

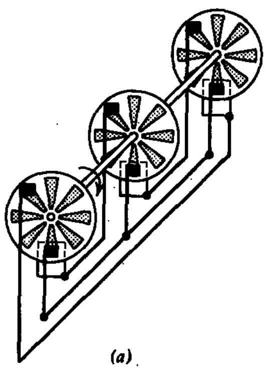
$$-nC_iv_1 = C\frac{dv_2}{dt}, \qquad v_1 = \hat{V}_1 e^{st}$$

$$-nC_iv_2 = C\frac{dv_3}{dt}, \qquad v_2 = \hat{V}_2 e^{st} \tag{6}$$

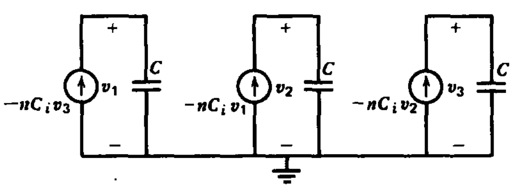
$$-nC_iv_3 = C\frac{dv_1}{dt}, \qquad v_3 = \hat{V}_3 e^{st}$$

equation (6) can be rewritten as

$$\begin{bmatrix} nC_i & Cs & 0 \\ 0 & nC_i & Cs \\ Cs & 0 & nC_i \end{bmatrix}\begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \\ \hat{V}_3 \end{bmatrix} = 0 \tag{7}$$



(a)




(b)

Figure 3-41 (*a*) Self-excited three-phase ac Wimshurst machine. (*b*) The coupled equivalent circuit is valid for any of the analogous machines discussed.

which requires that the determinant of the coefficients of $\hat{V}_1$, $\hat{V}_2$, and $\hat{V}_3$ be zero:

$$(nC_i)^3+(Cs)^3=0 \Rightarrow s=\left(\frac{nC_i}{C}\right)^{1/3}(-1)^{1/3}$$
$$=\left(\frac{nC_i}{C}\right)^{1/3} e^{j(\pi/3)(2r-1)}, \qquad r=1,2,3 \tag{8}$$
$$\Rightarrow s_1=-\frac{nC_i}{C}$$
$$s_{2,3}=\frac{nC_i}{2C}[1\pm\sqrt{3}\,j]$$

where we realized that $(-1)^{1/3}$ has three roots in the complex plane. The first root is an exponentially decaying solution, but the other two are complex conjugates where the positive real part means exponential growth with time while the imaginary part gives the frequency of oscillation. We have a self-excited three-phase generator as each voltage for the unstable modes is 120° apart in phase from the others:

$$\frac{\hat{V}_2}{\hat{V}_1}=\frac{\hat{V}_3}{\hat{V}_2}=\frac{\hat{V}_1}{\hat{V}_3}=-\frac{nC_i}{Cs_{2,3}}=-\tfrac{1}{2}(1\pm\sqrt{3}j)=e^{\pm j(2/3)\pi} \tag{9}$$

Using our earlier typical values following (5), we see that the oscillation frequencies are very low, $f=(1/2\pi)\,\text{Im}\,(s)\approx 0.28$ Hz.

### 3-10-4 Self-Excited Multi-frequency Generators

If we have $N$ such generators, as in Figure 3-42, with the last one connected to the first one, the $k$th equivalent circuit yields

$$-nC_i\hat{V}_k=Cs\hat{V}_{k+1} \tag{10}$$

This is a linear constant coefficient difference equation. Analogously to the exponential time solutions in (3) valid for linear constant coefficient differential equations, solutions to (10) are of the form

$$\hat{V}_k=A\lambda^k \tag{11}$$

where the characteristic root $\lambda$ is found by substitution back into (10) to yield

$$-nC_iA\lambda^k=CsA\lambda^{k+1} \Rightarrow \lambda=-nC_i/Cs \tag{12}$$

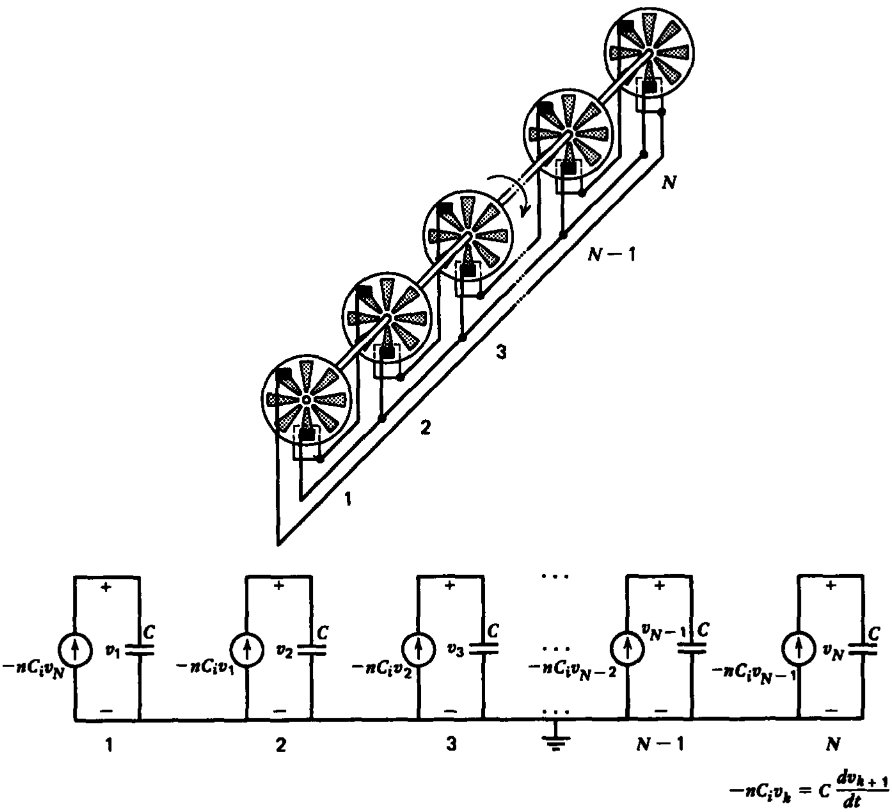



Figure 3-42 Multi-frequency, polyphase self-excited Wimshurst machine with equivalent circuit.

Since the last generator is coupled to the first one, we must have that

$$V_{N+1} = V_1 \Rightarrow \lambda^{N+1} = \lambda^1$$
$$\Rightarrow \lambda^N = 1$$
$$\Rightarrow \lambda = 1^{1/N} = e^{j2\pi r/N}, \qquad r = 1, 2, 3, \ldots, N \tag{13}$$

where we realize that unity has $N$ complex roots.

The system natural frequencies are then obtained from (12) and (13) as

$$s = -\frac{nC_i}{C\lambda} = -\frac{nC_i}{C} e^{-j2\pi r/N}, \qquad r = 1, 2, \ldots, N \tag{14}$$

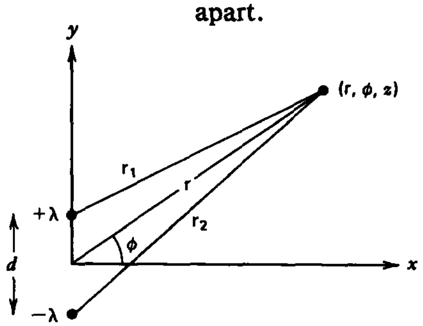
We see that for $N = 2$ and $N = 3$ we recover the results of (4) and (8). All the roots with a positive real part of $s$ are unstable and the voltages spontaneously build up in time with oscillation frequencies $\omega_0$ given by the imaginary part of $s$.

$$\omega_0 = |\mathrm{Im}\,(s)| = \frac{nC_i}{C} |\sin 2\pi r/N| \tag{15}$$

## PROBLEMS

Section 3-1

1. A two-dimensional dipole is formed by two infinitely long parallel line charges of opposite polarity $\pm\lambda$ a small distance $d\mathbf{i}_y$ apart.

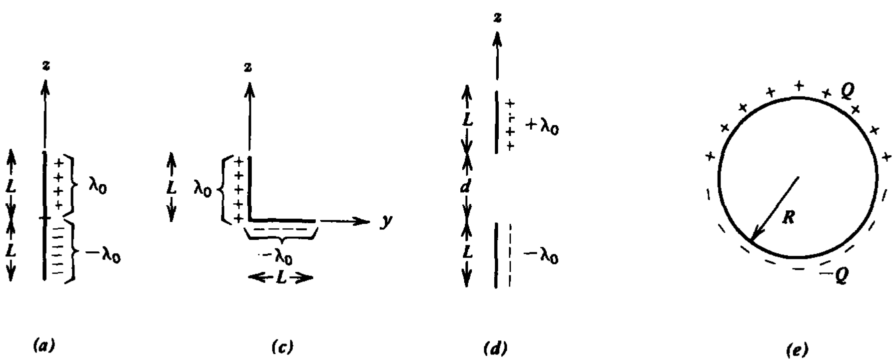



(a) What is the potential at any coordinate $(r, \phi, z)$?

(b) What are the potential and electric field far from the dipole ($r \gg d$)? What is the dipole moment per unit length?

(c) What is the equation of the field lines?

2. Find the dipole moment for each of the following charge distributions:




(a) Two uniform colinear opposite polarity line charges $\pm\lambda_0$ each a small distance $L$ along the $z$ axis.

(b) Same as (a) with the line charge distribution as

$$\lambda(z) = \begin{cases} \lambda_0(1 - z/L), & 0 < z < L \\ -\lambda_0(1 + z/L), & -L < z < 0 \end{cases}$$

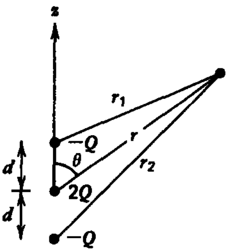
(c) Two uniform opposite polarity line charges $\pm\lambda_0$ each of length $L$ but at right angles.

(d) Two parallel uniform opposite polarity line charges $\pm\lambda_0$ each of length $L$ a distance $d\mathbf{i}_z$ apart.

(e) A spherical shell with total uniformly distributed surface charge $Q$ on the upper half and $-Q$ on the lower half. (**Hint:** $\mathbf{i}_r = \sin\theta\cos\phi\,\mathbf{i}_x + \sin\theta\sin\phi\,\mathbf{i}_y + \cos\theta\,\mathbf{i}_z$.)

(f) A spherical volume with total uniformly distributed volume charge of $Q$ in the upper half and $-Q$ on the lower half. (**Hint:** Integrate the results of (e).)

3. The linear quadrapole consists of two superposed dipoles along the $z$ axis. Find the potential and electric field for distances far away from the charges ($r \gg d$).




$$\frac{1}{r_1} \approx \frac{1}{r}\left\{1 + \frac{d}{r}\cos\theta - \frac{1}{2}\left(\frac{d}{r}\right)^2(1 - 3\cos^2\theta)\right\}$$

$$\frac{1}{r_2} \approx \frac{1}{r}\left\{1 - \frac{d}{r}\cos\theta - \frac{1}{2}\left(\frac{d}{r}\right)^2(1 - 3\cos^2\theta)\right\}$$

Linear quadrapole

4. Model an atom as a fixed positive nucleus of charge $Q$ with a surrounding spherical negative electron cloud of nonuniform charge density:

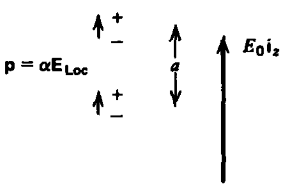
$$\rho = -\rho_0(1 - r/R_0), \qquad r < R_0$$

(a) If the atom is neutral, what is $\rho_0$?

(b) An electric field is applied with local field $\mathbf{E}_{Loc}$ causing a slight shift $\mathbf{d}$ between the center of the spherical cloud and the positive nucleus. What is the equilibrium dipole spacing?

(c) What is the approximate polarizability $\alpha$ if $9\varepsilon_0 E_{Loc}/(\rho_0 R_0) \ll 1$?

5. Two colinear dipoles with polarizability $\alpha$ are a distance $a$ apart along the $z$ axis. A uniform field $E_0\mathbf{i}_z$ is applied.



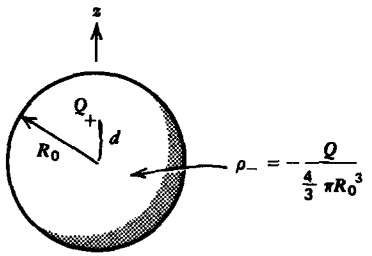

(a) What is the total local field seen by each dipole?

(b) Repeat (a) if we have an infinite array of dipoles with constant spacing $a$. (**Hint:** $\sum_{n=1}^{\infty} 1/n^3 \approx 1.2$.)

(c) If we assume that we have one such dipole within each volume of $a^3$, what is the permittivity of the medium?

6. A dipole is modeled as a point charge $Q$ surrounded by a spherical cloud of electrons with radius $R_0$. Then the local

field $\mathbf{E}_{Loc}$ differs from the applied field $\mathbf{E}$ by the field due to the dipole itself. Since $\mathbf{E}_{dip}$ varies within the spherical cloud, we use the average field within the sphere.




(a) Using the center of the cloud as the origin, show that the dipole electric field within the cloud is

$$\mathbf{E}_{dip} = -\frac{Qr\mathbf{i}_r}{4\pi\varepsilon_0 R_0^3} + \frac{Q(r\mathbf{i}_r - d\mathbf{i}_z)}{4\pi\varepsilon_0[d^2 + r^2 - 2rd\cos\theta]^{3/2}}$$

(b) Show that the average $x$ and $y$ field components are zero. (**Hint:** $\mathbf{i}_r = \sin\theta\cos\phi\mathbf{i}_x + \sin\theta\sin\phi\mathbf{i}_y + \cos\theta\mathbf{i}_z$.)

(c) What is the average $z$ component of the field? (**Hint:** Change variables to $u = r^2 + d^2 - 2rd\cos\theta$ and remember $\sqrt{(r-d)^2} = |r-d|$.)

(d) If we have one dipole within every volume of $\frac{4}{3}\pi R_0^3$, how is the polarization $\mathbf{P}$ related to the applied field $\mathbf{E}$?

7. Assume that in the dipole model of Figure 3-5$a$ the mass of the positive charge is so large that only the electron cloud moves as a solid mass $m$.

(a) The local electric field is $\mathbf{E}_0$. What is the dipole spacing?

(b) At $t = 0$, the local field is turned off ($\mathbf{E}_0 = 0$). What is the subsequent motion of the electron cloud?

(c) What is the oscillation frequency if $Q$ has the charge and mass of an electron with $R_0 = 10^{-10}$ m?

(d) In a real system there is always some damping that we take to be proportional to the velocity ($\mathbf{f}_{damping} = -\beta\mathbf{v}$). What is the equation of motion of the electron cloud for a sinusoidal electric field $\text{Re}(\hat{E}_0 e^{j\omega t})$?

(e) Writing the driven displacement of the dipole as

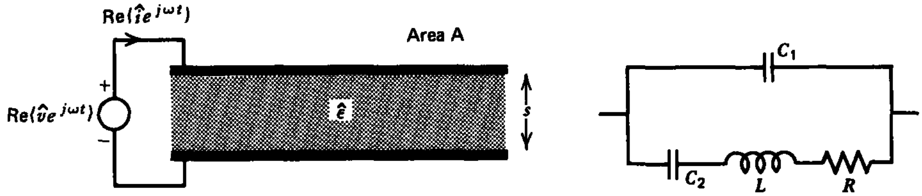
$$d = \text{Re}(\hat{d}e^{j\omega t}).$$

what is the complex polarizability $\hat{\alpha}$, where $\hat{p} = Q\hat{d} = \hat{\alpha}E_0$?

(f) What is the complex dielectric constant $\hat{\varepsilon} = \varepsilon_r + j\varepsilon_i$ of the system? (**Hint:** Define $\omega_p^2 = Q^2N/(m\varepsilon_0)$.)

(g) Such a dielectric is placed between parallel plate electrodes. Show that the equivalent circuit is a series $R$, $L$, $C$ shunted by a capacitor. What are $C_1$, $C_2$, $L$, and $R$?

(h) Consider the limit where the electron cloud has no mass ($m = 0$). With the frequency $\omega$ as a parameter show that

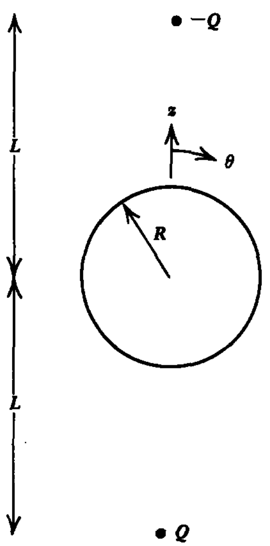



(g)

a plot of $\varepsilon_r$ versus $\varepsilon_i$ is a circle. Where is the center of the circle and what is its radius? Such a diagram is called a Cole-Cole plot.

(i) What is the maximum value of $\varepsilon_i$ and at what frequency does it occur?

8. Two point charges of opposite sign $\pm Q$ are a distance $L$ above and below the center of a grounded conducting sphere of radius $R$.




(a) What is the electric field everywhere along the $z$ axis and in the $\theta = \pi/2$ plane? (**Hint:** Use the method of images.)

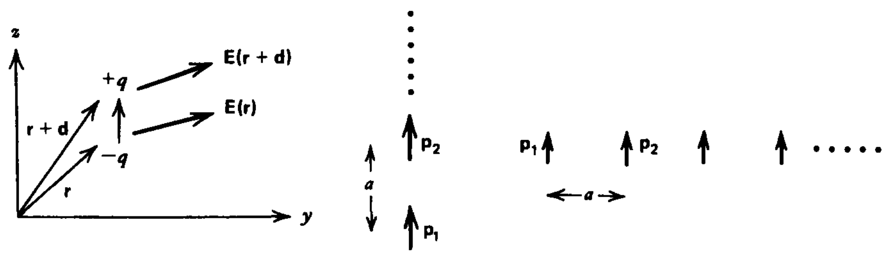
(b) We would like this problem to model the case of a conducting sphere in a uniform electric field by bringing the point charges $\pm Q$ out to infinity ($L \to \infty$). What must the ratio $Q/L^2$ be such that the field far from the sphere in the $\theta = \pi/2$ plane is $E_0\mathbf{i}_z$?

(c) In this limit, what is the electric field everywhere?

9. A dipole with moment $\mathbf{p}$ is placed in a nonuniform electric field.

(a) Show that the force on a dipole is

$$\mathbf{f} = (\mathbf{p} \cdot \nabla)\mathbf{E}$$

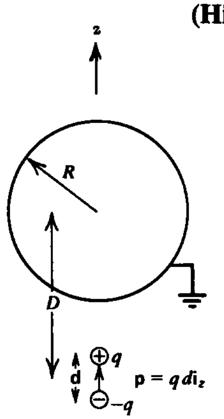



*(a)* *(b), (c)*

(b) Find the force on dipole 1 due to dipole 2 when the two dipoles are colinear, or are adjacent a distance $a$ apart.

(c) Find the force on dipole 1 if it is the last dipole in an infinite array of identical colinear or adjacent dipoles with spacing $a$. (**Hint:** $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$.)

10. A point dipole with moment $p\mathbf{i}_z$ is a distance $D$ from the center of a grounded sphere of radius $R$.
(**Hint:** $d \ll D$.)




(a) What is the induced dipole moment of the sphere?
(b) What is the electric field everywhere along the $z$ axis?
(c) What is the force on the sphere? (**Hint:** See Problem 9$a$.)

Section 3-2
11. Find the potential, electric field, and charge density distributions for each of the following charges placed within a medium of infinite extent, described by drift-diffusion conduction in the limit when the electrical potential is much less than the thermal voltage ($qV/kT \ll 1$):

(a) Sheet of surface charge $\sigma_f$ placed at $x = 0$.
(b) Infinitely long line charge with uniform density $\lambda$. (**Hint:** Bessel's equation results.)
(c) Conducting sphere of radius $R$ carrying a total surface charge $Q$.

12. Two electrodes at potential $\pm V_0/2$ located at $x=\pm l$ enclose a medium described by drift-diffusion conduction for two oppositely charged carriers, where $qV_0/kT \ll 1$.

(a) Find the approximate solutions of the potential, electric field, and charge density distributions. What is the charge polarity near each electrode?

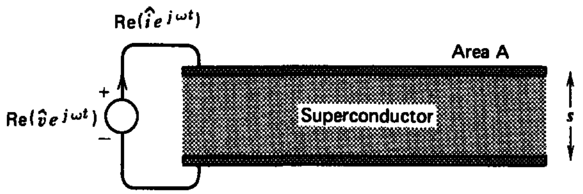
(b) What is the total charge per unit area within the volume of the medium and on each electrode?

13. (a) Neglecting diffusion effects but including charge inertia and collisions, what is the time dependence of the velocity of charge carriers when an electric field $E_0\mathbf{i}_x$ is instantaneously turned on at $t=0$?

(b) After the charge carriers have reached their steady-state velocity, the electric field is suddenly turned off. What is their resulting velocity?

(c) This material is now placed between parallel plate electrodes of area $A$ and spacing $s$. A sinusoidal voltage is applied $\text{Re}\,(V_0\, e^{j\omega t})$. What is the equivalent circuit?

14. Parallel plate electrodes enclose a superconductor that only has free electrons with plasma frequency $\omega_{pe}$.




(a) What is the terminal current when a sinusoidal voltage is applied?

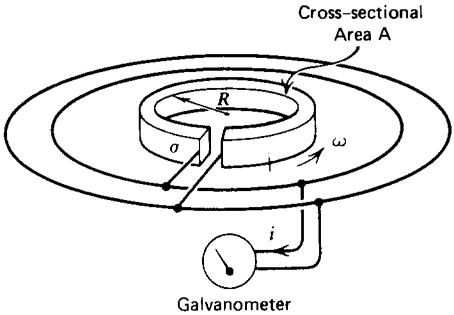
(b) What is the equivalent circuit?

15. A conducting ring of radius $R$ is rotated at constant angular speed. The ring has Ohmic conductivity $\sigma$ and cross sectional area $A$. A galvanometer is connected to the ends of the ring to indicate the passage of any charge. The connection is made by slip rings so that the rotation of the ring is unaffected by the galvanometer. The ring is instantly stopped, but the electrons within the ring continue to move a short time until their momentum is dissipated by collisions. For a particular electron of charge $q$ and mass $m$ conservation of momentum requires

$$\Delta(mv) = \int F\, dt$$

where $F = qE$ is the force on the electron.

(a) For the Ohmic conductor, relate the electric field to the current in the wire.

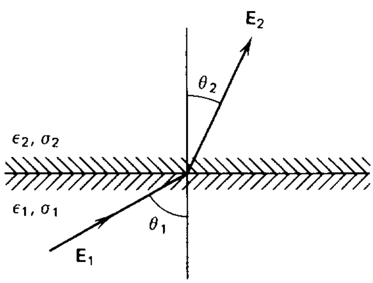



(b) When the ring is instantly stopped, what is the charge $Q$ through the galvanometer? (**Hint:** $Q=\int i\,dt$. This experiment is described by R. C. Tolman and T. D. Stewart, *Phys. Rev.* **8**, No. 2 (1916), p. 97.)

(c) If the ring is an electron superconductor with plasma frequency $\omega_p$, what is the resulting current in the loop when it stops?

Section 3.3

16. An electric field with magnitude $E_1$ is incident upon the interface between two materials at angle $\theta_1$ from the normal. For each of the following material properties find the magnitude and direction of the field $\mathbf{E}_2$ in region 2.




(a) Lossless dielectrics with respective permittivities $\varepsilon_1$ and $\varepsilon_2$. There is no interfacial surface charge.

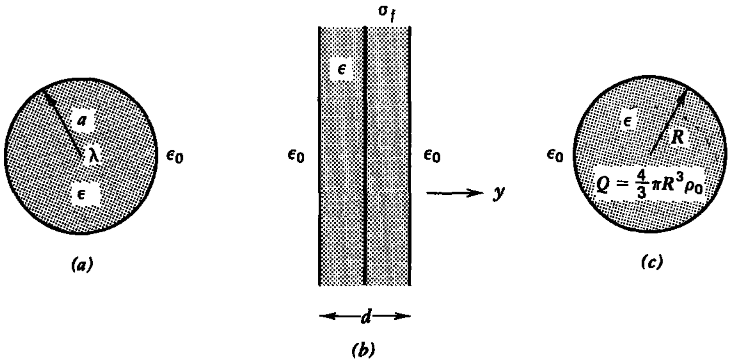
(b) Ohmic materials with respective conductivities $\sigma_1$ and $\sigma_2$ in the dc steady state. What is the free surface charge density $\sigma_f$ on the interface?

(c) Lossy dielectrics $(\varepsilon_1, \sigma_1)$ and $(\varepsilon_2, \sigma_2)$ with a sinusoidally varying electric field

$$E_1 = \text{Re}\,(\hat{E}_1 e^{j\omega t})$$

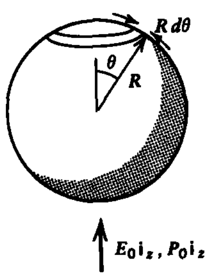
What is the free surface charge density $\sigma_f$ on the interface?

17. Find the electric, displacement, and polarization fields and the polarization charge everywhere for each of the following configurations:




(a) An infinitely long line charge $\lambda$ placed at the center of a dielectric cylinder of radius $a$ and permittivity $\varepsilon$.
(b) A sheet of surface charge $\sigma_f$ placed at the center of a dielectric slab with permittivity $\varepsilon$ and thickness $d$.
(c) A uniformly charged dielectric sphere with permittivity $\varepsilon$ and radius $R$ carrying a total free charge $Q$.

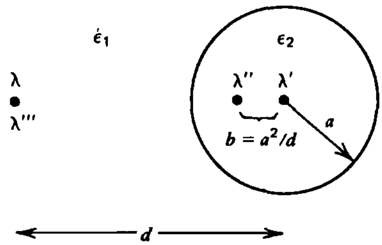
18. Lorentz calculated the local field acting on a dipole due to a surrounding uniformly polarized medium stressed by a macroscopic field $E_0\mathbf{i}_z$ by encircling the dipole with a small spherical free space cavity of radius $R$.




(a) If the medium outside the cavity has polarization $P_0\mathbf{i}_z$, what is the surface polarization charge on the spherical interface? (**Hint:** $\mathbf{i}_z = \mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta$)
(b) Break this surface polarization charge distribution into hoop line charge elements of thickness $d\theta$. What is the total charge on a particular shell at angle $\theta$?
(c) What is the electric field due to this shell at the center of the sphere where the dipole is?
(d) By integrating over all shells, find the total electric field acting on the dipole. This is called the Lorentz field. (**Hint:** Let $u = \cos\theta$).

19. A line charge $\lambda$ within a medium of permittivity $\varepsilon_1$ is outside a dielectric cylinder of radius $a$ and permittivity $\varepsilon_2$.

The line charge is parallel to the cylinder axis and a distance $d$ from it.



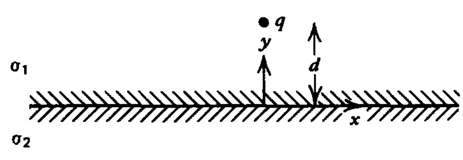

(a) Try using the method of images by placing a line charge $\lambda'$ at the center and another image $\lambda''$ within the cylinder at distance $b = a^2/d$ from the axis along the line joining the axis to the line charge. These image charges together with the original line charge will determine the electric field outside the cylinder. Put another line charge $\lambda'''$ at the position of the original line charge to determine the field within the cylinder. What values of $\lambda'$, $\lambda''$, and $\lambda'''$ satisfy the boundary conditions?

(b) Check your answers with that of Section 3-3-3 in the limit as the radius of the cylinder becomes large so that it looks like a plane.

(c) What is the force per unit length on the line charge $\lambda$?

(d) Repeat (a)–(c) when the line charge $\lambda$ is within the dielectric cylinder.

20. A point charge $q$ is a distance $d$ above a planar boundary separating two Ohmic materials with respective conductivities $\sigma_1$ and $\sigma_2$.




(a) What steady-state boundary conditions must the electric field satisfy?

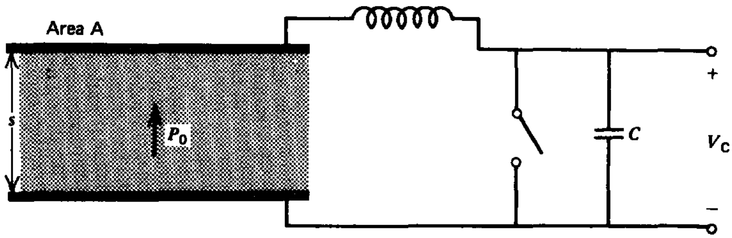
(b) What image charge configuration will satisfy these boundary conditions? (**Hint:** See Section 3-3-3.)

(c) What is the force on $q$?

21. The polarization of an electret is measured by placing it between parallel plate electrodes that are shorted together.

(a) What is the surface charge on the upper electrode?

(b) The switch is then opened and the upper electrode is taken far away from the electret. What voltage is measured across the capacitor?



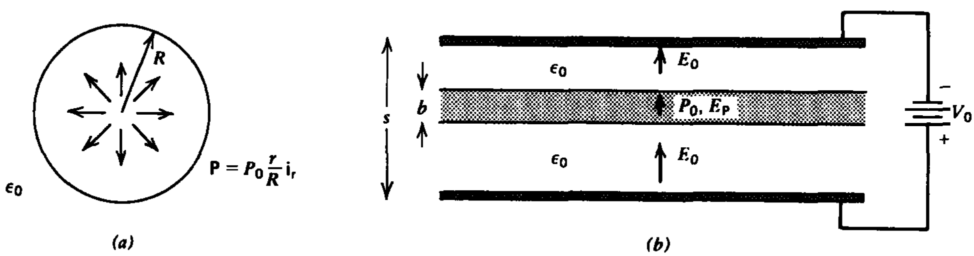

22. A cylinder of radius $a$ and height $L$ as in Figure 3-14, has polarization

$$\mathbf{P}=\frac{P_0 z}{L}\mathbf{i}_z$$

(a) What is the polarization charge distribution?

(b) Find the electric and displacement fields everywhere along the $z$ axis. (**Hint:** Use the results of Sections 2-3-5*b* and 2-3-5*d*.)

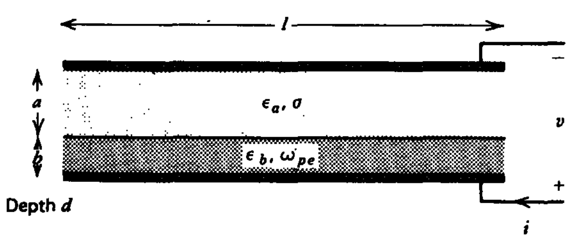
23. Find the electric field everywhere for the following permanently polarized structures which do not support any free charge:




(a) Sphere of radius $R$ with polarization $\mathbf{P}=(P_0 r/R)\mathbf{i}_r$.

(b) Permanently polarized slab $P_0\mathbf{i}_x$ of thickness $b$ placed between parallel plate electrodes in free space at potential difference $V_0$.

24. Parallel plate electrodes enclose the series combination of an Ohmic conductor of thickness $a$ with conductivity $\sigma$ and a superconductor that only has free electrons with plasma



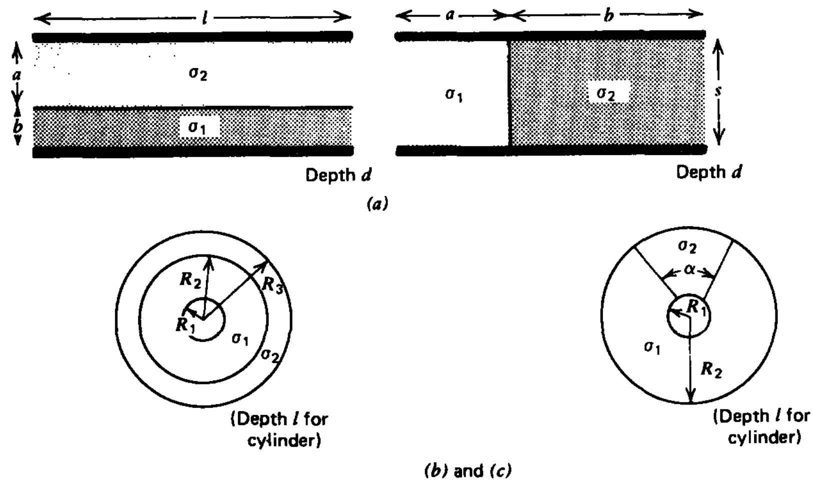
frequency $\omega_{pe}$. What is the time dependence of the terminal current, the electric field in each region, and the surface charge at the interface separating the two conductors for each of the following terminal constraints:

(a) A step voltage $V_0$ is applied at $t=0$. For what values of $\omega_{pe}$ are the fields critically damped?

(b) A sinusoidal voltage $v(t)=V_0\cos\omega t$ has been applied for a long time.

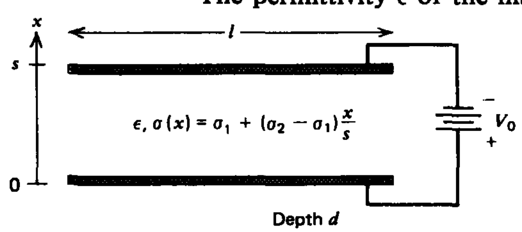
Section 3-4

25. Find the series and parallel resistance between two materials with conductivities $\sigma_1$ and $\sigma_2$ for each of the following electrode geometries:




(a) Parallel plates.
(b) Coaxial cylinders.
(c) Concentric spheres.

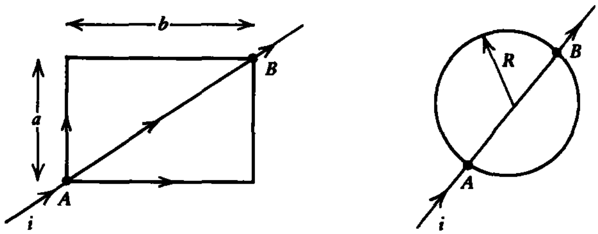
26. A pair of parallel plate electrodes at voltage difference $V_0$ enclose an Ohmic material whose conductivity varies linearly from $\sigma_1$ at the lower electrode to $\sigma_2$ at the upper electrode. The permittivity $\epsilon$ of the material is a constant.




(a) Find the fields and the resistance.
(b) What are the volume and surface charge distributions?

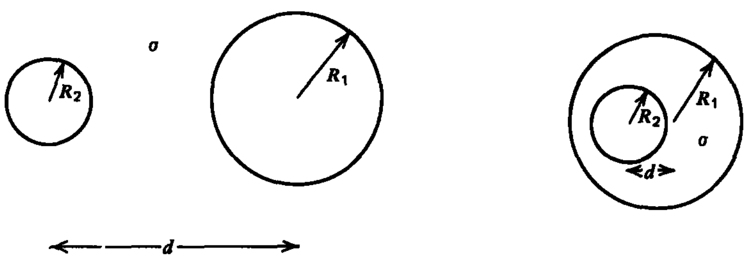
(c) What is the total volume charge in the system and how is it related to the surface charge on the electrodes?

27. A wire of Ohmic conductivity $\sigma$ and cross sectional area A is twisted into the various shapes shown. What is the resistance $R$ between the points $A$ and $B$ for each of the configurations?



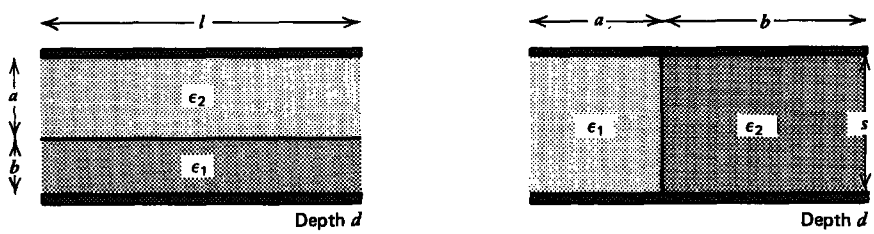

Section 3-5

28. Two conducting cylinders of length $l$ and differing radii $R_1$ and $R_2$ within an Ohmic medium with conductivity $\sigma$ have their centers a distance $d$ apart. What is the resistance between cylinders when they are adjacent and when the smaller one is inside the larger one? (**Hint:** See Section 2-6-4c.)

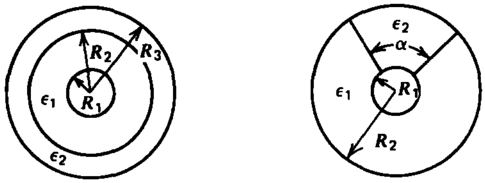



29. Find the series and parallel capacitance for each of the following geometries:

(a) Parallel plate.
(b) Coaxial cylinders.
(c) Concentric spheres.

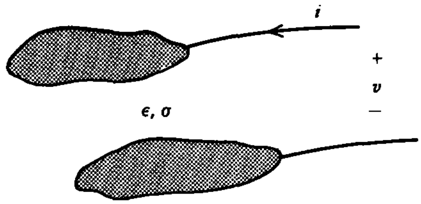



(a)




(Depth $l$ for cylinders)
(b), (c)

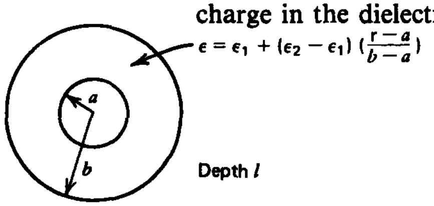
30. Two arbitrarily shaped electrodes are placed within a medium of constant permittivity $\varepsilon$ and Ohmic conductivity $\sigma$. When a dc voltage $V$ is applied across the system, a current $I$ flows.




(a) What is the current $i(t)$ when a sinusoidal voltage $\text{Re}\,(V_0 e^{j\omega t})$ is applied?

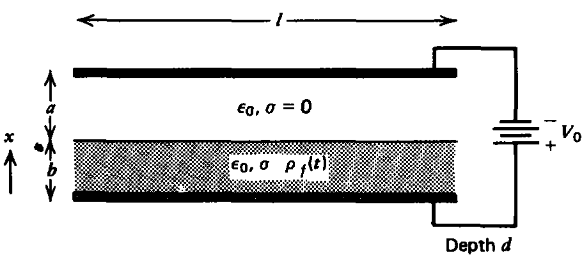
(b) What is the equivalent circuit of the system?

31. Concentric cylindrical electrodes of length $l$ with respective radii $a$ and $b$ enclose an Ohmic material whose permittivity varies linearly with radius from $\varepsilon_1$ at the inner cylinder to $\epsilon_2$ at the outer. What is the capacitance? There is no volume charge in the dielectric.




Section 3.6

32. A lossy material with the permittivity $\varepsilon_0$ of free space and conductivity $\sigma$ partially fills the region between parallel plate electrodes at constant potential difference $V_0$ and is initially



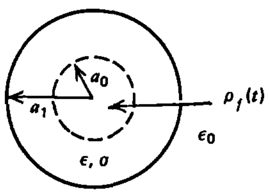
uniformly charged with density $\rho_0$ at $t=0$ with zero surface charge at $x=b$. What is the time dependence of the following:

(a) the electric field in each region? (**Hint:** See Section 3-3-5.)

(b) the surface charge at $x=b$?

(c) the force on the conducting material?

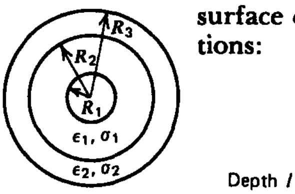
33. An infinitely long cylinder of radius $a_1$, permittivity $\varepsilon$, and conductivity $\sigma$ is nonuniformly charged at $t=0$:




$$\rho_f(t=0)=\begin{cases}\rho_0 r/a_0, & 0<r<a_0\\ 0, & r>a_0\end{cases}$$

What is the time dependence of the electric field everywhere and the surface charge at $r=a_1$?

34. Concentric cylindrical electrodes enclose two different media in series. Find the electric field, current density, and surface charges everywhere for each of the following conditions:




(a) at $t=0_+$ right after a step voltage $V_0$ is applied to the initially unexcited system;

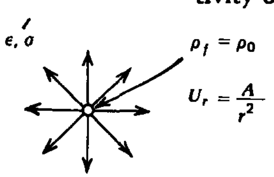
(b) at $t=\infty$ when the fields have reached their dc steady-state values;

(c) during the in-between transient interval. (What is the time constant $\tau$?);

(d) a sinusoidal voltage $V_0 \cos \omega t$ is applied and has been on a long time;

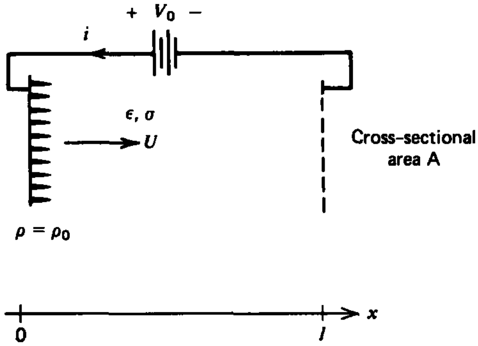
(e) what is the equivalent circuit for this system?

35. A fluid flow emanates radially from a point outlet with velocity distribution $U_r = A/r^2$. The fluid has Ohmic conductivity $\sigma$ and permittivity $\varepsilon$. An external source maintains the




charge density $\rho_0$ at $r=0$. What are the steady-state charge and electric field distributions throughout space?

36. Charge maintained at constant density $\rho_0$ at $x=0$ is carried away by a conducting fluid travelling at constant velocity $U\mathbf{i}_x$ and is collected at $x=l$.




(a) What are the field and charge distributions within the fluid if the electrodes are at potential difference $V_0$?

(b) What is the force on the fluid?

(c) Repeat (a) and (b) if the voltage source is replaced by a load resistor $R_L$.

37. A dc voltage has been applied a long time to an open circuited resistive-capacitive structure so that the voltage and current have their steady-state distributions as given by (44). Find the resulting discharging transients for voltage and current if at $t=0$ the terminals at $z=0$ are suddenly:

(a) open circuited. **Hint:**

$$\int_0^l \sinh a(z-l)\sin\left(\frac{m\pi z}{l}\right)dz = -\frac{m\pi}{l}\frac{\sinh al}{[a^2+(m\pi/l)^2]}$$

(b) Short circuited. **Hint:**

$$\int_0^l \cosh a(z-l)\sin\left(\frac{(2n+1)\pi z}{2l}\right)dz = \frac{(2n+1)\pi\cosh al}{2l\left[a^2+\left[\frac{(2n+1)\pi}{2l}\right]^2\right]}$$

38. At $t=0$ a distributed resistive line as described in Section 3-6-4 has a step dc voltage $V_0$ applied at $z=0$. The other end at $z=l$ is short circuited.

(a) What are the steady-state voltage and current distributions?

(b) What is the time dependence of the voltage and current during the transient interval? **Hint:**

$$\int_0^l \sinh a(z-l)\sin\left(\frac{m\pi z}{l}\right)dz = -\frac{m\pi\sinh al}{l\left[a^2+(m\pi/l)^2\right]}$$

39. A distributed resistive line is excited at $z=0$ with a sinusoidal voltage source $v(t)=V_0\cos\omega t$ that has been on for a long time. The other end at $z=l$ is either open or short circuited.

(a) Using complex phasor notation of the form

$$v(z,t)=\text{Re}\,(\hat{v}(z)e^{j\omega t})$$

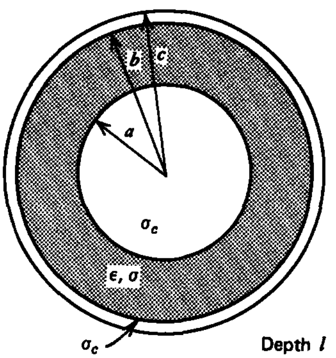
find the sinusoidal steady-state voltage and current distributions for each termination.

(b) What are the complex natural frequencies of the system?

(c) How much time average power is delivered by the source?

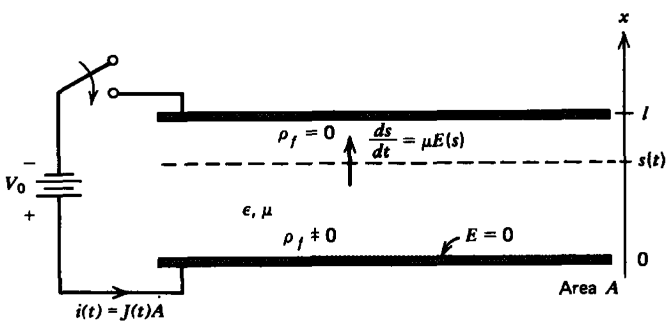
40. A lossy dielectric with permittivity $\varepsilon$ and Ohmic conductivity $\sigma$ is placed between coaxial cylindrical electrodes with large Ohmic conductivity $\sigma_c$ and length $l$.

What is the series resistance per unit length 2R of the electrodes, and the capacitance $C$ and conductance $G$ per unit length of the dielectric?




Section 3.7

41. Two parallel plate electrodes of spacing $l$ enclosing a dielectric with permittivity $\varepsilon$ are stressed by a step voltage at $t=0$. Positive charge is then injected at $t=0$ from the lower electrode with mobility $\mu$ and travels towards the opposite electrode.



(a) Using the charge conservation equation of Section 3-2-1, show that the governing equation is

$$\frac{\partial E}{\partial t}+\mu E\frac{\partial E}{\partial x}=\frac{J(t)}{\varepsilon}$$

where $J(t)$ is the current per unit electrode area through the terminal wires. This current does not depend on $x$.

(b) By integrating (a) between the electrodes, relate the current $J(t)$ solely to the voltage and the electric field at the two electrodes.

(c) For space charge limited conditions ($E(x=0)=0$), find the time dependence of the electric field at the other electrode $E(x=l,t)$ before the charge front reaches it. (**Hint:** With constant voltage, $J(t)$ from (b) only depends on $E(x=l,t)$. Using (a) at $x=l$ with no charge, $\partial E/\partial x=0$, we have a single differential equation in $E(x=l,t)$.)

(d) What is the electric field acting on the charge front? (**Hint:** There is no charge ahead of the front.)

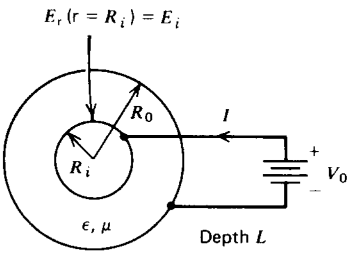
(e) What is the position of the front $s(t)$ as a function of time?

(f) At what time does the front reach the other electrode?

(g) What are the steady-state distribution of potential, electric field, and charge density? What is the steady-state current density $J(t\to\infty)$?

(h) Repeat (g) for nonspace charge limited conditions when the emitter electric field $E(x=0)=E_0$ is nonzero.

42. In a coaxial cylindrical geometry of length $L$, the inner electrode at $r=R_i$ is a source of positive ions with mobility $\mu$ in the dielectric medium. The inner cylinder is at a dc voltage $V_0$ with respect to the outer cylinder.




(a) The electric field at the emitter electrode is given as $E_r(r=R_i)=E_i$. If a current $I$ flows, what are the steady-state electric field and space charge distributions?

(b) What is the dc current $I$ in terms of the voltage under space charge limited conditions ($E_i=0$)? **Hint:**

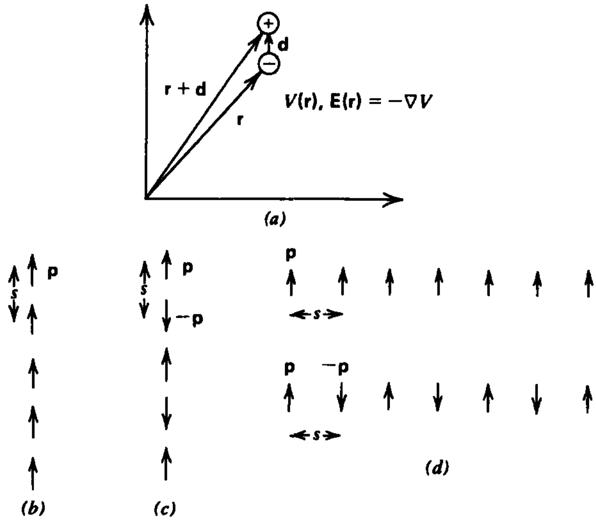
$$\int\frac{[r^2-R_i^2]^{1/2}}{r}\,dr=[r^2-R_i^2]^{1/2}-R_i\cos^{-1}\left(\frac{R_i}{r}\right)$$

(c) For what value of $E_i$ is the electric field constant between electrodes? What is the resulting current?

(d) Repeat (a)–(b) for concentric spherical electrodes.

Section 3.8

43. (a) How much work does it take to bring a point dipole from infinity to a position where the electric field is $\mathbf{E}$?




(b) A crystal consists of an infinitely long string of dipoles a constant distance $s$ apart. What is the binding energy of the crystal? (**Hint:** $\sum_{n=1}^{\infty} 1/n^3 \approx 1.2$.)

(c) Repeat (b) if the dipole moments alternate in sign. (**Hint:** $\sum_{n=1}^{\infty}(-1)^n/n^3 \approx -0.90$.)

(d) Repeat (b) and (c) if the dipole moments are perpendicular to the line of dipoles for identical or alternating polarity dipoles.

44. What is the energy stored in the field of a point dipole with moment $\mathbf{p}$ outside an encircling concentric sphere with molecular radius $R$? **Hint:**

$$\int \cos^2\theta \sin\theta\, d\theta = -\frac{\cos^3\theta}{3}$$

$$\int \sin^3\theta\, d\theta = -\tfrac{1}{3}\cos\theta\,(\sin^2\theta + 2)$$

45. A spherical droplet of radius $R$ carrying a total charge $Q$ on its surface is broken up into $N$ identical smaller droplets.

(a) What is the radius of each droplet and how much charge does it carry?

(b) Assuming the droplets are very far apart and do not interact, how much electrostatic energy is stored?

(c) Because of their surface tension the droplets also have a constant surface energy per unit area $w_s$. What is the total energy (electrostatic plus surface) in the system?

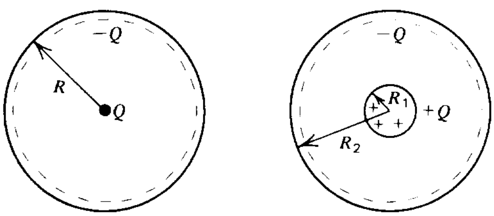
(d) How much work was required to form the droplets and to separate them to infinite spacing.

(e) What value of $N$ minimizes this work? Evaluate for a water droplet with original radius of 1 mm and charge of $10^{-6}$ coul. (For water $w_s \approx 0.072$ joule/m$^2$.)

46. Two coaxial cylinders of radii $a$ and $b$ carry uniformly distributed charge either on their surfaces or throughout the volume. Find the energy stored per unit length in the $z$ direction for each of the following charge distributions that have a total charge of zero:

(a) Surface charge on each cylinder with $\sigma_a 2\pi a = -\sigma_b 2\pi b$.
(b) Inner cylinder with volume charge $\rho_a$ and outer cylinder with surface charge $\sigma_b$ where $\sigma_b 2\pi b = -\rho_a \pi a^2$.
(c) Inner cylinder with volume charge $\rho_a$ with the region between cylinders having volume charge $\rho_b$ where $\rho_a \pi a^2 = -\rho_b \pi (b^2 - a^2)$.

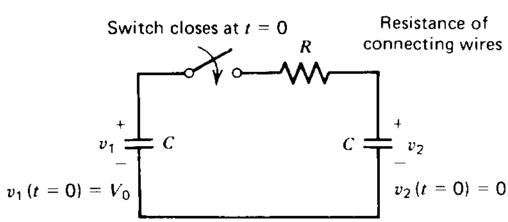
47. Find the binding energy in the following atomic models:




(a) (b)

(a) A point charge $Q$ surrounded by a uniformly distributed surface charge $-Q$ of radius $R$.
(b) A uniformly distributed volume charge $Q$ within a sphere of radius $R_1$ surrounded on the outside by a uniformly distributed surface charge $-Q$ at radius $R_2$.

48. A capacitor $C$ is charged to a voltage $V_0$. At $t = 0$ another initially uncharged capacitor of equal capacitance $C$ is



connected across the charged capacitor through some lossy wires having an Ohmic conductivity $\sigma$, cross-sectional area $A$, and total length $l$.

(a) What is the initial energy stored in the system?

(b) What is the circuit current $i$ and voltages $v_1$ and $v_2$ across each capacitor as a function of time?

(c) What is the total energy stored in the system in the dc steady state and how does it compare with (a)?

(d) How much energy has been dissipated in the wire resistance and how does it compare with (a)?

(e) How do the answers of (b)–(d) change if the system is lossless so that $\sigma = \infty$? How is the power dissipated?

(f) If the wires are superconducting Section 3-2-5*d* showed that the current density is related to the electric field as

$$\frac{\partial J}{\partial t} = \omega_p^2 \varepsilon E$$

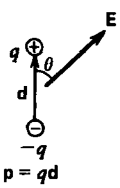
where the plasma frequency $\omega_p$ is a constant. What is the equivalent circuit of the system?

(g) What is the time dependence of the current now?

(h) How much energy is stored in each element as a function of time?

(i) At any time $t$ what is the total circuit energy and how does it compare with (a)?

Section 3.9




49. A permanently polarized dipole with moment $\mathbf{p}$ is at an angle $\theta$ to a uniform electric field $\mathbf{E}$.

(a) What is the torque $T$ on the dipole?

(b) How much incremental work $dW$ is necessary to turn the dipole by a small angle $d\theta$? What is the total work required to move the dipole from $\theta = 0$ to any value of $\theta$? (**Hint:** $dW = T\,d\theta$.)

(c) In general, thermal agitation causes the dipoles to be distributed over all angles of $\theta$. Boltzmann statistics tell us that the number density of dipoles having energy $W$ are

$$n = n_0\, e^{-W/kT}$$

where $n_0$ is a constant. If the total number of dipoles within a sphere of radius $R$ is $N$, what is $n_0$? (**Hint:** Let $u = (pE/kT)\cos\theta$.)

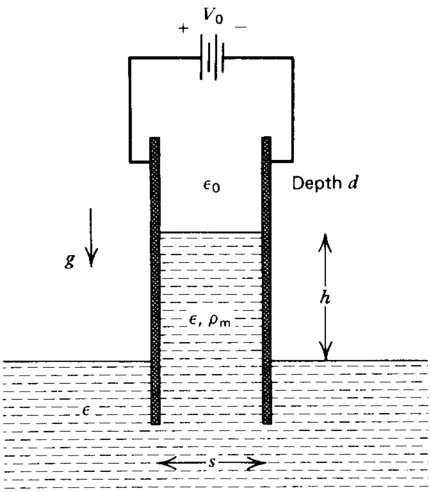
(d) Consider a shell of dipoles within the range of $\theta$ to $\theta + d\theta$. What is the magnitude and direction of the net polarization due to this shell?

(e) What is the total polarization integrated over $\theta$? This is known as the Langevin equation. (**Hint:** $\int ue^u\,du = (u-1)e^u$.)

(f) Even with a large field of $E \approx 10^6$ v/m with a dipole composed of one proton and electron a distance of 10 Å ($10^{-9}$ m) apart, show that at room temperature the quantity $(pE/kT)$ is much less than unity and expand the results of (e). (**Hint:** It will be necessary to expand (e) up to third order in $(pE/kT)$.

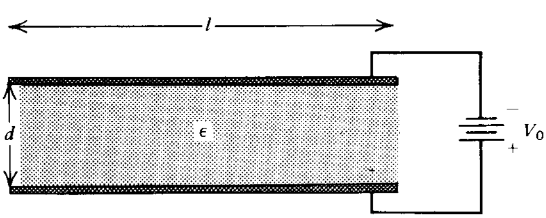
(g) In this limit what is the orientational polarizability?

50. A pair of parallel plate electrodes a distance $s$ apart at a voltage difference $V_0$ is dipped into a dielectric fluid of permittivity $\varepsilon$. The fluid has a mass density $\rho_m$ and gravity acts downward. How high does the liquid rise between the plates?




51. Parallel plate electrodes at voltage difference $V_0$ enclose an elastic dielectric with permittivity $\varepsilon$. The electric force of attraction between the electrodes is balanced by the elastic force of the dielectric.

(a) When the electrode spacing is $d$ what is the free surface charge density on the upper electrode?



(b) What is the electric force per unit area that the electrode exerts on the dielectric interface?

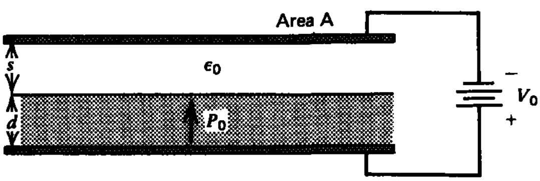
(c) The elastic restoring force per unit area is given by the relation

$$F_A = Y \ln \frac{d}{d_0}$$

where $Y$ is the modulus of elasticity and $d_0$ is the unstressed ($V_0 = 0$) thickness of the dielectric. Write a transcendental expression for the equilibrium thickness of the dielectric.

(d) What is the minimum equilibrium dielectric thickness and at what voltage does it occur? If a larger voltage is applied there is no equilibrium and the dielectric fractures as the electric stress overcomes the elastic restoring force. This is called the theory of electromechanical breakdown. [See K. H. Stark and C. G. Garton, Electric Strength of Irradiated Polythene, *Nature* **176** (1955) 1225–26.]

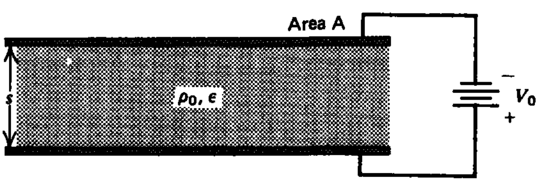
52. An electret with permanent polarization $P_0\mathbf{i}_x$ and thickness $d$ partially fills a free space capacitor. There is no surface charge on the electret free space interface.




(a) What are the electric fields in each region?

(b) What is the force on the upper electrode?

53. A uniform distribution of free charge with density $\rho_0$ is between parallel plate electrodes at potential difference $V_0$.




(a) What is the energy stored in the system?

(b) Compare the capacitance to that when $\rho_0 = 0$.

(c) What is the total force on each electrode and on the volume charge distribution?

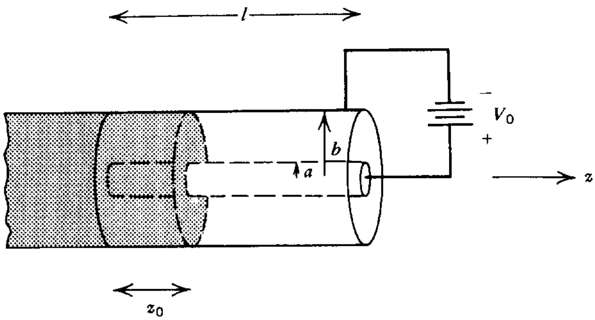
(d) What is the total force on the system?

54. Coaxial cylindrical electrodes at voltage difference $V_0$ are partially filled with a polarized material. Find the force on this

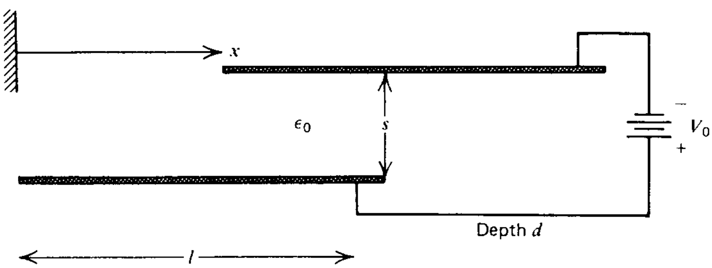
material if it is

(a) permanently polarized as $P_0\mathbf{i}_r$;

(b) linearly polarized with permittivity $\varepsilon$.




55. The upper electrode of a pair at constant potential difference $V_0$ is free to slide in the $x$ direction. What is the $x$ component of the force on the upper electrode?




56. A capacitor has a moveable part that can rotate through the angle $\theta$ so that the capacitance $C(\theta)$ depends on $\theta$.

(a) What is the torque on the moveable part?

(b) An electrostatic voltmeter consists of $N+1$ fixed pie-shaped electrodes at the same potential interspersed with $N$ plates mounted on a shaft that is free to rotate for $-\theta_0<\theta<\theta_0$. What is the capacitance as a function of $\theta$?

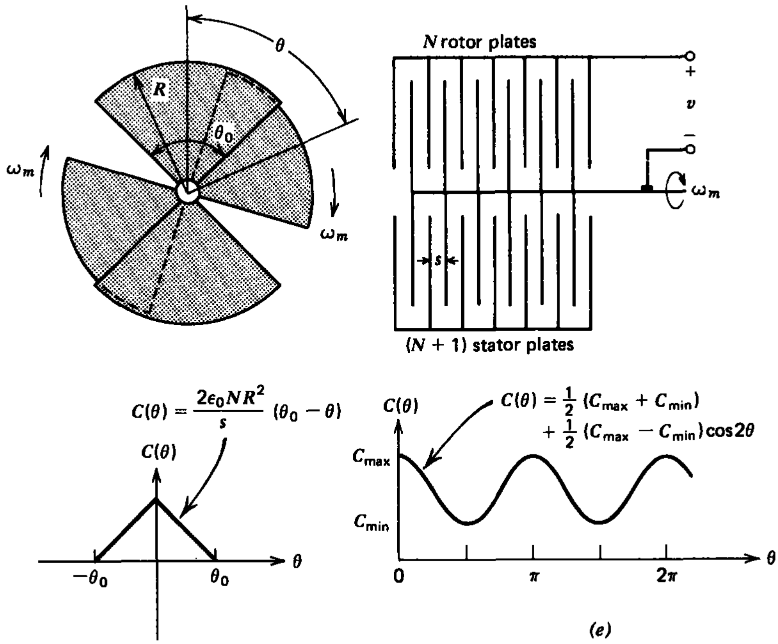
(c) A voltage $v$ is applied. What is the electric torque on the shaft?

(d) A torsional spring exerts a restoring torque on the shaft

$$T_s=-K(\theta-\theta_s)$$

where $K$ is the spring constant and $\theta_s$ is the equilibrium position of the shaft at zero voltage. What is the equilibrium position of the shaft when the voltage $v$ is applied? If a sinusoidal voltage is applied, what is the time average angular deflection $\langle\theta\rangle$?

(e) The torsional spring is removed so that the shaft is free to continuously rotate. Fringing field effects cause the




(b)

(e)

capacitance to vary smoothly between minimum and maximum values of a dc value plus a single sinusoidal spatial term

$$C(\theta)=\tfrac{1}{2}[C_{\max}+C_{\min}]+\tfrac{1}{2}[C_{\max}-C_{\min}]\cos 2\theta$$

A sinusoidal voltage $V_0 \cos \omega t$ is applied. What is the instantaneous torque on the shaft?

(f) If the shaft is rotating at constant angular speed $\omega_m$ so that

$$\theta = \omega_m t+\delta$$

where $\delta$ is the angle of the shaft at $t=0$, under what conditions is the torque in (e) a constant? **Hint:**

$$\sin 2\theta \cos^2 \omega t=\tfrac{1}{2}\sin 2\theta(1+\cos 2\omega t)$$
$$=\tfrac{1}{2}\sin 2\theta+\tfrac{1}{4}[\sin (2(\omega t+\theta))-\sin (2(\omega t-\theta))]$$

(g) A time average torque $T_0$ is required of the shaft. What is the torque angle $\delta$?

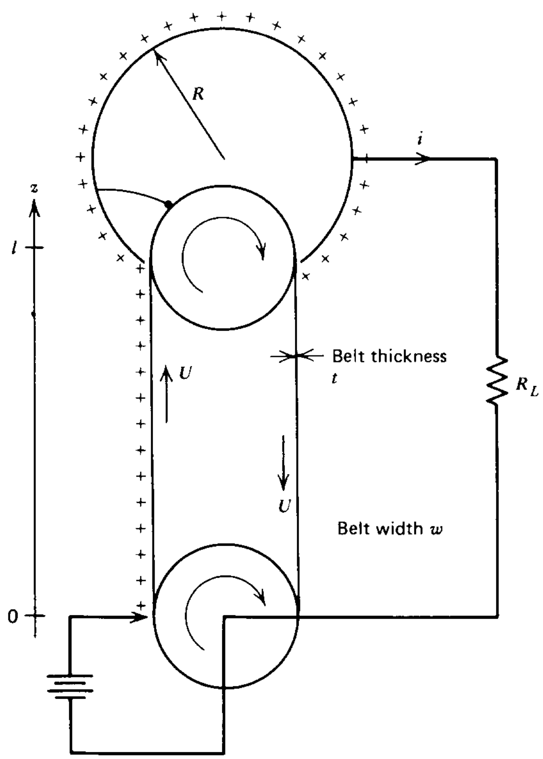
(h) What is the maximum torque that can be delivered? This is called the pull-out torque. At what angle $\delta$ does this occur?

Section 3-10

57. The belt of a Van de Graaff generator has width $w$ and moves with speed $U$ carrying a surface charge $\sigma_f$ up to the spherical dome of radius $R$.

(a) What is the time dependence of the dome voltage?

(b) Assuming that the electric potential varies linearly between the charging point and the dome, how much power as a function of time is required for the motor to rotate the belt?



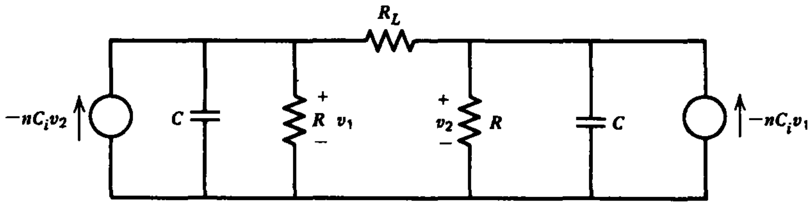

58. A Van de Graaff generator has a lossy belt with Ohmic conductivity $\sigma$ traveling at constant speed $U$. The charging point at $z=0$ maintains a constant volume charge density $\rho_0$ on the belt at $z=0$. The dome is loaded by a resistor $R_L$ to ground.

(a) Assuming only one-dimensional variations with $z$, what are the steady-state volume charge, electric field, and current density distributions on the belt?

(b) What is the steady-state dome voltage?

59. A pair of coupled electrostatic induction machines have their inducer electrodes connected through a load resistor $R_L$. In addition, each electrode has a leakage resistance $R$ to ground.

(a) For what values of $n$, the number of conductors per second passing the collector, will the machine self-excite?




(b) If $n = 10$, $C_i = 2$ pf, and $C = 10$ pf with $R_L = R$, what is the minimum value of $R$ for self-excitation?

(c) If we have three such coupled machines, what is the condition for self-excitation and what are the oscillation frequencies if $R_L = \infty$?

(d) Repeat (c) for $N$ such coupled machines with $R_L = \infty$. The last machine is connected to the first.

# chapter 4

## *electric field boundary value problems*

The electric field distribution due to external sources is disturbed by the addition of a conducting or dielectric body because the resulting induced charges also contribute to the field. The complete solution must now also satisfy boundary conditions imposed by the materials.

## 4-1 THE UNIQUENESS THEOREM

Consider a linear dielectric material where the permittivity may vary with position:

$$\mathbf{D}=\varepsilon(r)\mathbf{E}=-\varepsilon(r)\nabla V \tag{1}$$

The special case of different constant permittivity media separated by an interface has $\varepsilon(r)$ as a step function. Using (1) in Gauss's law yields

$$\nabla\cdot[\varepsilon(r)\nabla V]=-\rho_f \tag{2}$$

which reduces to Poisson's equation in regions where $\varepsilon(r)$ is a constant. Let us call $V_p$ a solution to (2).

The solution $V_L$ to the homogeneous equation

$$\nabla\cdot[\varepsilon(r)\nabla V]=0 \tag{3}$$

which reduces to Laplace's equation when $\varepsilon(r)$ is constant, can be added to $V_p$ and still satisfy (2) because (2) is linear in the potential:

$$\nabla\cdot[\varepsilon(r)\nabla(V_p+V_L)]=\nabla\cdot[\varepsilon(r)\nabla V_p]+\underbrace{\nabla\cdot[\varepsilon(r)\nabla V_L]}_{0}=-\rho_f \tag{4}$$

Any linear physical problem must only have one solution yet (3) and thus (2) have many solutions. We need to find what boundary conditions are necessary to uniquely specify this solution. Our method is to consider two different solutions $V_1$ and $V_2$ for the same charge distribution

$$\nabla\cdot(\varepsilon\nabla V_1)=-\rho_f,\qquad \nabla\cdot(\varepsilon\nabla V_2)=-\rho_f \tag{5}$$

so that we can determine what boundary conditions force these solutions to be identical, $V_1=V_2$.

The difference of these two solutions $V_T = V_1 - V_2$ obeys the homogeneous equation

$$\nabla \cdot (\varepsilon \nabla V_T) = 0 \tag{6}$$

We examine the vector expansion

$$\nabla \cdot (\varepsilon V_T \nabla V_T) = V_T \underbrace{\nabla \cdot (\varepsilon \nabla V_T)}_{0} + \varepsilon \nabla V_T \cdot \nabla V_T = \varepsilon |\nabla V_T|^2 \tag{7}$$

noting that the first term in the expansion is zero from (6) and that the second term is never negative.

We now integrate (7) over the volume of interest V, which may be of infinite extent and thus include all space

$$\int_V \nabla \cdot (\varepsilon V_T \nabla V_T)\, dV = \oint_S \varepsilon V_T \nabla V_T \cdot \mathbf{dS} = \int_V \varepsilon |\nabla V_T|^2\, dV \tag{8}$$

The volume integral is converted to a surface integral over the surface bounding the region using the divergence theorem. Since the integrand in the last volume integral of (8) is never negative, the integral itself can only be zero if $V_T$ is zero at every point in the volume making the solution unique ($V_T = 0 \Rightarrow V_1 = V_2$). To force the volume integral to be zero, the surface integral term in (8) must be zero. This requires that on the surface $S$ the two solutions must have the same value ($V_1 = V_2$) or their normal derivatives must be equal $[\nabla V_1 \cdot \mathbf{n} = \nabla V_2 \cdot \mathbf{n}]$. This last condition is equivalent to requiring that the normal components of the electric fields be equal ($\mathbf{E} = -\nabla V$).

Thus, a problem is uniquely posed when in addition to giving the charge distribution, the potential or the normal component of the electric field on the bounding surface surrounding the volume is specified. The bounding surface can be taken in sections with some sections having the potential specified and other sections having the normal field component specified.

If a particular solution satisfies (2) but it does not satisfy the boundary conditions, additional homogeneous solutions where $\rho_f = 0$, must be added so that the boundary conditions are met. No matter how a solution is obtained, even if guessed, if it satisfies (2) and all the boundary conditions, it is the only solution.

## 4-2 BOUNDARY VALUE PROBLEMS IN CARTESIAN GEOMETRIES

For most of the problems treated in Chapters 2 and 3 we restricted ourselves to one-dimensional problems where the electric field points in a single direction and only depends on that coördinate. For many cases, the volume is free of charge so that the system is described by Laplace's equation. Surface charge is present only on interfacial boundaries separating dissimilar conducting materials. We now consider such volume charge-free problems with two- and three dimensional variations.

### 4-2-1 Separation of Variables

Let us assume that within a region of space of constant permittivity with no volume charge, that solutions do not depend on the $z$ coordinate. Then Laplace's equation reduces to

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0 \tag{1}$$

We try a solution that is a product of a function only of the $x$ coordinate and a function only of $y$:

$$V(x, y)=X(x)\, Y(y) \tag{2}$$

This assumed solution is often convenient to use if the system boundaries lay in constant $x$ or constant $y$ planes. Then along a boundary, one of the functions in (2) is constant. When (2) is substituted into (1) we have

$$Y\frac{d^2X}{dx^2}+X\frac{d^2Y}{dy^2}=0 \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0 \tag{3}$$

where the partial derivatives become total derivatives because each function only depends on a single coordinate. The second relation is obtained by dividing through by $XY$ so that the first term is only a function of $x$ while the second is only a function of $y$.

The only way the sum of these two terms can be zero for all values of $x$ and $y$ is if each term is separately equal to a constant so that (3) separates into two equations,

$$\frac{1}{X}\frac{d^2X}{dx^2}=k^2, \qquad \frac{1}{Y}\frac{d^2Y}{dy^2}=-k^2 \tag{4}$$

where $k^2$ is called the separation constant and in general can be a complex number. These equations can then be rewritten as the ordinary differential equations:

$$\frac{d^2X}{dx^2}-k^2X=0, \qquad \frac{d^2Y}{dy^2}+k^2Y=0 \tag{5}$$

### 4-2-2 Zero Separation Constant Solutions

When the separation constant is zero ($k^2 = 0$) the solutions to (5) are

$$X = a_1 x + b_1, \qquad Y = c_1 y + d_1 \tag{6}$$

where $a_1$, $b_1$, $c_1$, and $d_1$ are constants. The potential is given by the product of these terms which is of the form

$$V = a_2 + b_2 x + c_2 y + d_2 xy \tag{7}$$

The linear and constant terms we have seen before, as the potential distribution within a parallel plate capacitor with no fringing, so that the electric field is uniform. The last term we have not seen previously.

**(a) Hyperbolic Electrodes**

A hyperbolically shaped electrode whose surface shape obeys the equation $xy = ab$ is at potential $V_0$ and is placed above a grounded right-angle corner as in Figure 4-1. The

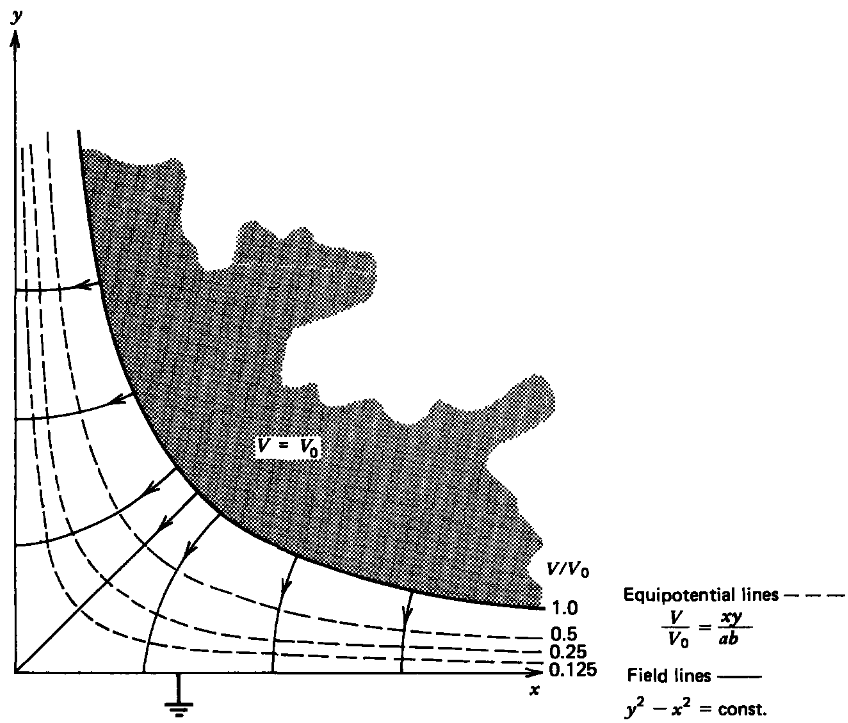



Figure 4-1 The equipotential and field lines for a hyperbolically shaped electrode at potential $V_0$ above a right-angle conducting corner are orthogonal hyperbolas.

boundary conditions are

$$V(x=0)=0, \qquad V(y=0)=0, \qquad V(xy=ab)=V_0 \tag{8}$$

so that the solution can be obtained from (7) as

$$V(x,y)=V_0xy/(ab) \tag{9}$$

The electric field is then

$$\mathbf{E}=-\nabla V=-\frac{V_0}{ab}[y\mathbf{i}_x+x\mathbf{i}_y] \tag{10}$$

The field lines drawn in Figure 4-1 are the perpendicular family of hyperbolas to the equipotential hyperbolas in (9):

$$\frac{dy}{dx}=\frac{E_y}{E_x}=\frac{x}{y}\Rightarrow y^2-x^2=\text{const} \tag{11}$$

**(b) Resistor in an Open Box**

A resistive medium is contained between two electrodes, one of which extends above and is bent through a right-angle corner as in Figure 4-2. We try zero separation constant

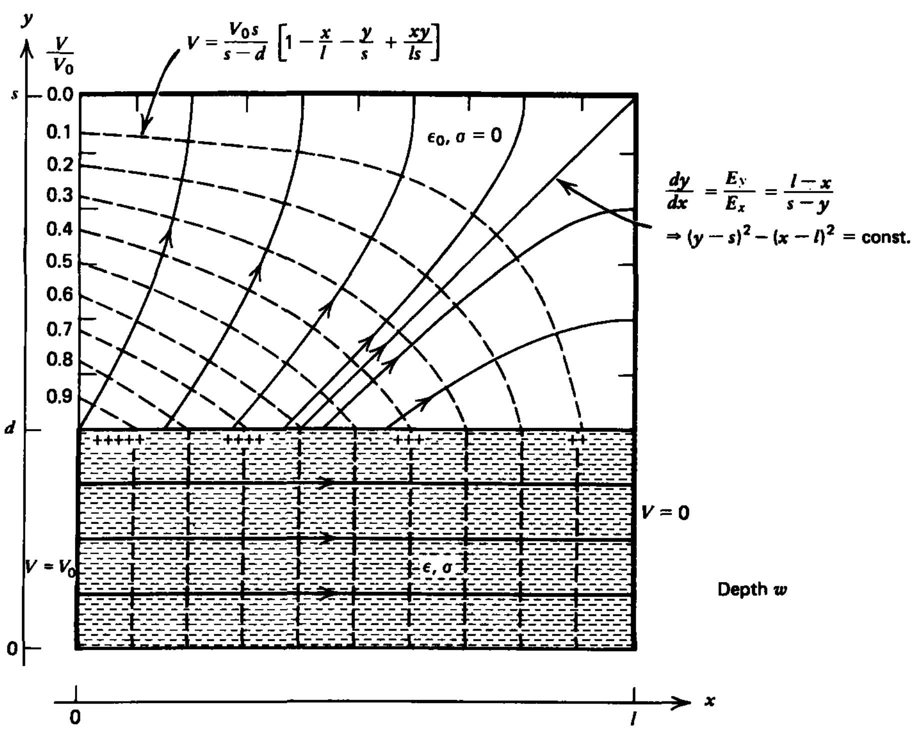



Figure 4-2 A resistive medium partially fills an open conducting box.

solutions given by (7) in each region enclosed by the electrodes:

$$V=\begin{cases} a_1+b_1x+c_1y+d_1xy, & o\le y\le d \\ a_2+b_2x+c_2y+d_2xy, & d\le y\le s \end{cases} \tag{12}$$

With the potential constrained on the electrodes and being continuous across the interface, the boundary conditions are

$$V(x=0)=V_0=a_1+c_1y \Rightarrow a_1=V_0, \quad c_1=0 \qquad (0\le y\le d)$$

$$V(x=l)=0=\begin{cases} \underbrace{a_1}_{V_0}+b_1l+\cancelto{0}{c_1}y+d_1ly \Rightarrow b_1=-V_0/l, \quad d_1=0 \\ \qquad\qquad\qquad\qquad (0\le y\le d) \\ a_2+b_2l+c_2y+d_2ly \Rightarrow a_2+b_2l=0, \quad c_2+d_2l=0 \\ \qquad\qquad\qquad\qquad (d\le y\le s) \end{cases}$$

$$V(y=s)=0=a_2+b_2x+c_2s+d_2xs \Rightarrow a_2+c_2s=0, \quad b_2+d_2s=0$$

$$\begin{aligned} V(y=d_+)=V(y=d_-)&=a_1+b_1x+\cancelto{0}{c_1}\,d+\cancelto{0}{d_1}\,xd \\ &=a_2+b_2x+c_2d+d_2xd \end{aligned} \tag{13}$$

$$\Rightarrow a_1=V_0=a_2+c_2d, \quad b_1=-V_0/l=b_2+d_2d$$

so that the constants in (12) are

$$\begin{aligned} &a_1=V_0, \quad b_1=-V_0/l, \quad c_1=0, \quad d_1=0 \\ &a_2=\frac{V_0}{(1-d/s)}, \quad b_2=-\frac{V_0}{l(1-d/s)}, \\ &c_2=-\frac{V_0}{s(1-d/s)}, \quad d_2=\frac{V_0}{ls(1-d/s)} \end{aligned} \tag{14}$$

The potential of (12) is then

$$V=\begin{cases} V_0(1-x/l), & 0\le y\le d \\ \dfrac{V_0s}{s-d}\left(1-\dfrac{x}{l}-\dfrac{y}{s}+\dfrac{xy}{ls}\right), & d\le y\le s \end{cases} \tag{15}$$

with associated electric field

$$\mathbf{E}=-\nabla V=\begin{cases} \dfrac{V_0}{l}\mathbf{i}_x, & 0\le y\le d \\ \dfrac{V_0s}{s-d}\left[\dfrac{\mathbf{i}_x}{l}\left(1-\dfrac{y}{s}\right)+\dfrac{\mathbf{i}_y}{s}\left(1-\dfrac{x}{l}\right)\right], & d<y<s \end{cases} \tag{16}$$

Note that in the dc steady state, the conservation of charge boundary condition of Section 3-3-5 requires that no current cross the interfaces at $y=0$ and $y=d$ because of the surrounding zero conductivity regions. The current and, thus, the electric field within the resistive medium must be purely tangential to the interfaces, $E_y(y=d_-)=E_y(y=0_+)=0$. The surface charge density on the interface at $y=d$ is then due only to the normal electric field above, as below, the field is purely tangential:

$$\sigma_f(y=d)=\varepsilon_0 E_y(y=d_+)-\cancel{\varepsilon E_y}^{\,0}(y=d_-)=\frac{\varepsilon_0 V_0}{s-d}\left(1-\frac{x}{l}\right) \quad (17)$$

The interfacial shear force is then

$$F_x=\int_0^l \sigma_f E_x(y=d)w\,dx=\frac{\varepsilon_0 V_0^2}{2(s-d)}w \quad (18)$$

If the resistive material is liquid, this shear force can be used to pump the fluid.*

### 4-2-3 Nonzero Separation Constant Solutions

Further solutions to (5) with nonzero separation constant ($k^2\neq 0$) are

$$\begin{aligned} X&=A_1\sinh kx+A_2\cosh kx=B_1\,e^{kx}+B_2\,e^{-kx}\\ Y&=C_1\sin ky+C_2\cos ky=D_1\,e^{jky}+D_2\,e^{-jky}\end{aligned} \quad (19)$$

When $k$ is real, the solutions of $X$ are hyperbolic or equivalently exponential, as drawn in Figure 4-3, while those of $Y$ are trigonometric. If $k$ is pure imaginary, then $X$ becomes trigonometric and $Y$ is hyperbolic (or exponential).

The solution to the potential is then given by the product of $X$ and $Y$:

$$\begin{aligned} V=E_1\sin ky\sinh kx+E_2\sin ky\cosh kx\\ +E_3\cos ky\sinh kx+E_4\cos ky\cosh kx\end{aligned} \quad (20)$$

or equivalently

$$V=F_1\sin ky\,e^{kx}+F_2\sin ky\,e^{-kx}+F_3\cos ky\,e^{kx}+F_4\cos ky\,e^{-kx} \quad (21)$$

We can always add the solutions of (7) or any other Laplacian solutions to (20) and (21) to obtain a more general

* *See J. R. Melcher and G. I. Taylor, Electrohydrodynamics: A Review of the Role of Interfacial Shear Stresses,* Annual Rev. Fluid Mech., *Vol. 1, Annual Reviews, Inc., Palo Alto, Calif., 1969, ed. by Sears and Van Dyke, pp. 111–146. See also J. R. Melcher, "Electric Fields and Moving Media", film produced for the National Committee on Electrical Engineering Films by the Educational Development Center, 39 Chapel St., Newton, Mass. 02160. This film is described in* IEEE Trans. Education ***E-17***, *(1974) pp. 100–110.*

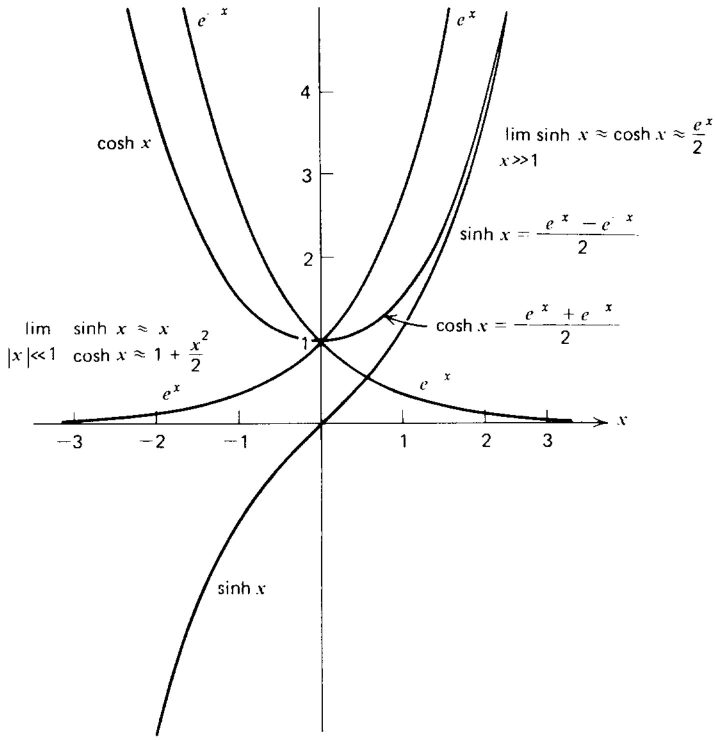



Figure 4-3 The exponential and hyperbolic functions for positive and negative arguments.

solution because Laplace's equation is linear. The values of the coefficients and of $k$ are determined by boundary conditions.

When regions of space are of infinite extent in the $x$ direction, it is often convenient to use the exponential solutions in (21) as it is obvious which solutions decay as $x$ approaches $\pm\infty$. For regions of finite extent, it is usually more convenient to use the hyperbolic expressions of (20). A general property of Laplace solutions are that they are oscillatory in one direction and decay in the perpendicular direction.

### 4-2-4 Spatially Periodic Excitation

A sheet in the $x=0$ plane has the imposed periodic potential, $V=V_0 \sin ay$ shown in Figure 4-4. In order to meet this boundary condition we use the solution of (21) with $k=a$. The potential must remain finite far away from the source so

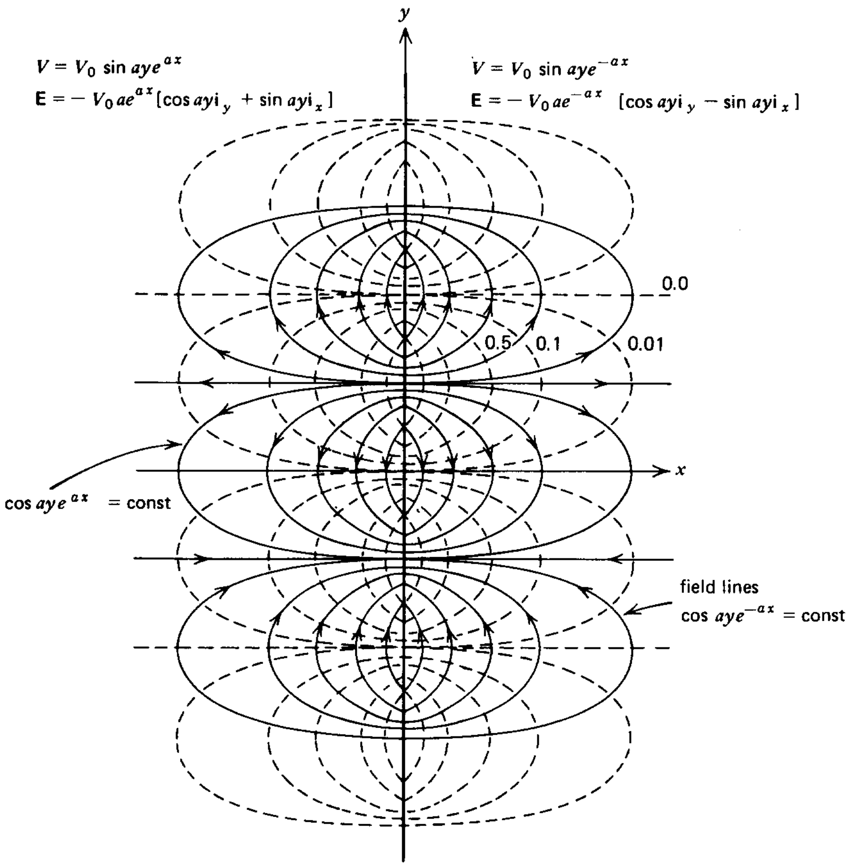



Figure 4-4 The potential and electric field decay away from an infinite sheet with imposed spatially periodic voltage. The field lines emanate from positive surface charge on the sheet and terminate on negative surface charge.

we write the solution separately for positive and negative $x$ as

$$V = \begin{cases} V_0 \sin ay\, e^{-ax}, & x \geq 0 \\ V_0 \sin ay\, e^{ax}, & x \leq 0 \end{cases} \tag{22}$$

where we picked the amplitude coefficients to be continuous and match the excitation at $x = 0$. The electric field is then

$$\mathbf{E} = -\nabla V = \begin{cases} -V_0 a\, e^{-ax}[\cos ay\mathbf{i}_y - \sin ay\mathbf{i}_x], & x > 0 \\ -V_0 a\, e^{ax}[\cos ay\mathbf{i}_y + \sin ay\mathbf{i}_x], & x < 0 \end{cases} \tag{23}$$

The surface charge density on the sheet is given by the discontinuity in normal component of $\mathbf{D}$ across the sheet:

$$\begin{aligned} \sigma_f(x=0) &= \varepsilon[E_x(x=0_+) - E_x(x=0_-)] \\ &= 2\varepsilon V_0 a \sin ay \end{aligned} \tag{24}$$

The field lines drawn in Figure 4-4 obey the equation

$$\frac{dy}{dx}=\frac{E_y}{E_x}=\mp\cot ay\Rightarrow\cos ay\, e^{\mp ax}=\text{const}\quad\begin{cases}x>0\\x<0\end{cases}\tag{25}$$

### 4-2-5 Rectangular Harmonics

When excitations are not sinusoidally periodic in space, they can be made so by expressing them in terms of a trigonometric Fourier series. Any periodic function of $y$ can be expressed as an infinite sum of sinusoidal terms as

$$f(y)=\tfrac{1}{2}b_0+\sum_{n=1}^{\infty}\left(a_n\sin\frac{2n\pi y}{\lambda}+b_n\cos\frac{2n\pi y}{\lambda}\right)\tag{26}$$

where $\lambda$ is the fundamental period of $f(y)$.

The Fourier coefficients $a_n$ are obtained by multiplying both sides of the equation by $\sin(2p\pi y/\lambda)$ and integrating over a period. Since the parameter $p$ is independent of the index $n$, we may bring the term inside the summation on the right hand side. Because the trigonometric functions are orthogonal to one another, they integrate to zero except when the function multiplies itself:

$$\begin{aligned}&\int_0^{\lambda}\sin\frac{2p\pi y}{\lambda}\sin\frac{2n\pi y}{\lambda}\,dy=\begin{cases}0, & p\neq n\\ \lambda/2, & p=n\end{cases}\\ &\int_0^{\lambda}\sin\frac{2p\pi y}{\lambda}\cos\frac{2n\pi y}{\lambda}\,dy=0\end{aligned}\tag{27}$$

Every term in the series for $n\neq p$ integrates to zero. Only the term for $n=p$ is nonzero so that

$$a_p=\frac{2}{\lambda}\int_0^{\lambda}f(y)\sin\frac{2p\pi y}{\lambda}\,dy\tag{28}$$

To obtain the coefficients $b_n$, we similarly multiply by $\cos(2p\pi y/\lambda)$ and integrate over a period:

$$b_p=\frac{2}{\lambda}\int_0^{\lambda}f(y)\cos\frac{2p\pi y}{\lambda}\,dy\tag{29}$$

Consider the conducting rectangular box of infinite extent in the $x$ and $z$ directions and of width $d$ in the $y$ direction shown in Figure 4-5. The potential along the $x=0$ edge is $V_0$ while all other surfaces are grounded at zero potential. Any periodic function can be used for $f(y)$ if over the interval $0\leq y\leq d, f(y)$ has the properties

$$f(y)=V_0,\ 0<y<d;\ f(y=0)=f(y=d)=0\tag{30}$$

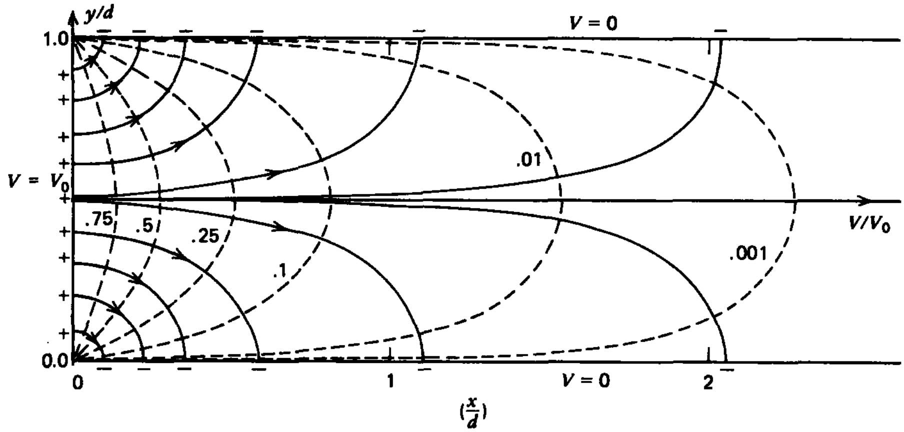



Figure 4-5 An open conducting box of infinite extent in the $x$ and $z$ directions and of finite width $d$ in the $y$ direction, has zero potential on all surfaces except the closed end at $x=0$, where $V=V_0$.

In particular, we choose the periodic square wave function with $\lambda=2d$ shown in Figure 4-6 so that performing the integrations in (28) and (29) yields

$$a_p=-\frac{2V_0}{p\pi}(\cos p\pi-1)$$
$$=\begin{cases}0, & p \text{ even}\\ 4V_0/p\pi, & p \text{ odd}\end{cases} \tag{31}$$
$$b_p=0$$

Thus the constant potential at $x=0$ can be written as the Fourier sine series

$$V(x=0)=V_0=\frac{4V_0}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{\sin(n\pi y/d)}{n} \tag{32}$$

In Figure 4-6 we plot various partial sums of the Fourier series to show that as the number of terms taken becomes large, the series approaches the constant value $V_0$ except for the Gibbs overshoot of about 18% at $y=0$ and $y=d$ where the function is discontinuous.

The advantage in writing $V_0$ in a Fourier sine series is that each term in the series has a similar solution as found in (22) where the separation constant for each term is $k_n=n\pi/d$ with associated amplitude $4V_0/(n\pi)$.

The solution is only nonzero for $x>0$ so we immediately write down the total potential solution as

$$V(x,y)=\frac{4V_0}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{1}{n}\sin\frac{n\pi y}{d}e^{-n\pi x/d} \tag{33}$$

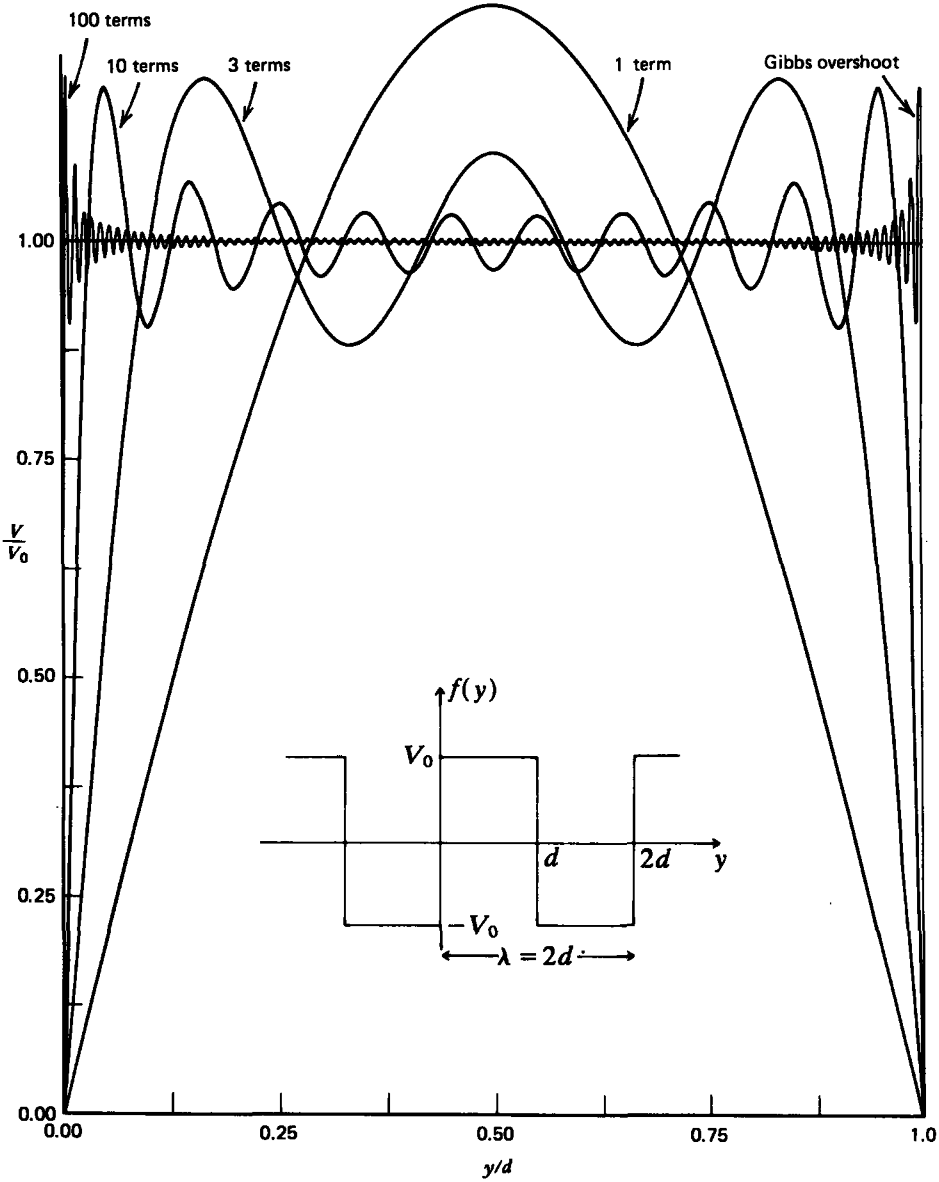



**Figure 4-6** Fourier series expansion of the imposed constant potential along the $x = 0$ edge in Figure 4-5 for various partial sums. As the number of terms increases, the series approaches a constant except at the boundaries where the discontinuity in potential gives rise to the Gibbs phenomenon of an 18% overshoot with narrow width.

The electric field is then

$$\mathbf{E} = -\nabla V = -\frac{4V_0}{d} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left( -\sin\frac{n\pi y}{d}\mathbf{i}_x + \cos\frac{n\pi y}{d}\mathbf{i}_y \right) e^{-n\pi x/d} \tag{34}$$

The field and equipotential lines are sketched in Figure 4-5. Note that for $x \gg d$, the solution is dominated by the first harmonic. Far from a source, Laplacian solutions are insensitive to the details of the source geometry.

### 4-2-6 Three-Dimensional Solutions

If the potential depends on the three coordinates $(x, y, z)$, we generalize our approach by trying a product solution of the form

$$V(x, y, z) = X(x)\, Y(y)\, Z(z) \tag{35}$$

which, when substituted into Laplace's equation, yields after division through by $XYZ$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \tag{36}$$

three terms each wholly a function of a single coordinate so that each term again must separately equal a constant:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2, \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2, \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = k_z^2 = k_x^2 + k_y^2 \tag{37}$$

We change the sign of the separation constant for the $z$ dependence as the sum of separation constants must be zero. The solutions for nonzero separation constants are

$$\begin{aligned} X &= A_1 \sin k_x x + A_2 \cos k_x x \\ Y &= B_1 \sin k_y y + B_2 \cos k_y y \\ Z &= C_1 \sinh k_z z + C_2 \cosh k_z z = D_1\, e^{k_z z} + D_2\, e^{-k_z z} \end{aligned} \tag{38}$$

The solutions are written as if $k_x$, $k_y$, and $k_z$ are real so that the $x$ and $y$ dependence is trigonometric while the $z$ dependence is hyperbolic or equivalently exponential. However, $k_x$, $k_y$, or $k_z$ may be imaginary converting hyperbolic functions to trigonometric and vice versa. Because the squares of the separation constants must sum to zero at least one of the solutions in (38) must be trigonometric and one must be hyperbolic. The remaining solution may be either trigonometric or hyperbolic depending on the boundary conditions. If the separation constants are all zero, in addition to the solutions of (6) we have the similar addition

$$Z = e_1 z + f_1 \tag{39}$$

## 4-3 SEPARATION OF VARIABLES IN CYLINDRICAL GEOMETRY

Product solutions to Laplace's equation in cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0 \tag{1}$$

also separate into solvable ordinary differential equations.

### 4-3-1 Polar Solutions

If the system geometry does not vary with z, we try a solution that is a product of functions which only depend on the radius r and angle $\phi$:

$$V(r,\phi)=R(r)\Phi(\phi) \tag{2}$$

which when substituted into (1) yields

$$\frac{\Phi}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+\frac{R}{r^2}\frac{d^2\Phi}{d\phi^2}=0 \tag{3}$$

This assumed solution is convenient when boundaries lay at a constant angle of $\phi$ or have a constant radius, as one of the functions in (2) is then constant along the boundary.

For (3) to separate, each term must only be a function of a single variable, so we multiply through by $r^2/R\Phi$ and set each term equal to a constant, which we write as $n^2$:

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)=n^2, \qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-n^2 \tag{4}$$

The solution for $\Phi$ is easily solved as

$$\Phi=\begin{cases}A_1\sin n\phi+A_2\cos n\phi, & n\neq 0\\ B_1\phi+B_2, & n=0\end{cases} \tag{5}$$

The solution for the radial dependence is not as obvious. However, if we can find two independent solutions by any means, including guessing, the total solution is uniquely given as a linear combination of the two solutions. So, let us try a power-law solution of the form

$$R=Ar^p \tag{6}$$

which when substituted into (4) yields

$$p^2=n^2\Rightarrow p=\pm n \tag{7}$$

For $n \neq 0$, (7) gives us two independent solutions. When $n = 0$ we refer back to (4) to solve

$$r\frac{dR}{dr} = \text{const} \Rightarrow R = D_1 \ln r + D_2 \tag{8}$$

so that the solutions are

$$R = \begin{cases} C_1 r^n + C_2 r^{-n}, & n \neq 0 \\ D_1 \ln r + D_2, & n = 0 \end{cases} \tag{9}$$

We recognize the $n = 0$ solution for the radial dependence as the potential due to a line charge. The $n = 0$ solution for the $\phi$ dependence shows that the potential increases linearly with angle. Generally $n$ can be any complex number, although in usual situations where the domain is periodic and extends over the whole range $0 \leq \phi \leq 2\pi$, the potential at $\phi = 2\pi$ must equal that at $\phi = 0$ since they are the same point. This requires that $n$ be an integer.

## EXAMPLE 4-1 SLANTED CONDUCTING PLANES

Two planes of infinite extent in the $z$ direction at an angle $\alpha$ to one another, as shown in Figure 4-7, are at a potential difference $v$. The planes do not intersect but come sufficiently close to one another that fringing fields at the electrode ends may be neglected. The electrodes extend from $r = a$ to $r = b$. What is the approximate capacitance per unit length of the structure?

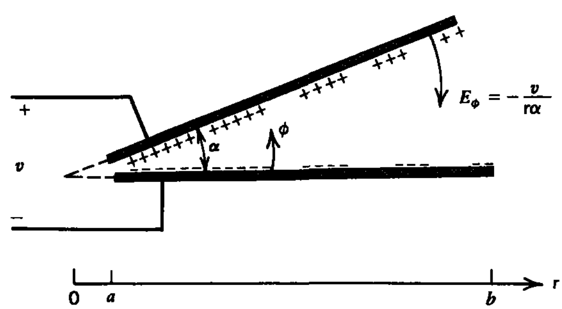



Figure 4-7 Two conducting planes at angle $\alpha$ stressed by a voltage $v$ have a $\phi$-directed electric field.

**SOLUTION**

---

We try the $n=0$ solution of (5) with no radial dependence as

$$V=B_1\phi+B_2$$

The boundary conditions impose the constraints

$$V(\phi=0)=0, \qquad V(\phi=\alpha)=v \Rightarrow V=v\phi/\alpha$$

The electric field is

$$E_\phi=-\frac{1}{r}\frac{dV}{d\phi}=-\frac{v}{r\alpha}$$

The surface charge density on the upper electrode is then

$$\sigma_f(\phi=\alpha)=-\varepsilon E_\phi(\phi=\alpha)=\frac{\varepsilon v}{r\alpha}$$

with total charge per unit length

$$\lambda(\phi=\alpha)=\int_{r=a}^{b}\sigma_f(\phi=\alpha)\,dr=\frac{\varepsilon v}{\alpha}\ln\frac{b}{a}$$

so that the capacitance per unit length is

$$C=\frac{\lambda}{v}=\frac{\varepsilon\ln(b/a)}{\alpha}$$

## 4-3-2 Cylinder in a Uniform Electric Field

**(a) Field Solutions**

An infinitely long cylinder of radius $a$, permittivity $\varepsilon_2$, and Ohmic conductivity $\sigma_2$ is placed within an infinite medium of permittivity $\varepsilon_1$ and conductivity $\sigma_1$. A uniform electric field at infinity $\mathbf{E}=E_0\mathbf{i}_x$ is suddenly turned on at $t=0$. This problem is analogous to the series lossy capacitor treated in Section 3-6-3. As there, we will similarly find that:

(i) At $t=0$ the solution is the same as for two lossless dielectrics, independent of the conductivities, with no interfacial surface charge, described by the boundary condition

$$\sigma_f(r=a)=D_r(r=a_+)-D_r(r=a_-)=0$$
$$\Rightarrow \varepsilon_1E_r(r=a_+)=\varepsilon_2E_r(r=a_-) \quad (10)$$

(ii) As $t\to\infty$, the steady-state solution depends only on the conductivities, with continuity of normal current at the cylinder interface,

$$J_r(r=a_+)=J_r(r=a_-)\Rightarrow \sigma_1 E_r(r=a_+)=\sigma_2 E_r(r=a_-) \tag{11}$$

(iii) The time constant describing the transition from the initial to steady-state solutions will depend on some weighted average of the ratio of permittivities to conductivities.

To solve the general transient problem we must find the potential both inside and outside the cylinder, joining the solutions in each region via the boundary conditions at $r=a$.

Trying the nonzero $n$ solutions of (5) and (9), $n$ must be an integer as the potential at $\phi=0$ and $\phi=2\pi$ must be equal, since they are the same point. For the most general case, an infinite series of terms is necessary, superposing solutions with $n=1,2,3,4,\cdots$. However, because of the form of the uniform electric field applied at infinity, expressed in cylindrical coordinates as

$$\mathbf{E}(r\to\infty)=E_0\mathbf{i}_x=E_0[\mathbf{i}_r\cos\phi-\mathbf{i}_\phi\sin\phi] \tag{12}$$

we can meet all the boundary conditions using only the $n=1$ solution.

Keeping the solution finite at $r=0$, we try solutions of the form

$$V(r,\phi)=\begin{cases} A(t)r\cos\phi, & r\le a \\ [B(t)r+C(t)/r]\cos\phi, & r\ge a \end{cases} \tag{13}$$

with associated electric field

$$\mathbf{E}=-\nabla V=\begin{cases} -A(t)[\cos\phi\,\mathbf{i}_r-\sin\phi\,\mathbf{i}_\phi]=-A(t)\mathbf{i}_x, & r<a \\ -[B(t)-C(t)/r^2]\cos\phi\,\mathbf{i}_r & \\ \quad +[B(t)+C(t)/r^2]\sin\phi\,\mathbf{i}_\phi, & r>a \end{cases} \tag{14}$$

We do not consider the $\sin\phi$ solution of (5) in (13) because at infinity the electric field would have to be $y$ directed:

$$V=Dr\sin\phi\Rightarrow\mathbf{E}=-\nabla V=-D[\mathbf{i}_r\sin\phi+\mathbf{i}_\phi\cos\phi]=-D\mathbf{i}_y \tag{15}$$

The electric field within the cylinder is $x$ directed. The solution outside is in part due to the imposed $x$-directed uniform field, so that as $r\to\infty$ the field of (14) must approach (12), requiring that $B(t)=-E_0$. The remaining contribution to the external field is equivalent to a two-dimensional line dipole (see Problem 3.1), with dipole moment per unit length:

$$p_x=\lambda d=2\pi\varepsilon C(t) \tag{16}$$

The other time-dependent amplitudes $A(t)$ and $C(t)$ are found from the following additional boundary conditions:

(i) the potential is continuous at $r=a$, which is the same as requiring continuity of the tangential component of $\mathbf{E}$:

$$V(r=a_+)=V(r=a_-)\Rightarrow E_\phi(r=a_-)=E_\phi(r=a_+)$$
$$\Rightarrow Aa=Ba+C/a \tag{17}$$

(ii) charge must be conserved on the interface:

$$J_r(r=a_+)-J_r(r=a_-)+\frac{\partial\sigma_f}{\partial t}=0$$
$$\Rightarrow \sigma_1 E_r(r=a_+)-\sigma_2 E_r(r=a_-)$$
$$+\frac{\partial}{\partial t}[\varepsilon_1 E_r(r=a_+)-\varepsilon_2 E_r(r=a_-)]=0 \tag{18}$$

In the steady state, (18) reduces to (11) for the continuity of normal current, while for $t=0$ the time derivative must be noninfinite so $\sigma_f$ is continuous and thus zero as given by (10).

Using (17) in (18) we obtain a single equation in $C(t)$:

$$\frac{dC}{dt}+\left(\frac{\sigma_1+\sigma_2}{\varepsilon_1+\varepsilon_2}\right)C=\frac{-a^2}{\varepsilon_1+\varepsilon_2}\left(E_0(\sigma_1-\sigma_2)+(\varepsilon_1-\varepsilon_2)\frac{dE_0}{dt}\right) \tag{19}$$

Since $E_0$ is a step function in time, the last term on the right-hand side is an impulse function, which imposes the initial condition

$$C(t=0)=-a^2\frac{(\varepsilon_1-\varepsilon_2)}{\varepsilon_1+\varepsilon_2}E_0 \tag{20}$$

so that the total solution to (19) is

$$C(t)=a^2E_0\left(\frac{\sigma_1-\sigma_2}{\sigma_1+\sigma_2}+\frac{2(\sigma_1\varepsilon_2-\sigma_2\varepsilon_1)}{(\sigma_1+\sigma_2)(\varepsilon_1+\varepsilon_2)}e^{-t/\tau}\right),\quad \tau=\frac{\varepsilon_1+\varepsilon_2}{\sigma_1+\sigma_2} \tag{21}$$

The interfacial surface charge is

$$\sigma_f(r=a,t)=\varepsilon_1 E_r(r=a_+)-\varepsilon_2 E_r(r=a_-)$$
$$=\left[-\varepsilon_1\left(B-\frac{C}{a^2}\right)+\varepsilon_2 A\right]\cos\phi$$
$$=\left[(\varepsilon_1-\varepsilon_2)E_0+(\varepsilon_1+\varepsilon_2)\frac{C}{a^2}\right]\cos\phi$$
$$=\frac{2(\sigma_2\varepsilon_1-\sigma_1\varepsilon_2)}{\sigma_1+\sigma_2}E_0[1-e^{-t/\tau}]\cos\phi \tag{22}$$

The upper part of the cylinder $(-\pi/2 \leq \phi \leq \pi/2)$ is charged of one sign while the lower half $(\pi/2 \leq \phi \leq \frac{3}{2}\pi)$ is charged with the opposite sign, the net charge on the cylinder being zero. The cylinder is uncharged at each point on its surface if the relaxation times in each medium are the same, $\varepsilon_1/\sigma_1 = \varepsilon_2/\sigma_2$

The solution for the electric field at $t=0$ is

$$\mathbf{E}(t=0) = \begin{cases} \dfrac{2\varepsilon_1 E_0}{\varepsilon_1+\varepsilon_2}[\cos\phi\mathbf{i}_r - \sin\phi\mathbf{i}_\phi] = \dfrac{2\varepsilon_1 E_0}{\varepsilon_1+\varepsilon_2}\mathbf{i}_x, & r<a \\ E_0\left[\left(1+\dfrac{a^2}{r^2}\dfrac{\varepsilon_2-\varepsilon_1}{\varepsilon_1+\varepsilon_2}\right)\cos\phi\mathbf{i}_r \right. & \\ \left. -\left(1-\dfrac{a^2}{r^2}\dfrac{\varepsilon_2-\varepsilon_1}{\varepsilon_1+\varepsilon_2}\right)\sin\phi\mathbf{i}_\phi\right], & r>a \end{cases} \tag{23}$$

The field inside the cylinder is in the same direction as the applied field, and is reduced in amplitude if $\varepsilon_2 > \varepsilon_1$ and increased in amplitude if $\varepsilon_2 < \varepsilon_1$, up to a limiting factor of two as $\varepsilon_1$ becomes large compared to $\varepsilon_2$. If $\varepsilon_2 = \varepsilon_1$, the solution reduces to the uniform applied field everywhere.

The dc steady-state solution is identical in form to (23) if we replace the permittivities in each region by their conductivities;

$$\mathbf{E}(t\to\infty) = \begin{cases} \dfrac{2\sigma_1 E_0}{\sigma_1+\sigma_2}[\cos\phi\mathbf{i}_r - \sin\phi\mathbf{i}_\phi] = \dfrac{2\sigma_1 E_0}{\sigma_1+\sigma_2}\mathbf{i}_x, & r<a \\ E_0\left[\left(1+\dfrac{a^2}{r^2}\dfrac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\cos\phi\mathbf{i}_r \right. & \\ \left. -\left(1-\dfrac{a^2}{r^2}\dfrac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\sin\phi\mathbf{i}_\phi\right], & r>a \end{cases} \tag{24}$$

**(b) Field Line Plotting**

Because the region outside the cylinder is charge free, we know that $\nabla\cdot\mathbf{E}=0$. From the identity derived in Section 1-5-4*b*, that the divergence of the curl of a vector is zero, we thus know that the polar electric field with no *z* component can be expressed in the form

$$\begin{aligned} \mathbf{E}(r,\phi) &= \nabla\times\Sigma(r,\phi)\mathbf{i}_z \\ &= \frac{1}{r}\frac{\partial\Sigma}{\partial\phi}\mathbf{i}_r - \frac{\partial\Sigma}{\partial r}\mathbf{i}_\phi \end{aligned} \tag{25}$$

where $\Sigma$ is called the stream function. Note that the stream function vector is in the direction perpendicular to the electric field so that its curl has components in the same direction as the field.

Along a field line, which is always perpendicular to the equipotential lines,

$$\frac{d\mathrm{r}}{\mathrm{r}\,d\phi}=\frac{E_{\mathrm{r}}}{E_{\phi}}=-\frac{1}{\mathrm{r}}\frac{\partial\Sigma/\partial\phi}{\partial\Sigma/\partial\mathrm{r}} \tag{26}$$

By cross multiplying and grouping terms on one side of the equation, (26) reduces to

$$d\Sigma=\frac{\partial\Sigma}{\partial\mathrm{r}}\,d\mathrm{r}+\frac{\partial\Sigma}{\partial\phi}\,d\phi=0\Rightarrow\Sigma=\mathrm{const} \tag{27}$$

Field lines are thus lines of constant $\Sigma$.

For the steady-state solution of (24), outside the cylinder

$$\begin{aligned}\frac{1}{\mathrm{r}}\frac{\partial\Sigma}{\partial\phi}&=E_{\mathrm{r}}=E_0\left(1+\frac{a^2}{\mathrm{r}^2}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\cos\phi\\-\frac{\partial\Sigma}{\partial\mathrm{r}}&=E_{\phi}=-E_0\left(1-\frac{a^2}{\mathrm{r}^2}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\sin\phi\end{aligned} \tag{28}$$

we find by integration that

$$\Sigma=E_0\left(\mathrm{r}+\frac{a^2}{\mathrm{r}}\frac{\sigma_2-\sigma_1}{\sigma_1+\sigma_2}\right)\sin\phi \tag{29}$$

The steady-state field and equipotential lines are drawn in Figure 4-8 when the cylinder is perfectly conducting ($\sigma_2\to\infty$) or perfectly insulating ($\sigma_2=0$).

If the cylinder is highly conducting, the internal electric field is zero with the external electric field incident radially, as drawn in Figure 4-8*a*. In contrast, when the cylinder is perfectly insulating, the external field lines must be purely tangential to the cylinder as the incident normal current is zero, and the internal electric field has double the strength of the applied field, as drawn in Figure 4-8*b*.

### 4-3-3 Three-Dimensional Solutions

If the electric potential depends on all three coordinates, we try a product solution of the form

$$V(\mathrm{r},\phi,z)=\mathrm{R}(\mathrm{r})\Phi(\phi)Z(z) \tag{30}$$

which when substituted into Laplace's equation yields

$$\frac{Z\Phi}{\mathrm{r}}\frac{d}{d\mathrm{r}}\left(\mathrm{r}\frac{d\mathrm{R}}{d\mathrm{r}}\right)+\frac{\mathrm{R}Z}{\mathrm{r}^2}\frac{d^2\Phi}{d\phi^2}+\mathrm{R}\Phi\frac{d^2Z}{dz^2}=0 \tag{31}$$

We now have a difficulty, as we cannot divide through by a factor to make each term a function only of a single variable.

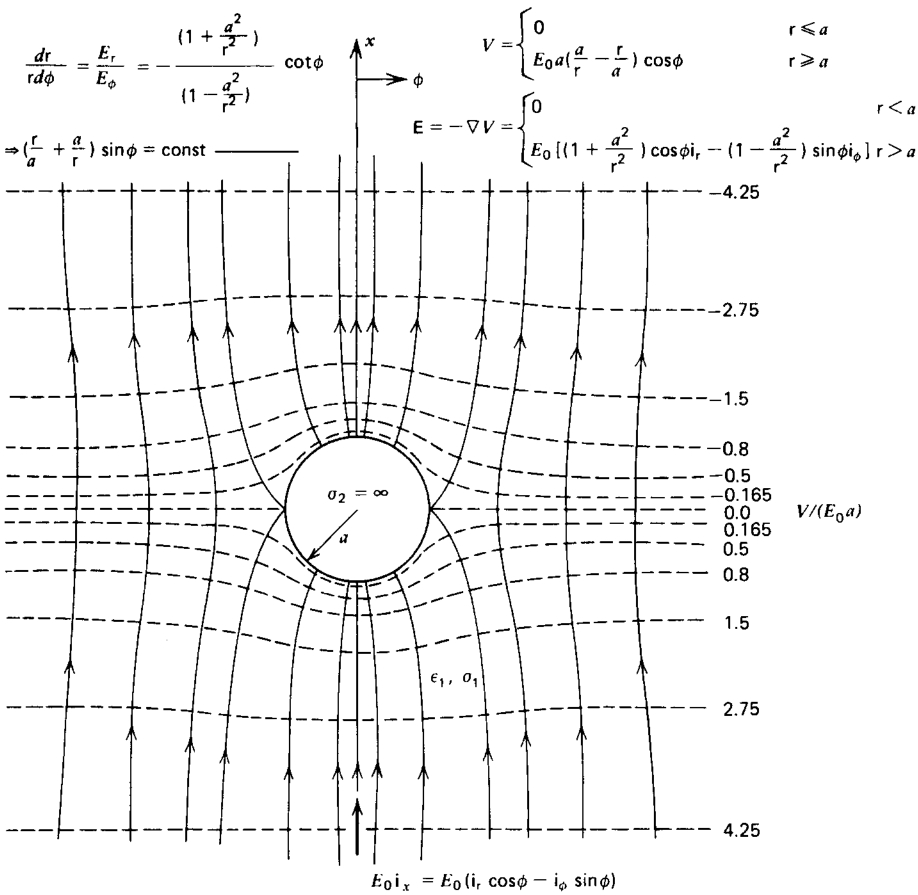



Figure 4-8 Steady-state field and equipotential lines about a (*a*) perfectly conducting or (*b*) perfectly insulating cylinder in a uniform electric field.

However, by dividing through by $V = \mathrm{R}\Phi Z$,

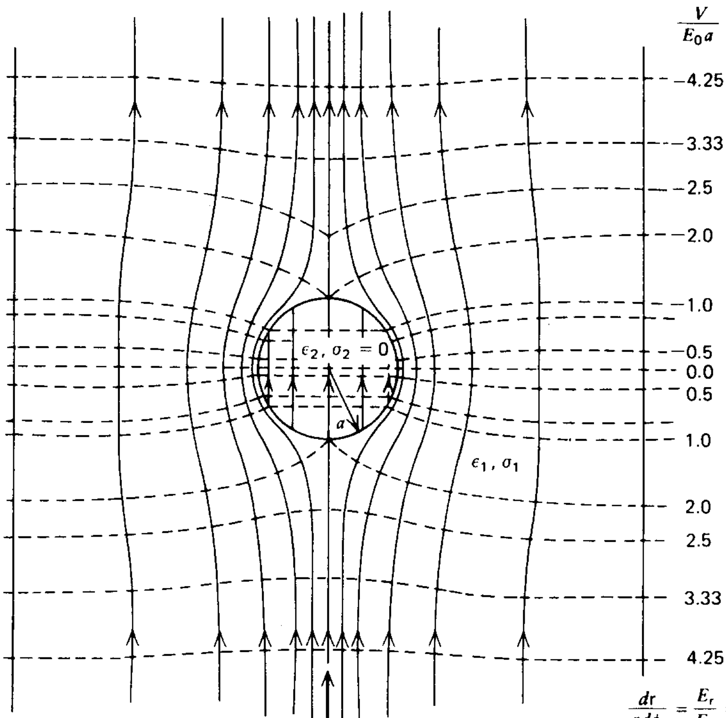
$$\underbrace{\frac{1}{\mathrm{R}r}\frac{d}{dr}\left(r\frac{d\mathrm{R}}{dr}\right)+\frac{1}{r^2\Phi}\frac{d^2\Phi}{d\phi^2}}_{-k^2}+\underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{k^2}=0 \tag{32}$$

we see that the first two terms are functions of r and $\phi$ while the last term is only a function of $z$. This last term must therefore equal a constant:

$$\frac{1}{Z}\frac{d^2Z}{dz^2}=k^2 \Rightarrow Z=\begin{cases} A_1\sinh kz + A_2\cosh kz, & k\neq 0 \\ A_3 z + A_4, & k=0 \end{cases} \tag{33}$$

$$V=\begin{cases}-2E_0 r\cos\phi & r\le a\\ -E_0 a\left(\dfrac{a}{r}+\dfrac{r}{a}\right)\cos\phi & r\ge a\end{cases}$$

$$\mathbf{E}=-\nabla V=\begin{cases}2E_0(\cos\phi\,\mathbf{i}_r-\sin\phi\,\mathbf{i}_\phi)=2E_0\,\mathbf{i}_x & r<a\\ E_0\left[\left(1-\dfrac{a^2}{r^2}\right)\cos\phi\,\mathbf{i}_r-\left(1+\dfrac{a^2}{r^2}\right)\sin\phi\,\mathbf{i}_\phi\right] & r>a\end{cases}$$




$E_0\mathbf{i}_x=E_0(\mathbf{i}_r\cos\phi-\mathbf{i}_\phi\sin\phi)$

$$\frac{dr}{r\,d\phi}=\frac{E_r}{E_\phi}=-\frac{\left(1-\dfrac{a^2}{r^2}\right)}{\left(1+\dfrac{a^2}{r^2}\right)}\cot\phi$$

$$\Rightarrow\left(\frac{r}{a}-\frac{a}{r}\right)\sin\phi=\text{const}$$

(*b*)

Figure 4-8*b*

The first two terms in (32) must now sum to $-k^2$ so that after multiplying through by $r^2$ we have

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+k^2r^2+\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=0 \tag{34}$$

Now again the first two terms are only a function of r, while the last term is only a function of $\phi$ so that (34) again separates:

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right)+k^2r^2=n^2,\qquad \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=-n^2 \tag{35}$$

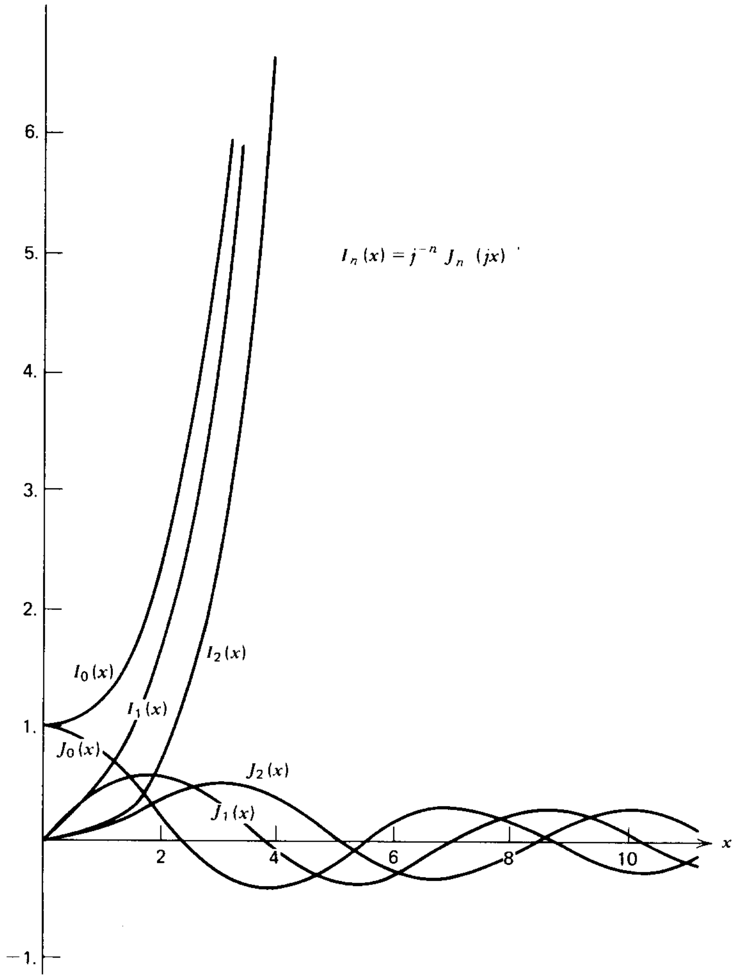
where $n^2$ is the second separation constant. The angular dependence thus has the same solutions as for the two-dimensional case

$$\Phi=\begin{cases}B_1 \sin n\phi + B_2 \cos n\phi, & n\neq 0\\ B_3\phi + B_4, & n=0\end{cases} \tag{36}$$

The resulting differential equation for the radial dependence

$$\mathrm{r}\frac{d}{d\mathrm{r}}\left(\mathrm{r}\frac{d\mathrm{R}}{d\mathrm{r}}\right)+(k^2\mathrm{r}^2-n^2)\mathrm{R}=0 \tag{37}$$

is Bessel's equation and for nonzero $k$ has solutions in terms




(a)

Figure 4-9 The Bessel functions (a) $J_n(x)$ and $I_n(x)$, and (b) $Y_n(x)$ and $K_n(x)$.

of tabulated functions:

$$R = \begin{cases} C_1 J_n(kr) + C_2 Y_n(kr), & k \neq 0 \\ C_3 r^n + C_4 r^{-n}, & k = 0, \quad n \neq 0 \\ C_5 \ln r + C_6, & k = 0, \quad n = 0 \end{cases} \qquad (38)$$

where $J_n$ is called a Bessel function of the first kind of order $n$ and $Y_n$ is called the $n$th-order Bessel function of the second kind. When $n = 0$, the Bessel functions are of zero order while if $k = 0$ the solutions reduce to the two-dimensional solutions of (9).

Some of the properties and limiting values of the Bessel functions are illustrated in Figure 4-9. Remember that $k$

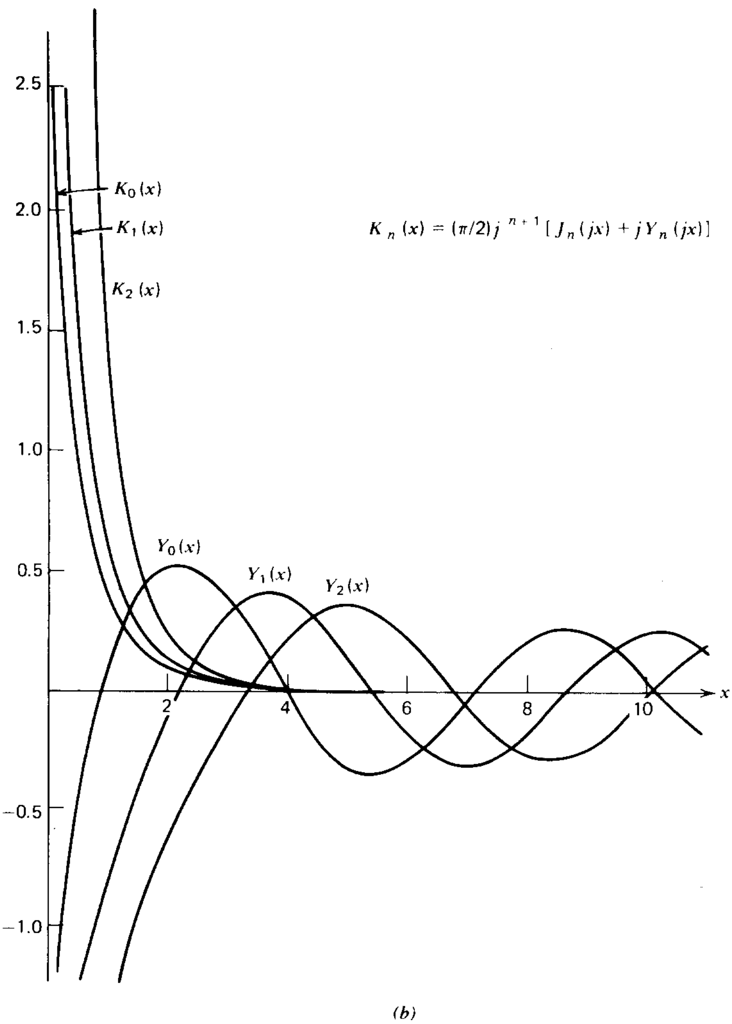



Figure 4-9b

can also be purely imaginary as well as real. When $k$ is real so that the $z$ dependence is hyperbolic or equivalently exponential, the Bessel functions are oscillatory while if $k$ is imaginary so that the axial dependence on $z$ is trigonometric, it is convenient to define the nonoscillatory modified Bessel functions as

$$\begin{aligned} I_n(k\mathrm{r}) &= j^{-n} J_n(jk\mathrm{r}) \\ K_n(k\mathrm{r}) &= \frac{\pi}{2} j^{n+1} [J_n(jk\mathrm{r}) + jY_n(jk\mathrm{r})] \end{aligned} \tag{39}$$

As in rectangular coordinates, if the solution to Laplace's equation decays in one direction, it is oscillatory in the perpendicular direction.

### 4-3-4 High Voltage Insulator Bushing

The high voltage insulator shown in Figure 4-10 consists of a cylindrical disk with Ohmic conductivity $\sigma$ supported by a perfectly conducting cylindrical post above a ground plane.*

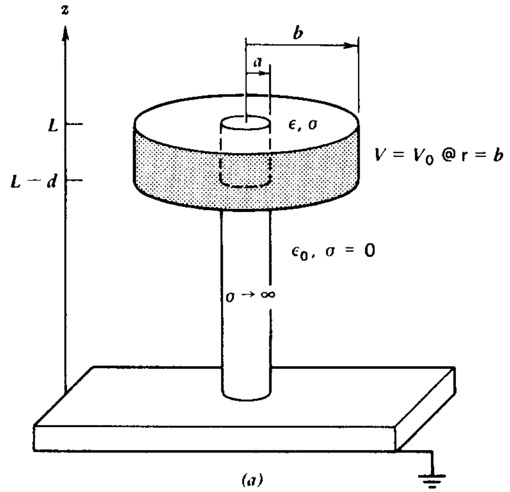
The plane at $z=0$ and the post at $\mathrm{r}=a$ are at zero potential, while a constant potential is imposed along the circumference of the disk at $\mathrm{r}=b$. The region below the disk is free space so that no current can cross the surfaces at $z=L$ and $z=L-d$. Because the boundaries lie along surfaces at constant $z$ or constant r we try the simple zero separation constant solutions in (33) and (38), which are independent of angle $\phi$:

$$V(\mathrm{r}, z) = \begin{cases} A_1 z + B_1 z \ln \mathrm{r} + C_1 \ln \mathrm{r} + D_1, & L-d < z < L \\ A_2 z + B_2 z \ln \mathrm{r} + C_2 \ln \mathrm{r} + D_2, & 0 \le z \le L-d \end{cases} \tag{40}$$

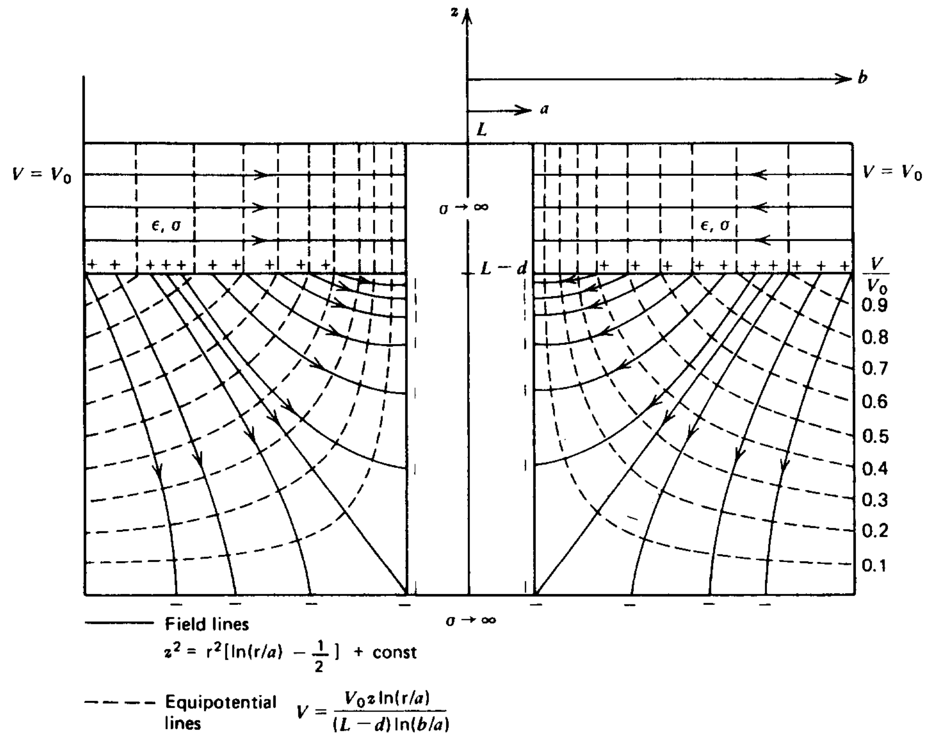
Applying the boundary conditions we relate the coefficients as

$$\begin{aligned} &V(z=0) = 0 \Rightarrow C_2 = D_2 = 0 \\ &V(\mathrm{r}=a) = 0 \Rightarrow \begin{cases} A_2 + B_2 \ln a = 0 \\ A_1 + B_1 \ln a = 0 \\ C_1 \ln a + D_1 = 0 \end{cases} \\ &V(\mathrm{r}=b, z > L-d) = V_0 \Rightarrow \begin{cases} A_1 + B_1 \ln b = 0 \\ C_1 \ln b + D_1 = V_0 \end{cases} \\ &V(z=(L-d)_-) = V(z=(L-d)_+) \Rightarrow (L-d)(A_2 + B_2 \ln \mathrm{r}) \\ &\qquad = (L-d)(A_1 + B_1 \ln \mathrm{r}) + C_1 \ln \mathrm{r} + D_1 \end{aligned} \tag{41}$$

* *M. N. Horenstein, "Particle Contamination of High Voltage DC Insulators," PhD thesis, Massachusetts Institute of Technology, 1978.*




(a)




(b)

Figure 4-10 (a) A finitely conducting disk is mounted upon a perfectly conducting cylindrical post and is placed on a perfectly conducting ground plane. (b) Field and equipotential lines.

which yields the values

$$A_1 = B_1 = 0, \qquad C_1 = \frac{V_0}{\ln(b/a)}, \qquad D_1 = -\frac{V_0 \ln a}{\ln(b/a)}$$
$$A_2 = -\frac{V_0 \ln a}{(L-d)\ln(b/a)}, \quad B_2 = \frac{V_0}{(L-d)\ln(b/a)}, \qquad C_2 = D_2 = 0 \tag{42}$$

The potential of (40) is then

$$V(\mathrm{r}, z) = \begin{cases} \dfrac{V_0 \ln(\mathrm{r}/a)}{\ln(b/a)}, & L-d \leq z \leq L \\[2ex] \dfrac{V_0 z \ln(\mathrm{r}/a)}{(L-d)\ln(b/a)}, & 0 \leq z \leq L-d \end{cases} \tag{43}$$

with associated electric field

$$\mathbf{E} = -\nabla V = \begin{cases} -\dfrac{V_0}{\mathrm{r}\ln(b/a)}\mathbf{i}_\mathrm{r}, & L-d < z < L \\[2ex] -\dfrac{V_0}{(L-d)\ln(b/a)}\left(\ln\dfrac{\mathrm{r}}{a}\mathbf{i}_z + \dfrac{z}{\mathrm{r}}\mathbf{i}_\mathrm{r}\right), & 0 < z < L-d \end{cases} \tag{44}$$

The field lines in the free space region are

$$\frac{d\mathrm{r}}{dz} = \frac{E_\mathrm{r}}{E_z} = \frac{z}{\mathrm{r}\ln(\mathrm{r}/a)} \Rightarrow z^2 = \mathrm{r}^2\left[\ln\frac{\mathrm{r}}{a} - \frac{1}{2}\right] + \text{const} \tag{45}$$

and are plotted with the equipotential lines in Figure 4-10b.

## 4-4 PRODUCT SOLUTIONS IN SPHERICAL GEOMETRY

In spherical coordinates, Laplace's equation is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0 \tag{1}$$

### 4-4-1 One-Dimensional Solutions

If the solution only depends on a single spatial coordinate, the governing equations and solutions for each of the three coordinates are

$$\text{(i)} \quad \frac{d}{dr}\left(r^2\frac{dV(r)}{dr}\right) = 0 \Rightarrow V(r) = \frac{A_1}{r} + A_2 \tag{2}$$

$$\text{(ii)}\quad \frac{d}{d\theta}\left(\sin\theta\frac{dV(\theta)}{d\theta}\right)=0\Rightarrow V(\theta)=B_1\ln\left(\tan\frac{\theta}{2}\right)+B_2 \tag{3}$$

$$\text{(iii)}\quad \frac{d^2V(\phi)}{d\phi^2}=0\Rightarrow V(\phi)=C_1\phi+C_2 \tag{4}$$

We recognize the radially dependent solution as the potential due to a point charge. The new solutions are those which only depend on $\theta$ or $\phi$.

## EXAMPLE 4-2 TWO CONES

Two identical cones with surfaces at angles $\theta=\alpha$ and $\theta=\pi-\alpha$ and with vertices meeting at the origin, are at a potential difference $v$, as shown in Figure 4-11. Find the potential and electric field.

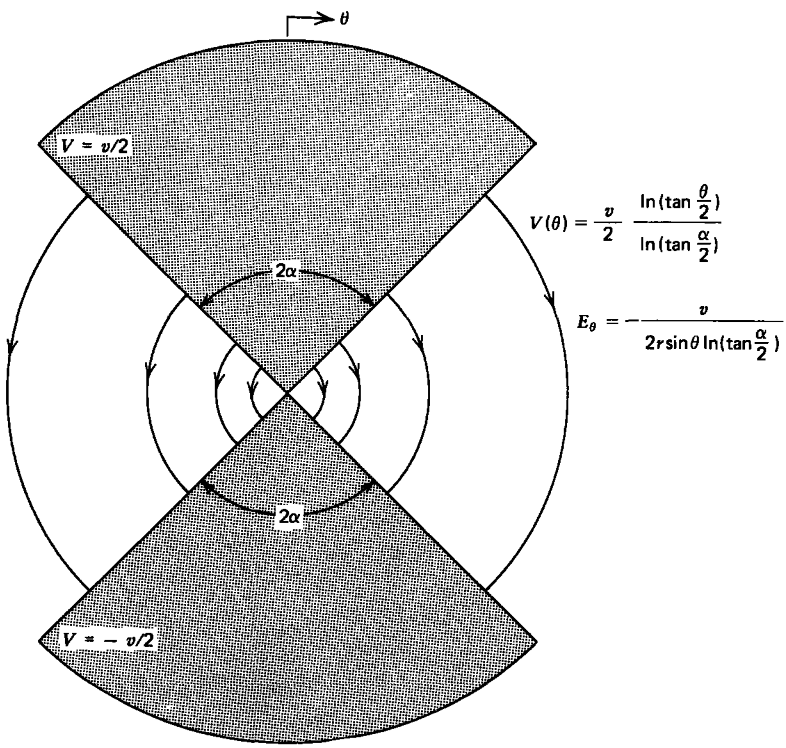



Figure 4-11 Two cones with vertices meeting at the origin are at a potential difference $v$.

**SOLUTION**

Because the boundaries are at constant values of $\theta$, we try (3) as a solution:

$$V(\theta)=B_1\ln[\tan(\theta/2)]+B_2$$

From the boundary conditions we have

$$V(\theta=\alpha)=\frac{v}{2}$$

$$V(\theta=\pi-\alpha)=\frac{-v}{2}\Rightarrow B_1=\frac{v}{2\ln[\tan(\alpha/2)]},\qquad B_2=0$$

so that the potential is

$$V(\theta)=\frac{v}{2}\frac{\ln[\tan(\theta/2)]}{\ln[\tan(\alpha/2)]}$$

with electric field

$$\mathbf{E}=-\nabla V=\frac{-v}{2r\sin\theta\ln[\tan(\alpha/2)]}\mathbf{i}_\theta$$

### 4-4-2 Axisymmetric Solutions

If the solution has no dependence on the coordinate $\phi$, we try a product solution

$$V(r,\theta)=R(r)\Theta(\theta) \tag{5}$$

which when substituted into (1), after multiplying through by $r^2/R\Theta$, yields

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0 \tag{6}$$

Because each term is again only a function of a single variable, each term is equal to a constant. Anticipating the form of the solution, we choose the separation constant as $n(n+1)$ so that (6) separates to

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-n(n+1)R=0 \tag{7}$$

$$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+n(n+1)\Theta\sin\theta=0 \tag{8}$$

For the radial dependence we try a power-law solution

$$R = Ar^p \tag{9}$$

which when substituted back into (7) requires

$$p(p+1) = n(n+1) \tag{10}$$

which has the two solutions

$$p = n, \qquad p = -(n+1) \tag{11}$$

When $n = 0$ we re-obtain the $1/r$ dependence due to a point charge.

To solve (8) for the $\theta$ dependence it is convenient to introduce the change of variable

$$\beta = \cos\theta \tag{12}$$

so that

$$\frac{d\Theta}{d\theta} = \frac{d\Theta}{d\beta}\frac{d\beta}{d\theta} = -\sin\theta\frac{d\Theta}{d\beta} = -(1-\beta^2)^{1/2}\frac{d\Theta}{d\beta} \tag{13}$$

Then (8) becomes

$$\frac{d}{d\beta}\left((1-\beta^2)\frac{d\Theta}{d\beta}\right) + n(n+1)\Theta = 0 \tag{14}$$

which is known as Legendre's equation. When $n$ is an integer, the solutions are written in terms of new functions:

$$\Theta = B_n P_n(\beta) + C_n Q_n(\beta) \tag{15}$$

where the $P_n(\beta)$ are called Legendre polynomials of the first kind and are tabulated in Table 4-1. The $Q_n$ solutions are called the Legendre functions of the second kind for which the first few are also tabulated in Table 4-1. Since all the $Q_n$ are singular at $\theta = 0$ and $\theta = \pi$, where $\beta = \pm 1$, for all problems which include these values of angle, the coefficients $C_n$ in (15) must be zero, so that many problems only involve the Legendre polynomials of first kind, $P_n(\cos\theta)$. Then using (9)–(11) and (15) in (5), the general solution for the potential with no $\phi$ dependence can be written as

$$V(r,\theta) = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-(n+1)}) P_n(\cos\theta) \tag{16}$$

**Table 4-1 Legendre polynomials of first and second kind**

| $n$ | $P_n(\beta=\cos\theta)$ | $Q_n(\beta=\cos\theta)$ |
|---|---|---|
| 0 | $1$ | $\frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)$ |
| 1 | $\beta=\cos\theta$ | $\frac{1}{2}\beta\ln\left(\frac{1+\beta}{1-\beta}\right)-1$ |
| 2 | $\frac{1}{2}(3\beta^2-1)$ $=\frac{1}{2}(3\cos^2\theta-1)$ | $\frac{1}{4}(3\beta^2-1)\ln\left(\frac{1+\beta}{1-\beta}\right)-\frac{3\beta}{2}$ |
| 3 | $\frac{1}{2}(5\beta^3-3\beta)$ $=\frac{1}{2}(5\cos^3\theta-3\cos\theta)$ | $\frac{1}{4}(5\beta^3-3\beta)\ln\left(\frac{1+\beta}{1-\beta}\right)-\frac{5}{2}\beta^2+\frac{2}{3}$ |
| ⋮ | | |
| $m$ | $\frac{1}{2^m m!}\frac{d^m}{d\beta^m}(\beta^2-1)^m$ | |

## 4-4-3 Conducting Sphere in a Uniform Field

### (a) Field Solution

A sphere of radius $R$, permittivity $\varepsilon_2$, and Ohmic conductivity $\sigma_2$ is placed within a medium of permittivity $\varepsilon_1$ and conductivity $\sigma_1$. A uniform dc electric field $E_0\mathbf{i}_z$ is applied at infinity. Although the general solution of (16) requires an infinite number of terms, the form of the uniform field at infinity in spherical coordinates,

$$\mathbf{E}(r\to\infty)=E_0\mathbf{i}_z=E_0(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta) \tag{17}$$

suggests that all the boundary conditions can be met with just the $n=1$ solution:

$$V(r,\theta)=\begin{cases}Ar\cos\theta, & r\le R\\ (Br+C/r^2)\cos\theta, & r\ge R\end{cases} \tag{18}$$

We do not include the $1/r^2$ solution within the sphere ($r<R$) as the potential must remain finite at $r=0$. The associated

electric field is

$$\mathbf{E}=-\nabla V=\begin{cases}-A(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta)=-A\,\mathbf{i}_z, & r<R\\ -(B-2C/r^3)\cos\theta\,\mathbf{i}_r+(B+C/r^3)\sin\theta\,\mathbf{i}_\theta, & r>R\end{cases} \tag{19}$$

The electric field within the sphere is uniform and $z$ directed while the solution outside is composed of the uniform $z$-directed field, for as $r\to\infty$ the field must approach (17) so that $B=-E_0$, plus the field due to a point dipole at the origin, with dipole moment

$$p_z=4\pi\varepsilon_1 C \tag{20}$$

Additional steady-state boundary conditions are the continuity of the potential at $r=R$ [equivalent to continuity of tangential $\mathbf{E}(r=R)$], and continuity of normal current at $r=R$,

$$\begin{aligned}&V(r=R_+)=V(r=R_-)\Rightarrow E_\theta(r=R_+)=E_\theta(r=R_-)\\ &\quad\Rightarrow AR=BR+C/R^2\\ &J_r(r=R_+)=J_r(r=R_-)\Rightarrow\sigma_1E_r(r=R_+)=\sigma_2E_r(r=R_-)\\ &\quad\Rightarrow\sigma_1(B-2C/R^3)=\sigma_2A\end{aligned} \tag{21}$$

for which solutions are

$$A=-\frac{3\sigma_1}{2\sigma_1+\sigma_2}E_0,\qquad B=-E_0,\qquad C=\frac{(\sigma_2-\sigma_1)R^3}{2\sigma_1+\sigma_2}E_0 \tag{22}$$

The electric field of (19) is then

$$\mathbf{E}=\begin{cases}\dfrac{3\sigma_1E_0}{2\sigma_1+\sigma_2}(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta)=\dfrac{3\sigma_1E_0}{2\sigma_1+\sigma_2}\mathbf{i}_z, & r<R\\ E_0\left[\left(1+\dfrac{2R^3(\sigma_2-\sigma_1)}{r^3(2\sigma_1+\sigma_2)}\right)\cos\theta\,\mathbf{i}_r\right.\\ \quad\left.-\left(1-\dfrac{R^3(\sigma_2-\sigma_1)}{r^3(2\sigma_1+\sigma_2)}\right)\sin\theta\,\mathbf{i}_\theta\right], & r>R\end{cases} \tag{23}$$

The interfacial surface charge is

$$\begin{aligned}\sigma_f(r=R)&=\varepsilon_1E_r(r=R_+)-\varepsilon_2E_r(r=R_-)\\ &=\frac{3(\sigma_2\varepsilon_1-\sigma_1\varepsilon_2)E_0}{2\sigma_1+\sigma_2}\cos\theta\end{aligned} \tag{24}$$

which is of one sign on the upper part of the sphere and of opposite sign on the lower half of the sphere. The total charge on the entire sphere is zero. The charge is zero at every point on the sphere if the relaxation times in each region are equal:

$$\frac{\varepsilon_1}{\sigma_1} = \frac{\varepsilon_2}{\sigma_2} \tag{25}$$

The solution if both regions were lossless dielectrics with no interfacial surface charge, is similar in form to (23) if we replace the conductivities by their respective permittivities.

**(b) Field Line Plotting**

As we saw in Section 4-3-2*b* for a cylindrical geometry, the electric field in a volume charge-free region has no divergence, so that it can be expressed as the curl of a vector. For an axisymmetric field in spherical coordinates we write the electric field as

$$\begin{aligned}\mathbf{E}(r,\theta) &= \nabla\times\left(\frac{\Sigma(r,\theta)}{r\sin\theta}\mathbf{i}_\phi\right)\\ &= \frac{1}{r^2\sin\theta}\frac{\partial\Sigma}{\partial\theta}\mathbf{i}_r - \frac{1}{r\sin\theta}\frac{\partial\Sigma}{\partial r}\mathbf{i}_\theta\end{aligned} \tag{26}$$

Note again, that for a two-dimensional electric field, the stream function vector points in the direction orthogonal to both field components so that its curl has components in the same direction as the field. The stream function $\Sigma$ is divided by $r\sin\theta$ so that the partial derivatives in (26) only operate on $\Sigma$.

The field lines are tangent to the electric field

$$\frac{dr}{r\,d\theta} = \frac{E_r}{E_\theta} = -\frac{1}{r}\frac{\partial\Sigma/\partial\theta}{\partial\Sigma/\partial r} \tag{27}$$

which after cross multiplication yields

$$d\Sigma = \frac{\partial\Sigma}{\partial r}dr + \frac{\partial\Sigma}{\partial\theta}d\theta = 0 \Rightarrow \Sigma = \text{const} \tag{28}$$

so that again $\Sigma$ is constant along a field line.

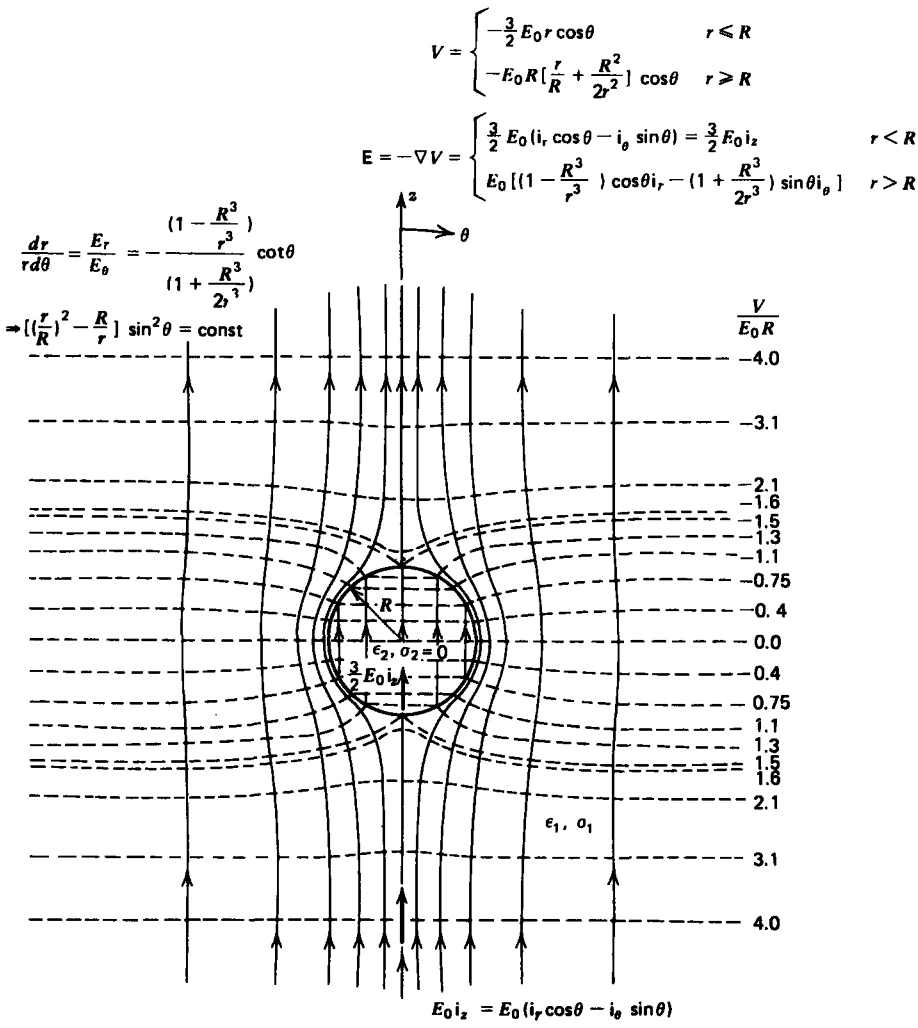
For the solution of (23) outside the sphere, we relate the field components to the stream function using (26) as

$$\begin{aligned}E_r &= \frac{1}{r^2\sin\theta}\frac{\partial\Sigma}{\partial\theta} = E_0\left(1+\frac{2R^3(\sigma_2-\sigma_1)}{r^3(2\sigma_1+\sigma_2)}\right)\cos\theta\\ E_\theta &= -\frac{1}{r\sin\theta}\frac{\partial\Sigma}{\partial r} = -E_0\left(1-\frac{R^3(\sigma_2-\sigma_1)}{r^3(2\sigma_1+\sigma_2)}\right)\sin\theta\end{aligned} \tag{29}$$

so that by integration the stream function is

$$\Sigma = E_0\left(\frac{r^2}{2}+\frac{R^3(\sigma_2-\sigma_1)}{r(2\sigma_1+\sigma_2)}\right)\sin^2\theta \tag{30}$$

The steady-state field and equipotential lines are drawn in Figure 4-12 when the sphere is perfectly insulating ($\sigma_2=0$) or perfectly conducting ($\sigma_2\to\infty$).




*(a)*

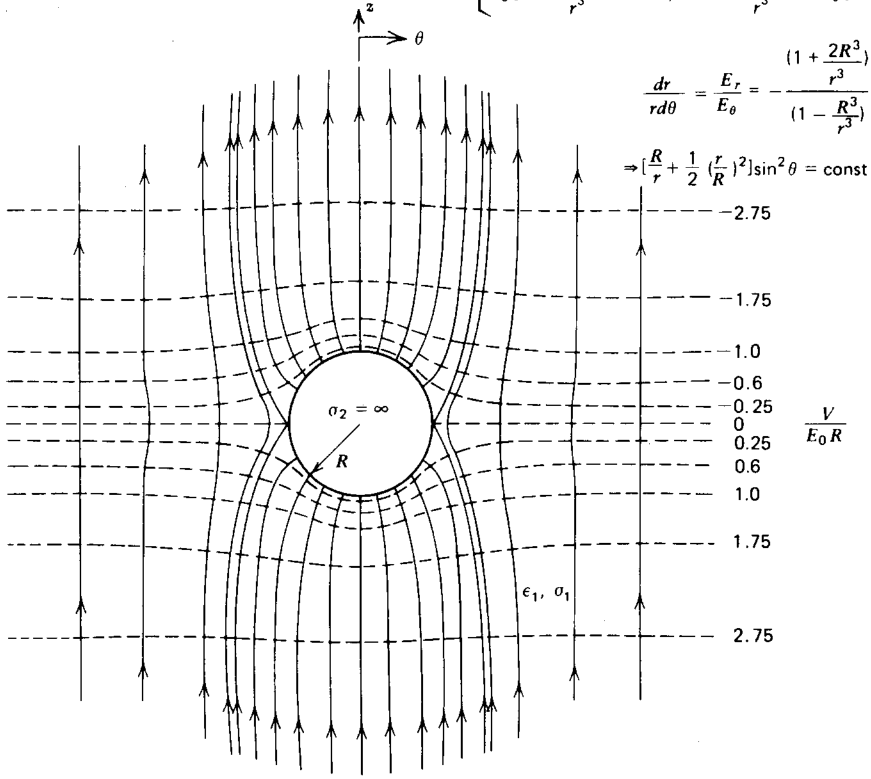
Figure 4-12 Steady-state field and equipotential lines about a (a) perfectly insulating or (b) perfectly conducting sphere in a uniform electric field.

$$V=\begin{cases}0 & r\leqslant R\\ -E_0R\left(\dfrac{r}{R}-\dfrac{R^2}{r^2}\right)\cos\theta & r\geqslant R\end{cases}$$

$$\mathbf{E}=-\nabla V=\begin{cases}0 & r<R\\ E_0\left[\left(1+\dfrac{2R^3}{r^3}\right)\cos\theta\,\mathbf{i}_r-\left(1-\dfrac{R^3}{r^3}\right)\sin\theta\,\mathbf{i}_\theta\right] & r>R\end{cases}$$

$$\frac{dr}{r\,d\theta}=\frac{E_r}{E_\theta}=-\frac{\left(1+\dfrac{2R^3}{r^3}\right)}{\left(1-\dfrac{R^3}{r^3}\right)}\cot\theta$$

$$\Rightarrow\left[\frac{R}{r}+\frac{1}{2}\left(\frac{r}{R}\right)^2\right]\sin^2\theta=\text{const}$$




$E_0\mathbf{i}_z=E_0(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta)$

(b)

Figure 4-12b

If the conductivity of the sphere is less than that of the surrounding medium ($\sigma_2<\sigma_1$), the electric field within the sphere is larger than the applied field. The opposite is true for ($\sigma_2>\sigma_1$). For the insulating sphere in Figure 4-12a, the field lines go around the sphere as no current can pass through.

For the conducting sphere in Figure 4-12b, the electric field lines must be incident perpendicularly. This case is used as a polarization model, for as we see from (23) with $\sigma_2\to\infty$, the external field is the imposed field plus the field of a point dipole with moment,

$$p_z = 4\pi\varepsilon_1 R^3 E_0 \tag{31}$$

If a dielectric is modeled as a dilute suspension of noninteracting, perfectly conducting spheres in free space with number density $N$, the dielectric constant is

$$\varepsilon = \frac{\varepsilon_0 E_0 + P}{E_0} = \frac{\varepsilon_0 E_0 + N p_z}{E_0} = \varepsilon_0(1 + 4\pi R^3 N) \tag{32}$$

### 4-4-4 Charged Particle Precipitation Onto a Sphere

The solution for a perfectly conducting sphere surrounded by free space in a uniform electric field has been used as a model for the charging of rain drops.* This same model has also been applied to a new type of electrostatic precipitator where small charged particulates are collected on larger spheres.†

Then, in addition to the uniform field $E_0\mathbf{i}_z$ applied at infinity, a uniform flux of charged particulate with charge density $\rho_0$, which we take to be positive, is also injected, which travels along the field lines with mobility $\mu$. Those field lines that start at infinity where the charge is injected and that approach the sphere with negative radial electric field, deposit charged particulate, as in Figure 4-13. The charge then redistributes itself uniformly on the equipotential surface so that the total charge on the sphere increases with time. Those field lines that do not intersect the sphere or those that start on the sphere do not deposit any charge.

We assume that the self-field due to the injected charge is very much less than the applied field $E_0$. Then the solution of (23) with $\sigma_2 = \infty$ is correct here, with the addition of the radial field of a uniformly charged sphere with total charge $Q(t)$:

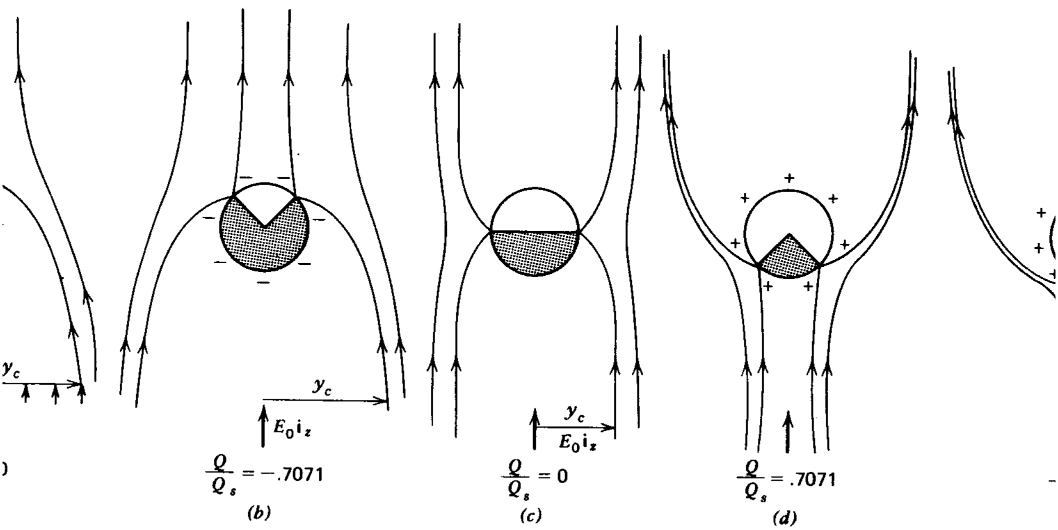
$$\mathbf{E} = \left[E_0\left(1 + \frac{2R^3}{r^3}\right)\cos\theta + \frac{Q}{4\pi\varepsilon r^2}\right]\mathbf{i}_r - E_0\left(1 - \frac{R^3}{r^3}\right)\sin\theta\,\mathbf{i}_\theta, \qquad r > R \tag{33}$$

Charge only impacts the sphere where $E_r(r = R)$ is negative:

$$E_r(r = R) = 3E_0\cos\theta + \frac{Q}{4\pi\varepsilon R^2} < 0 \tag{34}$$

* *See: F. J. W. Whipple and J. A. Chalmers, On Wilson's Theory of the Collection of Charge by Falling Drops,* Quart. J. Roy. Met. Soc. ***70**, (1944), p. 103.*

† *See: H. J. White,* Industrial Electrostatic Precipitation *Addison-Wesley, Reading. Mass. 1963, pp. 126–137.*




$$E_0 R^2 \left[\frac{R}{r} + \frac{1}{2}\left(\frac{r}{R}\right)^2\right] \sin^2\theta - \frac{Q\cos\theta}{4\pi\epsilon} = \text{constant}$$

ric field lines around a uniformly charged perfectly conducting sphere in a uniform electric field
ction from $z = -\infty$. Only those field lines that impact on the sphere with the electric field radially in
If the total charge on the sphere starts out as negative charge with magnitude greater or equal to th
ı the distance $y_c$ of the $z$ axis impact over the entire sphere. (b)–(d) As the sphere charges up it tends
and only part of the sphere collects charge. With increasing charge the angular window for c
. (e) For $Q \geq Q_s$ no further charge collects on the sphere so that the charge remains constant thereaf
shrunk to zero.

which gives us a window for charge collection over the range of angle, where

$$\cos\theta \leq -\frac{Q}{12\pi\varepsilon E_0 R^2} \tag{35}$$

Since the magnitude of the cosine must be less than unity, the maximum amount of charge that can be collected on the sphere is

$$Q_s = 12\pi\varepsilon E_0 R^2 \tag{36}$$

As soon as this saturation charge is reached, all field lines emanate radially outward from the sphere so that no more charge can be collected. We define the critical angle $\theta_c$ as the angle where the radial electric field is zero, defined when (35) is an equality $\cos\theta_c = -Q/Q_s$. The current density charging the sphere is

$$\begin{aligned} J_r &= \rho_0\mu E_r(r=R) \\ &= 3\rho_0\mu E_0(\cos\theta + Q/Q_s), \qquad \theta_c < \theta < \pi \end{aligned} \tag{37}$$

The total charging current is then

$$\begin{aligned} \frac{dQ}{dt} &= -\int_{\theta=\theta_c}^{\pi} J_r 2\pi R^2 \sin\theta\, d\theta \\ &= -6\pi\rho_0\mu E_0 R^2 \int_{\theta=\theta_c}^{\pi} (\cos\theta + Q/Q_s)\sin\theta\, d\theta \\ &= -6\pi\rho_0\mu E_0 R^2 (-\tfrac{1}{4}\cos 2\theta - (Q/Q_s)\cos\theta)\big|_{\theta=\theta_c}^{\pi} \\ &= -6\pi\rho_0\mu E_0 R^2 (-\tfrac{1}{4}(1-\cos 2\theta_c) + (Q/Q_s)(1+\cos\theta_c)) \end{aligned} \tag{38}$$

As long as $|Q| < Q_s$, $\theta_c$ is defined by the equality in (35). If $Q$ exceeds $Q_s$, which can only occur if the sphere is intentionally overcharged, then $\theta_c = \pi$ and no further charging can occur as $dQ/dt$ in (38) is zero. If $Q$ is negative and exceeds $Q_s$ in magnitude, $Q < -Q_s$, then the whole sphere collects charge as $\theta_c = 0$. Then for these conditions we have

$$\cos\theta_c = \begin{cases} -1, & Q > Q_s \\ -Q/Q_s & -Q_s < Q < Q_s \\ 1, & Q < -Q_s \end{cases} \tag{39}$$

$$\cos 2\theta_c = 2\cos^2\theta_c - 1 = \begin{cases} 1, & |Q| > Q_s \\ 2(Q/Q_s)^2 - 1, & |Q| < Q_s \end{cases} \tag{40}$$

so that (38) becomes

$$\frac{d\dfrac{Q}{Q_s}}{dt}=\begin{cases}0, & Q>Q_s\\ \dfrac{\rho_0\mu}{4\varepsilon}\left(1-\dfrac{Q}{Q_s}\right)^2, & -Q_s<Q<Q_s\\ -\dfrac{\rho_0\mu}{\varepsilon}\dfrac{Q}{Q_s}, & Q<-Q_s\end{cases} \tag{41}$$

with integrated solutions

$$\frac{Q}{Q_s}=\begin{cases}\dfrac{Q_0}{Q_s}, & Q>Q_s\\ \dfrac{\dfrac{Q_0}{Q_s}+\dfrac{(t-t_0)}{4\tau}\left(1-\dfrac{Q_0}{Q_s}\right)}{1+\dfrac{(t-t_0)}{4\tau}\left(1-\dfrac{Q_0}{Q_s}\right)}, & -Q_s<Q<Q_s\\ \dfrac{Q_0}{Q_s}e^{-t/\tau}, & Q<-Q_s\end{cases} \tag{42}$$

where $Q_0$ is the initial charge at $t=0$ and the characteristic charging time is

$$\tau=\varepsilon/(\rho_0\mu) \tag{43}$$

If the initial charge $Q_0$ is less than $-Q_s$, the charge magnitude decreases with the exponential law in (42) until the total charge reaches $-Q_s$ at $t=t_0$. Then the charging law switches to the next regime with $Q_0=-Q_s$, where the charge passes through zero and asymptotically slowly approaches $Q=Q_s$. The charge can never exceed $Q_s$ unless externally charged. It then remains constant at this value repelling any additional charge. If the initial charge $Q_0$ has magnitude less than $Q_s$, then $t_0=0$. The time dependence of the charge is plotted in Figure 4-14 for various initial charge values $Q_0$. No matter the initial value of $Q_0$ for $Q<Q_s$, it takes many time constants for the charge to closely approach the saturation value $Q_s$. The force of repulsion on the injected charge increases as the charge on the sphere increases so that the charging current decreases.

The field lines sketched in Figure 4-13 show how the fields change as the sphere charges up. The window for charge collection decreases with increasing charge. The field lines are found by adding the stream function of a uniformly charged sphere with total charge $Q$ to the solution of (30)

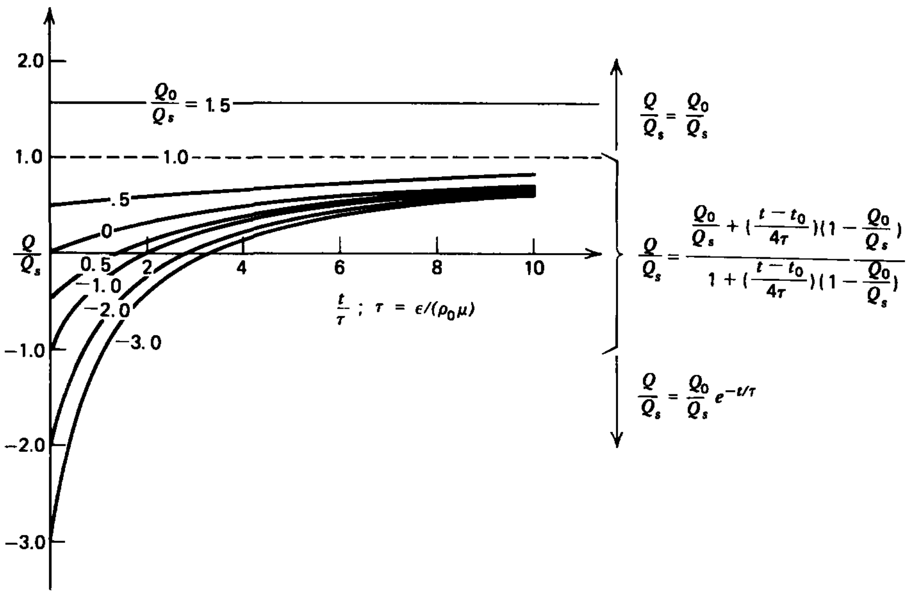



**Figure 4-14** There are three regimes describing the charge build-up on the sphere. It takes many time constants $[\tau = \varepsilon/(\rho_0\mu)]$ for the charge to approach the saturation value $Q_s$, because as the sphere charges up the Coulombic repulsive force increases so that most of the charge goes around the sphere. If the sphere is externally charged to a value in excess of the saturation charge, it remains constant as all additional charge is completely repelled.

with $\sigma_2 \to \infty$:

$$\Sigma = E_0 R^2 \left[\frac{R}{r} + \frac{1}{2}\left(\frac{r}{R}\right)^2\right] \sin^2\theta - \frac{Q\cos\theta}{4\pi\varepsilon} \tag{44}$$

The streamline intersecting the sphere at $r = R, \theta = \theta_c$ separates those streamlines that deposit charge onto the sphere from those that travel past.

## 4-5 A NUMERICAL METHOD—SUCCESSIVE RELAXATION

In many cases, the geometry and boundary conditions are irregular so that closed form solutions are not possible. It then becomes necessary to solve Poisson's equation by a computational procedure. In this section we limit ourselves to dependence on only two Cartesian coordinates.

### 4-5-1 Finite Difference Expansions

The Taylor series expansion to second order of the potential $V$, at points a distance $\Delta x$ on either side of the coordinate

$(x, y)$, is

$$\begin{aligned} V(x+\Delta x, y) &\approx V(x, y)+\frac{\partial V}{\partial x}\bigg|_{x,y} \Delta x+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}\bigg|_{x,y} (\Delta x)^2 \\ V(x-\Delta x, y) &\approx V(x, y)-\frac{\partial V}{\partial x}\bigg|_{x,y} \Delta x+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}\bigg|_{x,y} (\Delta x)^2 \end{aligned} \tag{1}$$

If we add these two equations and solve for the second derivative, we have

$$\frac{\partial^2 V}{\partial x^2} \approx \frac{V(x+\Delta x, y)+V(x-\Delta x, y)-2V(x, y)}{(\Delta x)^2} \tag{2}$$

Performing similar operations for small variations from $y$ yields

$$\frac{\partial^2 V}{\partial y^2} \approx \frac{V(x, y+\Delta y)+V(x, y-\Delta y)-2V(x, y)}{(\Delta y)^2} \tag{3}$$

If we add (2) and (3) and furthermore let $\Delta x=\Delta y$, Poisson's equation can be approximated as

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2} \approx \frac{1}{(\Delta x)^2}[&V(x+\Delta x, y)+V(x-\Delta x, y) \\ &+V(x, y+\Delta y)+V(x, y-\Delta y)-4V(x, y)]=-\frac{\rho_f(x, y)}{\varepsilon} \end{aligned} \tag{4}$$

so that the potential at $(x, y)$ is equal to the average potential of its four nearest neighbors plus a contribution due to any volume charge located at $(x, y)$:

$$\begin{aligned} V(x, y)=\tfrac{1}{4}[&V(x+\Delta x, y)+V(x-\Delta x, y) \\ &+V(x, y+\Delta y)+V(x, y-\Delta y)]+\frac{\rho_f(x, y)\,(\Delta x)^2}{4\varepsilon} \end{aligned} \tag{5}$$

The components of the electric field are obtained by taking the difference of the two expressions in (1)

$$\begin{aligned} E_x(x, y) &= -\frac{\partial V}{\partial x}\bigg|_{x,y} \approx -\frac{1}{2\Delta x}[V(x+\Delta x, y)-V(x-\Delta x, y)] \\ E_y(x, y) &= -\frac{\partial V}{\partial y}\bigg|_{x,y} \approx -\frac{1}{2\Delta y}[V(x, y+\Delta y)-V(x, y-\Delta y)] \end{aligned} \tag{6}$$

### 4-5-2 Potential Inside a Square Box

Consider the square conducting box whose sides are constrained to different potentials, as shown in Figure (4-15). We discretize the system by drawing a square grid with four

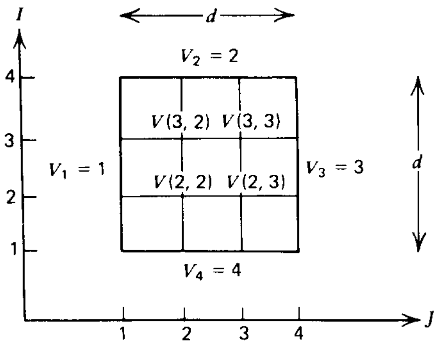



Figure 4-15 The potentials at the four interior points of a square conducting box with imposed potentials on its surfaces are found by successive numerical relaxation. The potential at any charge free interior grid point is equal to the average potential of the four adjacent points.

interior points. We must supply the potentials along the boundaries as proved in Section 4-1:

$$\begin{aligned} V_1 &= \sum_{I=1}^{4} V(I, J=1) = 1, \qquad & V_3 &= \sum_{I=1}^{4} V(I, J=4) = 3 \\ V_2 &= \sum_{J=1}^{4} V(I=4, J) = 2, \qquad & V_4 &= \sum_{J=1}^{4} V(I=1, J) = 4 \end{aligned} \tag{7}$$

Note the discontinuity in the potential at the corners.

We can write the charge-free discretized version of (5) as

$$V(I, J) = \tfrac{1}{4}[V(I+1, J) + V(I-1, J) + V(I, J+1) + V(I, J-1)] \tag{8}$$

We then guess any initial value of potential for all interior grid points not on the boundary. The boundary potentials must remain unchanged. Taking the interior points one at a time, we then improve our initial guess by computing the average potential of the four surrounding points.

We take our initial guess for all interior points to be zero inside the box:

$$\begin{aligned} V(2,2) &= 0, \qquad & V(3,3) &= 0 \\ V(3,2) &= 0, \qquad & V(2,3) &= 0 \end{aligned} \tag{9}$$

Then our first improved estimate for $V(2, 2)$ is

$$\begin{aligned} V(2,2) &= \tfrac{1}{4}[V(2,1) + V(2,3) + V(1,2) + V(3,2)] \\ &= \tfrac{1}{4}[1 + 0 + 4 + 0] = 1.25 \end{aligned} \tag{10}$$

Using this value of $V(2,2)$ we improve our estimate for $V(3,2)$ as

$$\begin{aligned} V(3,2)&=\tfrac{1}{4}[V(2,2)+V(4,2)+V(3,1)+V(3,3)]\\ &=\tfrac{1}{4}[1.25+2+1+0]=1.0625 \end{aligned} \tag{11}$$

Similarly for $V(3,3)$,

$$\begin{aligned} V(3,3)&=\tfrac{1}{4}[V(3,2)+V(3,4)+V(2,3)+V(4,3)]\\ &=\tfrac{1}{4}[1.0625+3+0+2]=1.5156 \end{aligned} \tag{12}$$

and $V(2,3)$

$$\begin{aligned} V(2,3)&=\tfrac{1}{4}[V(2,2)+V(2,4)+V(1,3)+V(3,3)]\\ &=\tfrac{1}{4}[1.25+3+4+1.5156]=2.4414 \end{aligned} \tag{13}$$

We then continue and repeat the procedure for the four interior points, always using the latest values of potential. As the number of iterations increase, the interior potential values approach the correct solutions. Table 4-2 shows the first ten iterations and should be compared to the exact solution to four decimal places, obtained by superposition of the rectangular harmonic solution in Section 4-2-5 (see problem 4-4):

$$\begin{aligned} V(x,y)=\sum_{\substack{n=1\\ n\ \text{odd}}}^{\infty}\frac{4}{n\pi\sinh n\pi}\Bigg[&\sin\frac{n\pi y}{d}\left(V_3\sinh\frac{n\pi x}{d}\right.\\ &\left.-V_1\sinh\frac{n\pi(x-d)}{d}\right)\\ &+\sin\frac{n\pi x}{d}\left(V_2\sinh\frac{n\pi y}{d}-V_4\sinh\frac{n\pi(y-d)}{d}\right)\Bigg] \end{aligned} \tag{14}$$

where $V_1$, $V_2$, $V_3$ and $V_4$ are the boundary potentials that for this case are

$$V_1=1,\qquad V_2=2,\qquad V_3=3,\qquad V_4=4 \tag{15}$$

To four decimal places the numerical solutions remain unchanged for further iterations past ten.

**Table 4-2 Potential values for the four interior points in Figure 4-15 obtained by successive relaxation for the first ten iterations**

| | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $V_1$ | 0 | 1.2500 | 2.1260 | 2.3777 | 2.4670 | 2.4911 |
| $V_2$ | 0 | 1.0625 | 1.6604 | 1.9133 | 1.9770 | 1.9935 |
| $V_3$ | 0 | 1.5156 | 2.2755 | 2.4409 | 2.4829 | 2.4952 |
| $V_4$ | 0 | 2.4414 | 2.8504 | 2.9546 | 2.9875 | 2.9966 |

| | 6 | 7 | 8 | 9 | 10 | Exact |
|---|---|---|---|---|---|---|
| $V_1$ | 2.4975 | 2.4993 | 2.4998 | 2.4999 | 2.5000 | 2.5000 |
| $V_2$ | 1.9982 | 1.9995 | 1.9999 | 2.0000 | 2.0000 | 1.9771 |
| $V_3$ | 2.4987 | 2.4996 | 2.4999 | 2.5000 | 2.5000 | 2.5000 |
| $V_4$ | 2.9991 | 2.9997 | 2.9999 | 3.0000 | 3.0000 | 3.0229 |

The results are surprisingly good considering the coarse grid of only four interior points. This relaxation procedure can be used for any values of boundary potentials, for any number of interior grid points, and can be applied to other boundary shapes. The more points used, the greater the accuracy. The method is easily implemented as a computer algorithm to do the repetitive operations.

## PROBLEMS

Section 4.2

1. The hyperbolic electrode system of Section 4-2-2$a$ only extends over the range $0 \leq x \leq x_0$, $0 \leq y \leq y_0$ and has a depth $D$.

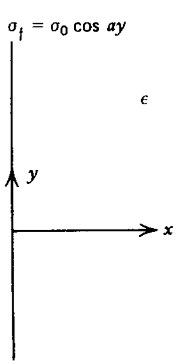
(a) Neglecting fringing field effects what is the approximate capacitance?

(b) A small positive test charge $q$ (image charge effects are negligible) with mass $m$ is released from rest from the surface of the hyperbolic electrode at $x = x_0$, $y = ab/x_0$. What is the velocity of the charge as a function of its position?

(c) What is the velocity of the charge when it hits the opposite electrode?

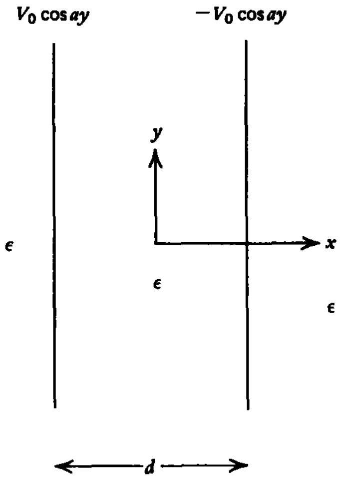
2. A sheet of free surface charge at $x = 0$ has charge distribution

$$\sigma_f = \sigma_0 \cos ay$$



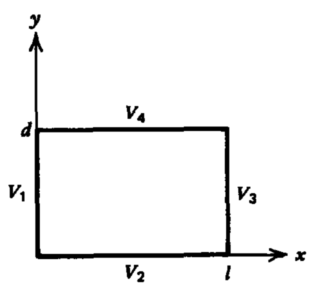
(a) What are the potential and electric field distributions?
(b) What is the equation of the field lines?

3. Two sheets of opposite polarity with their potential distributions constrained are a distance $d$ apart.




(a) What are the potential and electric field distributions everywhere?
(b) What are the surface charge distributions on each sheet?

4. A conducting rectangular box of width $d$ and length $l$ is of infinite extent in the $z$ direction. The potential along the $x = 0$ edge is $V_1$ while all other surfaces are grounded ($V_2 = V_3 = V_4 = 0$).

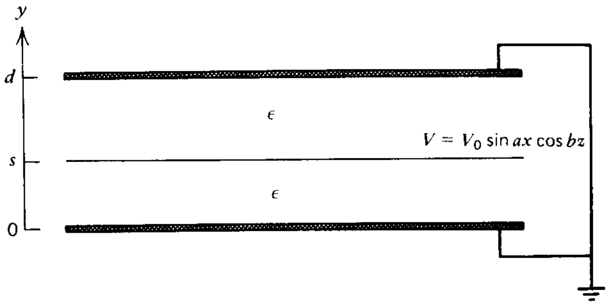



(a) What are the potential and electric field distributions?
(b) The potential at $y = 0$ is now raised to $V_2$ while the surface at $x = 0$ remains at potential $V_1$. The other two surfaces remain at zero potential ($V_3 = V_4 = 0$). What are the potential and electric field distributions? (**Hint:** Use superposition.)
(c) What is the potential distribution if each side is respectively at nonzero potentials $V_1$, $V_2$, $V_3$, and $V_4$?

5. A sheet with potential distribution

$$V = V_0 \sin ax \cos bz$$

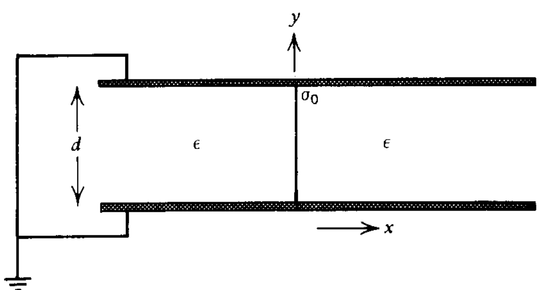
is placed parallel and between two parallel grounded conductors a distance $d$ apart. It is a distance $s$ above the lower plane.




(a) What are the potential and electric field distributions? (**Hint:** You can write the potential distribution by inspection using a spatially shifted hyperbolic function $\sinh c(y-d)$.)

(b) What is the surface charge distribution on each plane at $y=0$, $y=s$, and $y=d$?

6. A uniformly distributed surface charge $\sigma_0$ of width $d$ and of infinite extent in the $z$ direction is placed at $x=0$ perpendicular to two parallel grounded planes of spacing $d$.



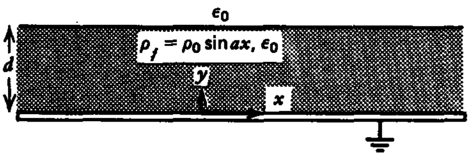

(a) What are the potential and electric field distributions? (**Hint:** Write $\sigma_0$ as a Fourier series.)

(b) What is the induced surface charge distribution on each plane?

(c) What is the total induced charge per unit length on each plane? **Hint:**

$$\sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

7. A slab of volume charge of thickness $d$ with volume charge density $\rho_f = \rho_0 \sin ax$ is placed upon a conducting ground plane.




(a) Find a particular solution to Poisson's equation. Are the boundary conditions satisfied?

(b) If the solution to (a) does not satisfy all the boundary conditions, add a Laplacian solution which does.

(c) What is the electric field distribution everywhere and the surface charge distribution on the ground plane?

(d) What is the force per unit length on the volume charge and on the ground plane for a section of width $2\pi/a$? Are these forces equal?

(e) Repeat (a)–(c), if rather than free charge, the slab is a permanently polarized medium with polarization

$$\mathbf{P} = P_0 \sin ax\,\mathbf{i}_y$$

8. Consider the Cartesian coordinates $(x, y)$ and define the complex quantity

$$z = x + jy, \qquad j = \sqrt{-1}$$

where $z$ is not to be confused with the Cartesian coordinate. Any function of $z$ also has real and imaginary parts

$$w(z) = u(x, y) + jv(x, y)$$

(a) Find $u$ and $v$ for the following functions:

(i) $z^2$
(ii) $\sin z$
(iii) $\cos z$
(iv) $e^z$
(v) $\ln z$

(b) Realizing that the partial derivatives of $w$ are

$$\frac{\partial w}{\partial x} = \frac{dw}{dz}\frac{\partial z}{\partial x} = \frac{dw}{dz} = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x}$$

$$\frac{\partial w}{\partial y} = \frac{dw}{dz}\frac{\partial z}{\partial y} = j\frac{dw}{dz} = \frac{\partial u}{\partial y} + j\frac{\partial v}{\partial y}$$

show that $u$ and $v$ must be related as

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

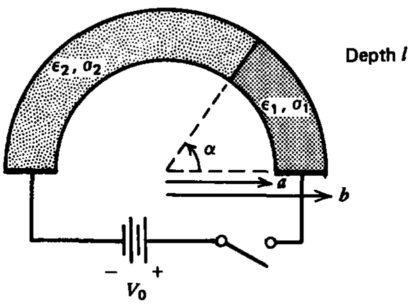
These relations are known as the Cauchy–Riemann equations and $u$ and $v$ are called conjugate functions.

(c) Show that both $u$ and $v$ obey Laplace's equation.

(d) Show that lines of constant $u$ and $v$ are perpendicular to each other in the $xy$ plane. (**Hint:** Are $\nabla u$ and $\nabla v$ perpendicular vectors?)

Section 4.3

9. A half cylindrical shell of length $l$ having inner radius $a$ and outer radius $b$ is composed of two different lossy dielectric materials $(\varepsilon_1, \sigma_1)$ for $0<\phi<\alpha$ and $(\varepsilon_2, \sigma_2)$ for $\alpha<\phi<\pi$. A step voltage $V_0$ is applied at $t=0$. Neglect variations with $z$.



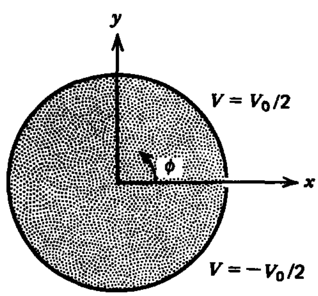

(a) What are the potential and electric field distributions within the shell at times $t=0$, $t=\infty$, and during the transient interval? (**Hint:** Assume potentials of the form $V(\phi)=A(t)\phi+B(t)$ and neglect effects of the region outside the half cylindrical shell.)

(b) What is the time dependence of the surface charge at $\phi=\alpha$?

(c) What is the resistance and capacitance?

10. The potential on an infinitely long cylinder is constrained to be

$$V(r=a)=V_0 \sin n\phi$$




(b)

(a) Find the potential and electric field everywhere.

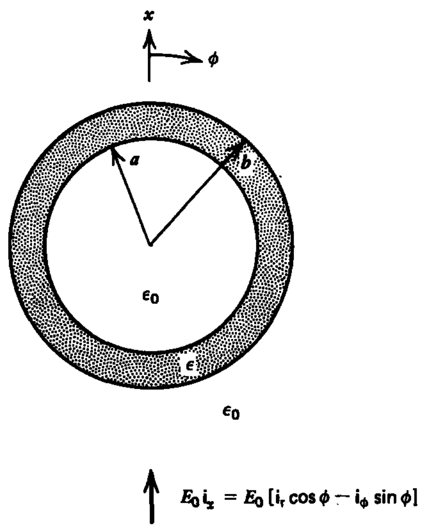
(b) The potential is now changed so that it is constant on each half of the cylinder:

$$V(r=a,\phi)=\begin{cases} V_0/2, & 0<\phi<\pi \\ -V_0/2, & \pi<\phi<2\pi \end{cases}$$

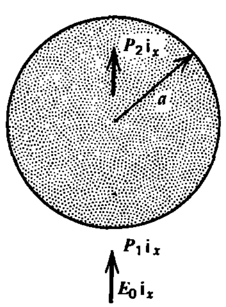
Write this square wave of potential in a Fourier series.

(c) Use the results of (a) and (b) to find the potential and electric field due to this square wave of potential.

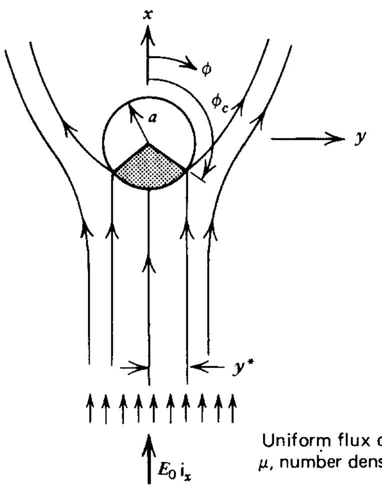
11. A cylindrical dielectric shell of inner radius $a$ and outer radius $b$ is placed in free space within a uniform electric field $E_0\mathbf{i}_x$. What are the potential and electric field distributions everywhere?




12. A permanently polarized cylinder $P_2\mathbf{i}_x$ of radius $a$ is placed within a polarized medium $P_1\mathbf{i}_x$ of infinite extent. A uniform electric field $E_0\mathbf{i}_x$ is applied at infinity. There is no free charge on the cylinder. What are the potential and electric field distributions?



13. One type of electrostatic precipitator has a perfectly conducting cylinder of radius $a$ placed within a uniform electric field $E_0\mathbf{i}_x$. A uniform flux of positive ions with charge $q_0$ and number density $n_0$ are injected at infinity and travel along the field lines with mobility $\mu$. Those field lines that approach the cylinder with $E_r < 0$ deposit ions, which redistribute themselves uniformly on the surface of the cylinder. The self-field due to the injected charge is negligible compared to $E_0$.




(a) If the uniformly distributed charge per unit length on the cylinder is $\lambda(t)$, what is the field distribution? Where is the electric field zero? This point is called a critical point because ions flowing past one side of this point miss the cylinder while those on the other side are collected. What equation do the field lines obey? (**Hint:** To the field solution of Section 4-3-2$a$, add the field due to a line charge $\lambda$.)

(b) Over what range of angle $\phi$, $\phi_c < \phi < 2\pi - \phi_c$, is there a window (shaded region in figure) for charge collection as a function of $\lambda(t)$? (**Hint:** $E_r < 0$ for charge collection.)

(c) What is the maximum amount of charge per unit length that can be collected on the cylinder?

(d) What is the cylinder charging current per unit length? (**Hint:** $dI = -q_0 n_0 \mu E_r a\, d\phi$)

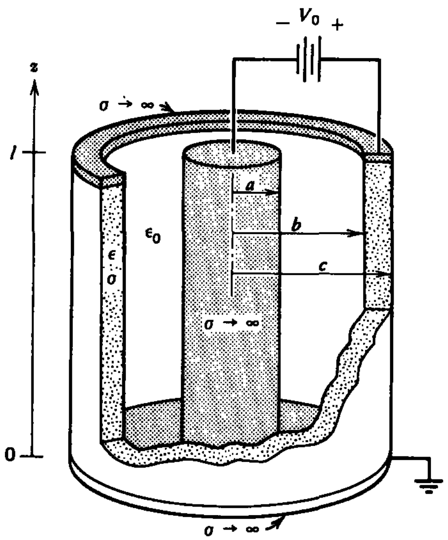
(e) Over what range of $y = y^*$ at $r = \infty, \phi = \pi$ do the injected ions impact on the cylinder as a function of $\lambda(t)$? What is this charging current per unit length? Compare to (d).

14. The cylinder of Section 4-3-2 placed within a lossy medium is allowed to reach the steady state.

(a) At $t = 0$ the imposed electric field at infinity is suddenly set to zero. What is the time dependence of the surface charge distribution at $r = a$?

(b) Find the surface charge distribution if the field at infinity is a sinusoidal function of time $E_0 \cos \omega t$.

15. A perfectly conducting cylindrical can of radius $c$ open at one end has its inside surface coated with a resistive layer. The bottom at $z = 0$ and a perfectly conducting center post of radius $a$ are at zero potential, while a constant potential $V_0$ is imposed at the top of the can.




(a) What are the potential and electric field distributions within the structure ($a < r < c$, $0 < z < l$)? (**Hint:** Try the zero separation constant solutions $n = 0$, $k = 0$.)

(b) What is the surface charge distribution and the total charge at $r = a$, $r = b$, and $z = 0$?

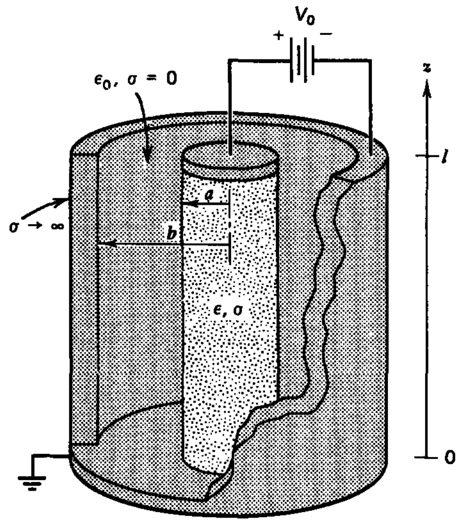
(c) What is the equation of the field lines in the free space region?

16. An Ohmic conducting cylinder of radius $a$ is surrounded by a grounded perfectly conducting cylindrical can of radius $b$ open at one end. A voltage $V_0$ is applied at the top of the resistive cylinder. Neglect variations with $\phi$.

(a) What are the potential and electric field distributions within the structure, $0 < z < l$, $0 < r < b$? (**Hint:** Try the zero separation constant solutions $n = 0$, $k = 0$ in each region.)

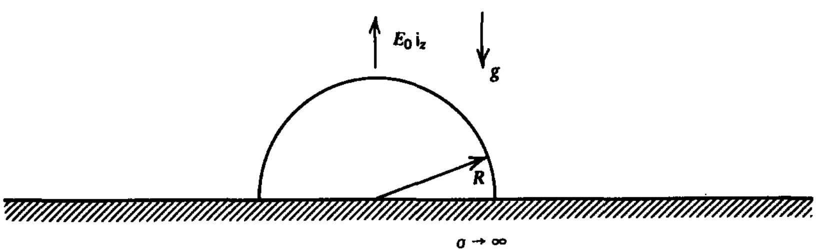
(b) What is the surface charge distribution and total charge on the interface at $r = a$?

(c) What is the equation of the field lines in the free space region?




Section 4.4

17. A perfectly conducting hemisphere of radius $R$ is placed upon a ground plane of infinite extent. A uniform field $E_0\mathbf{i}_z$ is applied at infinity.



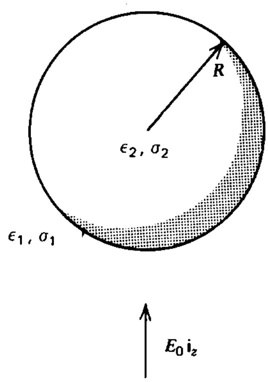

(a) How much more charge is on the hemisphere than would be on the plane over the area occupied by the hemisphere.

(b) If the hemisphere has mass density $\rho_m$ and is in a gravity field $-g\mathbf{i}_z$, how large must $E_0$ be to lift the hemisphere? **Hint:**

$$\int \sin\theta \cos^m\theta \, d\theta = -\frac{\cos^{m+1}\theta}{m+1}$$

18. A sphere of radius $R$, permittivity $\varepsilon_2$, and Ohmic conductivity $\sigma_2$ is placed within a medium of permittivity $\varepsilon_1$ and conductivity $\sigma_1$. A uniform electric field $E_0\mathbf{i}_z$ is suddenly turned on at $t=0$.

(a) What are the necessary boundary and initial conditions?




(b) What are the potential and electric field distributions as a function of time?

(c) What is the surface charge at $r = R$?

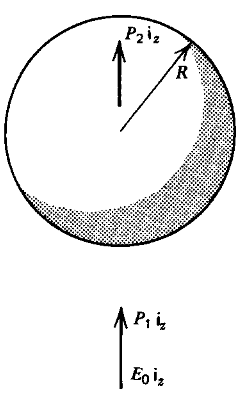
(d) Repeat (b) and (c) if the applied field varies sinusoidally with time as $E_0 \cos \omega t$ and has been on a long time.

19. The surface charge distribution on a dielectric sphere with permittivity $\varepsilon_2$ and radius $R$ is

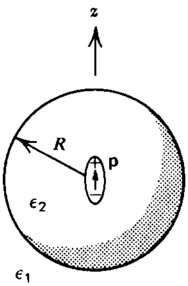
$$\sigma_f = \sigma_0(3 \cos^2 \theta - 1)$$

The surrounding medium has permittivity $\varepsilon_1$. What are the potential and electric field distributions? (**Hint:** Try the $n = 2$ solutions.)

20. A permanently polarized sphere $P_2\mathbf{i}_z$ of radius $R$ is placed within a polarized medium $P_1\mathbf{i}_z$. A uniform electric field $E_0\mathbf{i}_z$ is applied at infinity. There is no free charge at $r = R$. What are the potential and electric field distributions?




21. A point dipole $\mathbf{p} = p\mathbf{i}_z$ is placed at the center of a dielectric sphere that is surrounded by a different dielectric medium. There is no free surface charge on the interface.




What are the potential and electric field distributions? **Hint:**

$$\lim_{r\to 0} V(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_2 r^2}$$

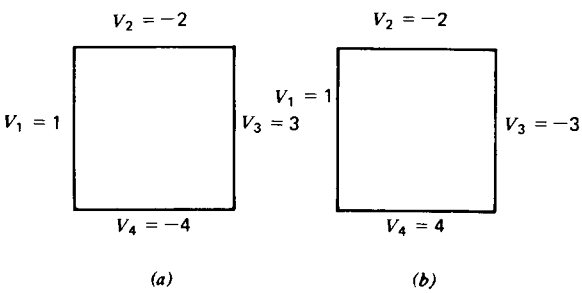
Section 4.5

22. The conducting box with sides of length $d$ in Section 4-5-2 is filled with a uniform distribution of volume charge with density

$$\rho_0 = -\frac{72\varepsilon_0}{d^2}\,[\text{coul-m}^{-3}]$$

What are the potentials at the four interior points when the outside of the box is grounded?

23. Repeat the relaxation procedure of Section 4-5-2 if the boundary potentials are:




(a) $V_1 = 1,\ V_2 = -2,\ V_3 = 3,\ V_4 = -4$

(b) $V_1 = 1,\ V_2 = -2,\ V_3 = -3,\ V_4 = 4$

(c) Compare to four decimal places with the exact solution.

# chapter 5

## *the magnetic field*

The ancient Chinese knew that the iron oxide magnetite ($Fe_3O_4$) attracted small pieces of iron. The first application of this effect was the navigation compass, which was not developed until the thirteenth century. No major advances were made again until the early nineteenth century when precise experiments discovered the properties of the magnetic field.

## 5-1 FORCES ON MOVING CHARGES

### 5-1-1 The Lorentz Force Law

It was well known that magnets exert forces on each other, but in 1820 Oersted discovered that a magnet placed near a current carrying wire will align itself perpendicular to the wire. Each charge $q$ in the wire, moving with velocity $\mathbf{v}$ in the magnetic field $\mathbf{B}$ [teslas, (kg-s$^{-2}$-A$^{-1}$)], felt the empirically determined Lorentz force perpendicular to both $\mathbf{v}$ and $\mathbf{B}$

$$\mathbf{f} = q(\mathbf{v}\times\mathbf{B}) \tag{1}$$

as illustrated in Figure 5-1. A distribution of charge feels a differential force $d\mathbf{f}$ on each moving incremental charge element $dq$:

$$d\mathbf{f} = dq(\mathbf{v}\times\mathbf{B}) \tag{2}$$

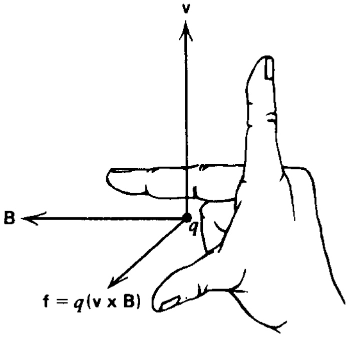



Figure 5-1 A charge moving through a magnetic field experiences the Lorentz force perpendicular to both its motion and the magnetic field.

Moving charges over a line, surface, or volume, respectively constitute line, surface, and volume currents, as in Figure 5-2, where (2) becomes

$$d\mathbf{f} = \begin{cases} \rho_f \mathbf{v} \times \mathbf{B}\, dV = \mathbf{J} \times \mathbf{B}\, dV & (\mathbf{J} = \rho_f \mathbf{v}, \text{ volume current density}) \\ \sigma_f \mathbf{v} \times \mathbf{B}\, dS = \mathbf{K} \times \mathbf{B}\, dS & \\ & (\mathbf{K} = \sigma_f \mathbf{v}, \text{ surface current density}) \\ \lambda_f \mathbf{v} \times \mathbf{B}\, dl = \mathbf{I} \times \mathbf{B}\, dl & (\mathbf{I} = \lambda_f \mathbf{v}, \text{ line current}) \end{cases} \quad (3)$$

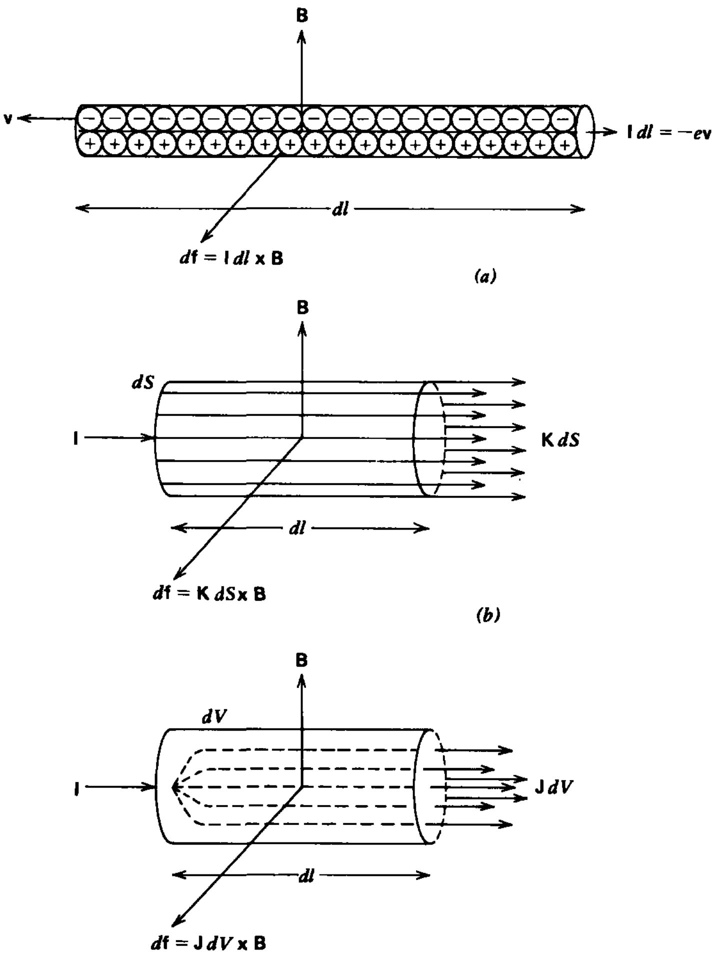



Figure 5-2 Moving line, surface, and volume charge distributions constitute currents. (*a*) In metallic wires the net charge is zero since there are equal amounts of negative and positive charges so that the Coulombic force is zero. Since the positive charge is essentially stationary, only the moving electrons contribute to the line current in the direction opposite to their motion. (*b*) Surface current. (*c*) Volume current.

The total magnetic force on a current distribution is then obtained by integrating (3) over the total volume, surface, or contour containing the current. If there is a net charge with its associated electric field $\mathbf{E}$, the total force densities include the Coulombic contribution:

$$\begin{aligned} \mathbf{f} &= q(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \quad \text{Newton} \\ \mathbf{F}_L &= \lambda_f(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \lambda_f\mathbf{E}+\mathbf{I}\times\mathbf{B} \quad \text{N/m} \\ \mathbf{F}_S &= \sigma_f(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \sigma_f\mathbf{E}+\mathbf{K}\times\mathbf{B} \quad \text{N/m}^2 \\ \mathbf{F}_V &= \rho_f(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \rho_f\mathbf{E}+\mathbf{J}\times\mathbf{B} \quad \text{N/m}^3 \end{aligned} \tag{4}$$

In many cases the net charge in a system is very small so that the Coulombic force is negligible. This is often true for conduction in metal wires. A net current still flows because of the difference in velocities of each charge carrier.

Unlike the electric field, the magnetic field cannot change the kinetic energy of a moving charge as the force is perpendicular to the velocity. It can alter the charge's trajectory but not its velocity magnitude.

### 5-1-2 Charge Motions in a Uniform Magnetic Field

The three components of Newton's law for a charge $q$ of mass $m$ moving through a uniform magnetic field $B_z\mathbf{i}_z$ are

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v}\times\mathbf{B} \Rightarrow \begin{cases} m\dfrac{dv_x}{dt} = qv_yB_z \\ m\dfrac{dv_y}{dt} = -qv_xB_z \\ m\dfrac{dv_z}{dt} = 0 \Rightarrow v_z = \text{const} \end{cases} \tag{5}$$

The velocity component along the magnetic field is unaffected. Solving the first equation for $v_y$ and substituting the result into the second equation gives us a single equation in $v_x$:

$$\frac{d^2v_x}{dt^2}+\omega_0^2 v_x = 0, \qquad v_y = \frac{1}{\omega_0}\frac{dv_x}{dt}, \qquad \omega_0 = \frac{qB_z}{m} \tag{6}$$

where $\omega_0$ is called the Larmor angular velocity or the cyclotron frequency (see Section 5-1-4). The solutions to (6) are

$$\begin{aligned} v_x &= A_1 \sin\omega_0 t + A_2\cos\omega_0 t \\ v_y &= \frac{1}{\omega_0}\frac{dv_x}{dt} = A_1\cos\omega_0 t - A_2 \sin\omega_0 t \end{aligned} \tag{7}$$

where $A_1$ and $A_2$ are found from initial conditions. If at $t=0$,

$$\mathbf{v}(t=0)=v_0\mathbf{i}_x \tag{8}$$

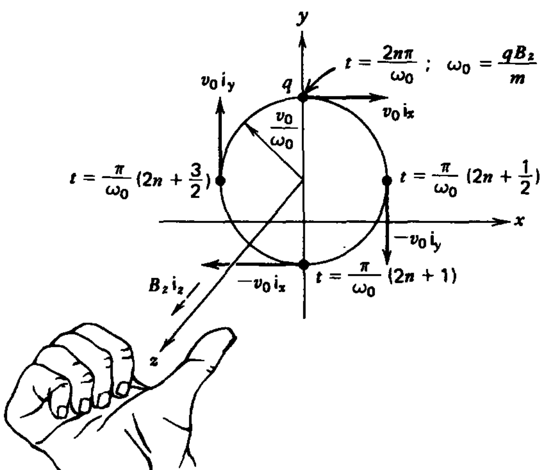
then (7) and Figure 5-3*a* show that the particle travels in a circle, with constant speed $v_0$ in the $xy$ plane:

$$v=v_0(\cos\omega_0 t\,\mathbf{i}_x-\sin\omega_0 t\,\mathbf{i}_y) \tag{9}$$

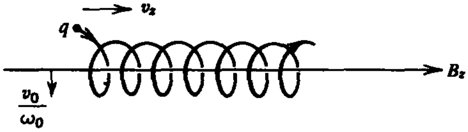
with radius

$$R=v_0/\omega_0 \tag{10}$$

If the particle also has a velocity component along the magnetic field in the $z$ direction, the charge trajectory becomes a helix, as shown in Figure 5-3*b*.




(a)




(b)

Figure 5-3 (*a*) A positive charge $q$, initially moving perpendicular to a magnetic field, feels an orthogonal force putting the charge into a circular motion about the magnetic field where the Lorentz force is balanced by the centrifugal force. Note that the charge travels in the direction (in this case clockwise) so that its self-field through the loop [see Section 5-2-1] is opposite in direction to the applied field. (*b*) A velocity component in the direction of the magnetic field is unaffected resulting in a helical trajectory.

### 5-1-3 The Mass Spectrograph

The mass spectrograph uses the circular motion derived in Section 5-1-2 to determine the masses of ions and to measure the relative proportions of isotopes, as shown in Figure 5-4. Charges enter between parallel plate electrodes with a $y$-directed velocity distribution. To pick out those charges with a particular magnitude of velocity, perpendicular electric and magnetic fields are imposed so that the net force on a charge is

$$f_x = q(E_x + v_y B_z) \tag{11}$$

For charges to pass through the narrow slit at the end of the channel, they must not be deflected by the fields so that the force in (11) is zero. For a selected velocity $v_y = v_0$ this requires a negatively $x$ directed electric field

$$E_x = \frac{V}{s} = -v_0 B_0 \tag{12}$$

which is adjusted by fixing the applied voltage $V$. Once the charge passes through the slit, it no longer feels the electric field and is only under the influence of the magnetic field. It thus travels in a circle of radius

$$r = \frac{v_0}{\omega_0} = \frac{v_0 m}{q B_0} \tag{13}$$

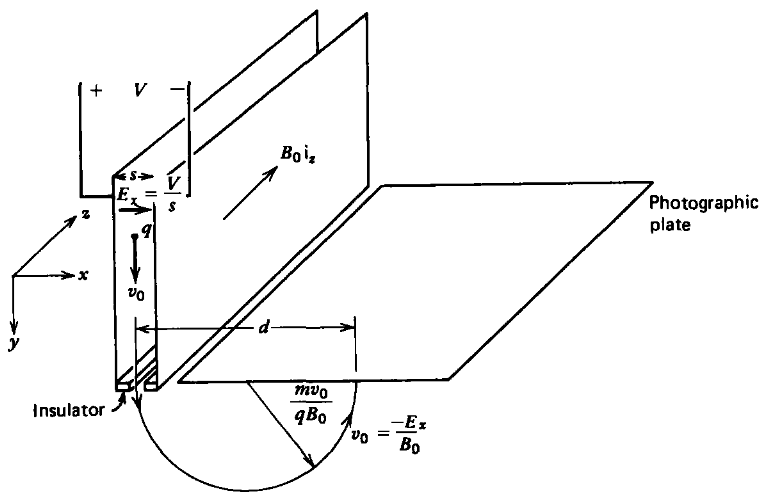



Figure 5-4 The mass spectrograph measures the mass of an ion by the radius of its trajectory when moving perpendicular to a magnetic field. The crossed uniform electric field selects the ion velocity that can pass through the slit.

which is directly proportional to the mass of the ion. By measuring the position of the charge when it hits the photographic plate, the mass of the ion can be calculated. Different isotopes that have the same number of protons but different amounts of neutrons will hit the plate at different positions.

For example, if the mass spectrograph has an applied voltage of $V = -100$ V across a 1-cm gap ($E_x = -10^4$ V/m) with a magnetic field of 1 tesla, only ions with velocity

$$v_y = -E_x/B_0 = 10^4 \text{ m/sec} \tag{14}$$

will pass through. The three isotopes of magnesium, $_{12}Mg^{24}$, $_{12}Mg^{25}$, $_{12}Mg^{26}$, each deficient of one electron, will hit the photographic plate at respective positions:

$$d = 2r = \frac{2\times 10^4 N(1.67\times 10^{-27})}{1.6\times 10^{-19}(1)} \approx 2\times 10^{-4} N$$
$$\Rightarrow \; 0.48, 0.50, 0.52 \text{ cm} \tag{15}$$

where $N$ is the number of protons and neutrons ($m = 1.67\times 10^{-27}$ kg) in the nucleus.

## 5-1-4 The Cyclotron

A cyclotron brings charged particles to very high speeds by many small repeated accelerations. Basically it is composed of a split hollow cylinder, as shown in Figure 5-5, where each half is called a "dee" because their shape is similar to the

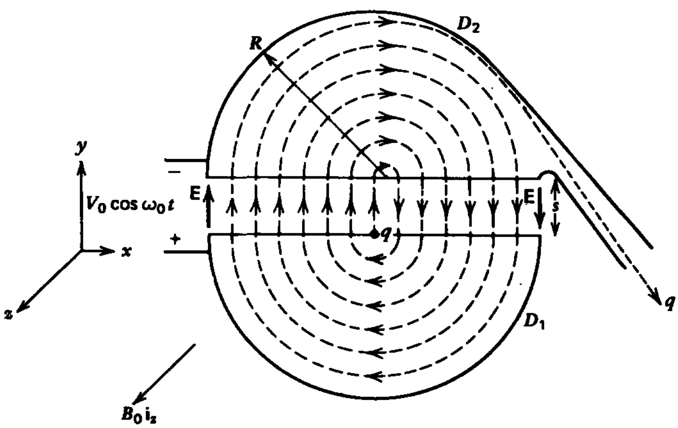



Figure 5-5 The cyclotron brings ions to high speed by many small repeated accelerations by the electric field in the gap between dees. Within the dees the electric field is negligible so that the ions move in increasingly larger circular orbits due to an applied magnetic field perpendicular to their motion.

fourth letter of the alphabet. The dees are put at a sinusoidally varying potential difference. A uniform magnetic field $B_0\mathbf{i}_z$ is applied along the axis of the cylinder. The electric field is essentially zero within the cylindrical volume and assumed uniform $E_y = v(t)/s$ in the small gap between dees. A charge source at the center of $D_1$ emits a charge $q$ of mass $m$ with zero velocity at the peak of the applied voltage at $t=0$. The electric field in the gap accelerates the charge towards $D_2$. Because the gap is so small the voltage remains approximately constant at $V_0$ while the charge is traveling between dees so that its displacement and velocity are

$$
\begin{aligned}
m\frac{dv_y}{dt} &= \frac{qV_0}{s} \Rightarrow v_y = \frac{qV_0}{sm}t \\
v_y &= \frac{dy}{dt} \Rightarrow y = \frac{qV_0 t^2}{2ms}
\end{aligned}
\tag{16}
$$

The charge thus enters $D_2$ at time $t = [2ms^2/qV_0]^{1/2}$ later with velocity $v_y = \sqrt{2qV_0/m}$. Within $D_2$ the electric field is negligible so that the charge travels in a circular orbit of radius $r = v_y/\omega_0 = mv_y/qB_0$ due to the magnetic field alone. The frequency of the voltage is adjusted to just equal the angular velocity $\omega_0 = qB_0/m$ of the charge, so that when the charge re-enters the gap between dees the polarity has reversed accelerating the charge towards $D_1$ with increased velocity. This process is continually repeated, since every time the charge enters the gap the voltage polarity accelerates the charge towards the opposite dee, resulting in a larger radius of travel. Each time the charge crosses the gap its velocity is increased by the same amount so that after $n$ gap traversals its velocity and orbit radius are

$$
v_n = \left(\frac{2qnV_0}{m}\right)^{1/2}, \qquad R_n = \frac{v_n}{\omega_0} = \left(\frac{2nmV_0}{qB_0^2}\right)^{1/2} \tag{17}
$$

If the outer radius of the dees is $R$, the maximum speed of the charge

$$
v_{\max} = \omega_0 R = \frac{qB_0}{m}R \tag{18}
$$

is reached after $2n = qB_0^2R^2/mV_0$ round trips when $R_n = R$. For a hydrogen ion ($q = 1.6\times10^{-19}$ coul, $m = 1.67\times10^{-27}$ kg), within a magnetic field of 1 tesla ($\omega_0 \approx 9.6\times10^7$ radian/sec) and peak voltage of 100 volts with a cyclotron radius of one meter, we reach $v_{\max} = 9.6\times10^7$ m/s (which is about 30% of the speed of light) in about $2n \approx 9.6\times10^5$ round-trips, which takes a time $\tau = 4n\pi/\omega_0 \approx 2\pi/100 \approx 0.06$ sec. To reach this speed with an electrostatic accelerator would require

$$\tfrac{1}{2}mv^2 = qV \Rightarrow V = \frac{mv_{\max}^2}{2q} \approx 48\times 10^6 \text{ Volts} \tag{19}$$

The cyclotron works at much lower voltages because the angular velocity of the ions remains constant for fixed $qB_0/m$ and thus arrives at the gap in phase with the peak of the applied voltage so that it is sequentially accelerated towards the opposite dee. It is not used with electrons because their small mass allows them to reach relativistic velocities close to the speed of light, which then greatly increases their mass, decreasing their angular velocity $\omega_0$, putting them out of phase with the voltage.

### 5-1-5 Hall Effect

When charges flow perpendicular to a magnetic field, the transverse displacement due to the Lorentz force can give rise to an electric field. The geometry in Figure 5-6 has a uniform magnetic field $B_0\mathbf{i}_z$ applied to a material carrying a current in the $y$ direction. For positive charges as for holes in a $p$-type semiconductor, the charge velocity is also in the positive $y$ direction, while for negative charges as occur in metals or in $n$-type semiconductors, the charge velocity is in the negative $y$ direction. In the steady state where the charge velocity does not vary with time, the net force on the charges must be zero,

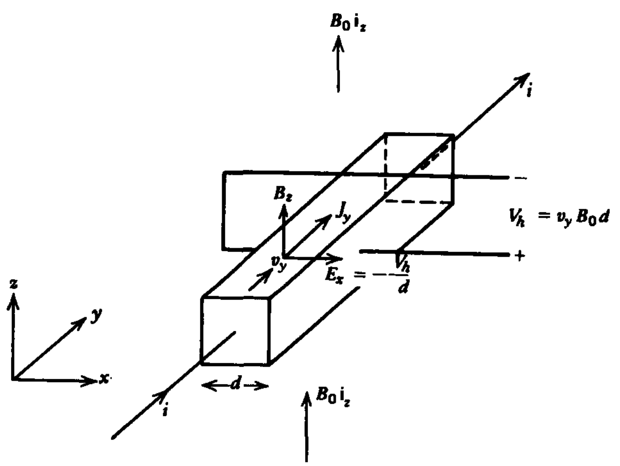



Figure 5-6 A magnetic field perpendicular to a current flow deflects the charges transversely giving rise to an electric field and the Hall voltage. The polarity of the voltage is the same as the sign of the charge carriers.

which requires the presence of an $x$-directed electric field

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=0\Rightarrow E_x=-v_yB_0 \tag{20}$$

A transverse potential difference then develops across the material called the Hall voltage:

$$V_h=-\int_0^d E_x\,dx=v_yB_0d \tag{21}$$

The Hall voltage has its polarity given by the sign of $v_y$; positive voltage for positive charge carriers and negative voltage for negative charges. This measurement provides an easy way to determine the sign of the predominant charge carrier for conduction.

## 5-2 MAGNETIC FIELD DUE TO CURRENTS

Once it was demonstrated that electric currents exert forces on magnets, Ampere immediately showed that electric currents also exert forces on each other and that a magnet could be replaced by an equivalent current with the same result. Now magnetic fields could be turned on and off at will with their strength easily controlled.

### 5-2-1 The Biot–Savart Law

Biot and Savart quantified Ampere's measurements by showing that the magnetic field $\mathbf{B}$ at a distance $\mathbf{r}$ from a moving charge is

$$\mathbf{B}=\frac{\mu_0 q\mathbf{v}\times\mathbf{i}_r}{4\pi r^2}\text{ teslas (kg-s}^{-2}\text{-A}^{-1}) \tag{1}$$

as in Figure 5-7$a$, where $\mu_0$ is a constant called the permeability of free space and in SI units is defined as having the exact numerical value

$$\mu_0\equiv 4\pi\times 10^{-7}\text{ henry/m (kg-m-A}^{-2}\text{-s}^{-2}) \tag{2}$$

The $4\pi$ is introduced in (1) for the same reason it was introduced in Coulomb's law in Section 2-2-1. It will cancel out a $4\pi$ contribution in frequently used laws that we will soon derive from (1). As for Coulomb's law, the magnetic field drops off inversely as the square of the distance, but its direction is now perpendicular both to the direction of charge flow and to the line joining the charge to the field point.

In the experiments of Ampere and those of Biot and Savart, the charge flow was constrained as a line current within a wire. If the charge is distributed over a line with

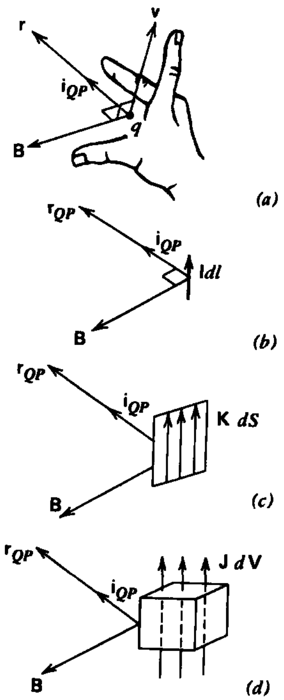



Figure 5-7 The magnetic field generated by a current is perpendicular to the current and the unit vector joining the current element to the field point; (*a*) point charge; (*b*) line current; (*c*) surface current; (*d*) volume current.

current **I**, or a surface with current per unit length **K**, or over a volume with current per unit area **J**, we use the differential-sized current elements, as in Figures 5-7*b*–5-7*d*:

$$dq\,\mathbf{v} = \begin{cases} \mathbf{I}\,dl & \text{(line current)} \\ \mathbf{K}\,dS & \text{(surface current)} \\ \mathbf{J}\,dV & \text{(volume current)} \end{cases} \tag{3}$$

The total magnetic field for a current distribution is then obtained by integrating the contributions from all the incremental elements:

$$\mathbf{B} = \begin{cases} \dfrac{\mu_0}{4\pi}\displaystyle\int_L \dfrac{\mathbf{I}\,dl\times\mathbf{i}_{QP}}{r_{QP}^2} & \text{(line current)} \\ \dfrac{\mu_0}{4\pi}\displaystyle\int_S \dfrac{\mathbf{K}\,dS\times\mathbf{i}_{QP}}{r_{QP}^2} & \text{(surface current)} \\ \dfrac{\mu_0}{4\pi}\displaystyle\int_V \dfrac{\mathbf{J}\,dV\times\mathbf{i}_{QP}}{r_{QP}^2} & \text{(volume current)} \end{cases} \tag{4}$$

The direction of the magnetic field due to a current element is found by the right-hand rule, where if the forefinger of the right hand points in the direction of current and the middle finger in the direction of the field point, then the thumb points in the direction of the magnetic field. This magnetic field **B** can then exert a force on other currents, as given in Section 5-1-1.

### 5-2-2 Line Currents

A constant current $I_1$ flows in the $z$ direction along a wire of infinite extent, as in Figure 5-8*a*. Equivalently, the right-hand rule allows us to put our thumb in the direction of current. Then the fingers on the right hand curl in the direction of **B**, as shown in Figure 5-8*a*. The unit vector in the direction of the line joining an incremental current element $I_1\,dz$ at $z$ to a field point $P$ is

$$\mathbf{i}_{QP}=\mathbf{i}_r\cos\theta-\mathbf{i}_z\sin\theta=\mathbf{i}_r\frac{r}{r_{QP}}-\mathbf{i}_z\frac{z}{r_{QP}} \tag{5}$$

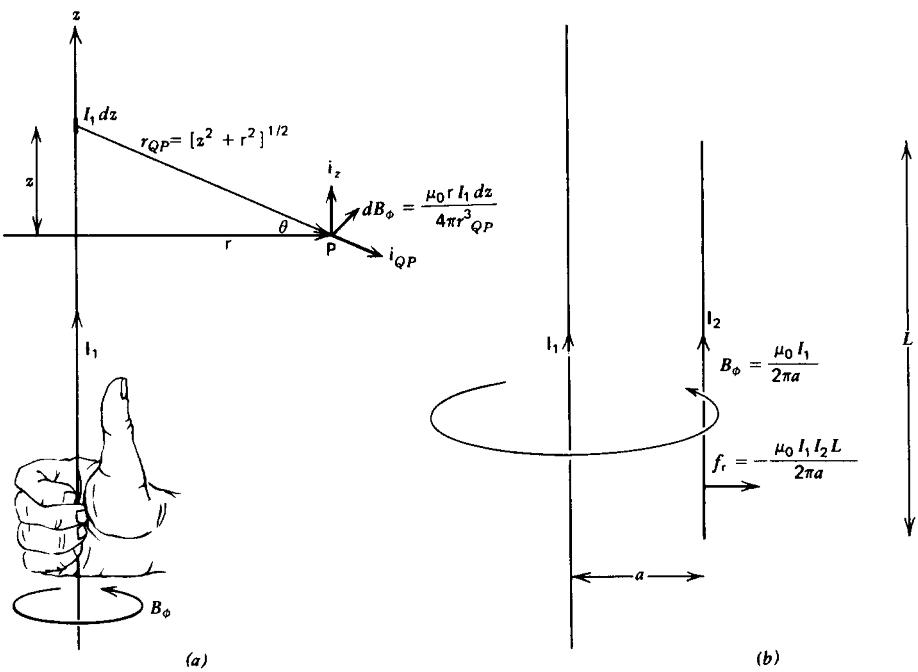



Figure 5-8 (*a*) The magnetic field due to an infinitely long $z$-directed line current is in the $\phi$ direction. (*b*) Two parallel line currents attract each other if flowing in the same direction and repel if oppositely directed.

with distance

$$r_{QP}=(z^2+r^2)^{1/2} \tag{6}$$

The magnetic field due to this current element is given by (4) as

$$d\mathbf{B}=\frac{\mu_0}{4\pi}\frac{I_1\,dz(\mathbf{i}_z\times\mathbf{i}_{QP})}{r_{QP}^2}=\frac{\mu_0 I_1 r\,dz}{4\pi(z^2+r^2)^{3/2}}\mathbf{i}_\phi \tag{7}$$

The total magnetic field from the line current is obtained by integrating the contributions from all elements:

$$\begin{aligned}B_\phi&=\frac{\mu_0 I_1 r}{4\pi}\int_{-\infty}^{+\infty}\frac{dz}{(z^2+r^2)^{3/2}}\\&=\frac{\mu_0 I_1 r}{4\pi}\frac{z}{r^2(z^2+r^2)^{1/2}}\bigg|_{z=-\infty}^{+\infty}\\&=\frac{\mu_0 I_1}{2\pi r}\end{aligned} \tag{8}$$

If a second line current $I_2$ of finite length $L$ is placed at a distance $a$ and parallel to $I_1$, as in Figure 5-8*b*, the force on $I_2$ due to the magnetic field of $I_1$ is

$$\begin{aligned}\mathbf{f}&=\int_{-L/2}^{+L/2}I_2\,dz\,\mathbf{i}_z\times\mathbf{B}\\&=\int_{-L/2}^{+L/2}I_2\,dz\frac{\mu_0 I_1}{2\pi a}(\mathbf{i}_z\times\mathbf{i}_\phi)\\&=-\frac{\mu_0 I_1 I_2 L}{2\pi a}\mathbf{i}_r\end{aligned} \tag{9}$$

If both currents flow in the same direction ($I_1I_2>0$), the force is attractive, while if they flow in opposite directions ($I_1I_2<0$), the force is repulsive. This is opposite in sense to the Coulombic force where opposite charges attract and like charges repel.

## 5-2-3 Current Sheets

### (a) Single Sheet of Surface Current

A constant current $K_0\mathbf{i}_z$ flows in the $y=0$ plane, as in Figure 5-9*a*. We break the sheet into incremental line currents $K_0\,dx$, each of which gives rise to a magnetic field as given by (8). From Table 1-2, the unit vector $\mathbf{i}_\phi$ is equivalent to the Cartesian components

$$\mathbf{i}_\phi=-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y \tag{10}$$

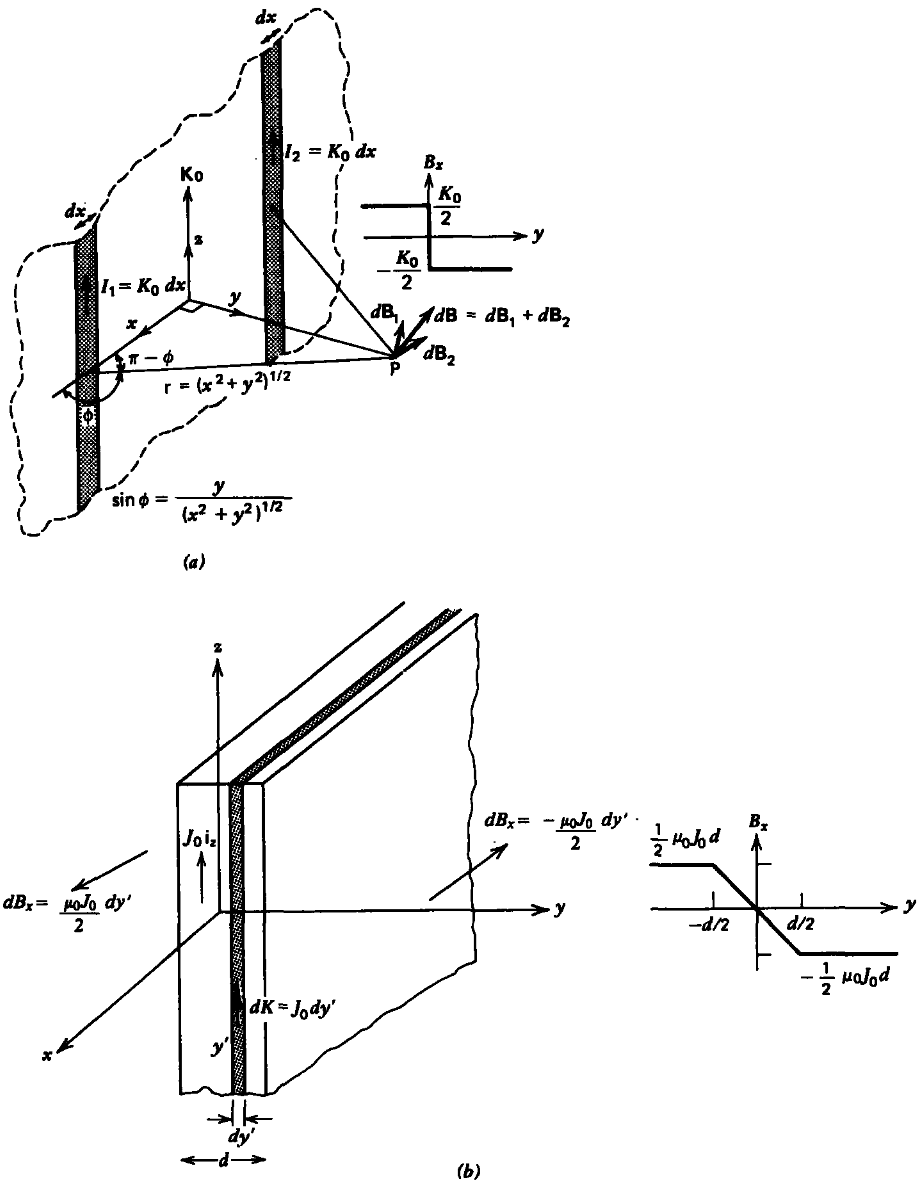



Figure 5-9 ($a$) A uniform surface current of infinite extent generates a uniform magnetic field oppositely directed on each side of the sheet. The magnetic field is perpendicular to the surface current but parallel to the plane of the sheet. ($b$) The magnetic field due to a slab of volume current is found by superimposing the fields due to incremental surface currents. ($c$) Two parallel but oppositely directed surface current sheets have fields that add in the region between the sheets but cancel outside the sheet. ($d$) The force on a current sheet is due to the average field on each side of the sheet as found by modeling the sheet as a uniform volume current distributed over an infinitesimal thickness $\Delta$.

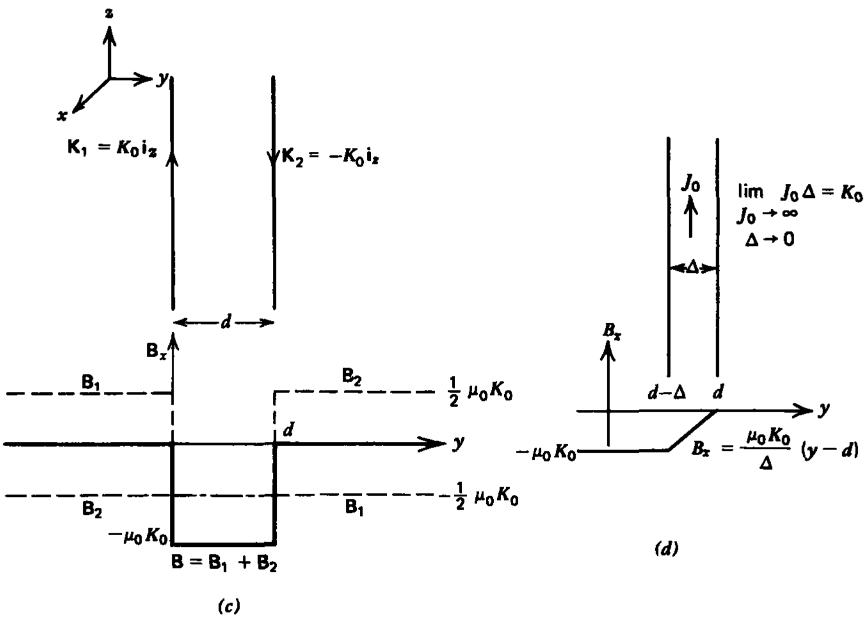



Figure 5-9

The symmetrically located line charge elements a distance $x$ on either side of a point $P$ have $y$ magnetic field components that cancel but $x$ components that add. The total magnetic field is then

$$\begin{aligned} B_x &= -\int_{-\infty}^{+\infty} \frac{\mu_0 K_0 \sin\phi}{2\pi(x^2+y^2)^{1/2}}\,dx \\ &= \frac{-\mu_0 K_0 y}{2\pi}\int_{-\infty}^{+\infty}\frac{dx}{(x^2+y^2)} \\ &= \frac{-\mu_0 K_0}{2\pi}\tan^{-1}\frac{x}{y}\bigg|_{-\infty}^{+\infty} \\ &= \begin{cases} -\mu_0 K_0/2, & y>0 \\ \mu_0 K_0/2, & y<0 \end{cases} \end{aligned} \tag{11}$$

The field is constant and oppositely directed on each side of the sheet.

**(b) Slab of Volume Current**

If the $z$-directed current $J_0\mathbf{i}_z$ is uniform over a thickness $d$, as in Figure 5-9$b$, we break the slab into incremental current sheets $J_0\,dy'$. The magnetic field from each current sheet is given by (11). When adding the contributions of all the differential-sized sheets, those to the left of a field point give a negatively $x$ directed magnetic field while those to the right contribute a positively $x$-directed field:

$$B_x = \begin{cases} \displaystyle\int_{-d/2}^{+d/2} \frac{-\mu_0 J_0\, dy'}{2} = \frac{-\mu_0 J_0 d}{2}, & y > \dfrac{d}{2} \\ \displaystyle\int_{-d/2}^{+d/2} \frac{\mu_0 J_0\, dy'}{2} = \frac{\mu_0 J_0 d}{2}, & y < -\dfrac{d}{2} \\ \displaystyle\int_{-d/2}^{y} \frac{-\mu_0 J_0\, dy'}{2} + \int_{y}^{d/2} \frac{\mu_0 J_0\, dy'}{2} = -\mu_0 J_0 y, & -\dfrac{d}{2} \le y \le \dfrac{d}{2} \end{cases} \tag{12}$$

The total force per unit area on the slab is zero:

$$\begin{aligned} F_{S_y} &= \int_{-d/2}^{+d/2} J_0 B_x\, dy = -\mu_o J_0^2 \int_{-d/2}^{+d/2} y\, dy \\ &= -\mu_0 J_0^2 \frac{y^2}{2}\bigg|_{-d/2}^{+d/2} = 0 \end{aligned} \tag{13}$$

A current distribution cannot exert a net force on itself.

### (c) Two Parallel Current Sheets

If a second current sheet with current flowing in the opposite direction $-K_0\mathbf{i}_z$ is placed at $y = d$ parallel to a current sheet $K_0\mathbf{i}_z$ at $y = 0$, as in Figure 5-9$c$, the magnetic field due to each sheet alone is

$$\mathbf{B}_1 = \begin{cases} \dfrac{-\mu_0 K_0}{2}\mathbf{i}_x, & y > 0 \\ \\ \dfrac{\mu_0 K_0}{2}\mathbf{i}_x, & y < 0 \end{cases} \qquad \mathbf{B}_2 = \begin{cases} \dfrac{\mu_0 K_0}{2}\mathbf{i}_x, & y > d \\ \\ \dfrac{-\mu_0 K_0}{2}\mathbf{i}_x, & y < d \end{cases} \tag{14}$$

Thus in the region outside the sheets, the fields cancel while they add in the region between:

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 = \begin{cases} -\mu_0 K_0 \mathbf{i}_x, & 0 < y < d \\ 0, & y < 0,\ y > d \end{cases} \tag{15}$$

The force on a surface current element on the second sheet is

$$\mathbf{df} = -K_0 \mathbf{i}_z\, dS \times \mathbf{B} \tag{16}$$

However, since the magnetic field is discontinuous at the current sheet, it is not clear which value of magnetic field to use. Tρ take the limit properly, we model the current sheet at $y = d$ as a thin volume current with density $J_0$ and thickness $\Delta$, as in Figure 5-9$d$, where $K_0 = J_0 \Delta$.

The results of (12) show that in a slab of uniform volume current, the magnetic field changes linearly to its values at the surfaces

$$\begin{aligned} B_x(y=d-\Delta) &= -\mu_0 K_0 \\ B_x(y=d) &= 0 \end{aligned} \tag{17}$$

so that the magnetic field within the slab is

$$B_x = \frac{\mu_0 K_0}{\Delta}(y-d) \tag{18}$$

The force per unit area on the slab is then

$$\begin{aligned} \mathbf{F}_S &= -\int_{d-\Delta}^{d} \frac{\mu_0 K_0}{\Delta} J_0(y-d)\mathbf{i}_y\, dy \\ &= \frac{-\mu_0 K_0 J_0}{\Delta} \frac{(y-d)^2}{2} \mathbf{i}_y \Big|_{d-\Delta}^{d} \\ &= \frac{\mu_0 K_0 J_0 \Delta}{2} \mathbf{i}_y = \frac{\mu_0 K_0^2}{2} \mathbf{i}_y \end{aligned} \tag{19}$$

The force acts to separate the sheets because the currents are in opposite directions and thus repel one another.

Just as we found for the electric field on either side of a sheet of surface charge in Section 3-9-1, when the magnetic field is discontinuous on either side of a current sheet **K**, being $\mathbf{B}_1$ on one side and $\mathbf{B}_2$ on the other, the average magnetic field is used to compute the force on the sheet:

$$\mathbf{df} = \mathbf{K}\, dS \times \frac{(\mathbf{B}_1 + \mathbf{B}_2)}{2} \tag{20}$$

In our case

$$\mathbf{B}_1 = -\mu_0 K_0 \mathbf{i}_x, \qquad \mathbf{B}_2 = 0 \tag{21}$$

### 5-2-4 Hoops of Line Current

**(a) Single hoop**

A circular hoop of radius $a$ centered about the origin in the $xy$ plane carries a constant current $I$, as in Figure 5-10$a$. The distance from any point on the hoop to a point at $z$ along the $z$ axis is

$$r_{QP} = (z^2 + a^2)^{1/2} \tag{22}$$

in the direction

$$\mathbf{i}_{QP} = \frac{(-a\mathbf{i}_r + z\mathbf{i}_z)}{(z^2 + a^2)^{1/2}} \tag{23}$$

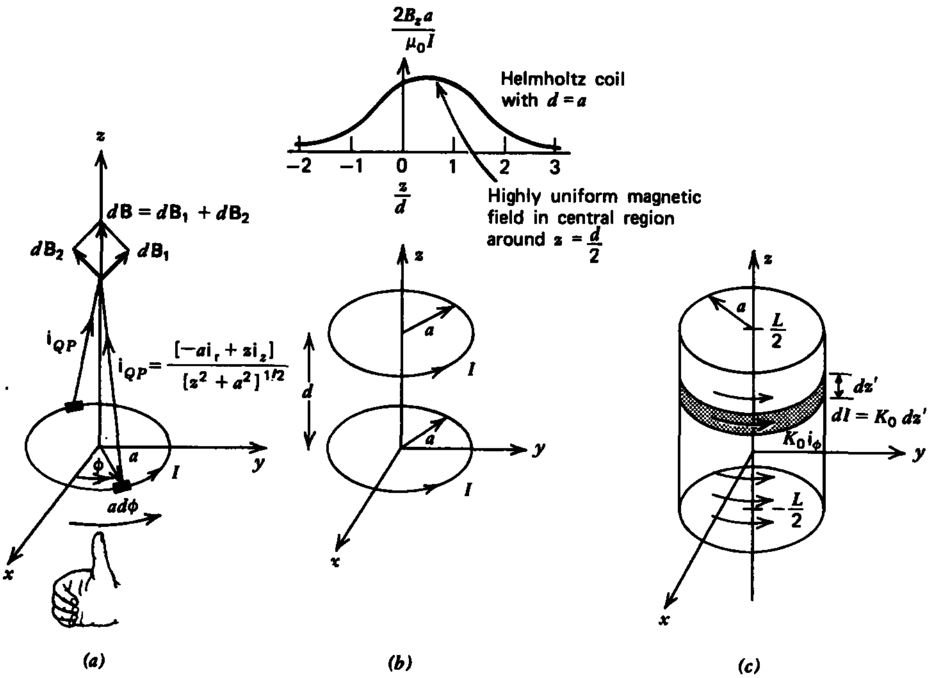



Figure 5-10 (*a*) The magnetic field due to a circular current loop is $z$ directed along the axis of the hoop. (*b*) A Helmholtz coil, formed by two such hoops at a distance apart $d$ equal to their radius, has an essentially uniform field near the center at $z = d/2$. (*c*) The magnetic field on the axis of a cylinder with a $\phi$-directed surface current is found by integrating the fields due to incremental current loops.

so that the incremental magnetic field due to a current element of differential size is

$$\begin{aligned} d\mathbf{B} &= \frac{\mu_0}{4\pi r_{QP}^2} Ia\, d\phi \mathbf{i}_\phi \times \mathbf{i}_{QP} \\ &= \frac{\mu_0 Ia\, d\phi}{4\pi(z^2+a^2)^{3/2}}(a\mathbf{i}_z + z\mathbf{i}_r) \end{aligned} \tag{24}$$

The radial unit vector changes direction as a function of $\phi$, being oppositely directed at $-\phi$, so that the total magnetic field due to the whole hoop is purely $z$ directed:

$$\begin{aligned} B_z &= \frac{\mu_0 I a^2}{4\pi(z^2+a^2)^{3/2}} \int_0^{2\pi} d\phi \\ &= \frac{\mu_0 I a^2}{2(z^2+a^2)^{3/2}} \end{aligned} \tag{25}$$

The direction of the magnetic field can be checked using the right-hand rule. Curling the fingers on the right hand in the direction of the current puts the thumb in the direction of the magnetic field. Note that the magnetic field along the $z$ axis is positively $z$ directed both above and below the hoop.

**(b) Two Hoops (Helmholtz Coil)**

Often it is desired to have an accessible region in space with an essentially uniform magnetic field. This can be arranged by placing another coil at $z = d$, as in Figure 5-10*b*. Then the total magnetic field along the $z$ axis is found by superposing the field of (25) for each hoop:

$$B_z = \frac{\mu_0 I a^2}{2}\left(\frac{1}{(z^2+a^2)^{3/2}} + \frac{1}{((z-d)^2+a^2)^{3/2}}\right) \tag{26}$$

We see then that the slope of $B_z$,

$$\frac{\partial B_z}{\partial z} = \frac{3\mu_0 I a^2}{2}\left(\frac{-z}{(z^2+a^2)^{5/2}} - \frac{(z-d)}{((z-d)^2+a^2)^{5/2}}\right) \tag{27}$$

is zero at $z = d/2$. The second derivative,

$$\frac{\partial^2 B_z}{\partial z^2} = \frac{3\mu_0 I a^2}{2}\left(\frac{5z^2}{(z^2+a^2)^{7/2}} - \frac{1}{(z^2+a^2)^{5/2}} + \frac{5(z-d)^2}{((z-d)^2+a^2)^{7/2}} - \frac{1}{((z-d)^2+a^2)^{5/2}}\right) \tag{28}$$

can also be set to zero at $z = d/2$, if $d = a$, giving a highly uniform field around the center of the system, as plotted in Figure 5-10*b*. Such a configuration is called a Helmholtz coil.

**(c) Hollow Cylinder of Surface Current**

A hollow cylinder of length $L$ and radius $a$ has a uniform surface current $K_0\mathbf{i}_\phi$ as in Figure 5-10*c*. Such a configuration is arranged in practice by tightly winding $N$ turns of a wire around a cylinder and imposing a current $I$ through the wire. Then the current per unit length is

$$K_0 = NI/L \tag{29}$$

The magnetic field along the $z$ axis at the position $z$ due to each incremental hoop at $z'$ is found from (25) by replacing $z$ by $(z - z')$ and $I$ by $K_0\,dz'$:

$$dB_z = \frac{\mu_0 a^2 K_0\,dz'}{2[(z-z')^2+a^2]^{3/2}} \tag{30}$$

The total axial magnetic field is then

$$
\begin{aligned}
B_z &= \int_{z'=-L/2}^{+L/2} \frac{\mu_0 a^2 K_0}{2} \frac{dz'}{[(z-z')^2+a^2]^{3/2}} \\
&= \frac{\mu_0 a^2 K_0}{2} \frac{(z'-z)}{a^2[(z-z')^2+a^2]^{1/2}}\bigg|_{z'=-L/2}^{+L/2} \\
&= \frac{\mu_0 K_0}{2}\left(\frac{-z+L/2}{[(z-L/2)^2+a^2]^{1/2}}+\frac{z+L/2}{[(z+L/2)^2+a^2]^{1/2}}\right)
\end{aligned} \tag{31}
$$

As the cylinder becomes very long, the magnetic field far from the ends becomes approximately constant

$$\lim_{L\to\infty} B_z = \mu_0 K_0 \tag{32}$$

## 5-3 DIVERGENCE AND CURL OF THE MAGNETIC FIELD

Because of our success in examining various vector operations on the electric field, it is worthwhile to perform similar operations on the magnetic field. We will need to use the following vector identities from Section 1-5-4, Problem 1-24 and Sections 2-4-1 and 2-4-2:

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{1}$$

$$\nabla \times (\nabla f) = 0 \tag{2}$$

$$\nabla\left(\frac{1}{r_{QP}}\right) = -\frac{\mathbf{i}_{QP}}{r_{QP}^2} \tag{3}$$

$$\int_V \nabla^2\left(\frac{1}{r_{QP}}\right) dV = \begin{cases} 0, & r_{QP} \neq 0 \\ -4\pi, & r_{QP} = 0 \end{cases} \tag{4}$$

$$\nabla \cdot (\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot\nabla\times\mathbf{B} \tag{5}$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + (\nabla\cdot\mathbf{B})\mathbf{A} - (\nabla\cdot\mathbf{A})\mathbf{B} \tag{6}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) \tag{7}$$

### 5-3-1 Gauss's Law for the Magnetic Field

Using (3) the magnetic field due to a volume distribution of current $\mathbf{J}$ is rewritten as

$$
\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}\times\mathbf{i}_{QP}}{r_{QP}^2}\, dV \\
&= \frac{-\mu_0}{4\pi}\int_V \mathbf{J}\times\nabla\left(\frac{1}{r_{QP}}\right) dV
\end{aligned} \tag{8}
$$

If we take the divergence of the magnetic field with respect to field coordinates, the del operator can be brought inside the integral as the integral is only over the source coordinates:

$$\nabla \cdot \mathbf{B} = \frac{-\mu_0}{4\pi} \int_V \nabla \cdot \left[\mathbf{J} \times \nabla\left(\frac{1}{r_{QP}}\right)\right] dV \tag{9}$$

The integrand can be expanded using (5)

$$\nabla \cdot \left[\mathbf{J} \times \nabla\left(\frac{1}{r_{QP}}\right)\right] = \nabla\left(\frac{1}{r_{QP}}\right) \cdot \underbrace{(\nabla \times \mathbf{J})}_{0} - \mathbf{J} \cdot \underbrace{\nabla \times \left[\nabla\left(\frac{1}{r_{QP}}\right)\right]}_{0} = 0 \tag{10}$$

The first term on the right-hand side in (10) is zero because $\mathbf{J}$ is not a function of field coordinates, while the second term is zero from (2), the curl of the gradient is always zero. Then (9) reduces to

$$\nabla \cdot \mathbf{B} = 0 \tag{11}$$

This contrasts with Gauss's law for the displacement field where the right-hand side is equal to the electric charge density. Since nobody has yet discovered any net magnetic charge, there is no source term on the right-hand side of (11).

The divergence theorem gives us the equivalent integral representation

$$\int_V \nabla \cdot \mathbf{B}\, dV = \oint_S \mathbf{B} \cdot \mathbf{dS} = 0 \tag{12}$$

which tells us that the net magnetic flux through a closed surface is always zero. As much flux enters a surface as leaves it. Since there are no magnetic charges to terminate the magnetic field, the field lines are always closed.

### 5-3-2 Ampere's Circuital Law

We similarly take the curl of (8) to obtain

$$\nabla \times \mathbf{B} = \frac{-\mu_0}{4\pi} \int_V \nabla \times \left[\mathbf{J} \times \nabla\left(\frac{1}{r_{QP}}\right)\right] dV \tag{13}$$

where again the del operator can be brought inside the integral and only operates on $r_{QP}$.

We expand the integrand using (6):

$$\nabla \times \left[\mathbf{J} \times \nabla\left(\frac{1}{r_{QP}}\right)\right] = \left[\nabla\left(\frac{1}{r_{QP}}\right) \cdot \underbrace{\nabla}_{0}\right]\mathbf{J} - (\mathbf{J} \cdot \nabla)\nabla\left(\frac{1}{r_{QP}}\right) + \left[\nabla^2\left(\frac{1}{r_{QP}}\right)\right]\mathbf{J} - \underbrace{(\nabla \cdot \mathbf{J})}_{0}\nabla\left(\frac{1}{r_{QP}}\right) \tag{14}$$

where two terms on the right-hand side are zero because $\mathbf{J}$ is not a function of the field coordinates. Using the identity of (7),

$$\nabla\left[\mathbf{J}\cdot\nabla\left(\frac{1}{r_{QP}}\right)\right]=\left[\nabla\left(\frac{1}{r_{QP}}\right)\underbrace{\cdot\nabla\Big]\mathbf{J}}_{0}+(\mathbf{J}\cdot\nabla)\nabla\left(\frac{1}{r_{QP}}\right)$$
$$+\nabla\left(\frac{1}{r_{QP}}\right)\times\underbrace{(\nabla\times\mathbf{J})}_{0}+\mathbf{J}\times\underbrace{\left[\nabla\times\nabla\left(\frac{1}{r_{QP}}\right)\right]}_{0} \tag{15}$$

the second term on the right-hand side of (14) can be related to a pure gradient of a quantity because the first and third terms on the right of (15) are zero since $\mathbf{J}$ is not a function of field coordinates. The last term in (15) is zero because the curl of a gradient is always zero. Using (14) and (15), (13) can be rewritten as

$$\nabla\times\mathbf{B}=\frac{\mu_0}{4\pi}\int_V\left\{\nabla\left[\mathbf{J}\cdot\nabla\left(\frac{1}{r_{QP}}\right)\right]-\mathbf{J}\nabla^2\left(\frac{1}{r_{QP}}\right)\right\}dV \tag{16}$$

Using the gradient theorem, a corollary to the divergence theorem, (see Problem 1-15$a$), the first volume integral is converted to a surface integral

$$\nabla\times\mathbf{B}=\frac{\mu_0}{4\pi}\left[\underbrace{\int_S\mathbf{J}\cdot\nabla\left(\frac{1}{r_{QP}}\right)d\mathbf{S}}_{0}-\int_V\mathbf{J}\nabla^2\left(\frac{1}{r_{QP}}\right)dV\right] \tag{17}$$

This surface completely surrounds the current distribution so that $S$ is outside in a zero current region where $\mathbf{J}=0$ so that the surface integral is zero. The remaining volume integral is nonzero only when $r_{QP}=0$, so that using (4) we finally obtain

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J} \tag{18}$$

which is known as Ampere's law.

Stokes' theorem applied to (18) results in Ampere's circuital law:

$$\int_S\nabla\times\frac{\mathbf{B}}{\mu_0}\cdot d\mathbf{S}=\oint_L\frac{\mathbf{B}}{\mu_0}\cdot d\mathbf{l}=\int_S\mathbf{J}\cdot d\mathbf{S} \tag{19}$$

Like Gauss's law, choosing the right contour based on symmetry arguments often allows easy solutions for $\mathbf{B}$.

If we take the divergence of both sides of (18), the left-hand side is zero because the divergence of the curl of a vector is always zero. This requires that magnetic field systems have divergence-free currents so that charge cannot accumulate. Currents must always flow in closed loops.

## 5-3-3 Currents With Cylindrical Symmetry

### (a) Surface Current

A surface current $K_0\mathbf{i}_z$ flows on the surface of an infinitely long hollow cylinder of radius $a$. Consider the two symmetrically located line charge elements $dI = K_0\, a d\phi$ and their effective fields at a point $P$ in Figure 5-11$a$. The magnetic field due to both current elements cancel in the radial direction but add in the $\phi$ direction. The total magnetic field can be found by doing a difficult integration over $\phi$. However,

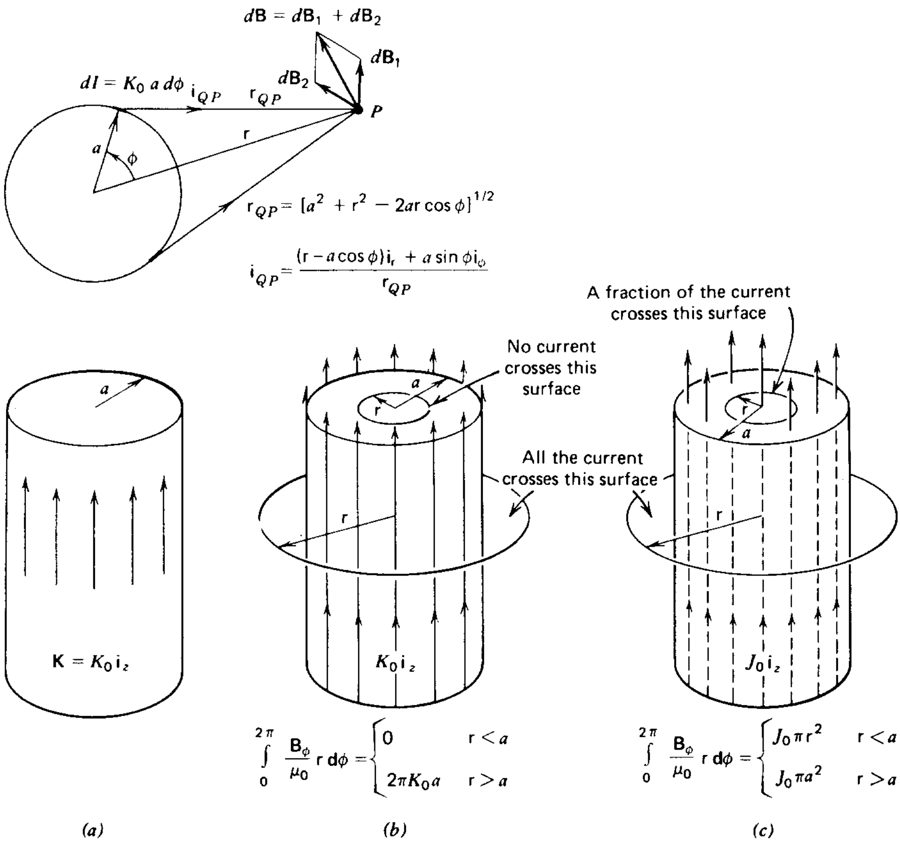



Figure 5-11 ($a$) The magnetic field of an infinitely long cylinder carrying a surface current parallel to its axis can be found using the Biot–Savart law for each incremental line current element. Symmetrically located elements have radial field components that cancel but $\phi$ field components that add. ($b$) Now that we know that the field is purely $\phi$ directed, it is easier to use Ampere's circuital law for a circular contour concentric with the cylinder. For $r < a$ no current passes through the contour while for $r > a$ all the current passes through the contour. ($c$) If the current is uniformly distributed over the cylinder the smaller contour now encloses a fraction of the current.

using Ampere's circuital law of (19) is much easier. Since we know the magnetic field is $\phi$ directed and by symmetry can only depend on r and not $\phi$ or $z$, we pick a circular contour of constant radius r as in Figure 5-11*b*. Since $\mathbf{dl} = \mathrm{r}\, d\phi\, \mathbf{i}_\phi$ is in the same direction as $\mathbf{B}$, the dot product between the magnetic field and $\mathbf{dl}$ becomes a pure multiplication. For $\mathrm{r} < a$ no current passes through the surface enclosed by the contour, while for $\mathrm{r} > a$ all the current is purely perpendicular to the normal to the surface of the contour:

$$\oint_L \frac{\mathbf{B}}{\mu_0} \cdot \mathbf{dl} = \int_0^{2\pi} \frac{B_\phi}{\mu_0} \mathrm{r}\, d\phi = \frac{2\pi \mathrm{r} B_\phi}{\mu_0} = \begin{cases} K_0 2\pi a = I, & \mathrm{r} > a \\ 0, & \mathrm{r} < a \end{cases} \tag{20}$$

where $I$ is the total current on the cylinder.

The magnetic field is thus

$$B_\phi = \begin{cases} \mu_0 K_0 a / r = \mu_0 I/(2\pi r), & r > a \\ 0, & r < a \end{cases} \tag{21}$$

Outside the cylinder, the magnetic field is the same as if all the current was concentrated along the axis as a line current.

**(b) Volume Current**

If the cylinder has the current uniformly distributed over the volume as $J_0 \mathbf{i}_z$, the contour surrounding the whole cylinder still has the total current $I = J_0 \pi a^2$ passing through it. If the contour has a radius smaller than that of the cylinder, only the fraction of current proportional to the enclosed area passes through the surface as shown in Figure 5-11c:

$$\oint_L \frac{B_\phi}{\mu_0} \mathrm{r}\, d\phi = \frac{2\pi \mathrm{r} B_\phi}{\mu_0} = \begin{cases} J_0 \pi a^2 = I, & \mathrm{r} > a \\ J_0 \pi \mathrm{r}^2 = I \mathrm{r}^2 / a^2, & \mathrm{r} < a \end{cases} \tag{22}$$

so that the magnetic field is

$$B_\phi = \begin{cases} \dfrac{\mu_0 J_0 a^2}{2\mathrm{r}} = \dfrac{\mu_0 I}{2\pi \mathrm{r}}, & \mathrm{r} > a \\[2ex] \dfrac{\mu_0 J_0 \mathrm{r}}{2} = \dfrac{\mu_0 I \mathrm{r}}{2\pi a^2}, & \mathrm{r} < a \end{cases} \tag{23}$$

## 5-4 THE VECTOR POTENTIAL

### 5-4-1 Uniqueness

Since the divergence of the magnetic field is zero, we may write the magnetic field as the curl of a vector,

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \mathbf{B} = \nabla \times \mathbf{A} \tag{1}$$

where **A** is called the vector potential, as the divergence of the curl of any vector is always zero. Often it is easier to calculate **A** and then obtain the magnetic field from (1).

From Ampere's law, the vector potential is related to the current density as

$$\nabla\times\mathbf{B}=\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}=\mu_0\mathbf{J} \tag{2}$$

We see that (1) does not uniquely define **A**, as we can add the gradient of any term to **A** and not change the value of the magnetic field, since the curl of the gradient of any function is always zero:

$$\mathbf{A}\rightarrow\mathbf{A}+\nabla f\Rightarrow\mathbf{B}=\nabla\times(\mathbf{A}+\nabla f)=\nabla\times\mathbf{A} \tag{3}$$

Helmholtz's theorem states that to uniquely specify a vector, both its curl and divergence must be specified and that far from the sources, the fields must approach zero. To prove this theorem, let's say that we are given the curl and divergence of **A** and we are to determine what **A** is. Is there any other vector **C**, different from **A** that has the same curl and divergence? We try **C** of the form

$$\mathbf{C}=\mathbf{A}+\mathbf{a} \tag{4}$$

and we will prove that **a** is zero.

By definition, the curl of **C** must equal the curl of **A** so that the curl of **a** must be zero:

$$\nabla\times\mathbf{C}=\nabla\times(\mathbf{A}+\mathbf{a})=\nabla\times\mathbf{A}\Rightarrow\nabla\times\mathbf{a}=0 \tag{5}$$

This requires that **a** be derivable from the gradient of a scalar function $f$:

$$\nabla\times\mathbf{a}=0\Rightarrow\mathbf{a}=\nabla f \tag{6}$$

Similarly, the divergence condition requires that the divergence of **a** be zero,

$$\nabla\cdot\mathbf{C}=\nabla\cdot(\mathbf{A}+\mathbf{a})=\nabla\cdot\mathbf{A}\Rightarrow\nabla\cdot\mathbf{a}=0 \tag{7}$$

so that the Laplacian of $f$ must be zero,

$$\nabla\cdot\mathbf{a}=\nabla^2 f=0 \tag{8}$$

In Chapter 2 we obtained a similar equation and solution for the electric potential that goes to zero far from the charge distribution:

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \Rightarrow V = \int_V \frac{\rho\, dV}{4\pi\varepsilon r_{QP}} \tag{9}$$

If we equate $f$ to $V$, then $\rho$ must be zero giving us that the scalar function $f$ is also zero. That is, the solution to Laplace's equation of (8) for zero sources everywhere is zero, even though Laplace's equation in a region does have nonzero solutions if there are sources in other regions of space. With $f$ zero, from (6) we have that the vector **a** is also zero and then $\mathbf{C} = \mathbf{A}$, thereby proving Helmholtz's theorem.

### 5-4-2 The Vector Potential of a Current Distribution

Since we are free to specify the divergence of the vector potential, we take the simplest case and set it to zero:

$$\nabla \cdot \mathbf{A} = 0 \tag{10}$$

Then (2) reduces to

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \tag{11}$$

Each vector component of (11) is just Poisson's equation so that the solution is also analogous to (9)

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}\, dV}{r_{QP}} \tag{12}$$

The vector potential is often easier to use since it is in the same direction as the current, and we can avoid the often complicated cross product in the Biot–Savart law. For moving point charges, as well as for surface and line currents, we use (12) with the appropriate current elements:

$$\mathbf{J}\, dV \rightarrow \mathbf{K}\, dS \rightarrow \mathbf{I}\, dL \rightarrow q\mathbf{v} \tag{13}$$

### 5-4-3 The Vector Potential and Magnetic Flux

Using Stokes' theorem, the magnetic flux through a surface can be expressed in terms of a line integral of the vector potential:

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS} = \int_S \nabla \times \mathbf{A} \cdot \mathbf{dS} = \oint_L \mathbf{A} \cdot \mathbf{dl} \tag{14}$$

**(a) Finite Length Line Current**

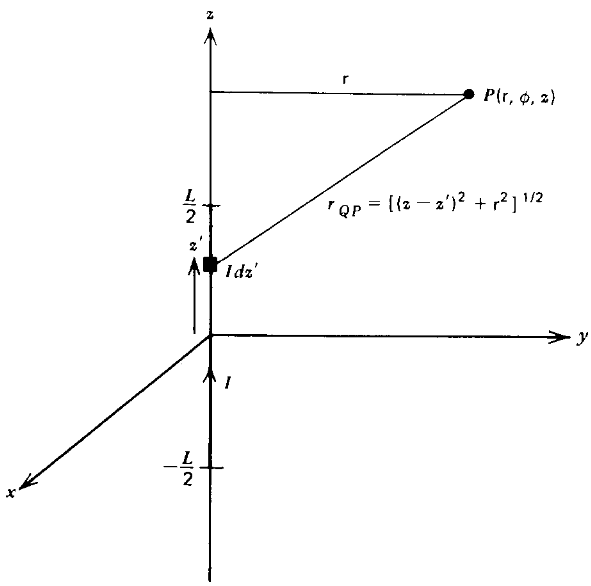
The problem of a line current $I$ of length $L$, as in Figure 5-12$a$ appears to be nonphysical as the current must be continuous. However, we can imagine this line current to be part of a closed loop and we calculate the vector potential and magnetic field from this part of the loop.

The distance $r_{QP}$ from the current element $I\,dz'$ to the field point at coordinate (r, $\phi$, z) is

$$r_{QP} = [(z-z')^2 + r^2]^{1/2} \tag{15}$$

The vector potential is then

$$\begin{aligned} A_z &= \frac{\mu_0 I}{4\pi}\int_{-L/2}^{L/2} \frac{dz'}{[(z-z')^2+r^2]^{1/2}} \\ &= \frac{\mu_0 I}{4\pi}\ln\left(\frac{-z+L/2+[(z-L/2)^2+r^2]^{1/2}}{-(z+L/2)+[(z+L/2)^2+r^2]^{1/2}}\right) \\ &= \frac{\mu_0 I}{4\pi}\left(\sinh^{-1}\frac{-z+L/2}{r}+\sinh^{-1}\frac{z+L/2}{r}\right) \end{aligned} \tag{16}$$




(a)

Figure 5-12 ($a$) The magnetic field due to a finite length line current is most easily found using the vector potential, which is in the direction of the current. This problem is physical only if the line current is considered to be part of a closed loop. ($b$) The magnetic field from a length $w$ of surface current is found by superposing the vector potential of ($a$) with $L\to\infty$. The field lines are lines of constant $A_z$. ($c$) The magnetic flux through a square current loop is in the $-x$ direction by the right-hand rule.

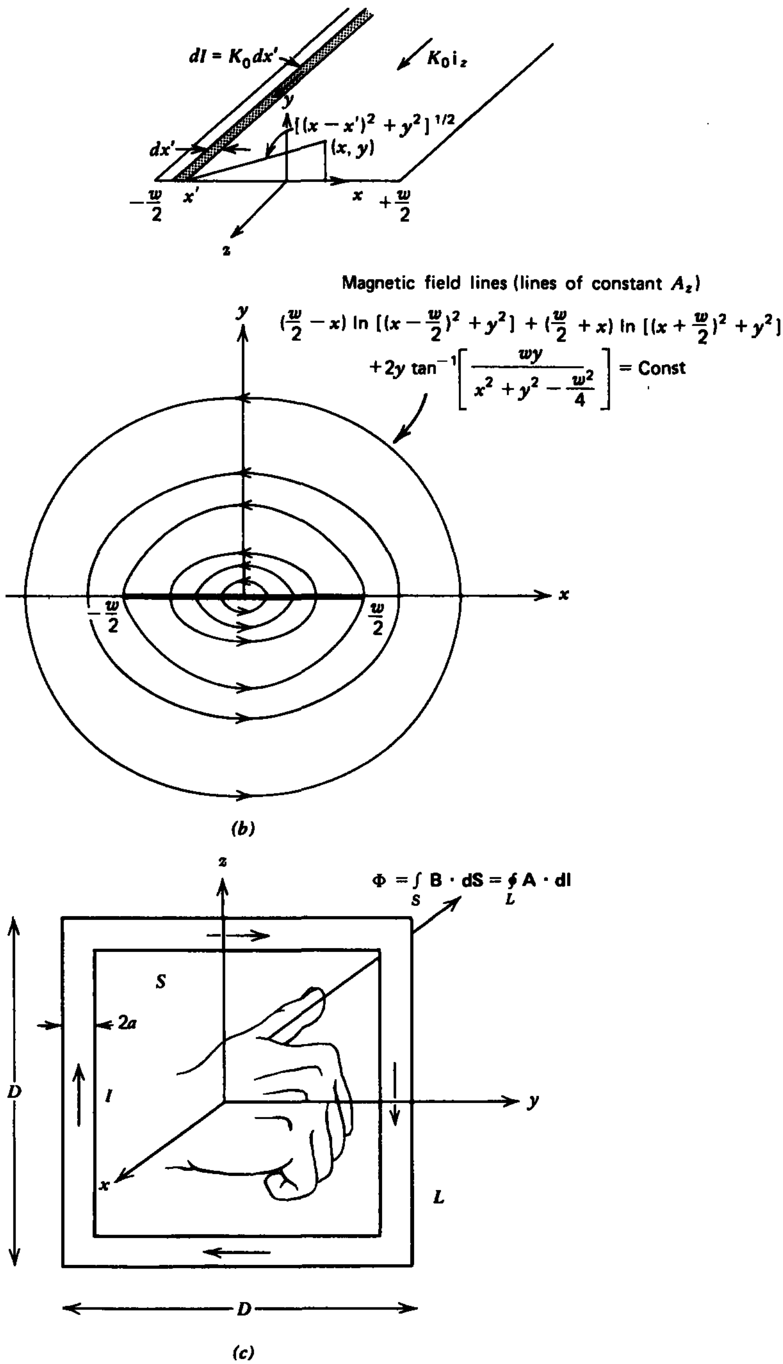

$dI = K_0\,dx'$
$K_0\mathbf{i}_z$
$[(x-x')^2+y^2]^{1/2}$
$dx'$
$(x, y)$
$-\frac{w}{2}$ $x'$ $x$ $+\frac{w}{2}$
$y$
$z$
Magnetic field lines (lines of constant $A_z$)
$(\frac{w}{2}-x)\ln[(x-\frac{w}{2})^2+y^2]+(\frac{w}{2}+x)\ln[(x+\frac{w}{2})^2+y^2]$
$+2y\tan^{-1}\left[\frac{wy}{x^2+y^2-\frac{w^2}{4}}\right]=\text{Const}$
$-\frac{w}{2}$
$\frac{w}{2}$
(b)
$\Phi=\int_S \mathbf{B}\cdot d\mathbf{S}=\oint_L \mathbf{A}\cdot d\mathbf{l}$
$S$
$2a$
$D$
$I$
$L$
(c)

Figure 5-12

with associated magnetic field

$$\begin{aligned}\mathbf{B}&=\nabla\times\mathbf{A}\\&=\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\mathbf{i}_r+\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)\mathbf{i}_\phi+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi)-\frac{\partial A_r}{\partial\phi}\right)\mathbf{i}_z\\&=-\frac{\partial A_z}{\partial r}\mathbf{i}_\phi\\&=\frac{-\mu_0 Ir}{4\pi}\left(\frac{1}{[(z-L/2)^2+r^2]^{1/2}\{-z+L/2+[(z-L/2)^2+r^2]^{1/2}\}}\right.\\&\qquad\left.-\frac{1}{[(z+L/2)^2+r^2]^{1/2}\{-(z+L/2)+[(z+L/2)^2+r^2]^{1/2}\}}\right)\mathbf{i}_\phi\\&=\frac{\mu_0 I}{4\pi r}\left(\frac{-z+L/2}{[r^2+(z-L/2)^2]^{1/2}}+\frac{z+L/2}{[r^2+(z+L/2)^2]^{1/2}}\right)\mathbf{i}_\phi\end{aligned}\tag{17}$$

For large $L$, (17) approaches the field of an infinitely long line current as given in Section 5-2-2:

$$\lim_{L\to\infty}\begin{cases}A_z=\dfrac{-\mu_0 I}{2\pi}\ln r+\text{const}\\[2ex] B_\phi=-\dfrac{\partial A_z}{\partial r}=\dfrac{\mu_0 I}{2\pi r}\end{cases}\tag{18}$$

Note that the vector potential constant in (18) is infinite, but this is unimportant as this constant has no contribution to the magnetic field.

**(b) Finite Width Surface Current**

If a surface current $K_0\mathbf{i}_z$, of width $w$, is formed by laying together many line current elements, as in Figure 5-12*b*, the vector potential at $(x, y)$ from the line current element $K_0\,dx'$ at position $x'$ is given by (18):

$$dA_z=\frac{-\mu_0 K_0\,dx'}{4\pi}\ln[(x-x')^2+y^2]\tag{19}$$

The total vector potential is found by integrating over all elements:

$$\begin{aligned}A_z&=\frac{-\mu_0K_0}{4\pi}\int_{-w/2}^{+w/2}\ln[(x-x')^2+y^2]\,dx'\\&=\frac{-\mu_0K_0}{4\pi}\Bigg((x'-x)\ln[(x-x')^2+y^2]-2(x'-x)\\&\quad+2y\tan^{-1}\frac{(x'-x)}{y}\Bigg)\Bigg|_{-w/2}^{+w/2}\\&=\frac{-\mu_0K_0}{4\pi}\Bigg\{\left(\frac{w}{2}-x\right)\ln\left[\left(x-\frac{w}{2}\right)^2+y^2\right]\\&\quad+\left(\frac{w}{2}+x\right)\ln\left[\left(x+\frac{w}{2}\right)^2+y^2\right]\\&\quad-2w+2y\tan^{-1}\frac{wy}{y^2+x^2-w^2/4}\Bigg\}\end{aligned}\tag{20*}$$

The magnetic field is then

$$\begin{aligned}\mathbf{B}&=\mathbf{i}_x\frac{\partial A_z}{\partial y}-\mathbf{i}_y\frac{\partial A_z}{\partial x}\\&=\frac{-\mu_0K_0}{4\pi}\left(2\tan^{-1}\frac{wy}{y^2+x^2-w^2/4}\mathbf{i}_x+\ln\frac{(x+w/2)^2+y^2}{(x-w/2)^2+y^2}\mathbf{i}_y\right)\end{aligned}\tag{21}$$

The vector potential in two-dimensional geometries is also useful in plotting field lines,

$$\frac{dy}{dx}=\frac{B_y}{B_x}=\frac{-\partial A_z/\partial x}{\partial A_z/\partial y}\tag{22}$$

for if we cross multiply (22),

$$\frac{\partial A_z}{\partial x}dx+\frac{\partial A_z}{\partial y}dy=dA_z=0\Rightarrow A_z=\text{const}\tag{23}$$

we see that it is constant on a field line. The field lines in Figure 5-12*b* are just lines of constant $A_z$. The vector potential thus plays the same role as the electric stream function in Sections 4.3.2*b* and 4.4.3*b*.

**(c) Flux Through a Square Loop**

The vector potential for the square loop in Figure 5-12*c* with very small radius $a$ is found by superposing (16) for each side with each component of **A** in the same direction as the current in each leg. The resulting magnetic field is then given by four

$^*\tan^{-1}(a-b)+\tan^{-1}(a+b)=\tan^{-1}\dfrac{2a}{1-a^2+b^2}$

terms like that in (17) so that the flux can be directly computed by integrating the normal component of **B** over the loop area. This method is straightforward but the algebra is cumbersome.

An easier method is to use (14) since we already know the vector potential along each leg. We pick a contour that runs along the inside wire boundary at small radius $a$. Since each leg is identical, we only have to integrate over one leg, then multiply the result by 4:

$$\begin{aligned}
\Phi &= 4\int_{\substack{a-D/2\\ r=a}}^{-a+D/2} A_z\,dz\\
&= \frac{\mu_0 I}{\pi}\int_{a-D/2}^{-a+D/2}\left(\sinh^{-1}\frac{-z+D/2}{a}+\sinh^{-1}\frac{z+D/2}{a}\right)dz\\
&= \frac{\mu_0 I}{\pi}\left\{-\left(\frac{D}{2}-z\right)\sinh^{-1}\frac{-z+D/2}{a}+\left[\left(\frac{D}{2}-z\right)^2+a^2\right]^{1/2}\right.\\
&\quad\left.+\left(\frac{D}{2}+z\right)\sinh^{-1}\frac{z+D/2}{a}-\left[\left(\frac{D}{2}+z\right)^2+a^2\right]^{1/2}\right\}\Bigg|_{a-D/2}^{-a+D/2}\\
&= 2\frac{\mu_0 I}{\pi}\left(-a\sinh^{-1}1+a\sqrt{2}+(D-a)\sinh^{-1}\frac{D-a}{a}\right.\\
&\quad\left.-[(D-a)^2+a^2]^{1/2}\right)
\end{aligned}\tag{24}$$

As $a$ becomes very small, (24) reduces to

$$\lim_{a\to 0}\Phi = 2\frac{\mu_0 I}{\pi}D\left(\sinh^{-1}\left(\frac{D}{a}\right)-1\right)\tag{25}$$

We see that the flux through the loop is proportional to the current. This proportionality constant is called the self-inductance and is only a function of the geometry:

$$L=\frac{\Phi}{I}=2\frac{\mu_0 D}{\pi}\left(\sinh^{-1}\left(\frac{D}{a}\right)-1\right)\tag{26}$$

Inductance is more fully developed in Chapter 6.

## 5-5 MAGNETIZATION

Our development thus far has been restricted to magnetic fields in free space arising from imposed current distributions. Just as small charge displacements in dielectric materials contributed to the electric field, atomic motions constitute microscopic currents, which also contribute to the magnetic field. There is a direct analogy between polarization and magnetization, so our development will parallel that of Section 3-1.

### 5-5-1 The Magnetic Dipole

Classical atomic models describe an atom as orbiting electrons about a positively charged nucleus, as in Figure 5-13.

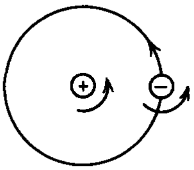


Figure 5-13 Atomic currents arise from orbiting electrons in addition to the spin contributions from the electron and nucleus.

The nucleus and electron can also be imagined to be spinning. The simplest model for these atomic currents is analogous to the electric dipole and consists of a small current loop of area $d\mathbf{S}$ carrying a current $I$, as in Figure 5-14. Because atomic dimensions are so small, we are only interested in the magnetic field far from this magnetic dipole. Then the shape of the loop is not important, thus for simplicity we take it to be rectangular.

The vector potential for this loop is then

$$\mathbf{A}=\frac{\mu_0 I}{4\pi}\left[dx\left(\frac{1}{r_3}-\frac{1}{r_1}\right)\mathbf{i}_x+dy\left(\frac{1}{r_4}-\frac{1}{r_2}\right)\mathbf{i}_y\right] \tag{1}$$

where we assume that the distance from any point on each side of the loop to the field point $P$ is approximately constant.

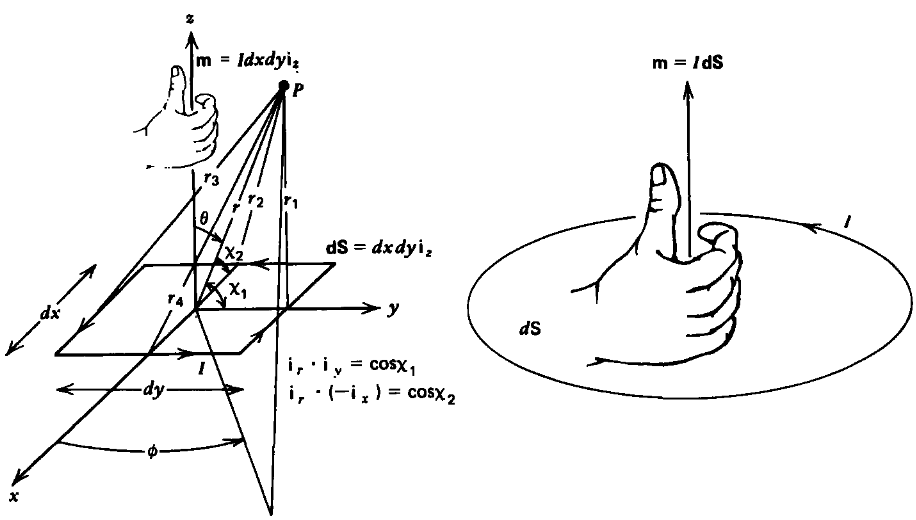



Figure 5-14 A magnetic dipole consists of a small circulating current loop. The magnetic moment is in the direction normal to the loop by the right-hand rule.

Using the law of cosines, these distances are related as

$$
\begin{aligned}
r_1^2 &= r^2 + \left(\frac{dy}{2}\right)^2 - r\,dy\cos\chi_1; \qquad & r_2^2 &= r^2 + \left(\frac{dx}{2}\right)^2 - r\,dx\cos\chi_2 \\
r_3^2 &= r^2 + \left(\frac{dy}{2}\right)^2 + r\,dy\cos\chi_1, \qquad & r_4^2 &= r^2 + \left(\frac{dx}{2}\right)^2 + r\,dx\cos\chi_2
\end{aligned}
\tag{2}
$$

where the angles $\chi_1$ and $\chi_2$ are related to the spherical coordinates from Table 1-2 as

$$
\begin{aligned}
\mathbf{i}_r \cdot \mathbf{i}_y &= \cos\chi_1 = \sin\theta\sin\phi \\
-\mathbf{i}_r \cdot \mathbf{i}_x &= \cos\chi_2 = -\sin\theta\cos\phi
\end{aligned}
\tag{3}
$$

In the far field limit (1) becomes

$$
\begin{aligned}
\lim_{\substack{r \gg dx \\ r \gg dy}} \mathbf{A} = \frac{\mu_0 I}{4\pi}\Bigg[\frac{dx}{r}\Bigg(&\frac{1}{\left[1+\frac{dy}{2r}\left(\frac{dy}{2r}+2\cos\chi_1\right)\right]^{1/2}} \\
&-\frac{1}{\left[1+\frac{dy}{2r}\left(\frac{dy}{2r}-2\cos\chi_1\right)\right]^{1/2}}\Bigg) \\
+\frac{dy}{r}\Bigg(&\frac{1}{\left[1+\frac{dx}{2r}\left(\frac{dx}{2r}+2\cos\chi_2\right)\right]^{1/2}} \\
&-\frac{1}{\left[1+\frac{dx}{2r}\left(\frac{dx}{2r}-2\cos\chi_2\right)\right]^{1/2}}\Bigg)\Bigg] \\
\approx \frac{-\mu_0 I}{4\pi r^2}\, dx\, dy&[\cos\chi_1 \mathbf{i}_x + \cos\chi_2 \mathbf{i}_y]
\end{aligned}
\tag{4}
$$

Using (3), (4) further reduces to

$$
\begin{aligned}
\mathbf{A} &= \frac{\mu_0 I\, dS}{4\pi r^2}\sin\theta[-\sin\phi\,\mathbf{i}_x + \cos\phi\,\mathbf{i}_y] \\
&= \frac{\mu_0 I\, dS}{4\pi r^2}\sin\theta\,\mathbf{i}_\phi
\end{aligned}
\tag{5}
$$

where we again used Table 1-2 to write the bracketed Cartesian unit vector term as $\mathbf{i}_\phi$. The magnetic dipole moment **m** is defined as the vector in the direction perpendicular to the loop (in this case $\mathbf{i}_z$) by the right-hand rule with magnitude equal to the product of the current and loop area:

$$
\mathbf{m} = I\, dS\, \mathbf{i}_z = I\, \mathbf{dS} \tag{6}
$$

Then the vector potential can be more generally written as

$$\mathbf{A} = \frac{\mu_0 m}{4\pi r^2} \sin\theta \mathbf{i}_\phi = \frac{\mu_0 \mathbf{m}}{4\pi r^2} \times \mathbf{i}_r \tag{7}$$

with associated magnetic field

$$\begin{aligned}\mathbf{B} = \nabla \times \mathbf{A} &= \frac{1}{r\sin\theta} \frac{\partial}{\partial\theta}(A_\phi \sin\theta)\mathbf{i}_r - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\mathbf{i}_\theta \\ &= \frac{\mu_0 m}{4\pi r^3}[2\cos\theta \mathbf{i}_r + \sin\theta \mathbf{i}_\theta]\end{aligned} \tag{8}$$

This field is identical in form to the electric dipole field of Section 3-1-1 if we replace $p/\varepsilon_0$ by $\mu_0 m$.

### 5-5-2 Magnetization Currents

Ampere modeled magnetic materials as having the volume filled with such infinitesimal circulating current loops with number density $N$, as illustrated in Figure 5-15. The magnetization vector $\mathbf{M}$ is then defined as the magnetic dipole density:

$$\mathbf{M} = N\mathbf{m} = NI\, \mathbf{dS} \text{ amp/m} \tag{9}$$

For the differential sized contour in the $xy$ plane shown in Figure 5-15, only those dipoles with moments in the $x$ or $y$ directions (thus $z$ components of currents) will give rise to currents crossing perpendicularly through the surface bounded by the contour. Those dipoles completely within the contour give no net current as the current passes through the contour twice, once in the positive $z$ direction and on its return in the negative $z$ direction. Only those dipoles on either side of the edges—so that the current only passes through the contour once, with the return outside the contour—give a net current through the loop.

Because the length of the contour sides $\Delta x$ and $\Delta y$ are of differential size, we assume that the dipoles along each edge do not change magnitude or direction. Then the net total current linked by the contour near each side is equal to the product of the current per dipole $I$ and the number of dipoles that just pass through the contour once. If the normal vector to the dipole loop (in the direction of $\mathbf{m}$) makes an angle $\theta$ with respect to the direction of the contour side at position $x$, the net current linked along the line at $x$ is

$$-IN\, dS\, \Delta y \cos\theta|_x = -M_y(x)\, \Delta y \tag{10}$$

The minus sign arises because the current within the contour adjacent to the line at coordinate $x$ flows in the $-z$ direction.

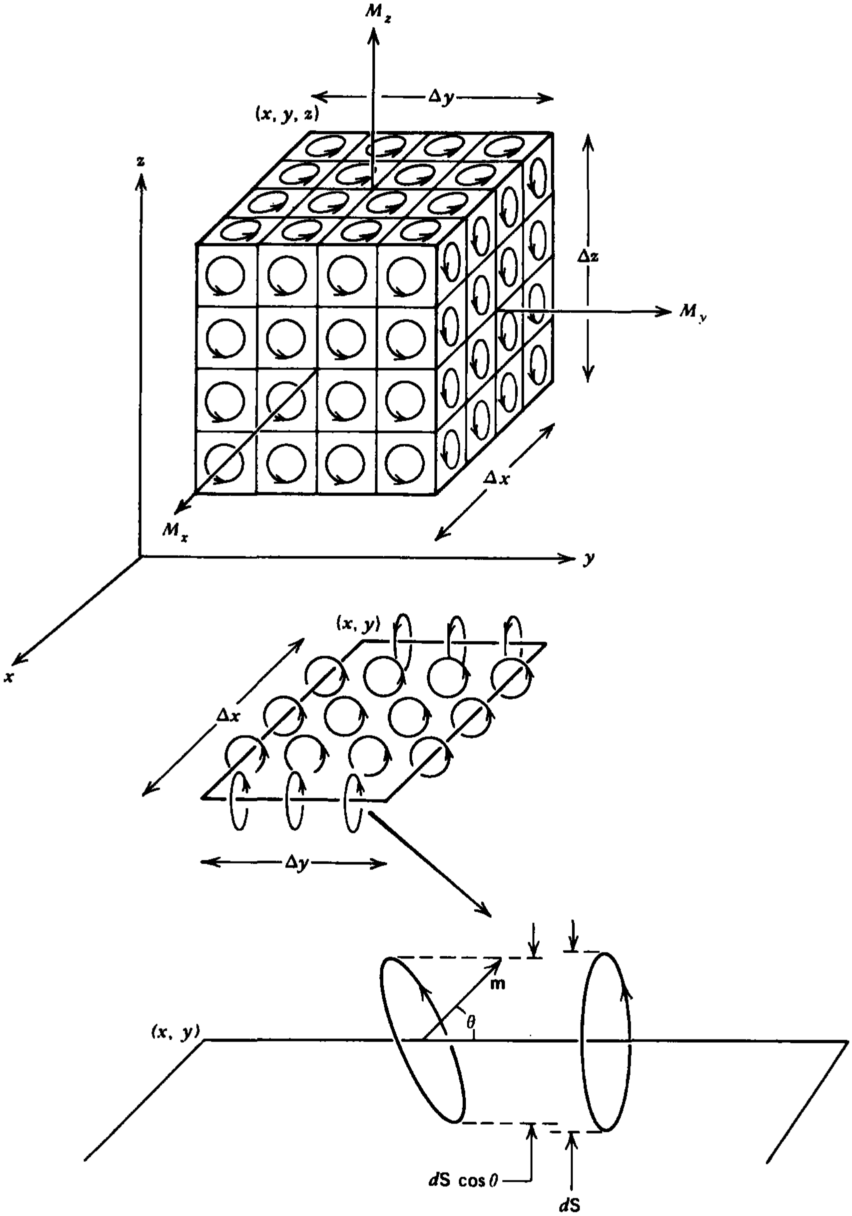



Figure 5-15 Many such magnetic dipoles within a material linking a closed contour gives rise to an effective magnetization current that is also a source of the magnetic field.

Similarly, near the edge at coordinate $x+\Delta x$, the net current linked perpendicular to the contour is

$$IN\,dS\,\Delta y\cos\theta|_{x+\Delta x}=M_y(x+\Delta x)\,\Delta y \qquad (11)$$

Along the edges at $y$ and $y+\Delta y$, the current contributions are

$$IN\,dS\,\Delta x\cos\theta|_y = M_x(y)\,\Delta x$$
$$-IN\,dS\,\Delta x\cos\theta|_{y+\Delta y} = -M_x(y+\Delta y)\,\Delta x \qquad (12)$$

The total current in the $z$ direction linked by this contour is thus the sum of contributions in (10)–(12):

$$I_{z\,\text{tot}} = \Delta x\,\Delta y\left(\frac{M_y(x+\Delta x)-M_y(x)}{\Delta x}-\frac{M_x(y+\Delta y)-M_x(y)}{\Delta y}\right) \qquad (13)$$

If the magnetization is uniform, the net total current is zero as the current passing through the loop at one side is canceled by the current flowing in the opposite direction at the other side. Only if the magnetization changes with position can there be a net current through the loop's surface. This can be accomplished if either the current per dipole, area per dipole, density of dipoles, or angle of orientation of the dipoles is a function of position.

In the limit as $\Delta x$ and $\Delta y$ become small, terms on the right-hand side in (13) define partial derivatives so that the current per unit area in the $z$ direction is

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} J_z = \frac{I_{z\,\text{tot}}}{\Delta x\,\Delta y} = \left(\frac{\partial M_y}{\partial x}-\frac{\partial M_x}{\partial y}\right) = (\nabla\times\mathbf{M})_z \qquad (14)$$

which we recognize as the $z$ component of the curl of the magnetization. If we had orientated our loop in the $xz$ or $yz$ planes, the current density components would similarly obey the relations

$$J_y = \left(\frac{\partial M_x}{\partial z}-\frac{\partial M_z}{\partial x}\right) = (\nabla\times\mathbf{M})_y$$
$$J_x = \left(\frac{\partial M_z}{\partial y}-\frac{\partial M_y}{\partial z}\right) = (\nabla\times\mathbf{M})_x \qquad (15)$$

so that in general

$$\mathbf{J}_m = \nabla\times\mathbf{M} \qquad (16)$$

where we subscript the current density with an $m$ to represent the magnetization current density, often called the Amperian current density.

These currents are also sources of the magnetic field and can be used in Ampere's law as

$$\nabla\times\frac{\mathbf{B}}{\mu_0} = \mathbf{J}_m+\mathbf{J}_f = \mathbf{J}_f+\nabla\times\mathbf{M} \qquad (17)$$

where $\mathbf{J}_f$ is the free current due to the motion of free charges as contrasted to the magnetization current $\mathbf{J}_m$, which is due to the motion of bound charges in materials.

As we can only impose free currents, it is convenient to define the vector **H** as the magnetic field intensity to be distinguished from **B**, which we will now call the magnetic flux density:

$$\mathbf{H}=\frac{\mathbf{B}}{\mu_0}-\mathbf{M} \Rightarrow \mathbf{B}=\mu_0(\mathbf{H}+\mathbf{M}) \tag{18}$$

Then (17) can be recast as

$$\nabla \times\left(\frac{\mathbf{B}}{\mu_0}-\mathbf{M}\right)=\nabla \times \mathbf{H}=\mathbf{J}_f \tag{19}$$

The divergence and flux relations of Section 5-3-1 are unchanged and are in terms of the magnetic flux density **B**. In free space, where $\mathbf{M}=0$, the relation of (19) between **B** and **H** reduces to

$$\mathbf{B}=\mu_0 \mathbf{H} \tag{20}$$

This is analogous to the development of the polarization with the relationships of **D**, **E**, and **P**. Note that in (18), the constant parameter $\mu_0$ multiplies both **H** and **M**, unlike the permittivity $\varepsilon_0$ which only multiplies **E**.

Equation (19) can be put into an equivalent integral form using Stokes' theorem:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}=\oint_L \mathbf{H} \cdot d\mathbf{l}=\int_S \mathbf{J}_f \cdot d\mathbf{S} \tag{21}$$

The free current density $\mathbf{J}_f$ is the source of the **H** field, the magnetization current density $\mathbf{J}_m$ is the source of the **M** field, while the total current, $\mathbf{J}_f+\mathbf{J}_m$, is the source of the **B** field.

## 5-5-3 Magnetic Materials

There are direct analogies between the polarization processes found in dielectrics and magnetic effects. The constitutive law relating the magnetization **M** to an applied magnetic field **H** is found by applying the Lorentz force to our atomic models.

**(a) Diamagnetism**

The orbiting electrons as atomic current loops is analogous to electronic polarization, with the current in the direction opposite to their velocity. If the electron ($e=1.6 \times 10^{-19}$ coul) rotates at angular speed $\omega$ at radius $R$, as in Figure 5-16, the current and dipole moment are

$$I=\frac{e\omega}{2\pi}, \qquad m=I\pi R^2=\frac{e\omega}{2} R^2 \tag{22}$$

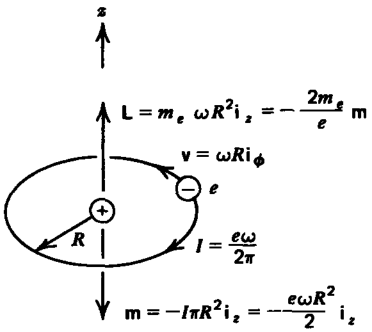



Figure 5-16 The orbiting electron has its magnetic moment **m** in the direction opposite to its angular momentum **L** because the current is opposite to the electron's velocity.

Note that the angular momentum **L** and magnetic moment **m** are oppositely directed and are related as

$$\mathbf{L} = m_e R\mathbf{i}_r \times \mathbf{v} = m_e\omega R^2\mathbf{i}_z = -\frac{2m_e}{e}\mathbf{m} \tag{23}$$

where $m_e = 9.1 \times 10^{-31}$ kg is the electron mass.

Since quantum theory requires the angular momentum to be quantized in units of $h/2\pi$, where Planck's constant is $h = 6.62 \times 10^{-34}$ joule-sec, the smallest unit of magnetic moment, known as the Bohr magneton, is

$$m_B = \frac{eh}{4\pi m_e} \approx 9.3 \times 10^{-24}\ \text{amp-m}^2 \tag{24}$$

Within a homogeneous material these dipoles are randomly distributed so that for every electron orbiting in one direction, another electron nearby is orbiting in the opposite direction so that in the absence of an applied magnetic field there is no net magnetization.

The Coulombic attractive force on the orbiting electron towards the nucleus with atomic number $Z$ is balanced by the centrifugal force:

$$m_e\omega^2 R = \frac{Ze^2}{4\pi\varepsilon_0 R^2} \tag{25}$$

Since the left-hand side is just proportional to the square of the quantized angular momentum, the orbit radius $R$ is also quantized for which the smallest value is

$$R = \frac{4\pi\varepsilon_0}{m_e Z e^2}\left(\frac{h}{2\pi}\right)^2 \approx \frac{5 \times 10^{-11}}{Z}\ \text{m} \tag{26}$$

with resulting angular speed

$$\omega = \frac{Z^2 e^4 m_e}{(4\pi\varepsilon_0)^2 (h/2\pi)^3} \approx 1.3\times 10^{16} Z^2 \tag{27}$$

When a magnetic field $H_0\mathbf{i}_z$ is applied, as in Figure 5-17, electron loops with magnetic moment opposite to the field feel an additional radial force inwards, while loops with colinear moment and field feel a radial force outwards. Since the orbital radius $R$ cannot change because it is quantized, this magnetic force results in a change of orbital speed $\Delta\omega$:

$$m_e(\omega+\Delta\omega_1)^2 R = e\left(\frac{Ze}{4\pi\varepsilon_0 R^2} + (\omega+\Delta\omega_1) R\mu_0 H_0\right)$$

$$m_e(\omega+\Delta\omega_2)^2 R = e\left(\frac{Ze}{4\pi\varepsilon_0 R^2} - (\omega+\Delta\omega_2) R\mu_0 H_0\right) \tag{28}$$

where the first electron speeds up while the second one slows down.

Because the change in speed $\Delta\omega$ is much less than the natural speed $\omega$, we solve (28) approximately as

$$\begin{aligned} \Delta\omega_1 &= \frac{e\omega\mu_0 H_0}{2m_e\omega - e\mu_0 H_0} \\ \Delta\omega_2 &= \frac{-e\omega\mu_0 H_0}{2m_e\omega + e\mu_0 H_0} \end{aligned} \tag{29}$$

where we neglect quantities of order $(\Delta\omega)^2$. However, even with very high magnetic field strengths of $H_0 = 10^6$ amp/m we see that usually

$$\begin{gathered} e\mu_0 H_0 \ll 2m_e\omega_0 \\ (1.6\times 10^{-19})(4\pi\times 10^{-7})10^6 \ll 2(9.1\times 10^{-31})(1.3\times 10^{16}) \end{gathered} \tag{30}$$

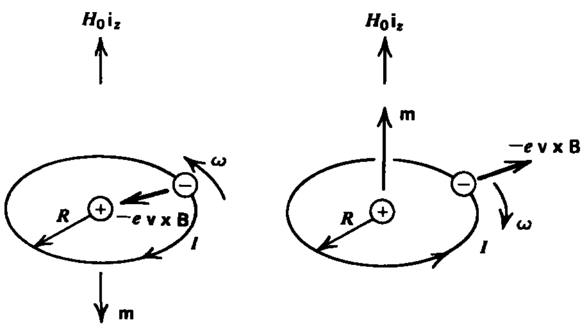



Figure 5-17 Diamagnetic effects, although usually small, arise in all materials because dipoles with moments parallel to the magnetic field have an increase in the orbiting electron speed while those dipoles with moments opposite to the field have a decrease in speed. The loop radius remains constant because it is quantized.

so that (29) further reduces to

$$\Delta\omega_1 \approx -\Delta\omega_2 \approx \frac{e\mu_0 H_0}{2m_e} \approx 1.1\times 10^5 H_0 \tag{31}$$

The net magnetic moment for this pair of loops,

$$m = \frac{eR^2}{2}(\omega_2-\omega_1) = -eR^2\,\Delta\omega_1 = \frac{-e^2\mu_0 R^2}{2m_e} H_0 \tag{32}$$

is opposite in direction to the applied magnetic field.

If we have $N$ such loop pairs per unit volume, the magnetization field is

$$\mathbf{M} = N\mathbf{m} = -\frac{Ne^2\mu_0 R^2}{2m_e} H_0 \mathbf{i}_z \tag{33}$$

which is also oppositely directed to the applied magnetic field.

Since the magnetization is linearly related to the field, we define the magnetic susceptibility $\chi_m$ as

$$\mathbf{M} = \chi_m \mathbf{H}, \qquad \chi_m = -\frac{Ne^2\mu_0 R^2}{2m_e} \tag{34}$$

where $\chi_m$ is negative. The magnetic flux density is then

$$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0(1+\chi_m)\mathbf{H} = \mu_0\mu_r\mathbf{H} = \mu\mathbf{H} \tag{35}$$

where $\mu_r = 1+\chi_m$ is called the relative permeability and $\mu$ is the permeability. In free space $\chi_m = 0$ so that $\mu_r = 1$ and $\mu = \mu_0$. The last relation in (35) is usually convenient to use, as all the results in free space are still correct within linear permeable material if we replace $\mu_0$ by $\mu$. In diamagnetic materials, where the susceptibility is negative, we have that $\mu_r < 1$, $\mu < \mu_0$. However, substituting in our typical values

$$\chi_m = -\frac{Ne^2\mu_0 R^2}{2m\cdot} \approx \frac{4.4\times 10^{-35}}{Z^2} N \tag{36}$$

we see that even with $N \approx 10^{30}$ atoms/m$^3$, $\chi_m$ is much less than unity so that diamagnetic effects are very small.

**(b) Paramagnetism**

As for orientation polarization, an applied magnetic field exerts a torque on each dipole tending to align its moment with the field, as illustrated for the rectangular magnetic dipole with moment at an angle $\theta$ to a uniform magnetic field $\mathbf{B}$ in Figure 5-18*a*. The force on each leg is

$$\begin{aligned} \mathbf{df}_1 &= -\mathbf{df}_2 = I\,\Delta x\,\mathbf{i}_x\times\mathbf{B} = I\,\Delta x[B_y\mathbf{i}_z - B_z\mathbf{i}_y] \\ \mathbf{df}_3 &= -\mathbf{df}_4 = I\,\Delta y\,\mathbf{i}_y\times\mathbf{B} = I\,\Delta y(-B_x\mathbf{i}_z + B_z\mathbf{i}_x) \end{aligned} \tag{37}$$

In a uniform magnetic field, the forces on opposite legs are equal in magnitude but opposite in direction so that the net

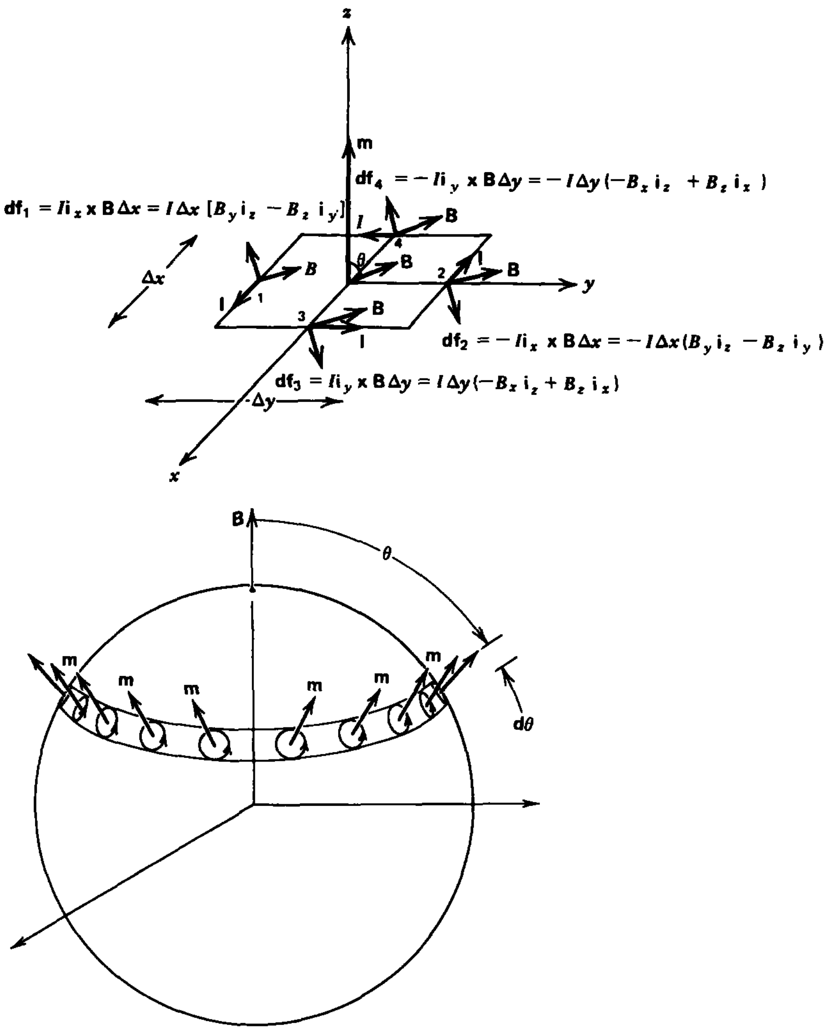



Figure 5-18 (*a*) A torque is exerted on a magnetic dipole with moment at an angle $\theta$ to an applied magnetic field. (*b*) From Boltzmann statistics, thermal agitation opposes the alignment of magnetic dipoles. All the dipoles at an angle $\theta$, together have a net magnetization in the direction of the applied field.

force on the loop is zero. However, there is a torque:

$$\begin{aligned}\mathbf{T} &= \sum_{n=1}^{4} \mathbf{r}\times d\mathbf{f}_n\\ &= \frac{\Delta y}{2}(-\mathbf{i}_y\times d\mathbf{f}_1+\mathbf{i}_y\times d\mathbf{f}_2)+\frac{\Delta x}{2}(\mathbf{i}_x\times d\mathbf{f}_3-\mathbf{i}_x\times d\mathbf{f}_4)\\ &= I\,\Delta x\,\Delta y(B_x\mathbf{i}_y-B_y\mathbf{i}_x)=\mathbf{m}\times\mathbf{B} \qquad (38)\end{aligned}$$

The incremental amount of work necessary to turn the dipole by a small angle $d\theta$ is

$$dW = T\,d\theta = m\mu_0 H_0 \sin\theta\, d\theta \tag{39}$$

so that the total amount of work necessary to turn the dipole from $\theta = 0$ to any value of $\theta$ is

$$W = \int_0^\theta T\,d\theta = -m\mu_0 H_0 \cos\theta\big|_0^\theta = m\mu_0 H_0(1-\cos\theta) \tag{40}$$

This work is stored as potential energy, for if the dipole is released it will try to orient itself with its moment parallel to the field. Thermal agitation opposes this alignment where Boltzmann statistics describes the number density of dipoles having energy $W$ as

$$n = n_1 e^{-W/kT} = n_1 e^{-m\mu_0 H_0(1-\cos\theta)/kT} = n_0 e^{m\mu_0 H_0 \cos\theta/kT} \tag{41}$$

where we lump the constant energy contribution in (40) within the amplitude $n_0$, which is found by specifying the average number density of dipoles $N$ within a sphere of radius $R$:

$$\begin{aligned} N &= \frac{1}{\frac{4}{3}\pi R^3}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{r=0}^{R} n_0\, e^{a\cos\theta}\, r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= \frac{n_0}{2}\int_{\theta=0}^{\pi} \sin\theta\, e^{a\cos\theta}\, d\theta \end{aligned} \tag{42}$$

where we let

$$a = m\mu_0 H_0/kT \tag{43}$$

With the change of variable

$$u = a\cos\theta, \qquad du = -a\sin\theta\, d\theta \tag{44}$$

the integration in (42) becomes

$$N = \frac{-n_0}{2a}\int_a^{-a} e^u\, du = \frac{n_0}{a}\sinh a \tag{45}$$

so that (41) becomes

$$n = \frac{Na}{\sinh a}\, e^{a\cos\theta} \tag{46}$$

From Figure 5-18*b* we see that all the dipoles in the shell over the interval $\theta$ to $\theta + d\theta$ contribute to a net magnetization. which is in the direction of the applied magnetic field:

$$dM = \frac{mn}{\frac{4}{3}\pi R^3}\cos\theta\, r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{47}$$

so that the total magnetization due to all the dipoles within the sphere is

$$M=\frac{maN}{2\sinh a}\int_{\theta=0}^{\pi}\sin\theta\cos\theta\, e^{a\cos\theta}\,d\theta \tag{48}$$

Again using the change of variable in (44), (48) integrates to

$$\begin{aligned}M&=\frac{-mN}{2a\sinh a}\int_{a}^{-a}ue^{u}\,du\\&=\frac{-mN}{2a\sinh a}e^{u}(u-1)\Big|_{a}^{-a}\\&=\frac{-mN}{2a\sinh a}[e^{-a}(-a-1)-e^{a}(a-1)]\\&=\frac{-mN}{a\sinh a}[-a\cosh a+\sinh a]\\&=mN[\coth a-1/a]\end{aligned} \tag{49}$$

which is known as the Langevin equation and is plotted as a function of reciprocal temperature in Figure 5-19. At low temperatures (high $a$) the magnetization saturates at $M=mN$ as all the dipoles have their moments aligned with the field. At room temperature, $a$ is typically very small. Using the parameters in (26) and (27) in a strong magnetic field of $H_0=10^6$ amps/m, $a$ is much less than unity:

$$a=\frac{m\mu_0H_0}{kT}=\frac{e\omega}{2}R^2\frac{\mu_0H_0}{kT}\approx 8\times10^{-4} \tag{50}$$

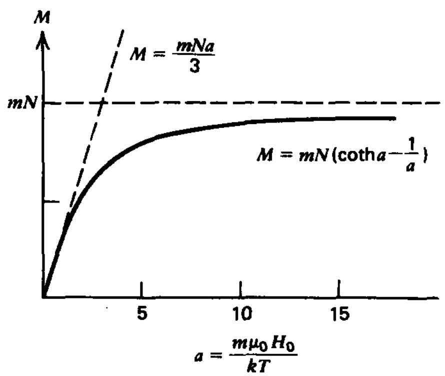



Figure 5-19 The Langevin equation describes the net magnetization. At low temperatures (high $a$) all the dipoles align with the field causing saturation. At high temperatures ($a\ll1$) the magnetization increases linearly with field.

In this limit, Langevin's equation simplifies to

$$\lim_{a\ll 1} M \approx mN\left[\frac{1+a^2/2}{a+a^3/6}-\frac{1}{a}\right]$$
$$\approx mN\left(\frac{(1+a^2/2)(1-a^3/6)}{a}-\frac{1}{a}\right]$$
$$\approx \frac{mNa}{3} \approx \frac{\mu_0 m^2 N}{3kT} H_0 \tag{51}$$

In this limit the magnetic susceptibility $\chi_m$ is positive:

$$\mathbf{M} = \chi_m \mathbf{H}, \qquad \chi_m = \frac{\mu_0 m^2 N}{3kT} \tag{52}$$

but even with $N \approx 10^{30}$ atoms/m$^3$, it is still very small:

$$\chi_m \approx 7 \times 10^{-4} \tag{53}$$

**(c) Ferromagnetism**

As for ferroelectrics (see Section 3-1-5), sufficiently high coupling between adjacent magnetic dipoles in some iron alloys causes them to spontaneously align even in the absence of an applied magnetic field. Each of these microscopic domains act like a permanent magnet, but they are randomly distributed throughout the material so that the macroscopic magnetization is zero. When a magnetic field is applied, the dipoles tend to align with the field so that domains with a magnetization along the field grow at the expense of non-aligned domains.

The friction-like behavior of domain wall motion is a lossy process so that the magnetization varies with the magnetic field in a nonlinear way, as described by the hysteresis loop in Figure 5-20. A strong field aligns all the domains to saturation. Upon decreasing **H**, the magnetization lags behind so that a remanent magnetization $M_r$ exists even with zero field. In this condition we have a permanent magnet. To bring the magnetization to zero requires a negative coercive field $-H_c$.

Although nonlinear, the main engineering importance of ferromagnetic materials is that the relative permeability $\mu_r$ is often in the thousands:

$$\mu = \mu_r \mu_0 = \mathbf{B}/\mathbf{H} \tag{54}$$

This value is often so high that in engineering applications we idealize it to be infinity. In this limit

$$\lim_{\mu\to\infty} \mathbf{B} = \mu \mathbf{H} \Rightarrow \mathbf{H} = 0, \qquad \mathbf{B} \text{ finite} \tag{55}$$

the **H** field becomes zero to keep the **B** field finite.

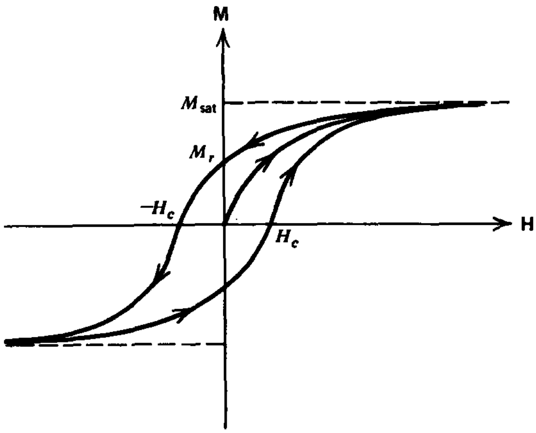



Figure 5-20 Ferromagnetic materials exhibit hysteresis where the magnetization saturates at high field strengths and retains a net remanent magnetization $\mathbf{M}_r$ even when $\mathbf{H}$ is zero. A coercive field $-\mathbf{H}_c$ is required to bring the magnetization back to zero.

## EXAMPLE 5-1 INFINITE LINE CURRENT WITHIN A MAGNETICALLY PERMEABLE CYLINDER

A line current $I$ of infinite extent is within a cylinder of radius $a$ that has permeability $\mu$, as in Figure 5-21. The cylinder is surrounded by free space. What are the $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{M}$ fields everywhere? What is the magnetization current?

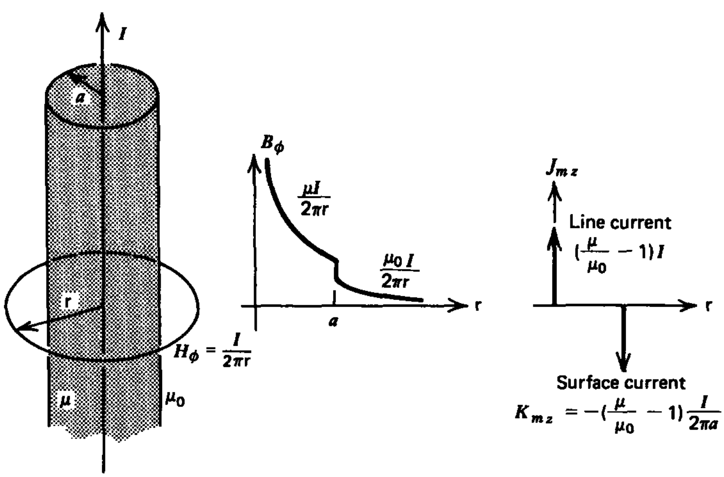



Figure 5-21 A free line current of infinite extent placed within a permeable cylinder gives rise to a line magnetization current along the axis and an oppositely directed surface magnetization current on the cylinder surface.

**SOLUTION**

---

Pick a circular contour of radius r around the current. Using the integral form of Ampere's law, (21), the **H** field is of the same form whether inside or outside the cylinder:

$$\oint_L \mathbf{H}\cdot d\mathbf{l} = H_\phi 2\pi r = I \Rightarrow H_\phi = \frac{I}{2\pi r}$$

The magnetic flux density differs in each region because the permeability differs:

$$B_\phi = \begin{cases} \mu H_\phi = \dfrac{\mu I}{2\pi r}, & 0<r<a \\ \mu_0 H_\phi = \dfrac{\mu_0 I}{2\pi r}, & r>a \end{cases}$$

The magnetization is obtained from the relation

$$\mathbf{M} = \frac{\mathbf{B}}{\mu_0} - \mathbf{H}$$

as

$$M_\phi = \begin{cases} \left(\dfrac{\mu}{\mu_0}-1\right)H_\phi = \dfrac{\mu-\mu_0}{\mu_0}\dfrac{I}{2\pi r}, & 0<r<a \\ 0, & r>a \end{cases}$$

The volume magnetization current can be found using (16):

$$\mathbf{J}_m = \nabla\times\mathbf{M} = -\frac{\partial M_\phi}{\partial z}\mathbf{i}_r + \frac{1}{r}\frac{\partial}{\partial r}(rM_\phi)\mathbf{i}_z = 0, \qquad 0<r<a$$

There is no bulk magnetization current because there are no bulk free currents. However, there is a line magnetization current at $r=0$ and a surface magnetization current at $r=a$. They are easily found using the integral form of (16) from Stokes' theorem:

$$\int_S \nabla\times\mathbf{M}\cdot d\mathbf{S} = \oint_L \mathbf{M}\cdot d\mathbf{l} = \int_S \mathbf{J}_m\cdot d\mathbf{S}$$

Pick a contour around the center of the cylinder with $r<a$:

$$M_\phi 2\pi r = \left(\frac{\mu-\mu_0}{\mu_0}\right)I = I_m$$

where $I_m$ is the magnetization line current. The result remains unchanged for any radius $r<a$ as no more current is enclosed since $\mathbf{J}_m=0$ for $0<r<a$. As soon as $r>a$, $M_\phi$ becomes zero so that the total magnetization current becomes zero. Therefore, at $r=a$ a surface magnetization current must flow whose total current is equal in magnitude but opposite in sign to the line magnetization current:

$$K_{zm}=\frac{-I_m}{2\pi a}=-\frac{(\mu-\mu_0)I}{\mu_0 2\pi a}$$

## 5-6 BOUNDARY CONDITIONS

At interfacial boundaries separating materials of differing properties, the magnetic fields on either side of the boundary must obey certain conditions. The procedure is to use the integral form of the field laws for differential sized contours, surfaces, and volumes in the same way as was performed for electric fields in Section 3-3.

To summarize our development thus far, the field laws for magnetic fields in differential and integral form are

$$\nabla\times\mathbf{H}=\mathbf{J}_f, \qquad \oint_L \mathbf{H}\cdot d\mathbf{l}=\int_S \mathbf{J}_f\cdot d\mathbf{S} \tag{1}$$

$$\nabla\times\mathbf{M}=\mathbf{J}_m, \qquad \oint_L \mathbf{M}\cdot d\mathbf{l}=\int_S \mathbf{J}_m\cdot d\mathbf{S} \tag{2}$$

$$\nabla\cdot\mathbf{B}=0, \qquad \oint_S \mathbf{B}\cdot d\mathbf{S}=0 \tag{3}$$

### 5-6-1 Tangential Component of H

We apply Ampere's circuital law of (1) to the contour of differential size enclosing the interface, as shown in Figure 5-22*a*. Because the interface is assumed to be infinitely thin, the short sides labelled *c* and *d* are of zero length and so offer

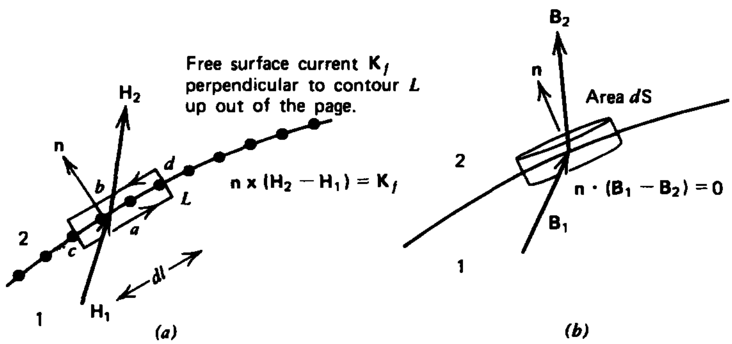



Figure 5-22 (*a*) The tangential component of **H** can be discontinuous in a free surface current across a boundary. (*b*) The normal component of **B** is always continuous across an interface.

no contribution to the line integral. The remaining two sides yield

$$\oint_L \mathbf{H}\cdot d\mathbf{l} = (H_{1t} - H_{2t})\, dl = K_{fn}\, dl \tag{4}$$

where $K_{fn}$ is the component of free surface current perpendicular to the contour by the right-hand rule in this case up out of the page. Thus, the tangential component of magnetic field can be discontinuous by a free surface current,

$$(H_{1t} - H_{2t}) = K_{fn} \Rightarrow \mathbf{n}\times(\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_f \tag{5}$$

where the unit normal points from region 1 towards region 2. If there is no surface current, the tangential component of **H** is continuous.

### 5-6-2 Tangential Component of M

Equation (2) is of the same form as (6) so we may use the results of (5) replacing **H** by **M** and $\mathbf{K}_f$ by $\mathbf{K}_m$, the surface magnetization current:

$$(M_{1t} - M_{2t}) = K_{mn}, \qquad \mathbf{n}\times(\mathbf{M}_2 - \mathbf{M}_1) = \mathbf{K}_m \tag{6}$$

This boundary condition confirms the result for surface magnetization current found in Example 5-1.

### 5-6-3 Normal Component of B

Figure 5-22*b* shows a small volume whose upper and lower surfaces are parallel and are on either side of the interface. The short cylindrical side, being of zero length, offers no contribution to (3), which thus reduces to

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = (B_{2n} - B_{1n})\, dS = 0 \tag{7}$$

yielding the boundary condition that the component of **B** normal to an interface of discontinuity is always continuous:

$$B_{1n} - B_{2n} = 0 \Rightarrow \mathbf{n}\cdot(\mathbf{B}_1 - \mathbf{B}_2) = 0 \tag{8}$$

### EXAMPLE 5-2 MAGNETIC SLAB WITHIN A UNIFORM MAGNETIC FIELD

A slab of infinite extent in the $x$ and $y$ directions is placed within a uniform magnetic field $H_0\mathbf{i}_z$ as shown in Figure 5-23.

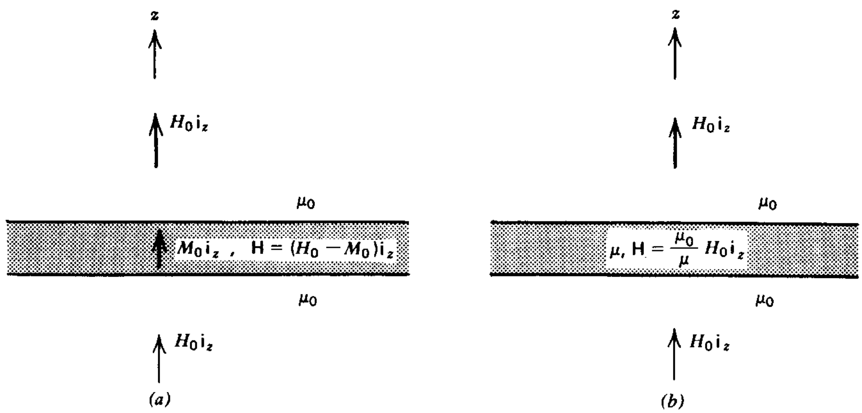



Figure 5-23 A (*a*) permanently magnetized or (*b*) linear magnetizable material is placed within a uniform magnetic field.

Find the **H** field within the slab when it is
(a) permanently magnetized with magnetization $M_0\mathbf{i}_z$,
(b) a linear permeable material with permeability $\mu$.

**SOLUTION**

---

For both cases, (8) requires that the **B** field across the boundaries be continuous as it is normally incident.

(a) For the permanently magnetized slab, this requires that

$$\mu_0 H_0 = \mu_0(H + M_0) \Rightarrow H = H_0 - M_0$$

Note that when there is no externally applied field ($H_0 = 0$), the resulting field within the slab is oppositely directed to the magnetization so that $\mathbf{B} = 0$.

(b) For a linear permeable medium (8) requires

$$\mu_0 H_0 = \mu H \Rightarrow H = \frac{\mu_0}{\mu} H_0$$

For $\mu > \mu_0$ the internal magnetic field is reduced. If $H_0$ is set to zero, the magnetic field within the slab is also zero.

## 5-7 MAGNETIC FIELD BOUNDARY VALUE PROBLEMS

### 5-7-1 The Method of Images

A line current $I$ of infinite extent in the $z$ direction is a distance $d$ above a plane that is either perfectly conducting or infinitely permeable, as shown in Figure 5-24. For both cases

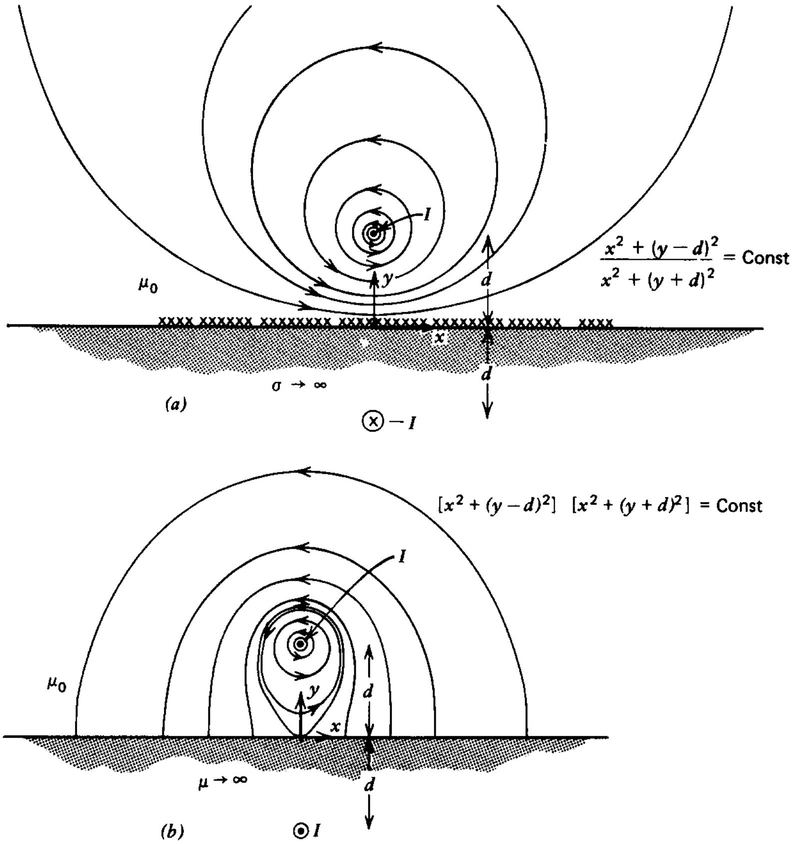



Figure 5-24 (*a*) A line current above a perfect conductor induces an oppositely directed surface current that is equivalent to a symmetrically located image line current. (*b*) The field due to a line current above an infinitely permeable medium is the same as if the medium were replaced by an image current now in the same direction as the original line current.

the **H** field within the material must be zero but the boundary conditions at the interface are different. In the perfect conductor both **B** and **H** must be zero, so that at the interface the normal component of **B** and thus **H** must be continuous and thus zero. The tangential component of **H** is discontinuous in a surface current.

In the infinitely permeable material **H** is zero but **B** is finite. No surface current can flow because the material is not a conductor, so the tangential component of **H** is continuous and thus zero. The **B** field must be normally incident.

Both sets of boundary conditions can be met by placing an image current $I$ at $y = -d$ flowing in the opposite direction for the conductor and in the same direction for the permeable material.

Using the upper sign for the conductor and the lower sign for the infinitely permeable material, the vector potential due to both currents is found by superposing the vector potential found in Section 5-4-3*a*, Eq. (18), for each infinitely long line current:

$$\begin{aligned} A_z &= \frac{-\mu_0 I}{2\pi}\{\ln [x^2+(y-d)^2]^{1/2} \mp \ln [x^2+(y+d)^2]^{1/2}\} \\ &= \frac{-\mu_0 I}{4\pi}\{\ln [x^2+(y-d)^2] \mp \ln [x^2+(y+d)^2]\} \end{aligned} \tag{1}$$

with resultant magnetic field

$$\begin{aligned} \mathbf{H} = \frac{1}{\mu_0}\nabla\times\mathbf{A} &= \frac{1}{\mu_0}\left(\mathbf{i}_x\frac{\partial A_z}{\partial y} - \mathbf{i}_y\frac{\partial A_z}{\partial x}\right) \\ &= \frac{-I}{2\pi}\left\{\frac{(y-d)\mathbf{i}_x - x\mathbf{i}_y}{[x^2+(y-d)^2]} \mp \frac{(y+d)\mathbf{i}_x - x\mathbf{i}_y}{[x^2+(y+d)^2]}\right\} \end{aligned} \tag{2}$$

The surface current distribution for the conducting case is given by the discontinuity in tangential **H**,

$$K_z = -H_x(y=0) = -\frac{Id}{\pi[d^2+x^2]} \tag{3}$$

which has total current

$$\begin{aligned} I_T = \int_{-\infty}^{+\infty} K_z\,dx &= -\frac{Id}{\pi}\int_{-\infty}^{+\infty}\frac{dx}{(x^2+d^2)} \\ &= -\frac{Id}{\pi}\frac{1}{d}\tan^{-1}\frac{x}{d}\bigg|_{-\infty}^{+\infty} = -I \end{aligned} \tag{4}$$

just equal to the image current.

The force per unit length on the current for each case is just due to the magnetic field from its image:

$$\mathbf{f} = \pm\frac{\mu_0 I^2}{4\pi d}\,\mathbf{i}_y \tag{5}$$

being repulsive for the conductor and attractive for the permeable material.

The magnetic field lines plotted in Figure 5-24 are just lines of constant $A_z$ as derived in Section 5-4-3*b*. Right next to the line current the self-field term dominates and the field lines are circles. The far field in Figure 5-24*b*, when the line and image current are in the same direction, is the same as if we had a single line current of $2I$.

### 5-7-2 Sphere in a Uniform Magnetic Field

A sphere of radius $R$ is placed within a uniform magnetic field $H_0\mathbf{i}_z$. The sphere and surrounding medium may have any of the following properties illustrated in Figure 5-25:

(i) Sphere has permeability $\mu_2$ and surrounding medium has permeability $\mu_1$.
(ii) Perfectly conducting sphere in free space.
(iii) Uniformly magnetized sphere $M_2\mathbf{i}_z$ in a uniformly magnetized medium $M_1\mathbf{i}_z$.

For each of these three cases, there are no free currents in either region so that the governing equations in each region are

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = 0 \tag{5}$$

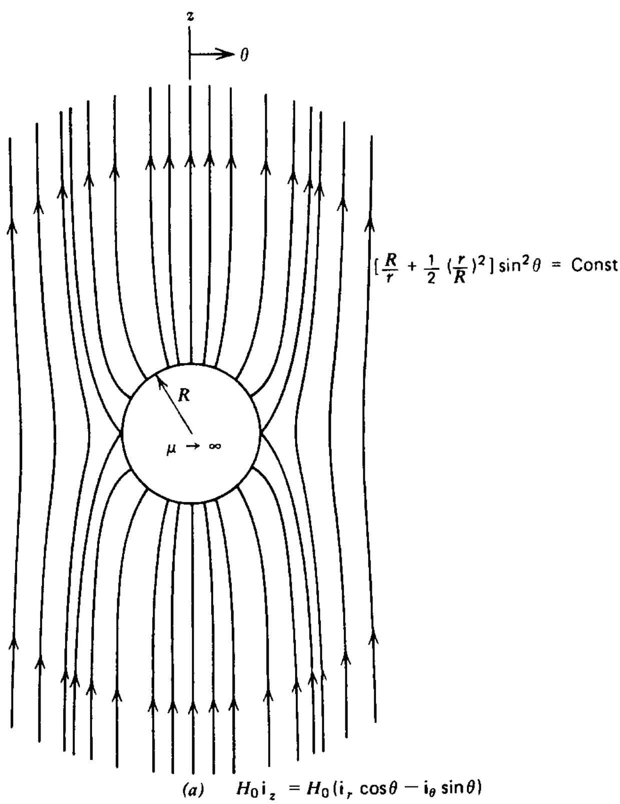



Figure 5-25 Magnetic field lines about an (*a*) infinitely permeable and (*b*) perfectly conducting sphere in a uniform magnetic field.

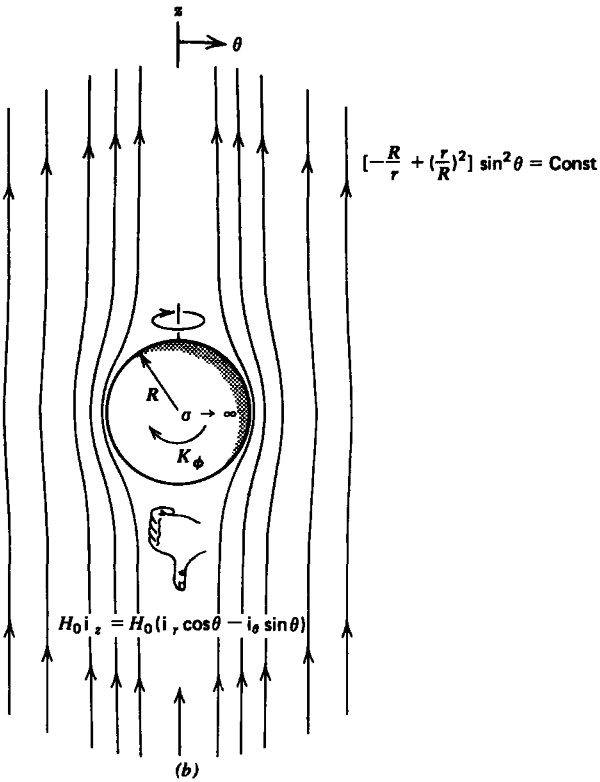



Figure 5-25

Because the curl of **H** is zero, we can define a scalar magnetic potential

$$\mathbf{H} = \nabla \chi \tag{6}$$

where we avoid the use of a negative sign as is used with the electric field since the potential $\chi$ is only introduced as a mathematical convenience and has no physical significance. With **B** proportional to **H** or for uniform magnetization, the divergence of **H** is also zero so that the scalar magnetic potential obeys Laplace's equation in each region:

$$\nabla^2 \chi = 0 \tag{7}$$

We can then use the same techniques developed for the electric field in Section 4-4 by trying a scalar potential in each region as

$$\chi = \begin{cases} Ar\cos\theta, & r < R \\ (Dr + C/r^2)\cos\theta & r > R \end{cases} \tag{8}$$

The associated magnetic field is then

$$\mathbf{H}=\nabla\chi=\frac{\partial\chi}{\partial r}\mathbf{i}_r+\frac{1}{r}\frac{\partial\chi}{\partial\theta}\mathbf{i}_\theta+\frac{1}{r\sin\theta}\frac{\partial\chi}{\partial\phi}\mathbf{i}_\phi$$

$$=\begin{cases}A(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta)=A\mathbf{i}_z, & r<R\\ (D-2C/r^3)\cos\theta\mathbf{i}_r-(D+C/r^3)\sin\theta\mathbf{i}_\theta, & r>R\end{cases} \tag{9}$$

For the three cases, the magnetic field far from the sphere must approach the uniform applied field:

$$\mathbf{H}(r=\infty)=H_0\mathbf{i}_z=H_0(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta)\Rightarrow D=H_0 \tag{10}$$

The other constants, $A$ and $C$, are found from the boundary conditions at $r=R$. The field within the sphere is uniform, in the same direction as the applied field. The solution outside the sphere is the imposed field plus a contribution as if there were a magnetic dipole at the center of the sphere with moment $m_z=4\pi C$.

(i) If the sphere has a different permeability from the surrounding region, both the tangential components of $\mathbf{H}$ and the normal components of $\mathbf{B}$ are continuous across the spherical surface:

$$\begin{aligned}&H_\theta(r=R_+)=H_\theta(r=R_-)\Rightarrow A=D+C/R^3\\ &B_r(r=R_+)=B_r(r=R_-)\Rightarrow \mu_1H_r(r=R_+)=\mu_2H_r(r=R_-)\end{aligned} \tag{11}$$

which yields solutions

$$A=\frac{3\mu_1H_0}{\mu_2+2\mu_1},\qquad C=-\frac{\mu_2-\mu_1}{\mu_2+2\mu_1}R^3H_0 \tag{12}$$

The magnetic field distribution is then

$$\mathbf{H}=\begin{cases}\dfrac{3\mu_1H_0}{\mu_2+2\mu_1}(\mathbf{i}_r\cos\theta-\mathbf{i}_\theta\sin\theta)=\dfrac{3\mu_1H_0\mathbf{i}_z}{\mu_2+2\mu_1}, & r<R\\ H_0\left\{\left[1+\dfrac{2R^3}{r^3}\left(\dfrac{\mu_2-\mu_1}{\mu_2+2\mu_1}\right)\right]\cos\theta\mathbf{i}_r\right.\\ \qquad\left.-\left[1-\dfrac{R^3}{r^3}\left(\dfrac{\mu_2-\mu_1}{\mu_2+2\mu_1}\right)\right]\sin\theta\mathbf{i}_\theta\right\}, & r>R\end{cases} \tag{13}$$

The magnetic field lines are plotted in Figure 5-25*a* when $\mu_2\to\infty$. In this limit, $\mathbf{H}$ within the sphere is zero, so that the field lines incident on the sphere are purely radial. The field lines plotted are just lines of constant stream function $\Sigma$, found in the same way as for the analogous electric field problem in Section 4-4-3*b*.

(ii) If the sphere is perfectly conducting, the internal magnetic field is zero so that $A=0$. The normal component of $\mathbf{B}$ right outside the sphere is then also zero:

$$H_r(r=R_+)=0 \Rightarrow C=H_0R^3/2 \tag{14}$$

yielding the solution

$$\mathbf{H}=H_0\left[\left(1-\frac{R^3}{r^3}\right)\cos\theta\,\mathbf{i}_r-\left(1+\frac{R^3}{2r^3}\right)\sin\theta\,\mathbf{i}_\theta\right],\quad r>R \tag{15}$$

The interfacial surface current at $r=R$ is obtained from the discontinuity in the tangential component of $\mathbf{H}$:

$$K_\phi=H_\theta(r=R)=-\tfrac{3}{2}H_0\sin\theta \tag{16}$$

The current flows in the negative $\phi$ direction around the sphere. The right-hand rule, illustrated in Figure 5-25*b*, shows that the resulting field from the induced current acts in the direction opposite to the imposed field. This opposition results in the zero magnetic field inside the sphere.

The field lines plotted in Figure 5-25*b* are purely tangential to the perfectly conducting sphere as required by (14).

(iii) If both regions are uniformly magnetized, the boundary conditions are

$$\begin{aligned} H_\theta(r=R_+)&=H_\theta(r=R_-)\Rightarrow A=D+C/R^3\\ B_r(r=R_+)&=B_r(r=R_-)\Rightarrow H_r(r=R_+)+M_1\cos\theta\\ &\qquad=H_r(r=R_-)+M_2\cos\theta \end{aligned} \tag{17}$$

with solutions

$$\begin{aligned} A&=H_0+\tfrac{1}{3}(M_1-M_2)\\ C&=\frac{R^3}{3}(M_1-M_2) \end{aligned} \tag{18}$$

so that the magnetic field is

$$\mathbf{H}=\begin{cases} [H_0+\frac{1}{3}(M_1-M_2)][\cos\theta\,\mathbf{i}_r-\sin\theta\,\mathbf{i}_\theta]\\ \qquad=[H_0+\frac{1}{3}(M_1-M_2)]\mathbf{i}_z \quad r<R\\ \left(H_0-\frac{2R^3}{3r^3}(M_1-M_2)\right)\cos\theta\,\mathbf{i}_r\\ \qquad-\left(H_0+\frac{R^3}{3r^3}(M_1-M_2)\right)\sin\theta\,\mathbf{i}_\theta,\quad r>R \end{cases} \tag{19}$$

Because the magnetization is uniform in each region, the curl of $\mathbf{M}$ is zero everywhere but at the surface of the sphere, so that the volume magnetization current is zero with a surface magnetization current at $r=R$ given by

$$\begin{aligned}\mathbf{K}_m &= \mathbf{n}\times(\mathbf{M}_1-\mathbf{M}_2)\\ &= \mathbf{i}_r\times(M_1-M_2)\mathbf{i}_z\\ &= \mathbf{i}_r\times(M_1-M_2)(\mathbf{i}_r\cos\theta-\sin\theta\,\mathbf{i}_\theta)\\ &= -(M_1-M_2)\sin\theta\,\mathbf{i}_\phi\end{aligned} \tag{20}$$

## 5-8 MAGNETIC FIELDS AND FORCES

### 5-8-1 Magnetizable Media

A magnetizable medium carrying a free current $\mathbf{J}_f$ is placed within a magnetic field $\mathbf{B}$, which is a function of position. In addition to the Lorentz force, the medium feels the forces on all its magnetic dipoles. Focus attention on the rectangular magnetic dipole shown in Figure 5-26. The force on each current carrying leg is

$$\begin{aligned}&\mathbf{f}=\mathbf{i}\,dl\times(B_x\mathbf{i}_x+B_y\mathbf{i}_y+B_z\mathbf{i}_z)\\ \Rightarrow\,&\mathbf{f}(x)=-i\,\Delta y[-B_x\mathbf{i}_z+B_z\mathbf{i}_x]|_x\\ &\mathbf{f}(x+\Delta x)=i\,\Delta y[-B_x\mathbf{i}_z+B_z\mathbf{i}_x]|_{x+\Delta x}\\ &\mathbf{f}(y)=i\,\Delta x[B_y\mathbf{i}_z-B_z\mathbf{i}_y]|_y\\ &\mathbf{f}(y+\Delta y)=-i\,\Delta x[B_y\mathbf{i}_z-B_z\mathbf{i}_y]|_{y+\Delta y}\end{aligned} \tag{1}$$

so that the total force on the dipole is

$$\begin{aligned}\mathbf{f}&=\mathbf{f}(x)+\mathbf{f}(x+\Delta x)+\mathbf{f}(y)+\mathbf{f}(y+\Delta y)\\ &=i\,\Delta x\,\Delta y\left[\frac{B_z(x+\Delta x)-B_z(x)}{\Delta x}\mathbf{i}_x-\frac{B_x(x+\Delta x)-B_x(x)}{\Delta x}\mathbf{i}_z\right.\\ &\qquad\left.+\frac{B_z(y+\Delta y)-B_z(y)}{\Delta y}\mathbf{i}_y-\frac{B_y(y+\Delta y)-B_y(y)}{\Delta y}\mathbf{i}_z\right]\end{aligned} \tag{2}$$

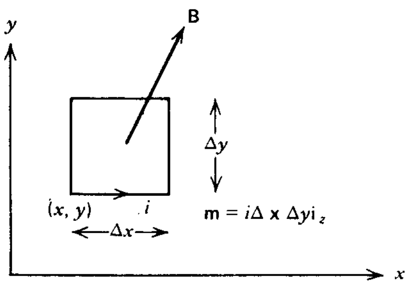



Figure 5-26 A magnetic dipole in a magnetic field **B**.

In the limit of infinitesimal $\Delta x$ and $\Delta y$ the bracketed terms define partial derivatives while the coefficient is just the magnetic dipole moment $\mathbf{m} = i\,\Delta x\,\Delta y\,\mathbf{i}_z$:

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \mathbf{f} = m_z\left[\frac{\partial B_z}{\partial x}\mathbf{i}_x - \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right)\mathbf{i}_z + \frac{\partial B_z}{\partial y}\mathbf{i}_y\right] \tag{3}$$

Ampere's and Gauss's law for the magnetic field relate the field components as

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \frac{\partial B_z}{\partial z} = -\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y}\right) \tag{4}$$

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_f + \nabla \times \mathbf{M}) = \mu_0 \mathbf{J}_T \Rightarrow \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_{Tx}$$

$$\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu_0 J_{Ty}$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu_0 J_{Tz} \tag{5}$$

which puts (3) in the form

$$\begin{aligned} \mathbf{f} &= m_z\left(\frac{\partial B_x}{\partial z}\mathbf{i}_x + \frac{\partial B_y}{\partial z}\mathbf{i}_y + \frac{\partial B_z}{\partial z}\mathbf{i}_z - \mu_0(J_{Ty}\mathbf{i}_x - J_{Tx}\mathbf{i}_y)\right) \\ &= (\mathbf{m} \cdot \nabla)\mathbf{B} + \mu_0 \mathbf{m} \times \mathbf{J}_T \end{aligned} \tag{6}$$

where $\mathbf{J}_T$ is the sum of free and magnetization currents.

If there are $N$ such dipoles per unit volume, the force density on the dipoles and on the free current is

$$\begin{aligned} \mathbf{F} = N\mathbf{f} &= (\mathbf{M} \cdot \nabla)\mathbf{B} + \mu_0 \mathbf{M} \times \mathbf{J}_T + \mathbf{J}_f \times \mathbf{B} \\ &= \mu_0(\mathbf{M} \cdot \nabla)(\mathbf{H} + \mathbf{M}) + \mu_0 \mathbf{M} \times (\mathbf{J}_f + \nabla \times \mathbf{M}) + \mu_0 \mathbf{J}_f \times (\mathbf{H} + \mathbf{M}) \\ &= \mu_0(\mathbf{M} \cdot \nabla)(\mathbf{H} + \mathbf{M}) + \mu_0 \mathbf{M} \times (\nabla \times \mathbf{M}) + \mu_0 \mathbf{J}_f \times \mathbf{H} \end{aligned} \tag{7}$$

Using the vector identity

$$\mathbf{M} \times (\nabla \times \mathbf{M}) = -(\mathbf{M} \cdot \nabla)\mathbf{M} + \tfrac{1}{2}\nabla(\mathbf{M} \cdot \mathbf{M}) \tag{8}$$

(7) can be reduced to

$$\mathbf{F} = \mu_0(\mathbf{M} \cdot \nabla)\mathbf{H} + \mu_0 \mathbf{J}_f \times \mathbf{H} + \nabla\left(\frac{\mu_0}{2}\mathbf{M} \cdot \mathbf{M}\right) \tag{9}$$

The total force on the body is just the volume integral of $\mathbf{F}$:

$$\mathbf{f} = \int_V \mathbf{F}\, dV \tag{10}$$

In particular, the last contribution in (9) can be converted to a surface integral using the gradient theorem, a corollary to the divergence theorem (see Problem 1-15*a*):

$$\int_V \nabla\left(\frac{\mu_0}{2}\mathbf{M}\cdot\mathbf{M}\right) dV = \oint_S \frac{\mu_0}{2}\mathbf{M}\cdot\mathbf{M}\, d\mathbf{S} \tag{11}$$

Since this surface $S$ surrounds the magnetizable medium, it is in a region where $\mathbf{M}=0$ so that the integrals in (11) are zero. For this reason the force density of (9) is written as

$$\mathbf{F} = \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H} + \mu_0\mathbf{J}_f \times \mathbf{H} \tag{12}$$

It is the first term on the right-hand side in (12) that accounts for an iron object to be drawn towards a magnet. Magnetizable materials are attracted towards regions of higher $\mathbf{H}$.

## 5-8-2 Force on a Current Loop

### (a) Lorentz Force Only

Two parallel wires are connected together by a wire that is free to move, as shown in Figure 5-27*a*. A current $I$ is imposed and the whole loop is placed in a uniform magnetic field $B_0\mathbf{i}_x$. The Lorentz force on the moveable wire is

$$f_y = IB_0 l \tag{13}$$

where we neglect the magnetic field generated by the current, assuming it to be much smaller than the imposed field $B_0$.

### (b) Magnetization Force Only

The sliding wire is now surrounded by an infinitely permeable hollow cylinder of inner radius $a$ and outer radius $b$, both being small compared to the wire's length $l$, as in Figure 5-27*b*. For distances near the cylinder, the solution is approximately the same as if the wire were infinitely long. For $r>0$ there is no current, thus the magnetic field is curl and divergence free within each medium so that the magnetic scalar potential obeys Laplace's equation as in Section 5-7-2. In cylindrical geometry we use the results of Section 4-3 and try a scalar potential of the form

$$\chi = \left(Ar + \frac{C}{r}\right)\cos\phi \tag{14}$$

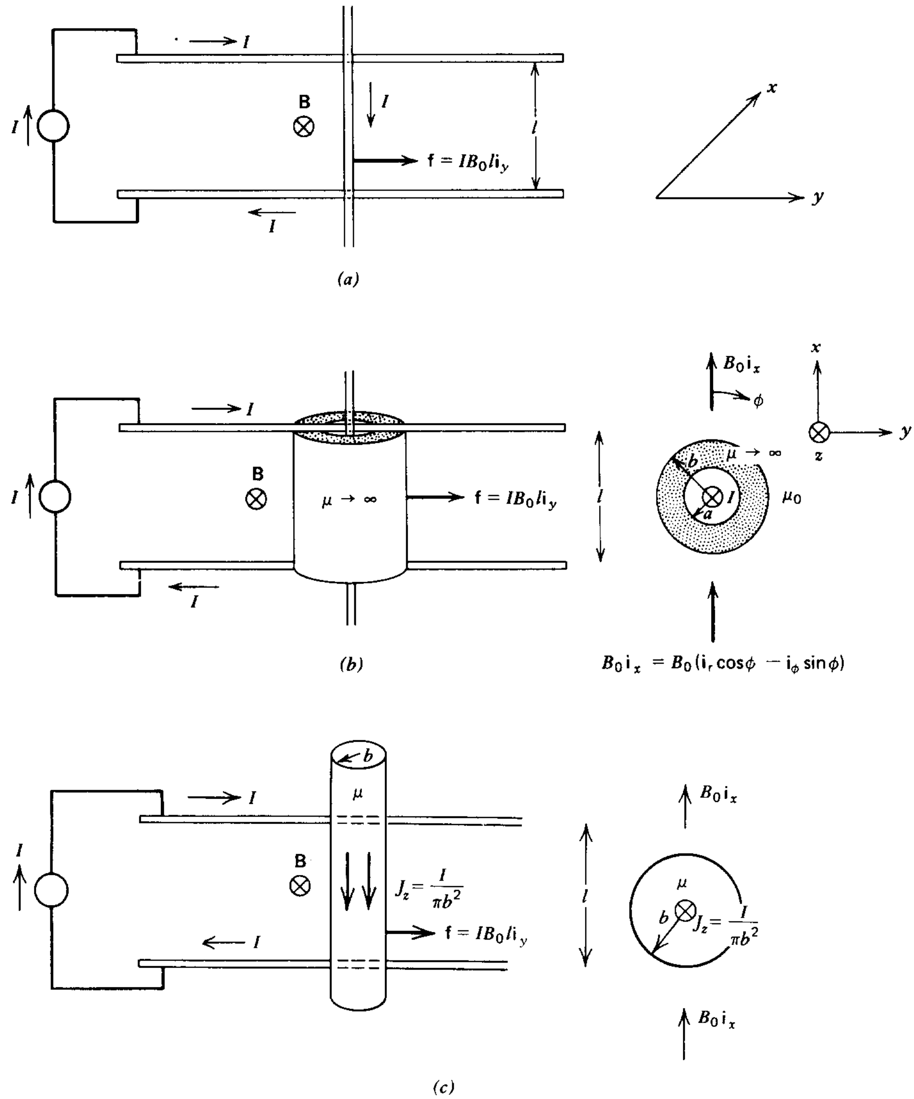



Figure 5-27 (*a*) The Lorentz-force on a current carrying wire in a magnetic field. (*b*) If the current-carrying wire is surrounded by an infinitely permeable hollow cylinder, there is no Lorentz force as the imposed magnetic field is zero where the current is. However, the magnetization force on the cylinder is the same as in (*a*). (*c*) The total force on a current-carrying magnetically permeable wire is also unchanged.

in each region, where $\mathbf{B} = \nabla\chi$ because $\nabla \times \mathbf{B} = 0$. The constants are evaluated by requiring that the magnetic field approach the imposed field $B_0\mathbf{i}_x$ at $r = \infty$ and be normally incident onto the infinitely permeable cylinder at $r = a$ and $r = b$. In addition, we must add the magnetic field generated by the line current. The magnetic field in each region is then (see Problem 32*a*):

$$\mathbf{B}=\begin{cases}\dfrac{\mu_0 I}{2\pi \mathrm{r}}\mathbf{i}_\phi, & 0<\mathrm{r}<a\\[2mm] \dfrac{2B_0b^2}{b^2-a^2}\left[\left(1-\dfrac{a^2}{\mathrm{r}^2}\right)\cos\phi\,\mathbf{i}_\mathrm{r}-\left(1+\dfrac{a^2}{\mathrm{r}^2}\right)\sin\phi\,\mathbf{i}_\phi\right]+\dfrac{\mu I}{2\pi\mathrm{r}}\mathbf{i}_\phi, & \\ & a<\mathrm{r}<b \\[2mm] B_0\left[\left(1+\dfrac{b^2}{\mathrm{r}^2}\right)\cos\phi\,\mathbf{i}_\mathrm{r}-\left(1-\dfrac{b^2}{\mathrm{r}^2}\right)\sin\phi\,\mathbf{i}_\phi\right]+\dfrac{\mu_0 I}{2\pi\mathrm{r}}\mathbf{i}_\phi, & \\ & \mathrm{r}>b\end{cases}\tag{15}$$

Note the infinite flux density in the iron ($\mu\to\infty$) due to the line current that sets up the finite **H** field. However, we see that none of the imposed magnetic field is incident upon the current carrying wire because it is shielded by the infinitely permeable cylindrical shell so that the Lorentz force contribution on the wire is zero. There is, however, a magnetization force on the cylindrical shell where the internal magnetic field **H** is entirely due to the line current, $H_\phi = I/2\pi\mathrm{r}$ because with $\mu\to\infty$, the contribution due to $B_0$ is negligibly small:

$$\begin{aligned}\mathbf{F}&=\mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}\\ &=\mu_0\left(M_\mathrm{r}\frac{\partial}{\partial \mathrm{r}}(H_\phi\mathbf{i}_\phi)+\frac{M_\phi}{\mathrm{r}}\frac{\partial}{\partial\phi}(H_\phi\mathbf{i}_\phi)\right)\end{aligned}\tag{16}$$

Within the infinitely permeable shell the magnetization and **H** fields are

$$\begin{aligned}H_\phi&=\frac{I}{2\pi\mathrm{r}}\\ \mu_0M_\mathrm{r}&=B_\mathrm{r}-\mu_0\cancelto{0}{H_\mathrm{r}}=\frac{2B_0b^2}{b^2-a^2}\left(1-\frac{a^2}{\mathrm{r}^2}\right)\cos\phi\\ \mu_0M_\phi&=B_\phi-\mu_0H_\phi=-\frac{2B_0b^2}{(b^2-a^2)}\left(1+\frac{a^2}{\mathrm{r}^2}\right)\sin\phi+\frac{(\mu-\mu_0)I}{2\pi\mathrm{r}}\end{aligned}\tag{17}$$

Although $H_\phi$ only depends on r, the unit vector $\mathbf{i}_\phi$ depends on $\phi$:

$$\mathbf{i}_\phi=(-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y)\tag{18}$$

so that the force density of (16) becomes

$$\begin{aligned}\mathbf{F}&=-\frac{B_\mathrm{r}I}{2\pi\mathrm{r}^2}\mathbf{i}_\phi+\frac{(B_\phi-\mu_0H_\phi)I}{2\pi\mathrm{r}^2}\frac{d}{d\phi}(\mathbf{i}_\phi)\\ &=\frac{I}{2\pi\mathrm{r}^2}[-B_\mathrm{r}(-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y)\\ &\quad+(B_\phi-\mu_0H_\phi)(-\cos\phi\,\mathbf{i}_x-\sin\phi\,\mathbf{i}_y)]\end{aligned}$$

$$
\begin{aligned}
&= \frac{I}{2\pi r^2}\left\{-\frac{2B_0 b^2}{b^2-a^2}\left[\left(1-\frac{a^2}{r^2}\right)\cos\phi(-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y)\right.\right.\\
&\quad\left.-\left(1+\frac{a^2}{r^2}\right)\sin\phi(\cos\phi\,\mathbf{i}_x+\sin\phi\,\mathbf{i}_y)\right]\\
&\quad\left.+\frac{(\mu-\mu_0)I}{2\pi r}(\cos\phi\,\mathbf{i}_x+\sin\phi\,\mathbf{i}_y)\right\}\\
&= \frac{I}{2\pi r^2}\left[-\frac{2B_0 b^2}{b^2-a^2}\left(-2\sin\phi\cos\phi\,\mathbf{i}_x-\frac{2a^2}{r^2}\mathbf{i}_y\right)\right.\\
&\quad\left.+\frac{(\mu-\mu_0)I}{2\pi r}(\cos\phi\,\mathbf{i}_x+\sin\phi\,\mathbf{i}_y)\right] \qquad (19)
\end{aligned}
$$

The total force on the cylinder is obtained by integrating (19) over r and $\phi$:

$$
\mathbf{f} = \int_{\phi=0}^{2\pi}\int_{r=a}^{b} \mathbf{F} l\, r\, dr\, d\phi \qquad (20)
$$

All the trigonometric terms in (19) integrate to zero over $\phi$ so that the total force is

$$
\begin{aligned}
f_y &= \frac{2B_0 b^2 I l}{(b^2-a^2)}\int_{r=a}^{b}\frac{a^2}{r^3}\,dr\\
&= -\frac{B_0 b^2 I l}{(b^2-a^2)}\frac{a^2}{r^2}\bigg|_a^b\\
&= IB_0 l \qquad (21)
\end{aligned}
$$

The force on the cylinder is the same as that of an unshielded current-carrying wire given by (13). If the iron core has a finite permeability, the total force on the wire (Lorentz force) and on the cylinder (magnetization force) is again equal to (13). This fact is used in rotating machinery where current-carrying wires are placed in slots surrounded by highly permeable iron material. Most of the force on the whole assembly is on the iron and not on the wire so that very little restraining force is necessary to hold the wire in place. The force on a current-carrying wire surrounded by iron is often calculated using only the Lorentz force, neglecting the presence of the iron. The correct answer is obtained but for the wrong reasons. Actually there is very little **B** field near the wire as it is almost surrounded by the high permeability iron so that the Lorentz force on the wire is very small. The force is actually on the iron core.

**(c) Lorentz and Magnetization Forces**

If the wire itself is highly permeable with a uniformly distributed current, as in Figure 5-27*c*, the magnetic field is (see Problem 32*a*)

$$\mathbf{H}=\begin{cases}\dfrac{2B_0}{\mu+\mu_0}(\mathbf{i}_r\cos\phi-\mathbf{i}_\phi\sin\phi)+\dfrac{Ir}{2\pi b^2}\mathbf{i}_\phi \\ \quad =\dfrac{2B_0}{\mu+\mu_0}\mathbf{i}_x+\dfrac{I}{2\pi b^2}(-y\mathbf{i}_x+x\mathbf{i}_y), \quad r<b \\ \dfrac{B_0}{\mu_0}\left[\left(1+\dfrac{b^2}{r^2}\dfrac{\mu-\mu_0}{\mu+\mu_0}\right)\cos\phi\,\mathbf{i}_r\right. \\ \quad \left.-\left(1-\dfrac{b^2}{r^2}\dfrac{\mu-\mu_0}{\mu+\mu_0}\right)\sin\phi\,\mathbf{i}_\phi\right]+\dfrac{I}{2\pi r}\mathbf{i}_\phi, \quad r>b\end{cases} \tag{22}$$

It is convenient to write the fields within the cylinder in Cartesian coordinates using (18) as then the force density given by (12) is

$$\begin{aligned}\mathbf{F}&=\mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}+\mu_0\mathbf{J}_f\times\mathbf{H}\\ &=(\mu-\mu_0)(\mathbf{H}\cdot\nabla)\mathbf{H}+\frac{\mu_0 I}{\pi b^2}\mathbf{i}_z\times\mathbf{H}\\ &=(\mu-\mu_0)\left(H_x\frac{\partial}{\partial x}+H_y\frac{\partial}{\partial y}\right)(H_x\mathbf{i}_x+H_y\mathbf{i}_y)+\frac{\mu_0 I}{\pi b^2}(H_x\mathbf{i}_y-H_y\mathbf{i}_x)\end{aligned} \tag{23}$$

Since within the cylinder ($r<b$) the partial derivatives of $\mathbf{H}$ are

$$\begin{aligned}\frac{\partial H_x}{\partial x}&=\frac{\partial H_y}{\partial y}=0\\ \frac{\partial H_x}{\partial y}&=-\frac{\partial H_y}{\partial x}=-\frac{I}{2\pi b^2}\end{aligned} \tag{24}$$

(23) reduces to

$$\begin{aligned}\mathbf{F}&=(\mu-\mu_0)\left(H_x\frac{\partial H_y}{\partial x}\mathbf{i}_y+H_y\frac{\partial H_x}{\partial y}\mathbf{i}_x\right)+\frac{\mu_0 I}{\pi b^2}(H_x\mathbf{i}_y-H_y\mathbf{i}_x)\\ &=\frac{I}{2\pi b^2}(\mu+\mu_0)(H_x\mathbf{i}_y-H_y\mathbf{i}_x)\\ &=\frac{I(\mu+\mu_0)}{2\pi b^2}\left[\left(\frac{2B_0}{\mu+\mu_0}-\frac{Iy}{2\pi b^2}\right)\mathbf{i}_y-\frac{Ix}{2\pi b^2}\mathbf{i}_x\right]\end{aligned} \tag{25}$$

Realizing from Table 1-2 that

$$y\mathbf{i}_y+x\mathbf{i}_x=r[\sin\phi\,\mathbf{i}_y+\cos\phi\,\mathbf{i}_x]=r\mathbf{i}_r \tag{26}$$

the force density can be written as

$$\mathbf{F}=\frac{IB_0}{\pi b^2}\mathbf{i}_y-\frac{I^2(\mu+\mu_0)}{(2\pi b^2)^2}\mathrm{r}\,(\sin\phi\,\mathbf{i}_y+\cos\phi\,\mathbf{i}_x) \tag{27}$$

The total force on the permeable wire is

$$\mathbf{f}=\int_{\phi=0}^{2\pi}\int_{\mathrm{r}=0}^{b}\mathbf{F}l\mathrm{r}\,d\mathrm{r}\,d\phi \tag{28}$$

We see that the trigonometric terms in (27) integrate to zero so that only the first term contributes:

$$\begin{aligned} f_y&=\frac{IB_0l}{\pi b^2}\int_{\phi=0}^{2\pi}\int_{\mathrm{r}=0}^{b}\mathrm{r}\,d\mathrm{r}\,d\phi\\ &=IB_0l \end{aligned} \tag{29}$$

The total force on the wire is independent of its magnetic permeability.

## PROBLEMS

Section 5-1

1. A charge $q$ of mass $m$ moves through a uniform magnetic field $B_o\mathbf{i}_z$. At $t=0$ its velocity and displacement are

$$\mathbf{v}(t=0)=v_{x0}\mathbf{i}_x+v_{y0}\mathbf{i}_y+v_{z0}\mathbf{i}_z$$

$$\mathbf{r}(t=0)=x_0\mathbf{i}_x+y_0\mathbf{i}_y+z_0\mathbf{i}_z$$

(a) What is the subsequent velocity and displacement?

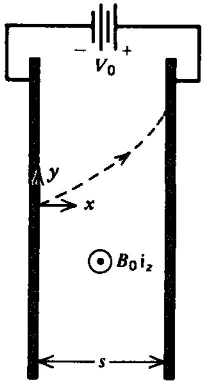
(b) Show that its motion projected onto the $xy$ plane is a circle. What is the radius of this circle and where is its center?

(c) What is the time dependence of the kinetic energy of the charge $\frac{1}{2}m|\mathbf{v}|^2$?

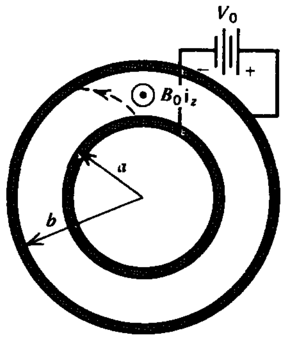
2. A magnetron is essentially a parallel plate capacitor stressed by constant voltage $V_0$ where electrons of charge $-e$ are emitted at $x=0$, $y=0$ with zero initial velocity. A transverse magnetic field $B_0\mathbf{i}_z$ is applied. Neglect the electric and magnetic fields due to the electrons in comparison to the applied field.

(a) What is the velocity and displacement of an electron, injected with zero initial velocity at $t=0$?

(b) What value of magnetic field will just prevent the electrons from reaching the other electrode? This is the cut-off magnetic field.



(a)



(c)

(c) A magnetron is built with coaxial electrodes where electrons are injected from $r = a$, $\phi = 0$ with zero initial velocity. Using the relations from Table 1-2,

$$\mathbf{i}_r = \cos\phi\,\mathbf{i}_x + \sin\phi\,\mathbf{i}_y$$

$$\mathbf{i}_\phi = -\sin\phi\,\mathbf{i}_x + \cos\phi\,\mathbf{i}_y$$

show that

$$\frac{d\mathbf{i}_r}{dt} = \mathbf{i}_\phi \frac{d\phi}{dt} = \frac{v_\phi}{r}\mathbf{i}_\phi$$

$$\frac{d\mathbf{i}_\phi}{dt} = -\mathbf{i}_r \frac{d\phi}{dt} = -\frac{v_\phi}{r}\mathbf{i}_r$$

What is the acceleration of a charge with velocity

$$\mathbf{v} = v_r \mathbf{i}_r + v_\phi \mathbf{i}_\phi?$$

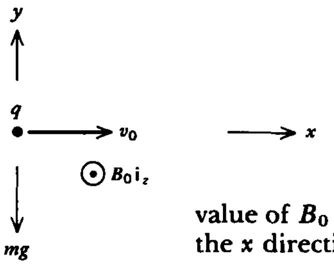
(d) Find the velocity of the electrons as a function of radial position.
**Hint:**

$$\frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r \frac{dv_r}{dr} = \frac{d}{dr}(\tfrac{1}{2}v_r^2)$$

$$\frac{dv_\phi}{dt} = \frac{dv_\phi}{dr}\frac{dr}{dt} = v_r \frac{dv_\phi}{dr}$$

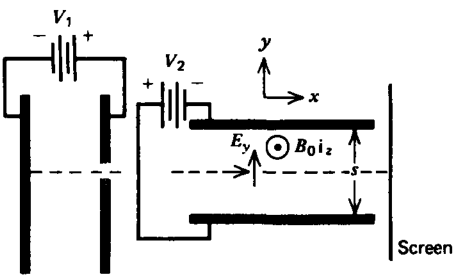
(e) What is the cutoff magnetic field? Check your answer with (b) in the limit $b = a + s$ where $s \ll a$.

3. A charge $q$ of mass $m$ within a gravity field $-g\mathbf{i}_y$ has an initial velocity $v_0\mathbf{i}_x$. A magnetic field $B_0\mathbf{i}_z$ is applied. What




value of $B_0$ will keep the particle moving at constant speed in the $x$ direction?

4. The charge to mass ratio of an electron $e/m$ was first measured by Sir J. J. Thomson in 1897 by the cathode-ray tube device shown. Electrons emitted by the cathode pass through a slit in the anode into a region with crossed electric and magnetic fields, both being perpendicular to the electrons velocity. The end of the tube is coated with a fluorescent material that produces a bright spot where the electron beam impacts.




(a) What is the velocity of the electrons when passing through the slit if their initial cathode velocity is $v_0$?

(b) The electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ are adjusted so that the vertical deflection of the beam is zero. What is the initial electron velocity? (Neglect gravity.)

(c) The voltage $V_2$ is now set to zero. What is the radius $R$ of the electrons motion about the magnetic field?

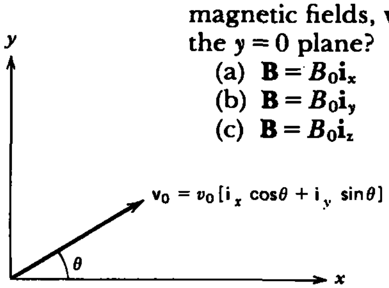
(d) What is $e/m$ in terms of $E$, $B$, and $R$?

5. A charge $q$ of mass $m$ at $t=0$ crosses the origin with velocity $\mathbf{v}_0 = \mathbf{v}_{x0}\mathbf{i}_x + v_{y0}\mathbf{i}_y$. For each of the following applied magnetic fields, where and when does the charge again cross the $y=0$ plane?

(a) $\mathbf{B} = B_0\mathbf{i}_x$

(b) $\mathbf{B} = B_0\mathbf{i}_y$

(c) $\mathbf{B} = B_0\mathbf{i}_z$



6. In 1896 Zeeman observed that an atom in a magnetic field had a fine splitting of its spectral lines. A classical theory of the Zeeman effect, developed by Lorentz, modeled the electron with mass $m$ as being bound to the nucleus by a spring-like force with spring constant $k$ so that in the absence of a magnetic field its natural frequency was $\omega_k=\sqrt{k/m}$.

(a) A magnetic field $B_0\mathbf{i}_z$ is applied. Write Newton's law for the $x$, $y$, and $z$ displacements of the electron including the spring and Lorentz forces.
(b) Because these equations are linear, guess exponential solutions of the form $e^{st}$. What are the natural frequencies?
(c) Because $\omega_k$ is typically in the optical range ($\omega_k \approx 10^{15}$ radian/sec), show that the frequency splitting is small compared to $\omega_k$ even for a strong field of $B_0=1$ tesla. In this limit, find approximate expressions for the natural frequencies of (b).

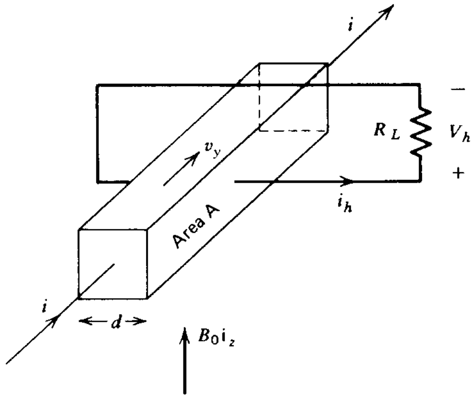
7. A charge $q$ moves through a region where there is an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$. The medium is very viscous so that inertial effects are negligible,

$$\beta\mathbf{v}=q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$$

where $\beta$ is the viscous drag coefficient. What is the velocity of the charge? (**Hint:** $(\mathbf{v}\times\mathbf{B})\times\mathbf{B}=-\mathbf{v}(\mathbf{B}\cdot\mathbf{B})+\mathbf{B}(\mathbf{v}\cdot\mathbf{B})$ and $\mathbf{v}\cdot\mathbf{B}=(q/\beta)\mathbf{E}\cdot\mathbf{B}$.)

8. Charges of mass $m$, charge $q$, and number density $n$ move through a conducting material and collide with the host medium with a collision frequency $\nu$ in the presence of an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$.

(a) Write Newton's first law for the charge carriers, along the same lines as developed in Section 3-2-2, with the addition of the Lorentz force.
(b) Neglecting particle inertia and diffusion, solve for the particle velocity $\mathbf{v}$.
(c) What is the constitutive law relating the current density $\mathbf{J}=qn\mathbf{v}$ to $\mathbf{E}$ and $\mathbf{B}$. This is the generalized Ohm's law in the presence of a magnetic field.
(d) What is the Ohmic conductivity $\sigma$? A current $i$ is passed through this material in the presence of a perpendicular magnetic field. A resistor $R_L$ is connected across the terminals. What is the Hall voltage? (See top of page 379).
(e) What value of $R_L$ maximizes the power dissipated in the load?




Section 5.2

9. A point charge $q$ is traveling within the magnetic field of an infinitely long line current $I$. At $r = r_0$ its velocity is

$$\mathbf{v}(t=0) = v_{r0}\mathbf{i}_r + v_{\phi 0}\mathbf{i}_\phi + v_{z0}\mathbf{i}_z$$

Its subsequent velocity is only a function of r.

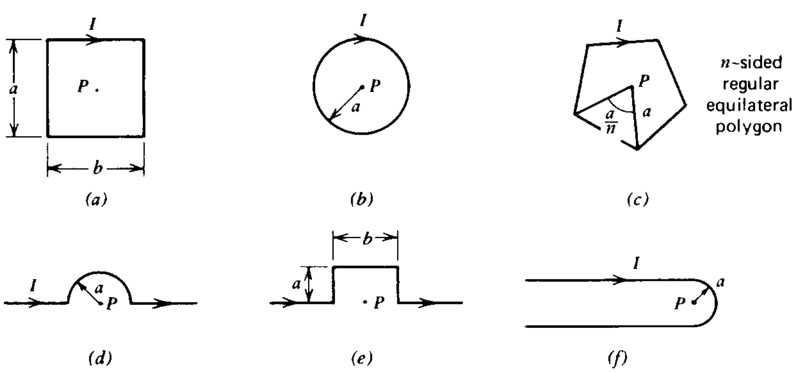
(a) What is the velocity of the charge as a function of position? **Hint:** See Problem 2*c* and 2*d*,

$$\int \frac{\ln x}{x}\,dx = \tfrac{1}{2}(\ln x)^2$$

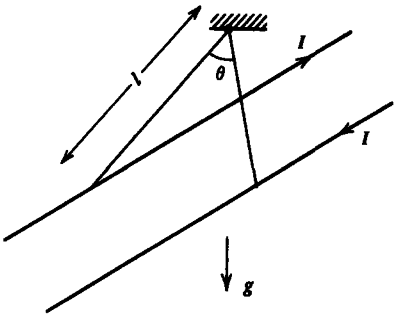
(b) What is the kinetic energy of the charge?

(c) What is the closest distance that the charge can approach the line current if $v_{\phi 0} = 0$?

10. Find the magnetic field at the point $P$ shown for the following line currents:

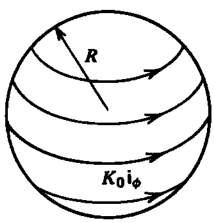



11. Two long parallel line currents of mass per unit length $m$ in a gravity field $g$ each carry a current $I$ in opposite




directions. They are suspended by cords of length $l$. What is the angle $\theta$ between the cords?

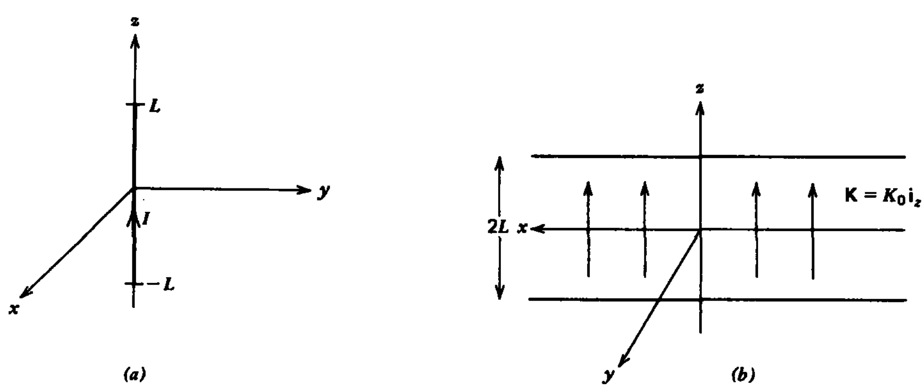
12. A constant current $K_0\mathbf{i}_\phi$ flows on the surface of a sphere of radius $R$.




(a) What is the magnetic field at the center of the sphere? (**HINT:** $\mathbf{i}_\phi \times \mathbf{i}_r = \mathbf{i}_\theta = \cos\theta\cos\phi\mathbf{i}_x + \cos\theta\sin\phi\mathbf{i}_y - \sin\theta\mathbf{i}_z$.)

(b) Use the results of (a) to find the magnetic field at the center of a spherical shell of inner radius $R_1$ and outer radius $R_2$ carrying a uniformly distributed volume current $J_0\mathbf{i}_\phi$.

13. A line current $I$ of length $2L$ flows along the $z$ axis.



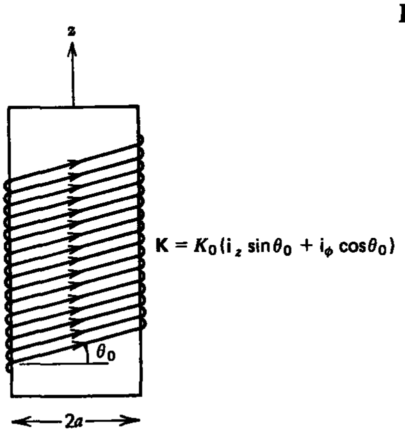
(a) What is the magnetic field everywhere in the $z=0$ plane?

(b) Use the results of (a) to find the magnetic field in the $z=0$ plane due to an infinitely long current sheet of height $2L$ and uniform current density $K_0\mathbf{i}_z$. **Hint:** Let $u=x^2+y^2$

$$\int \frac{du}{u(u^2+bu-a)^{1/2}}=\frac{1}{\sqrt{a}}\sin^{-1}\left(\frac{bu+2a}{u\sqrt{b^2+4a}}\right)$$

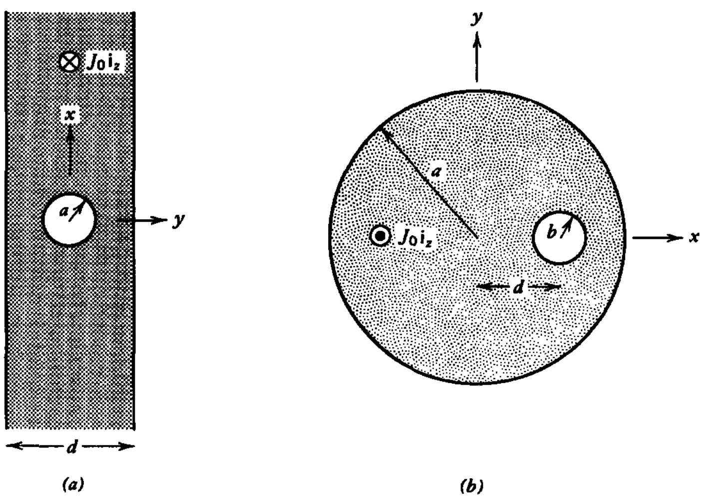
14. Closely spaced wires are wound about an infinitely long cylindrical core at pitch angle $\theta_0$. A current flowing in the wires then approximates a surface current

$$\mathbf{K}=K_0(\mathbf{i}_z\sin\theta_0+\mathbf{i}_\phi\cos\theta_0)$$




What is the magnetic field everywhere?

15. An infinite slab carries a uniform current $J_0\mathbf{i}_z$ except within a cylindrical hole of radius $a$ centered within the slab.



(a) Find the magnetic field everywhere? (**Hint:** Use superposition replacing the hole by two oppositely directed currents.)

(b) An infinitely long cylinder of radius $a$ carrying a uniform current $J_0\mathbf{i}_z$ has an off-axis hole of radius $b$ with center a distance $d$ from the center of the cylinder. What is the magnetic field within the hole? (**Hint:** Convert to Cartesian coordinates $r\mathbf{i}_\phi = x\mathbf{i}_y - y\mathbf{i}_x$.)

Section 5.3

16. Which of the following vectors can be a magnetic field $\mathbf{B}$? If so, what is the current density $\mathbf{J}$?

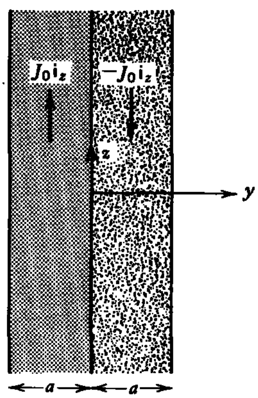
(a) $\mathbf{B} = ar\mathbf{i}_r$

(b) $\mathbf{B} = a(x\mathbf{i}_y - y\mathbf{i}_x)$

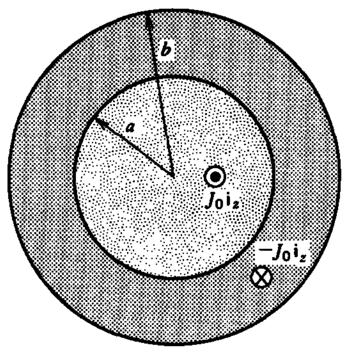
(c) $\mathbf{B} = a(x\mathbf{i}_x - y\mathbf{i}_y)$

(d) $\mathbf{B} = ar\mathbf{i}_\phi$

17. Find the magnetic field everywhere for each of the following current distributions:



(a)



(c)

(a) $\mathbf{J} = \begin{cases} J_0\mathbf{i}_z, & -a < y < 0 \\ -J_0\mathbf{i}_z, & 0 < y < a \end{cases}$

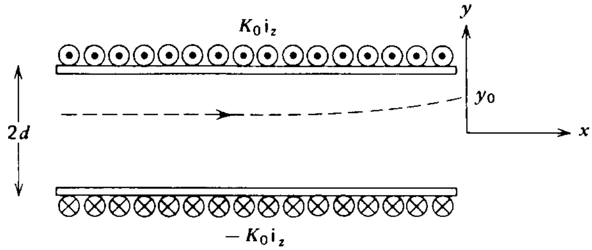
(b) $\mathbf{J} = \dfrac{J_0 y}{a}\mathbf{i}_z, \quad -a < y < a$

(c) $\mathbf{J} = \begin{cases} J_0\mathbf{i}_z, & 0 < r < a \\ -J_0\mathbf{i}_z, & a < r < b \end{cases}$

(d) $\mathbf{J} = \begin{cases} \dfrac{J_0 r}{a}\mathbf{i}_z, & r < a \\ 0, & r > a \end{cases}$

Section 5.4

18. Two parallel semi-infinite current sheets a distance $d$ apart have their currents flowing in opposite directions and extend over the interval $-\infty < x < 0$.



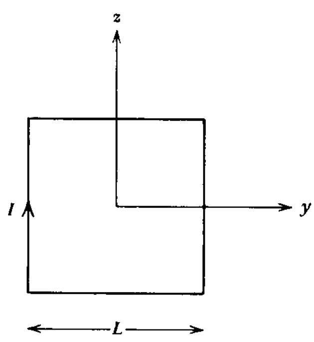

(a) What is the vector potential? (**Hint:** Use superposition of the results in Section 5-3-4*b*.)

(b) What is the magnetic field everywhere?

(c) How much magnetic flux per unit length emanates through the open face at $x = 0$? How much magnetic flux per unit length passes through each current sheet?

(d) A magnetic field line emanates at the position $y_0(0 < y_0 < d)$ in the $x = 0$ plane. At what value of $y$ is this field line at $x = -\infty$?

19. (a) Show that $\nabla \cdot \mathbf{A} \neq 0$ for the finite length line current in Section 5-4-3a. Why is this so?

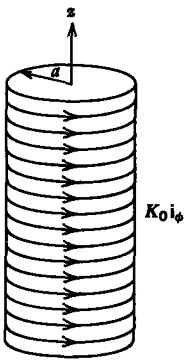



(b) Find the vector potential for a square loop.

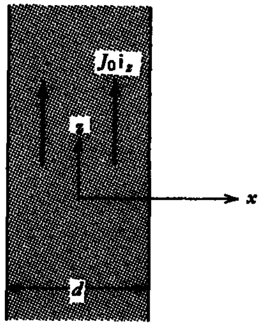
(c) What is $\nabla \cdot \mathbf{A}$ now?

20. Find the magnetic vector potential and magnetic field for the following current distributions: (**Hint:** $\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$)

(i) Infinitely long cylinder of radius $a$ carrying a
  (a) surface current $K_0 \mathbf{i}_\phi$
  (b) surface current $K_0 \mathbf{i}_z$
  (c) volume current $J_0 \mathbf{i}_z$




(a)




(d)

(ii) Infinitely long slab of thickness $d$ carrying a

(d) volume current $J_0\mathbf{i}_z$

(e) volume current $\dfrac{J_0 x}{d}\mathbf{i}_z$

Section 5.5

21. A general definition for the magnetic dipole moment for any shaped current loop is

$$\mathbf{m}=\frac{1}{2}\oint \mathbf{r}\times\mathbf{I}\,dl$$

If the current is distributed over a surface or volume or is due to a moving point charge we use

$$\mathbf{I}\,dl\to q\mathbf{v}\to\mathbf{K}\,dS\to\mathbf{J}\,dV$$

What is the magnetic dipole moment for the following current distributions:

(a) a point charge $q$ rotated at constant angular speed $\omega$ at radius $a$;

(b) a circular current loop of radius $a$ carrying a current $I$;

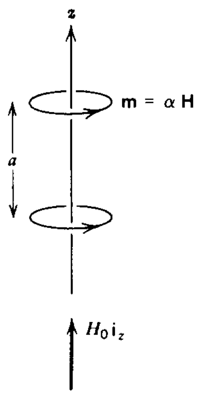
(c) a disk of radius $a$ with surface current $K_0\mathbf{i}_\phi$;

(d) a uniformly distributed sphere of surface or volume charge with total charge $Q$ and radius $R$ rotating in the $\phi$ direction at constant angular speed $\omega$. (**Hint:** $\mathbf{i}_r\times\mathbf{i}_\phi=-\mathbf{i}_\theta=-[\cos\theta\cos\phi\,\mathbf{i}_x+\cos\theta\sin\phi\,\mathbf{i}_y-\sin\theta\,\mathbf{i}_z]$)

22. Two identical point magnetic dipoles $\mathbf{m}$ with magnetic polarizability $\alpha\,(\mathbf{m}=\alpha\mathbf{H})$ are a distance $a$ apart along the $z$ axis. A macroscopic field $H_0\mathbf{i}_z$ is applied.

(a) What is the local magnetic field acting on each dipole?

(b) What is the force on each dipole?




(c) Repeat (a) and (b) if we have an infinite array of such dipoles. **Hint:**

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.2$$

(d) If we assume that there is one such dipole within each volume of $a^3$, what is the permeability of the medium?

23. An orbiting electron with magnetic moment $m_z \mathbf{i}_z$ is in a uniform magnetic field $B_0 \mathbf{i}_z$ when at $t=0$ it is slightly displaced so that its angular momentum $\mathbf{L} = -(2m_e/e)\mathbf{m}$ now also has $x$ and $y$ components.

(a) Show that the torque equation can be put in terms of the magnetic moment

$$\frac{d\mathbf{m}}{dt} = -\gamma \mathbf{m} \times \mathbf{B}$$

where $\gamma$ is called the gyromagnetic ratio. What is $\gamma$?

(b) Write out the three components of (a) and solve for the magnetic moment if at $t=0$ the moment is initially

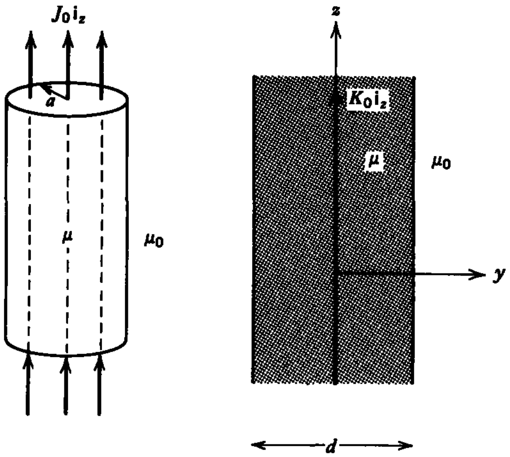
$$\mathbf{m}(t=0) = m_{x0}\mathbf{i}_x + m_{y0}\mathbf{i}_y + m_{z0}\mathbf{i}_z$$

(c) Show that the magnetic moment precesses about the applied magnetic field. What is the precessional frequency?

24. What are the **B**, **H**, and **M** fields and the resulting magnetization currents for the following cases:

(a) A uniformly distributed volume current $J_0 \mathbf{i}_z$ through a cylinder of radius $a$ and permeability $\mu$ surrounded by free space.

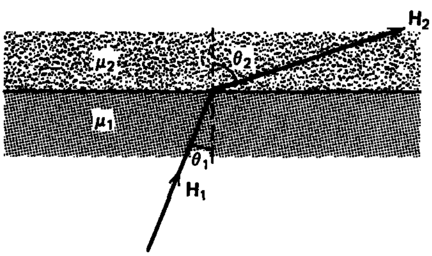
(b) A current sheet $K_0 \mathbf{i}_z$ centered within a permeable slab of thickness $d$ surrounded by free space.




(a) (b)

Section 5.6

25. A magnetic field with magnitude $H_1$ is incident upon the flat interface separating two different linearly permeable materials at an angle $\theta_1$ from the normal. There is no surface




current on the interface. What is the magnitude and angle of the magnetic field in region 2?

26. A cylinder of radius $a$ and length $L$ is permanently magnetized as $M_0\mathbf{i}_z$.

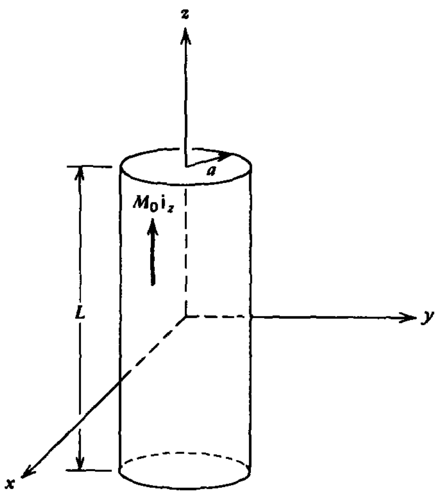
(a) What are the **B** and **H** fields everywhere along its axis?

(b) What are the fields far from the magnet ($r \gg a$, $r \gg L$)?

(c) Use the results of (a) to find the **B** and **H** fields everywhere due to a permanently magnetized slab $M_0\mathbf{i}_z$ of infinite $xy$ extent and thickness $L$.

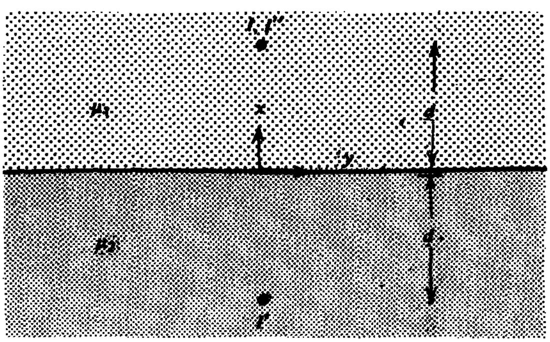
(d) Repeat (a) and (b) if the cylinder has magnetization $M_0(1 - r/a)\mathbf{i}_z$. **Hint:**

$$\int \frac{dr}{(a^2 + r^2)^{1/2}} = \ln\left(r + \sqrt{a^2 + r^2}\right)$$




Section 5.7

27. A $z$-directed line current $I$ is a distance $d$ above the interface separating two different magnetic materials with permeabilities $\mu_1$ and $\mu_2$.

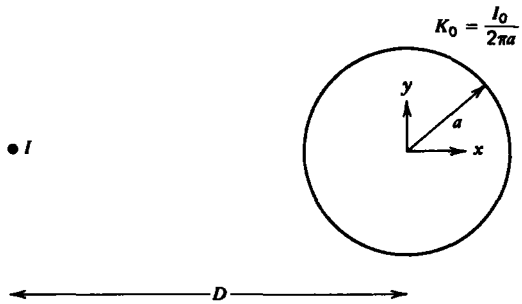


(a) Find the image currents $I'$ at position $x=-d$ and $I''$ at $x=d$ that satisfy all the boundary conditions. The field in region 1 is due to $I$ and $I'$ while the field in region 2 is due to $I''$. (**Hint:** See the analogous dielectric problem in Section 3-3-3.)

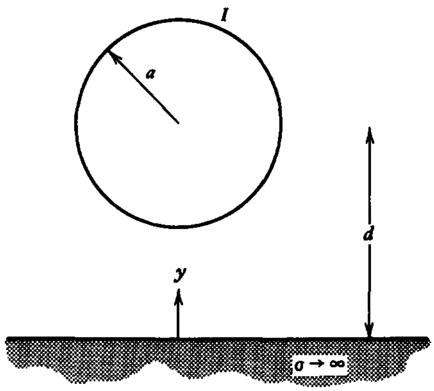
(b) What is the force per unit length on the line current $I$?

28. An infinitely long line current $I$ is parallel to and a distance $D$ from the axis of a perfectly conducting cylinder of radius $a$ carrying a total surface current $I_0$.

(a) Find suitable image currents and verify that the boundary conditions are satisfied. (**Hint:** $x\mathbf{i}_y - y\mathbf{i}_x = r\mathbf{i}_\phi$; $\mathbf{i}_y = \sin\phi\,\mathbf{i}_r + \cos\phi\,\mathbf{i}_\phi$; $x = r\cos\phi$.)




(a)




(d)

(b) What is the surface current distribution on the cylinder? What total current flows on the cylinder? **Hint:**

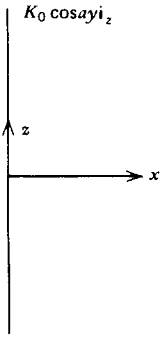
$$\int \frac{d\phi}{a+b\cos\phi} = \frac{2}{[a^2-b^2]^{1/2}} \tan^{-1}\left(\frac{[a^2-b^2]^{1/2}\tan\left(\frac{1}{2}\phi\right)}{(a+b)}\right)$$

(c) What is the force per unit length on the cylinder?

(d) A perfectly conducting cylinder of radius $a$ carrying a total current $I$ has its center a distance $d$ above a perfectly conducting plane. What image currents satisfy the boundary conditions?

(e) What is the force per unit length on the cylinder?

29. A current sheet $K_0 \cos ay\,\mathbf{i}_z$ is placed at $x=0$. Because there are no volume currents for $x \neq 0$, a scalar magnetic potential can be defined $\mathbf{H} = \nabla \chi$.



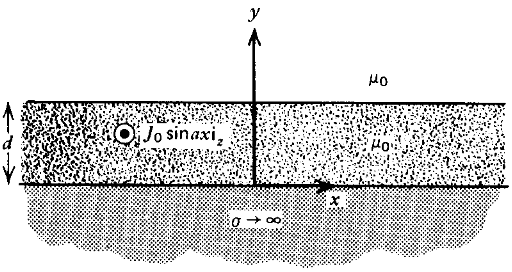

(a) What is the general form of solution for $\chi$? (**Hint:** See Section 4-2-3.)

(b) What boundary conditions must be satisfied?

(c) What is the magnetic field and vector potential everywhere?

(d) What is the equation of the magnetic field lines?

30. A slab of thickness $d$ carries a volume current distribution $J_0 \sin ax\mathbf{i}_z$ and is placed upon a perfectly conducting ground plane.




(a) Find a particular solution for the vector potential. Are all the boundary conditions satisfied?

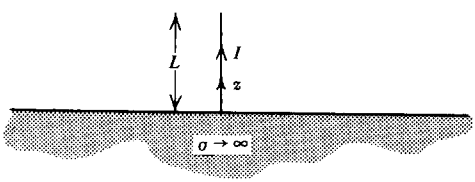
(b) Show that additional solutions to Laplace's equations can be added to the vector potential to satisfy the boundary conditions. What is the magnetic field everywhere?

(c) What is the surface current distribution on the ground plane?

(d) What is the force per unit length on a section of ground plane of width $2\pi/a$? What is the body force per unit length on a section of the current carrying slab of width $2\pi/a$?

(e) What is the magnetic field if the slab carries no current but is permanently magnetized as $M_0 \sin ax\mathbf{i}_y$ Repeat (c) and (d).

31. A line current of length $L$ stands perpendicularly upon a perfectly conducting ground plane.

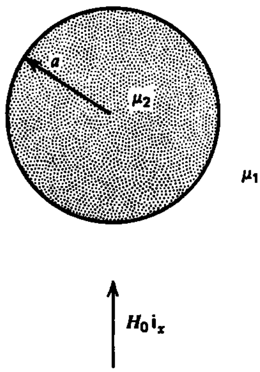


(a) Find a suitable image current that is equivalent to the induced current on the $z = 0$ plane. Does the direction of the image current surprise you?

(b) What is the magnetic field everywhere? (**Hint:** See Section 5-4-3*a*.)

(c) What is the surface current distribution on the conducting plane?

32. A cylinder of radius $a$ is placed within a uniform magnetic field $H_0\mathbf{i}_x$. Find the magnetic field for each of the following cases:



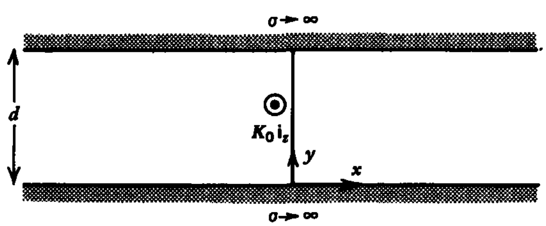

(a)

(a) Cylinder has permeability $\mu_2$ and surrounding medium has permeability $\mu_1$.

(b) Perfectly conducting cylinder in free space.

(c) Uniformly magnetized cylinder $M_2\mathbf{i}_x$ in a uniformly magnetized medium $M_1\mathbf{i}_x$.

33. A current sheet $K_0\mathbf{i}_z$ is placed along the $y$ axis at $x = 0$ between two parallel perfectly conducting planes a distance $d$ apart.




(a) Write the constant current at $x = 0$ as an infinite Fourier series of fundamental period $2d$. (**Hint:** See Section 4-2-5.)

(b) What general form of a scalar potential $\chi$, where $\mathbf{H} = \nabla\chi$, will satisfy the boundary conditions?

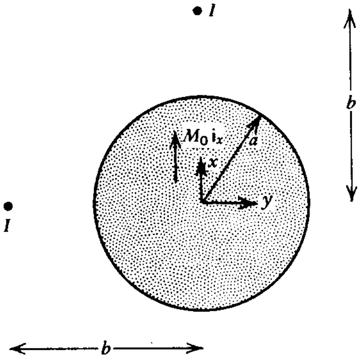
(c) What is the magnetic field everywhere?

(d) What is the surface current distribution and the total current on the conducting planes? **Hint:**

$$\sum_{\substack{n=1 \\ (n \text{ odd})}}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8}$$

Section 5.8

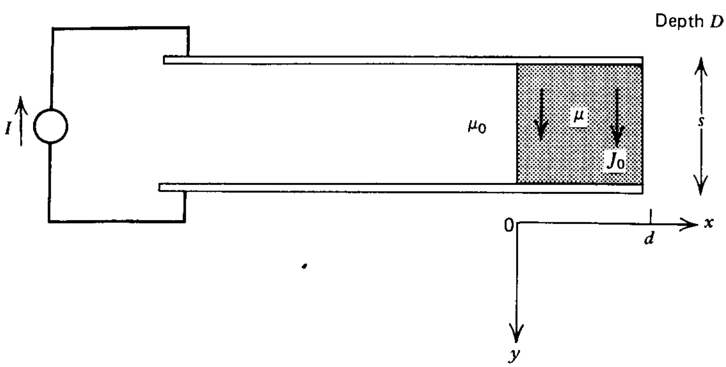
34. An infinitely long cylinder of radius $a$ is permanently magnetized as $M_0\mathbf{i}_x$.




(a) Find the magnetic field everywhere.

(b) An infinitely long line current $I$ is placed either at $y=-b$ or at $x=b$ $(b>a)$. For each of these cases, what is the force per unit length on the line current? (**Hint:** See problem 32c.)

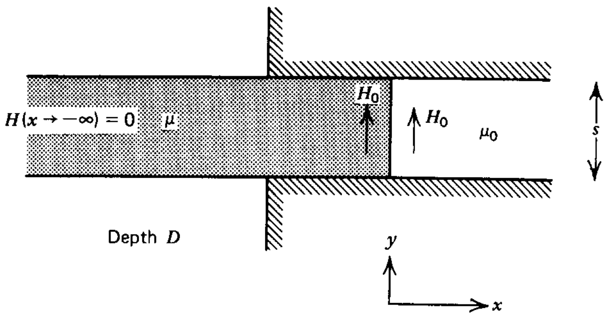
35. Parallel plate electrodes are separated by a rectangular conducting slab that has a permeability $\mu$. The system is driven by a dc current source.




(a) Neglecting fringing field effects assume the magnetic field is $H_z(x)\mathbf{i}_z$. If the current is uniformly distributed throughout the slab, find the magnetic field everywhere.

(b) What is the total force on the slab? Does the force change with different slab permeability? Why not?

36. A permeable slab is partially inserted into the air gap of a magnetic circuit with uniform field $H_0$. There is a nonuniform fringing field right outside the magnetic circuit near the edges.




(a) What is the total force on the slab in the $x$ direction?

(b) Repeat (a) if the slab is permanently magnetized $\mathbf{M}=M_0\mathbf{i}_y$. (**Hint:** What is $H_x(x=-\infty)$? See Example 5-2a.)

# chapter 6

*electromagnetic induction*

In our development thus far, we have found the electric and magnetic fields to be uncoupled. A net charge generates an electric field while a current is the source of a magnetic field. In 1831 Michael Faraday experimentally discovered that a time varying magnetic flux through a conducting loop also generated a voltage and thus an electric field, proving that electric and magnetic fields are coupled.

# 6-1 FARADAY'S LAW OF INDUCTION

## 6-1-1 The Electromotive Force (EMF)

Faraday's original experiments consisted of a conducting loop through which he could impose a dc current via a switch. Another short circuited loop with no source attached was nearby, as shown in Figure 6-1. When a dc current flowed in loop 1, no current flowed in loop 2. However, when the voltage was first applied to loop 1 by closing the switch, a transient current flowed in the opposite direction in loop 2.

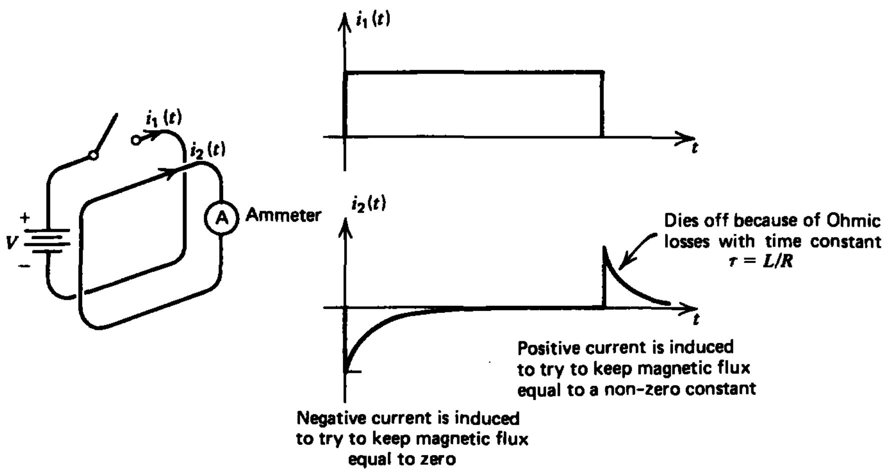



Figure 6-1 Faraday's experiments showed that a time varying magnetic flux through a closed conducting loop induced a current in the direction so as to keep the flux through the loop constant.

When the switch was later opened, another transient current flowed in loop 2, this time in the same direction as the original current in loop 1. Currents are induced in loop 2 whenever a time varying magnetic flux due to loop 1 passes through it.

In general, a time varying magnetic flux can pass through a circuit due to its own or nearby time varying current or by the motion of the circuit through a magnetic field. For any loop, as in Figure 6-2, Faraday's law is

$$\text{EMF} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \qquad (1)$$

where EMF is the electromotive force defined as the line integral of the electric field. The minus sign is introduced on the right-hand side of (1) as we take the convention that positive flux flows in the direction perpendicular to the direction of the contour by the right-hand rule.

### 6-1-2 Lenz's Law

The direction of induced currents is always such as to oppose any changes in the magnetic flux already present. Thus in Faraday's experiment, illustrated in Figure 6-1, when the switch in loop 1 is first closed there is no magnetic flux in loop 2 so that the induced current flows in the opposite direction with its self-magnetic field opposite to the imposed field. The induced current tries to keep a zero flux through

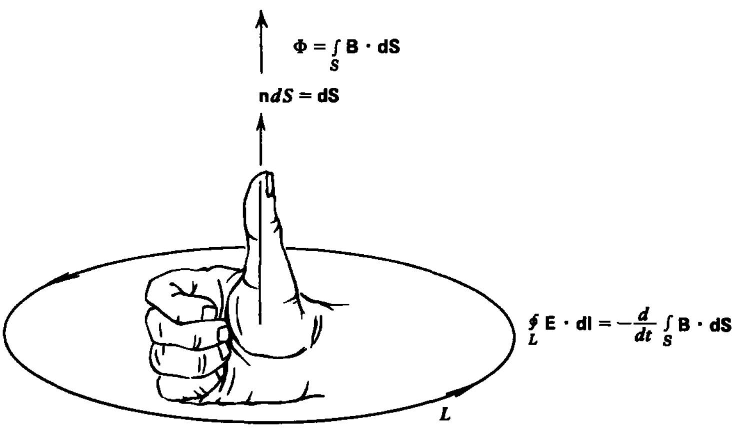



Figure 6-2 Faraday's law states that the line integral of the electric field around a closed loop equals the time rate of change of magnetic flux through the loop. The positive convention for flux is determined by the right-hand rule of curling the fingers on the right hand in the direction of traversal around the loop. The thumb then points in the direction of positive magnetic flux.

loop 2. If the loop is perfectly conducting, the induced current flows as long as current flows in loop 1, with zero net flux through the loop. However, in a real loop, resistive losses cause the current to exponentially decay with an $L/R$ time constant, where $L$ is the self-inductance of the loop and $R$ is its resistance. Thus, in the dc steady state the induced current has decayed to zero so that a constant magnetic flux passes through loop 2 due to the current in loop 1.

When the switch is later opened so that the current in loop 1 goes to zero, the second loop tries to maintain the constant flux already present by inducing a current flow in the same direction as the original current in loop 1. Ohmic losses again make this induced current die off with time.

If a circuit or any part of a circuit is made to move through a magnetic field, currents will be induced in the direction such as to try to keep the magnetic flux through the loop constant. The force on the moving current will always be opposite to the direction of motion.

Lenz's law is clearly demonstrated by the experiments shown in Figure 6-3. When a conducting ax is moved into a magnetic field, eddy currents are induced in the direction where their self-flux is opposite to the applied magnetic field. The Lorentz force is then in the direction opposite to the motion of the ax. This force decreases with time as the currents decay with time due to Ohmic dissipation. If the ax was slotted, effectively creating a very high resistance to the eddy currents, the reaction force becomes very small as the induced current is small.

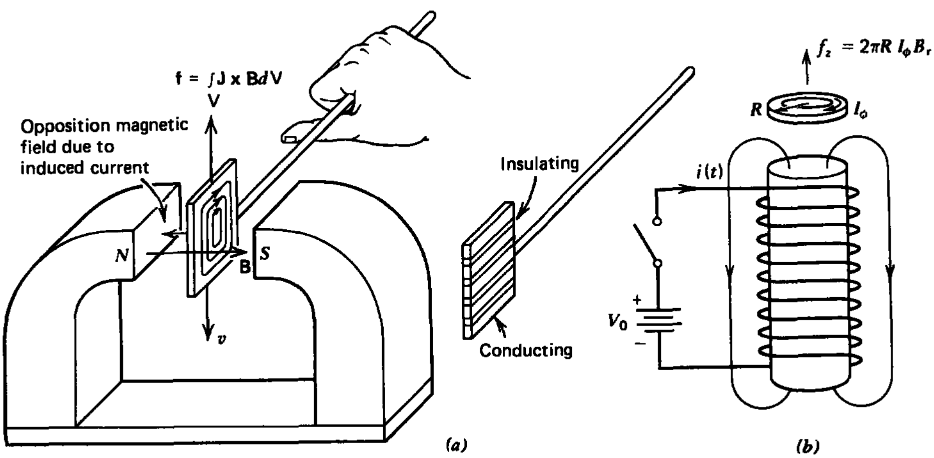



**Figure 6-3** Lenz's law. (*a*) Currents induced in a conductor moving into a magnetic field exert a force opposite to the motion. The induced currents can be made small by slotting the ax. (*b*) A conducting ring on top of a coil is flipped off when a current is suddenly applied, as the induced currents try to keep a zero flux through the ring.

When the current is first turned on in the coil in Figure 6-3*b*, the conducting ring that sits on top has zero flux through it. Lenz's law requires that a current be induced opposite to that in the coil. Instantaneously there is no $z$ component of magnetic field through the ring so the flux must return radially. This creates an upwards force:

$$\mathbf{f} = 2\pi R\mathbf{I}\times\mathbf{B} = 2\pi R I_\phi B_r \mathbf{i}_z \tag{2}$$

which flips the ring off the coil. If the ring is cut radially so that no circulating current can flow, the force is zero and the ring does not move.

**(a) Short Circuited Loop**

To be quantitative, consider the infinitely long time varying line current $I(t)$ in Figure 6-4, a distance r from a rectangular loop of wire with Ohmic conductivity $\sigma$, cross-sectional area $A$, and total length $l = 2(D+d)$. The magnetic flux through the loop due to $I(t)$ is

$$\Phi_m = \int_{z=-D/2}^{D/2}\int_{\mathrm{r}}^{\mathrm{r}+d} \mu_0 H_\phi(\mathrm{r}')\, d\mathrm{r}'\, dz$$

$$= \frac{\mu_0 ID}{2\pi}\int_{\mathrm{r}}^{\mathrm{r}+d}\frac{d\mathrm{r}'}{\mathrm{r}'} = \frac{\mu_0 ID}{2\pi}\ln\frac{\mathrm{r}+d}{\mathrm{r}} \tag{3}$$

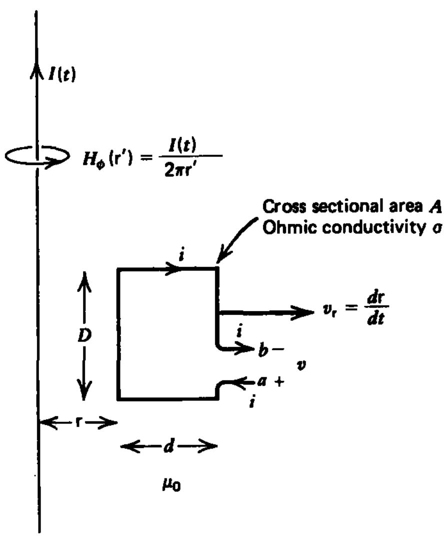



Figure 6-4 A rectangular loop near a time varying line current. When the terminals are short circuited the electromotive force induces a current due to the time varying mutual flux and/or because of the motion of the circuit through the imposed nonuniform magnetic field of the line current. If the loop terminals are open circuited there is no induced current but a voltage develops.

The mutual inductance $M$ is defined as the flux to current ratio where the flux through the loop is due to an external current. Then (3) becomes

$$\Phi_m = M(\mathrm{r})I, \qquad M(\mathrm{r}) = \frac{\mu_0 D}{2\pi}\ln\frac{\mathrm{r}+d}{\mathrm{r}} \tag{4}$$

When the loop is short circuited ($v=0$), the induced Ohmic current $i$ gives rise to an electric field $[E = J/\sigma = i/(A\sigma)]$ so that Faraday's law applied to a contour within the wire yields an electromotive force just equal to the Ohmic voltage drop:

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \frac{il}{\sigma A} = iR = -\frac{d\Phi}{dt} \tag{5}$$

where $R = l/(\sigma A)$ is the resistance of the loop. By convention, the current is taken as positive in the direction of the line integral.

The flux in (5) has contributions both from the imposed current as given in (3) and from the induced current proportional to the loop's self-inductance $L$, which for example is given in Section 5-4-3$c$ for a square loop ($D=d$):

$$\Phi = M(\mathrm{r})I + Li \tag{6}$$

If the loop is also moving radially outward with velocity $v_\mathrm{r} = d\mathrm{r}/dt$, the electromotively induced Ohmic voltage is

$$\begin{aligned} -iR &= \frac{d\Phi}{dt} \\ &= M(\mathrm{r})\frac{dI}{dt} + I\frac{dM(\mathrm{r})}{dt} + L\frac{di}{dt} \\ &= M(\mathrm{r})\frac{dI}{dt} + I\frac{dM}{d\mathrm{r}}\frac{d\mathrm{r}}{dt} + L\frac{di}{dt} \end{aligned} \tag{7}$$

where $L$ is not a function of the loop's radial position.

If the loop is stationary, only the first and third terms on the right-hand side contribute. They are nonzero only if the currents change with time. The second term is due to the motion and it has a contribution even for dc currents.

**Turn-on Transient.** If the loop is stationary ($d\mathrm{r}/dt = 0$) at $\mathrm{r} = \mathrm{r}_0$, (7) reduces to

$$L\frac{di}{dt} + iR = -M(\mathrm{r}_0)\frac{dI}{dt} \tag{8}$$

If the applied current $I$ is a dc step turned on at $t=0$, the solution to (8) is

$$i(t) = -\frac{M(\mathrm{r}_0)I}{L}e^{-(R/L)t}, \quad t>0 \tag{9}$$

where the impulse term on the right-hand side of (8) imposes the initial condition $i(t=0)=-M(\mathrm{r}_0)I/L$. The current is negative, as Lenz's law requires the self-flux to oppose the applied flux.

**Turn-off Transient.** If after a long time $T$ the current $I$ is instantaneously turned off, the solution is

$$i(t)=\frac{M(\mathrm{r}_0)I}{L}e^{-(R/L)(t-T)}, \quad t>T \tag{10}$$

where now the step decrease in current $I$ at $t=T$ reverses the direction of the initial current.

**Motion with a dc Current.** With a dc current, the first term on the right-hand side in (7) is zero yielding

$$L\frac{di}{dt}+iR=\frac{\mu_0IDd}{2\pi\mathrm{r}(\mathrm{r}+d)}\frac{d\mathrm{r}}{dt} \tag{11}$$

To continue, we must specify the motion so that we know how r changes with time. Let's consider the simplest case when the loop has no resistance ($R=0$). Then (11) can be directly integrated as

$$Li=-\frac{\mu_0ID}{2\pi}\ln\frac{1+d/\mathrm{r}}{1+d/\mathrm{r}_0} \tag{12}$$

where we specify that the current is zero when $\mathrm{r}=\mathrm{r}_0$. This solution for a lossless loop only requires that the total flux of (6) remain constant. The current is positive when $\mathrm{r}>\mathrm{r}_0$ as the self-flux must aid the decreasing imposed flux. The current is similarly negative when $\mathrm{r}<\mathrm{r}_0$ as the self-flux must cancel the increasing imposed flux.

The force on the loop for all these cases is only due to the force on the $z$-directed current legs at r and $\mathrm{r}+d$:

$$\begin{aligned} f_\mathrm{r}&=\frac{\mu_0DiI}{2\pi}\left(\frac{1}{\mathrm{r}+d}-\frac{1}{\mathrm{r}}\right)\\ &=-\frac{\mu_0DiId}{2\pi\mathrm{r}(\mathrm{r}+d)} \end{aligned} \tag{13}$$

being attractive if $iI>0$ and repulsive if $iI<0$.

### (b) Open Circuited Loop

If the loop is open circuited, no induced current can flow and thus the electric field within the wire is zero ($\mathbf{J}=\sigma\mathbf{E}=0$). The electromotive force then only has a contribution from the gap between terminals equal to the negative of the voltage:

$$\oint_L \mathbf{E}\cdot d\mathbf{l}=\int_b^a \mathbf{E}\cdot d\mathbf{l}=-v=-\frac{d\Phi}{dt}\Rightarrow v=\frac{d\Phi}{dt} \tag{14}$$

Note in Figure 6-4 that our convention is such that the current $i$ is always defined positive flowing out of the positive voltage terminal into the loop. The flux $\Phi$ in (14) is now only due to the mutual flux given by (3), as with $i=0$ there is no self-flux. The voltage on the moving open circuited loop is then

$$v = M(\mathrm{r})\frac{dI}{dt} + I\frac{dM}{d\mathrm{r}}\frac{d\mathrm{r}}{dt} \tag{15}$$

**(c) Reaction Force**

The magnetic force on a short circuited moving loop is always in the direction opposite to its motion. Consider the short circuited loop in Figure 6-5, where one side of the loop moves with velocity $v_x$. With a uniform magnetic field applied normal to the loop pointing out of the page, an expansion of the loop tends to link more magnetic flux requiring the induced current to flow clockwise so that its self-flux is in the direction given by the right-hand rule, opposite to the applied field. From (1) we have

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \frac{il}{\sigma A} = iR = -\frac{d\Phi}{dt} = B_0 D\frac{dx}{dt} = B_0 D v_x \tag{16}$$

where $l = 2(D+x)$ also changes with time. The current is then

$$i = \frac{B_0 D v_x}{R} \tag{17}$$

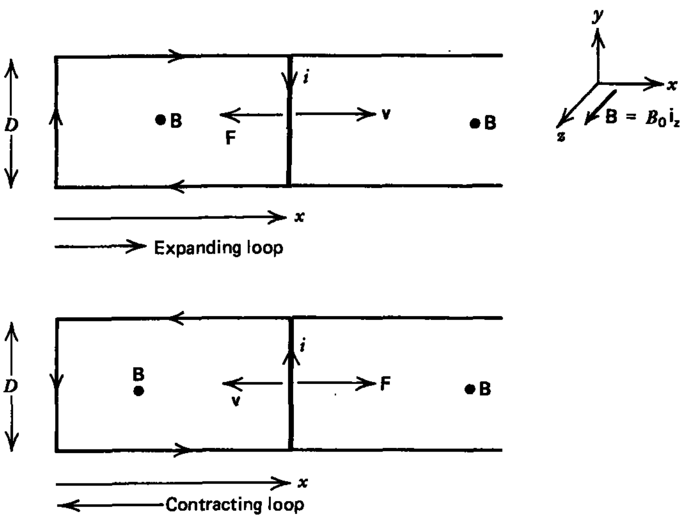



Figure 6-5 If a conductor moves perpendicular to a magnetic field a current is induced in the direction to cause the Lorentz force to be opposite to the motion. The total flux through the closed loop, due to both the imposed field and the self-field generated by the induced current, tries to remain constant.

where we neglected the self-flux generated by $i$, assuming it to be much smaller than the applied flux due to $B_0$. Note also that the applied flux is negative, as the right-hand rule applied to the direction of the current defines positive flux into the page, while the applied flux points outwards.

The force on the moving side is then to the left,

$$\mathbf{f} = -iD\mathbf{i}_y \times B_0\mathbf{i}_z = -iDB_0\mathbf{i}_x = -\frac{B_0^2D^2v_x}{R}\mathbf{i}_x \qquad (18)$$

opposite to the velocity.

However if the side moves to the left ($v_x < 0$), decreasing the loop's area thereby linking less flux, the current reverses direction as does the force.

### 6-1-3 Laminations

The induced eddy currents in Ohmic conductors results in Ohmic heating. This is useful in induction furnaces that melt metals, but is undesired in many iron core devices. To reduce this power loss, the cores are often sliced into many thin sheets electrically insulated from each other by thin oxide coatings. The current flow is then confined to lie within a thin sheet and cannot cross over between sheets. The insulating laminations serve the same purpose as the cuts in the slotted ax in Figure 6-3$a$.

The rectangular conductor in Figure 6-6$a$ has a time varying magnetic field $B(t)$ passing through it. We approximate the current path as following the rectangular shape so that

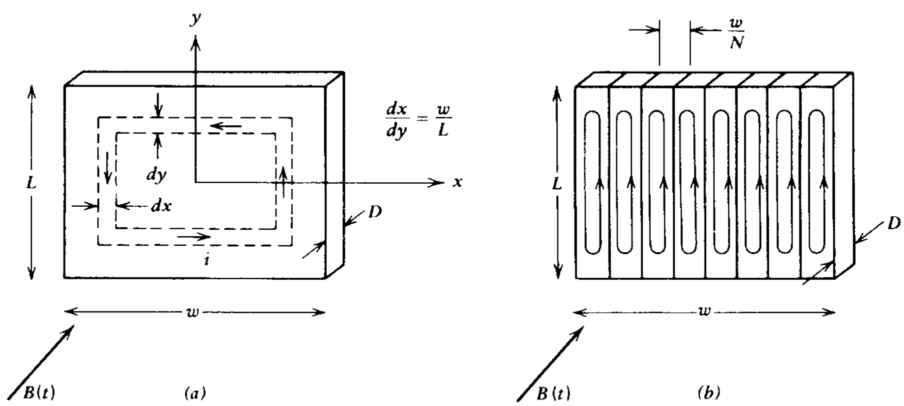



Figure 6-6 ($a$) A time varying magnetic field through a conductor induces eddy currents that cause Ohmic heating. ($b$) If the conductor is laminated so that the induced currents are confined to thin strips, the dissipated power decreases.

the flux through the loop of incremental width $dx$ and $dy$ of area $4xy$ is

$$\Phi = -4xyB(t) \tag{19}$$

where we neglect the reaction field of the induced current assuming it to be much smaller than the imposed field. The minus sign arises because, by the right-hand rule illustrated in Figure 6-2, positive flux flows in the direction opposite to $B(t)$. The resistance of the loop is

$$R_x = \frac{4}{\sigma D}\left(\frac{y}{dx} + \frac{x}{dy}\right) = \frac{4}{\sigma D}\frac{L}{w}\frac{x}{dx}\left[1 + \left(\frac{w}{L}\right)^2\right] \tag{20}$$

The electromotive force around the loop then just results in an Ohmic current:

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = iR_x = \frac{-d\Phi}{dt} = 4xy\frac{dB}{dt} = \frac{4L}{w}x^2\frac{dB}{dt} \tag{21}$$

with dissipated power

$$dp = i^2 R_x = \frac{4Dx^3\sigma L(dB/dt)^2\, dx}{w[1+(w/L)^2]} \tag{22}$$

The total power dissipated over the whole sheet is then found by adding the powers dissipated in each incremental loop:

$$\begin{aligned} P &= \int_0^{w/2} dp \\ &= \frac{4D(dB/dt)^2\sigma L}{w[1+(w/L)^2]}\int_0^{w/2} x^3\, dx \\ &= \frac{LDw^3\sigma(dB/dt)^2}{16[1+(w/L)^2]} \end{aligned} \tag{23}$$

If the sheet is laminated into $N$ smaller ones, as in Figure 6-6$b$, each section has the same solution as (23) if we replace $w$ by $w/N$. The total power dissipated is then $N$ times the power dissipated in a single section:

$$P = \frac{LD(w/N)^3\sigma(dB/dt)^2 N}{16[1+(w/NL)^2]} = \frac{\sigma LDw^3(dB/dt)^2}{16N^2[1+(w/NL)^2]} \tag{24}$$

As $N$ becomes large so that $w/NL \ll 1$, the dissipated power decreases as $1/N^2$.

### 6-1-4 Betatron

The cyclotron, discussed in Section 5-1-4, is not used to accelerate electrons because their small mass allows them to reach relativistic speeds, thereby increasing their mass and decreasing their angular speed. This puts them out of phase with the applied voltage. The betatron in Figure 6-7 uses the transformer principle where the electrons circulating about the evacuated toroid act like a secondary winding. The imposed time varying magnetic flux generates an electric field that accelerates the electrons.

Faraday's law applied to a contour following the charge's trajectory at radius $R$ yields

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = E_\phi 2\pi R = -\frac{d\Phi}{dt} \tag{25}$$

which accelerates the electrons as

$$m\frac{dv_\phi}{dt} = -eE_\phi = \frac{e}{2\pi R}\frac{d\Phi}{dt} \Rightarrow v_\phi = \frac{e}{2\pi m R}\Phi \tag{26}$$

The electrons move in the direction so that their self-magnetic flux is opposite to the applied flux. The resulting Lorentz force is radially inward. A stable orbit of constant radius $R$ is achieved if this force balances the centrifugal force:

$$m\frac{dv_r}{dt} = \frac{mv_\phi^2}{R} - ev_\phi B_z(R) = 0 \tag{27}$$

which from (26) requires the flux and magnetic field to be related as

$$\Phi = 2\pi R^2 B_z(R) \tag{28}$$

This condition cannot be met by a uniform field (as then $\Phi = \pi R^2 B_z$) so in practice the imposed field is made to approximately vary with radial position as

$$B_z(r) = B_0\left(\frac{R}{r}\right) \Rightarrow \Phi = 2\pi \int_{r=0}^{R} B_z(r) r\,dr = 2\pi R^2 B_0 \tag{29}$$

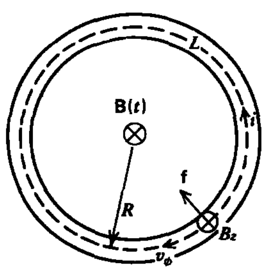



Figure 6-7 The betatron accelerates electrons to high speeds using the electric field generated by a time varying magnetic field.

where $R$ is the equilibrium orbit radius, so that (28) is satisfied.

The magnetic field must remain curl free where there is no current so that the spatial variation in (29) requires a radial magnetic field component:

$$\nabla \times \mathbf{B} = \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right)\mathbf{i}_\phi = 0 \Rightarrow B_r = -\frac{B_0 R}{r^2} z \tag{30}$$

Then any $z$-directed perturbation displacements

$$\frac{d^2 z}{dt^2} = \frac{ev_\phi}{m} B_r(R) = -\left(\frac{eB_0}{m}\right)^2 z$$

$$\Rightarrow z = A_1 \sin \omega_0 t + A_2 \cos \omega_0 t, \qquad \omega_0 = \frac{eB_0}{m} \tag{31}$$

have sinusoidal solutions at the cyclotron frequency $\omega_0 = eB_0/m$, known as betatron oscillations.

### 6-1-5 Faraday's Law and Stokes' Theorem

The integral form of Faraday's law in (1) shows that with magnetic induction the electric field is no longer conservative as its line integral around a closed path is non-zero. We may convert (1) to its equivalent differential form by considering a stationary contour whose shape does not vary with time. Because the area for the surface integral does not change with time, the time derivative on the right-hand side in (1) may be brought inside the integral but becomes a partial derivative because $\mathbf{B}$ is also a function of position:

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{32}$$

Using Stokes' theorem, the left-hand side of (32) can be converted to a surface integral,

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{33}$$

which is equivalent to

$$\int_S \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right) \cdot d\mathbf{S} = 0 \tag{34}$$

Since this last relation is true for any surface, the integrand itself must be zero, which yields Faraday's law of induction in differential form as

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{35}$$

## 6-2 MAGNETIC CIRCUITS

Various alloys of iron having very high values of relative permeability are typically used in relays and machines to constrain the magnetic flux to mostly lie within the permeable material.

### 6-2-1 Self-Inductance

The simple magnetic circuit in Figure 6-8 has an $N$ turn coil wrapped around a core with very high relative permeability idealized to be infinite. There is a small air gap of length $s$ in the core. In the core, the magnetic flux density $\mathbf{B}$ is proportional to the magnetic field intensity $\mathbf{H}$ by an infinite permeability $\mu$. The $\mathbf{B}$ field must remain finite to keep the flux and coil voltage finite so that the $\mathbf{H}$ field in the core must be zero:

$$\lim_{\mu\to\infty} \mathbf{B} = \mu\mathbf{H} \Rightarrow \begin{cases} \mathbf{H} = 0 \\ \mathbf{B} \text{ finite} \end{cases} \tag{1}$$

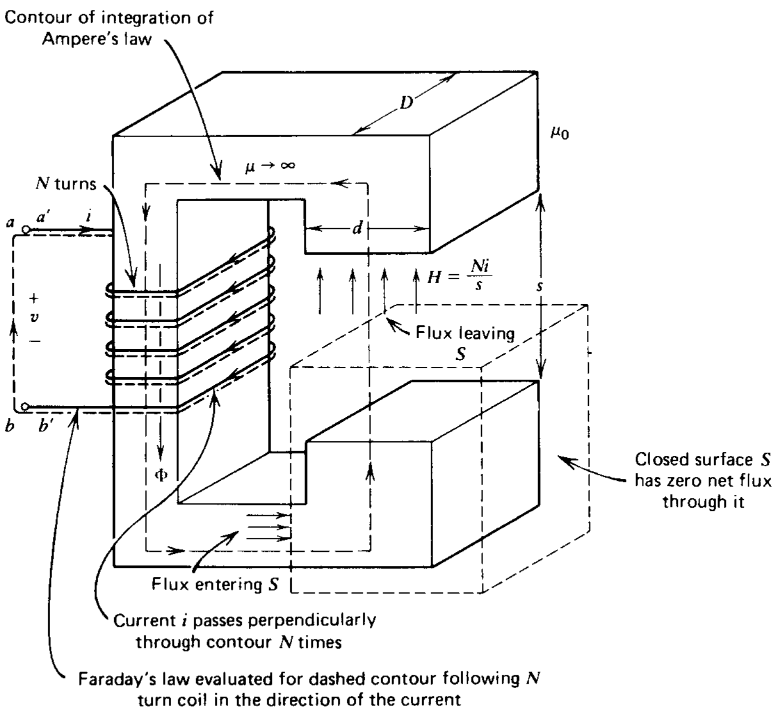



Figure 6-8 The magnetic field is zero within an infinitely permeable magnetic core and is constant in the air gap if we neglect fringing. The flux through the air gap is constant at every cross section of the magnetic circuit and links the $N$ turn coil $N$ times.

The **H** field can then only be nonzero in the air gap. This field emanates perpendicularly from the pole faces as no surface currents are present so that the tangential component of **H** is continuous and thus zero. If we neglect fringing field effects, assuming the gap $s$ to be much smaller than the width $d$ or depth $D$, the **H** field is uniform throughout the gap. Using Ampere's circuital law with the contour shown, the only nonzero contribution is in the air gap,

$$\oint_L \mathbf{H}\cdot d\mathbf{l} = Hs = I_{\text{total enclosed}} = Ni \tag{2}$$

where we realize that the coil current crosses perpendicularly through our contour $N$ times. The total flux in the air gap is then

$$\Phi = \mu_0 HDd = \frac{\mu_0 NDd}{s} i \tag{3}$$

Because the total flux through any closed surface is zero,

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = 0 \tag{4}$$

all the flux leaving $S$ in Figure 6-8 on the air gap side enters the surface through the iron core, as we neglect leakage flux in the fringing field. The flux at any cross section in the iron core is thus constant, given by (3).

If the coil current $i$ varies with time, the flux in (3) also varies with time so that a voltage is induced across the coil. We use the integral form of Faraday's law for a contour that lies within the winding with Ohmic conductivity $\sigma$, cross sectional area $A$, and total length $l$. Then the current density and electric field within the wire is

$$J = \frac{i}{A}, \qquad E = \frac{J}{\sigma} = \frac{i}{\sigma A} \tag{5}$$

so that the electromotive force has an Ohmic part as well as a contribution due to the voltage across the terminals:

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = \underbrace{\int_{a'}^{b'} \frac{i}{\sigma A}\, dl}_{\substack{iR \\ \text{in wire}}} + \underbrace{\int_b^a \mathbf{E}\cdot d\mathbf{l}}_{\substack{-v \\ \text{across} \\ \text{terminals}}} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \tag{6}$$

The surface $S$ on the right-hand side is quite complicated because of the spiral nature of the contour. If the coil only had one turn, the right-hand side of (6) would just be the time derivative of the flux of (3). For two turns, as in Figure 6-9, the flux links the coil twice, while for $N$ turns the total flux

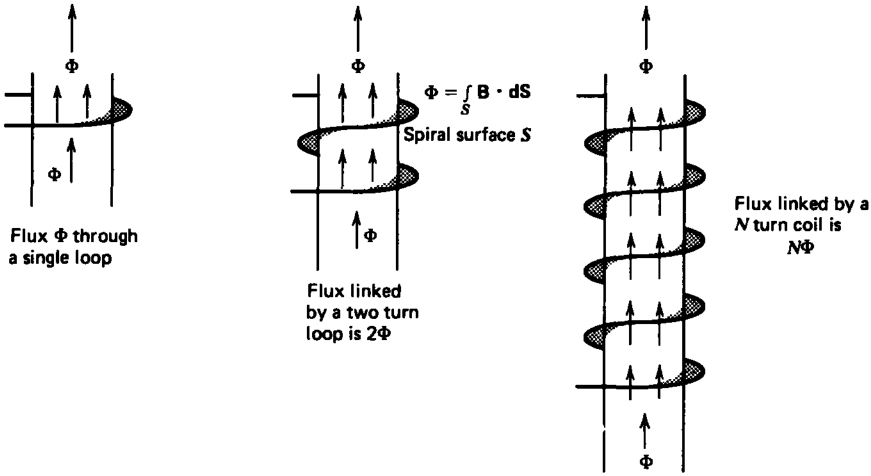



Figure 6-9 The complicated spiral surface for computation of the linked flux by an $N$ turn coil can be considered as $N$ single loops each linking the same flux $\Phi$.

linked by the coil is $N\Phi$. Then (6) reduces to

$$v = iR + L\frac{di}{dt} \tag{7}$$

where the self-inductance is defined as

$$L = \frac{N\Phi}{i} = \frac{N\int_S \mathbf{B}\cdot d\mathbf{S}}{\oint_L \mathbf{H}\cdot d\mathbf{l}} = \frac{\mu_0 N^2 Dd}{s}\ \text{henry [kg-m}^2\text{-A}^{-2}\text{-s}^{-2}] \tag{8}$$

For linearly permeable materials, the inductance is always independent of the excitations and only depends on the geometry. Because of the fixed geometry, the inductance is a constant and thus was taken outside the time derivative in (7). In geometries that change with time, the inductance will also be a function of time and must remain under the derivative. The inductance is always proportional to the square of the number of coil turns. This is because the flux $\Phi$ in the air gap is itself proportional to $N$ and it links the coil $N$ times.

## EXAMPLE 6-1 SELF-INDUCTANCES

Find the self-inductances for the coils shown in Figure 6-10.

**(a) Solenoid**

An $N$ turn coil is tightly wound upon a cylindrical core of radius $a$, length $l$, and permeability $\mu$.

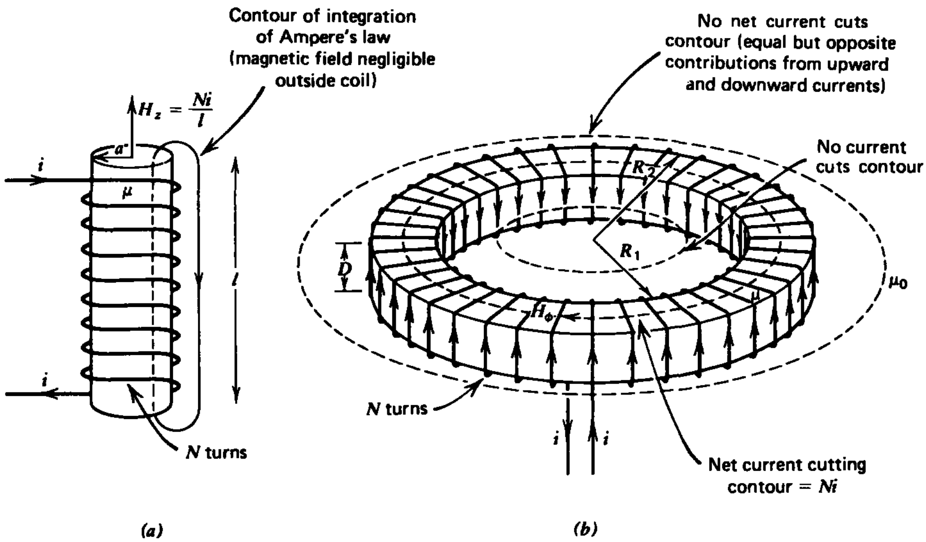



Figure 6-10 Inductances. (*a*) Solenoidal coil; (*b*) toroidal coil.

## SOLUTION

A current $i$ flowing in the wire approximates a surface current

$$K_\phi = Ni/l$$

If the length $l$ is much larger than the radius $a$, we can neglect fringing field effects at the ends and the internal magnetic field is approximately uniform and equal to the surface current,

$$H_z = K_\phi = \frac{Ni}{l}$$

as we assume the exterior magnetic field is negligible. The same result is obtained using Ampere's circuital law for the contour shown in Figure 6-10*a*. The flux links the coil $N$ times:

$$L = \frac{N\Phi}{i} = \frac{N\mu H_z \pi a^2}{i} = \frac{N^2\mu\pi a^2}{l}$$

**(b) Toroid**

An $N$ turn coil is tightly wound around a donut-shaped core of permeability $\mu$ with a rectangular cross section and inner and outer radii $R_1$ and $R_2$.

**SOLUTION**

---

Applying Ampere's circuital law to the three contours shown in Figure 6-10*b*, only the contour within the core has a net current passing through it:

$$\oint_L \mathbf{H}\cdot d\mathbf{l} = H_\phi 2\pi r = \begin{cases} 0, & r < R_1 \\ Ni, & R_1 < r < R_2 \\ 0, & r > R_2 \end{cases}$$

The inner contour has no current through it while the outer contour enclosing the whole toroid has equal but opposite contributions from upward and downward currents.

The flux through any single loop is

$$\begin{aligned} \Phi &= \mu D \int_{R_1}^{R_2} H_\phi \, dr \\ &= \frac{\mu D N i}{2\pi} \int_{R_1}^{R_2} \frac{dr}{r} \\ &= \frac{\mu D N i}{2\pi} \ln \frac{R_2}{R_1} \end{aligned}$$

so that the self-inductance is

$$L = \frac{N\Phi}{i} = \frac{\mu D N^2}{2\pi} \ln \frac{R_2}{R_1}$$

### 6-2-2 Reluctance

Magnetic circuits are analogous to resistive electronic circuits if we define the magnetomotive force (MMF) $\mathscr{F}$ analogous to the voltage (EMF) as

$$\mathscr{F} = Ni \tag{9}$$

The flux then plays the same role as the current in electronic circuits so that we define the magnetic analog to resistance as the reluctance:

$$\mathscr{R} = \frac{\mathscr{F}}{\Phi} = \frac{N^2}{L} = \frac{\text{(length)}}{\text{(permeability)(cross-sectional area)}} \tag{10}$$

which is proportional to the reciprocal of the inductance.

The advantage to this analogy is that the rules of adding reluctances in series and parallel obey the same rules as resistances.

**(a) Reluctances in Series**

For the iron core of infinite permeability in Figure 6-11*a*, with two finitely permeable gaps the reluctance of each gap is found from (8) and (10) as

$$\mathcal{R}_1=\frac{s_1}{\mu_1 a_1 D}, \qquad \mathcal{R}_2=\frac{s_2}{\mu_2 a_2 D} \tag{11}$$

so that the flux is

$$\Phi=\frac{\mathcal{F}}{\mathcal{R}_1+\mathcal{R}_2}=\frac{Ni}{\mathcal{R}_1+\mathcal{R}_2}\Rightarrow L=\frac{N\Phi}{i}=\frac{N^2}{\mathcal{R}_1+\mathcal{R}_2} \tag{12}$$

The iron core with infinite permeability has zero reluctance. If the permeable gaps were also iron with infinite permeability, the reluctances of (11) would also be zero so that the flux

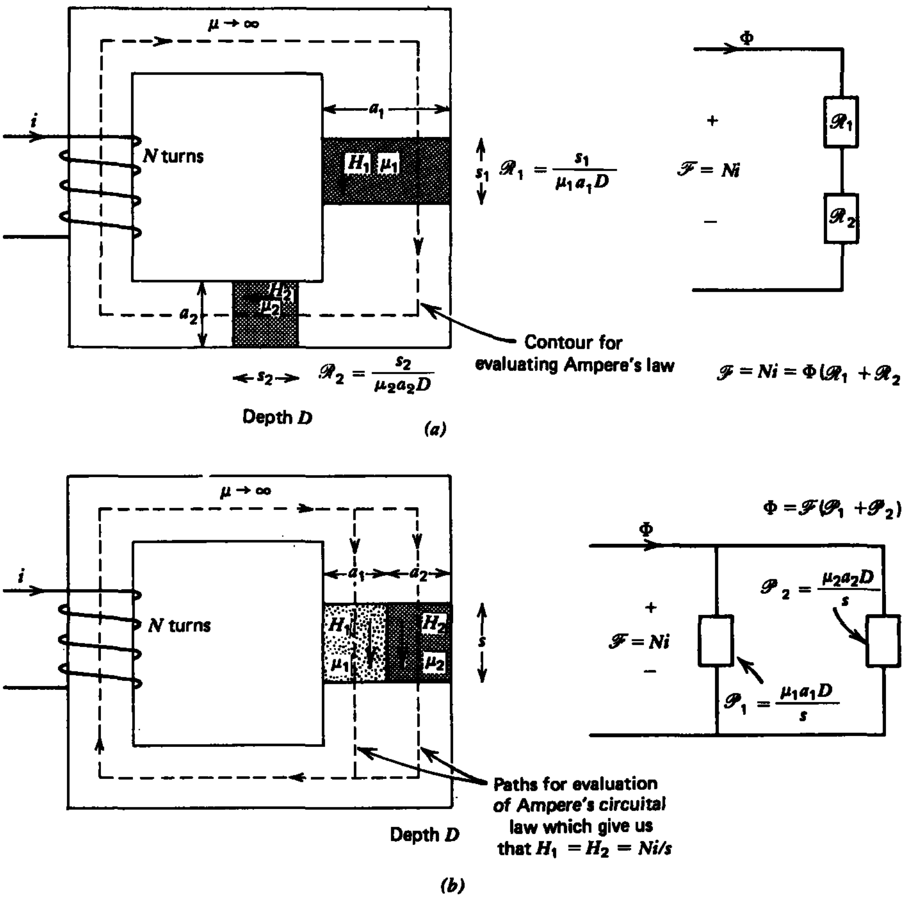



Figure 6-11 Magnetic circuits are most easily analyzed from a circuit approach where (*a*) reluctances in series add and (*b*) permeances in parallel add.

in (12) becomes infinite. This is analogous to applying a voltage across a short circuit resulting in an infinite current. Then the small resistance in the wires determines the large but finite current. Similarly, in magnetic circuits the small reluctance of a closed iron core of high permeability with no gaps limits the large but finite flux determined by the saturation value of magnetization.

The **H** field is nonzero only in the permeable gaps so that Ampere's law yields

$$H_1 s_1 + H_2 s_2 = Ni \tag{13}$$

Since the flux must be continuous at every cross section,

$$\Phi = \mu_1 H_1 a_1 D = \mu_2 H_2 a_2 D \tag{14}$$

we solve for the **H** fields as

$$H_1 = \frac{\mu_2 a_2 Ni}{\mu_1 a_1 s_2 + \mu_2 a_2 s_1}, \qquad H_2 = \frac{\mu_1 a_1 Ni}{\mu_1 a_1 s_2 + \mu_2 a_2 s_1} \tag{15}$$

**(b) Reluctances in Parallel**

If a gap in the iron core is filled with two permeable materials, as in Figure 6-11*b*, the reluctance of each material is still given by (11). Since each material sees the same magnetomotive force, as shown by applying Ampere's circuital law to contours passing through each material,

$$H_1 s = H_2 s = Ni \Rightarrow H_1 = H_2 = \frac{Ni}{s} \tag{16}$$

the $H$ fields in each material are equal. The flux is then

$$\Phi = (\mu_1 H_1 a_1 + \mu_2 H_2 a_2) D = \frac{Ni(\mathcal{R}_1 + \mathcal{R}_2)}{\mathcal{R}_1 \mathcal{R}_2} = Ni(\mathcal{P}_1 + \mathcal{P}_2) \tag{17}$$

where the permeances $\mathcal{P}_1$ and $\mathcal{P}_2$ are just the reciprocal reluctances analogous to conductance.

### 6-2-3 Transformer Action

**(a) Voltages are not Unique**

Consider two small resistors $R_1$ and $R_2$ forming a loop enclosing one leg of a closed magnetic circuit with permeability $\mu$, as in Figure 6-12. An $N$ turn coil excited on one leg with a time varying current generates a time varying flux that is approximately

$$\Phi(t) = \mu N A i_1 / l \tag{18}$$

where $l$ is the average length around the core.

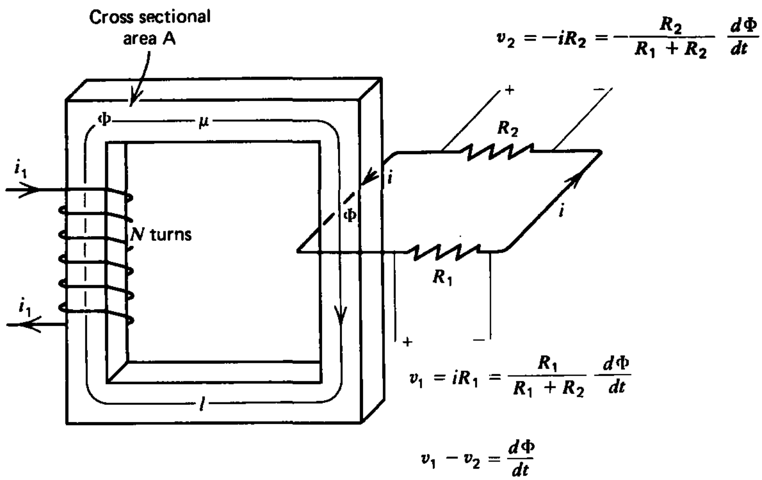



Figure 6-12 Voltages are not unique in the presence of a time varying magnetic field. A resistive loop encircling a magnetic circuit has different measured voltages across the same node pair. The voltage difference is equal to the time rate of magnetic flux through the loop.

Applying Faraday's law to the resistive loop we have

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = i(R_1+R_2) = +\frac{d\Phi(t)}{dt} \Rightarrow i = \frac{1}{R_1+R_2}\frac{d\Phi}{dt} \tag{19}$$

where we neglect the self-flux produced by the induced current $i$ and reverse the sign on the magnetic flux term because $\Phi$ penetrates the loop in Figure 6-12 in the direction opposite to the positive convention given by the right-hand rule illustrated in Figure 6-2.

If we now measured the voltage across each resistor, we would find different values and opposite polarities even though our voltmeter was connected to the same nodes:

$$\begin{aligned} v_1 &= iR_1 = +\frac{R_1}{R_1+R_2}\frac{d\Phi}{dt} \\ v_2 &= -iR_2 = \frac{-R_2}{R_1+R_2}\frac{d\Phi}{dt} \end{aligned} \tag{20}$$

This nonuniqueness of the voltage arises because the electric field is no longer curl free. The voltage difference between two points depends on the path of the connecting wires. If any time varying magnetic flux passes through the contour defined by the measurement, an additional contribution results.

### (b) Ideal Transformers

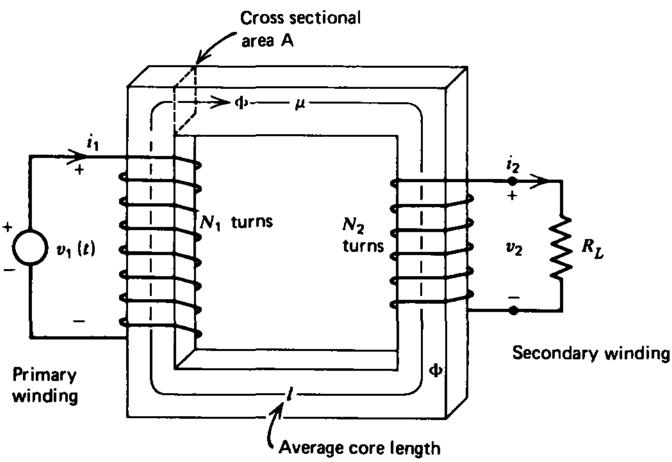
Two coils tightly wound on a highly permeable core, so that all the flux of one coil links the other, forms an ideal transformer, as in Figure 6-13. Because the iron core has an infinite permeability, all the flux is confined within the core. The currents flowing in each coil, $i_1$ and $i_2$, are defined so that when they are positive the fluxes generated by each coil are in the opposite direction. The total flux in the core is then

$$\Phi = \frac{N_1 i_1 - N_2 i_2}{\mathcal{R}}, \qquad \mathcal{R} = \frac{l}{\mu A} \tag{21}$$

where $\mathcal{R}$ is the reluctance of the core and $l$ is the average length of the core.

The flux linked by each coil is then

$$\begin{aligned} \lambda_1 &= N_1 \Phi = \frac{\mu A}{l}(N_1^2 i_1 - N_1 N_2 i_2) \\ \lambda_2 &= N_2 \Phi = \frac{\mu A}{l}(N_1 N_2 i_1 - N_2^2 i_2) \end{aligned} \tag{22}$$




$$\left.\begin{aligned} \frac{v_1}{v_2} &= \frac{N_1}{N_2} \\ \frac{i_1}{i_2} &= \frac{N_2}{N_1} \end{aligned}\right\} \Rightarrow v_1 i_1 = v_2 i_2$$

(*a*)

Figure 6-13 (*a*) An ideal transformer relates primary and secondary voltages by the ratio of turns while the currents are in the inverse ratio so that the input power equals the output power. The **H** field is zero within the infinitely permeable core. (*b*) In a real transformer the nonlinear B-H hysteresis loop causes a nonlinear primary current $i_1$ with an open circuited secondary ($i_2 = 0$) even though the imposed sinusoidal voltage $v_1 = V_0 \cos \omega t$ fixes the flux to be sinusoidal. (*c*) A more complete transformer equivalent circuit.

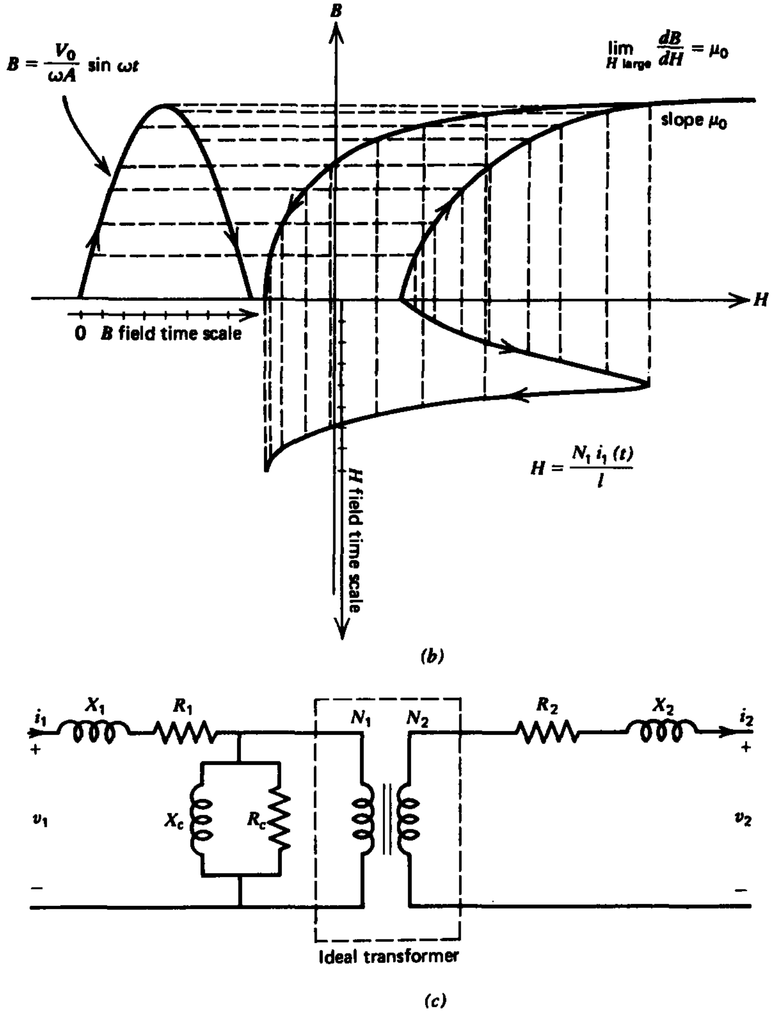



Figure 6.13.

which can be written as

$$\begin{aligned} \lambda_1 &= L_1 i_1 - M i_2 \\ \lambda_2 &= M i_1 - L_2 i_2 \end{aligned} \tag{23}$$

where $L_1$ and $L_2$ are the self-inductances of each coil alone and $M$ is the mutual inductance between coils:

$$L_1 = N_1^2 L_0, \qquad L_2 = N_2^2 L_0, \qquad M = N_1 N_2 L_0, \qquad L_0 = \mu A / l \tag{24}$$

In general, the mutual inductance obeys the equality:

$$M = k(L_1 L_2)^{1/2}, \qquad 0 \le k \le 1 \tag{25}$$

where $k$ is called the coefficient of coupling. For a noninfinite core permeability, $k$ is less than unity because some of the flux of each coil goes into the free space region and does not link the other coil. In an ideal transformer, where the permeability is infinite, there is no leakage flux so that $k = 1$.

From (23), the voltage across each coil is

$$\begin{aligned} v_1 &= \frac{d\lambda_1}{dt} = L_1\frac{di_1}{dt} - M\frac{di_2}{dt} \\ v_2 &= \frac{d\lambda_2}{dt} = M\frac{di_1}{dt} - L_2\frac{di_2}{dt} \end{aligned} \tag{26}$$

Because with no leakage, the mutual inductance is related to the self-inductances as

$$M = \frac{N_2}{N_1}L_1 = \frac{N_1}{N_2}L_2 \tag{27}$$

the ratio of coil voltages is the same as the turns ratio:

$$\frac{v_1}{v_2} = \frac{d\lambda_1/dt}{d\lambda_2/dt} = \frac{N_1}{N_2} \tag{28}$$

In the ideal transformer of infinite core permeability, the inductances of (24) are also infinite. To keep the voltages and fluxes in (26) finite, the currents must be in the inverse turns ratio

$$\frac{i_1}{i_2} = \frac{N_2}{N_1} \tag{29}$$

The electrical power delivered by the source to coil 1, called the primary winding, just equals the power delivered to the load across coil 2, called the secondary winding:

$$v_1 i_1 = v_2 i_2 \tag{30}$$

If $N_2 > N_1$, the voltage on winding 2 is greater than the voltage on winding 1 but current $i_2$ is less than $i_1$ keeping the powers equal.

If primary winding 1 is excited by a time varying voltage $v_1(t)$ with secondary winding 2 loaded by a resistor $R_L$ so that

$$v_2 = i_2 R_L \tag{31}$$

the effective resistance seen by the primary winding is

$$R_{\text{eff}} = \frac{v_1}{i_1} = \frac{N_1}{N_2}\frac{v_2}{(N_2/N_1)i_2} = \left(\frac{N_1}{N_2}\right)^2 R_L \tag{32}$$

A transformer is used in this way as an impedance transformer where the effective resistance seen at the primary winding is increased by the square of the turns ratio.

**(c) Real Transformers**

When the secondary is open circuited ($i_2 = 0$), (29) shows that the primary current of an ideal transformer is also zero. In practice, applying a primary sinusoidal voltage $V_0 \cos \omega t$ will result in a small current due to the finite self-inductance of the primary coil. Even though this self-inductance is large if the core permeability $\mu$ is large, we must consider its effect because there is no opposing flux as a result of the open circuited secondary coil. Furthermore, the nonlinear hysteresis curve of the iron as discussed in Section 5-5-3$c$ will result in a nonsinusoidal current even though the voltage is sinusoidal. In the magnetic circuit of Figure 6.13$a$ with $i_2 = 0$, the magnetic field is

$$\mathbf{H} = \frac{N_1 i_1}{l} \tag{33}$$

while the imposed sinusoidal voltage also fixes the magnetic flux to be sinusoidal

$$v_1 = \frac{d\Phi}{dt} = V_0 \cos \omega t \Rightarrow \Phi = BA = \frac{V_0}{\omega} \sin \omega t \tag{34}$$

Thus the upper half of the nonlinear B-H magnetization characteristic in Figure 6-13$b$ has the same shape as the flux-current characteristic with proportionality factors related to the geometry. Note that in saturation the B-H curve approaches a straight line with slope $\mu_0$. For a half-cycle of flux given by (34), the nonlinear open circuit magnetizing current is found graphically as a function of time in Figure 6-13$b$. The current is symmetric over the negative half of the flux cycle. Fourier analysis shows that this nonlinear current is composed of all the odd harmonics of the driving frequency dominated by the third and fifth harmonics. This causes problems in power systems and requires extra transformer windings to trap the higher harmonic currents, thus preventing their transmission.

A more realistic transformer equivalent circuit is shown in Figure 6-13$c$ where the leakage reactances $X_1$ and $X_2$ represent the fact that all the flux produced by one coil does not link the other. Some small amount of flux is in the free space region surrounding the windings. The nonlinear inductive reactance $X_c$ represents the nonlinear magnetization characteristic illustrated in Figure 6-13$b$, while $R_c$ represents the power dissipated in traversing the hysteresis loop over a cycle. This dissipated power per cycle equals the area enclosed by the hysteresis loop. The winding resistances are $R_1$ and $R_2$.

## 6-3 FARADAY'S LAW FOR MOVING MEDIA

### 6-3-1 The Electric Field Transformation

If a point charge $q$ travels with a velocity $\mathbf{v}$ through a region with electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, it experiences the combined Coulomb–Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{1}$$

Now consider another observer who is travelling at the same velocity $\mathbf{v}$ as the charge carrier so that their relative velocity is zero. This moving observer will then say that there is no Lorentz force, only a Coulombic force

$$\mathbf{F}' = q\mathbf{E}' \tag{2}$$

where we indicate quantities measured by the moving observer with a prime. A fundamental postulate of mechanics is that all physical laws are the same in every inertial coordinate system (systems that travel at constant relative velocity). This requires that the force measured by two inertial observers be the same so that $\mathbf{F}' = \mathbf{F}$:

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} \tag{3}$$

The electric field measured by the two observers in relative motion will be different. This result is correct for material velocities much less than the speed of light and is called a Galilean field transformation. The complete relativistically correct transformation slightly modifies (3) and is called a Lorentzian transformation but will not be considered here.

In using Faraday's law of Section 6-1-1, the question remains as to which electric field should be used if the contour $L$ and surface $S$ are moving. One uses the electric field that is measured by an observer moving at the same velocity as the convecting contour. The time derivative of the flux term cannot be brought inside the integral if the surface $S$ is itself a function of time.

### 6-3-2 Ohm's Law for Moving Conductors

The electric field transformation of (3) is especially important in modifying Ohm's law for moving conductors. For nonrelativistic velocities, an observer moving along at the same velocity as an Ohmic conductor measures the usual Ohm's law in his reference frame,

$$\mathbf{J}_f' = \sigma \mathbf{E}' \tag{4}$$

where we assume the conduction process is unaffected by the motion. Then in Galilean relativity for systems with no free charge, the current density in all inertial frames is the same so that (3) in (4) gives us the generalized Ohm's law as

$$\mathbf{J}_f' = \mathbf{J}_f = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{5}$$

where $\mathbf{v}$ is the velocity of the conductor.

The effects of material motion are illustrated by the parallel plate geometry shown in Figure 6-14. A current source is applied at the left-hand side that distributes itself uniformly as a surface current $K_x = \pm I/D$ on the planes. The electrodes are connected by a conducting slab that moves to the right with constant velocity $U$. The voltage across the current source can be computed using Faraday's law with the contour shown. Let us have the contour continually expanding with the 2–3 leg moving with the conductor. Applying Faraday's law we have

$$\oint_L \mathbf{E}' \cdot d\mathbf{l} = \int_1^2 \overset{0}{\mathbf{E}' \cdot d\mathbf{l}} + \int_2^3 \underbrace{\mathbf{E}' \cdot d\mathbf{l}}_{iR} + \int_3^4 \overset{0}{\mathbf{E}' \cdot d\mathbf{l}} + \int_4^1 \underbrace{\mathbf{E} \cdot d\mathbf{l}}_{-v}$$

$$= -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \tag{6}$$

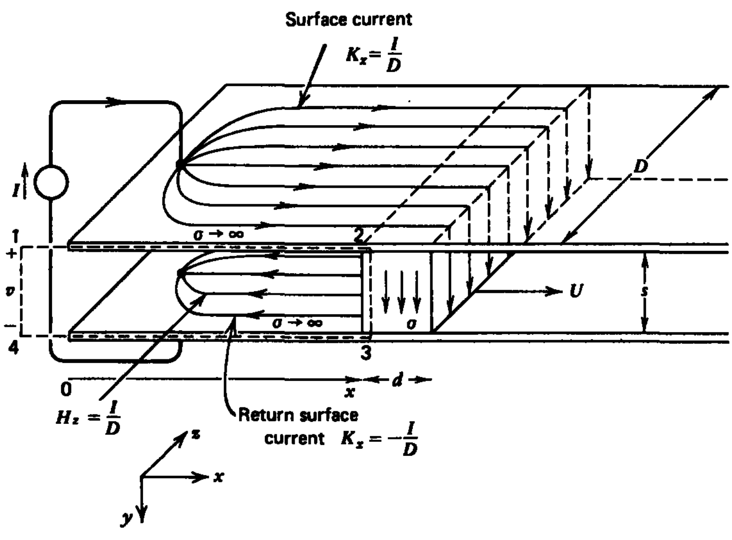



Figure 6-14 A moving, current-carrying Ohmic conductor generates a speed voltage as well as the usual resistive voltage drop.

where the electric field used along each leg is that measured by an observer in the frame of reference of the contour. Along the 1–2 and 3–4 legs, the electric field is zero within the stationary perfect conductors. The second integral within the moving Ohmic conductor uses the electric field $\mathbf{E}'$, as measured by a moving observer because the contour is also expanding at the same velocity, and from (4) and (5) is related to the terminal current as

$$\mathbf{E}' = \frac{\mathbf{J}'}{\sigma} = \frac{I}{\sigma D d}\mathbf{i}_y \tag{7}$$

In (6), the last line integral across the terminals defines the voltage.

$$\frac{Is}{\sigma D d} - v = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} = -\frac{d}{dt}(\mu_0 H_z x s) \tag{8}$$

The first term is just the resistive voltage drop across the conductor, present even if there is no motion. The term on the right-hand side in (8) only has a contribution due to the linearly increasing area ($dx/dt = U$) in the free space region with constant magnetic field,

$$H_z = I/D \tag{9}$$

The terminal voltage is then

$$v = I\left(R + \frac{\mu_0 U s}{D}\right), \qquad R = \frac{s}{\sigma D d} \tag{10}$$

We see that the speed voltage contribution arose from the flux term in Faraday's law. We can obtain the same solution using a contour that is stationary and does not expand with the conductor. We pick the contour to just lie within the conductor at the time of interest. Because the contour does not expand with time so that both the magnetic field and the contour area does not change with time, the right-hand side of (6) is zero. The only difference now is that along the 2–3 leg we use the electric field as measured by a stationary observer,

$$\mathbf{E} = \mathbf{E}' - \mathbf{v}\times\mathbf{B} \tag{11}$$

so that (6) becomes

$$IR + \frac{\mu_0 U I s}{D} - v = 0 \tag{12}$$

which agrees with (10) but with the speed voltage term now arising from the electric field side of Faraday's law.

This speed voltage contribution is the principle of electric generators converting mechanical work to electric power when moving a current-carrying conductor through a magnetic field. The resistance term accounts for the electric power dissipated. Note in (10) that the speed voltage contribution just adds with the conductor's resistance so that the effective terminal resistance is $v/I = R + (\mu_o Us/D)$. If the slab moves in the opposite direction such that $U$ is negative, the terminal resistance can also become negative for sufficiently large $U$ $(U < -RD/\mu_0 s)$. Such systems are unstable where the natural modes grow rather than decay with time with any small perturbation, as illustrated in Section 6-3-3*b*.

### 6-3-3 Faraday's Disk (Homopolar Generator)*

**(a) Imposed Magnetic Field**

A disk of conductivity $\sigma$ rotating at angular velocity $\omega$ transverse to a uniform magnetic field $B_0\mathbf{i}_z$, illustrates the basic principles of electromechanical energy conversion. In Figure 6-15*a* we assume that the magnetic field is generated by an $N$ turn coil wound on the surrounding magnetic circuit,

$$B_0 = \frac{\mu_0 N i_f}{s} \tag{13}$$

The disk and shaft have a permeability of free space $\mu_0$, so that the applied field is not disturbed by the assembly. The shaft and outside surface at $\mathrm{r} = R_0$ are highly conducting and make electrical connection to the terminals via sliding contacts.

We evaluate Faraday's law using the contour shown in Figure 6-15*a* where the 1–2 leg within the disk is stationary so the appropriate electric field to be used is given by (11):

$$E_\mathrm{r} = \frac{J_\mathrm{r}}{\sigma} - \omega \mathrm{r} B_0 = \frac{i_r}{2\pi\sigma d\mathrm{r}} - \omega \mathrm{r} B_0 \tag{14}$$

where the electric field and current density are radial and $i_r$ is the total rotor terminal current. For the stationary contour with a constant magnetic field, there is no time varying flux through the contour:

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \int_1^2 E_\mathrm{r}\, d\mathrm{r} + \int_3^4 \underbrace{\mathbf{E} \cdot d\mathbf{l}}_{-v_r} = 0 \tag{15}$$

* *Some of the treatment in this section is similar to that developed in: H. H. Woodson and J. R. Melcher,* Electromechanical Dynamics, *Part I, Wiley, N.Y., 1968, Ch. 6.*

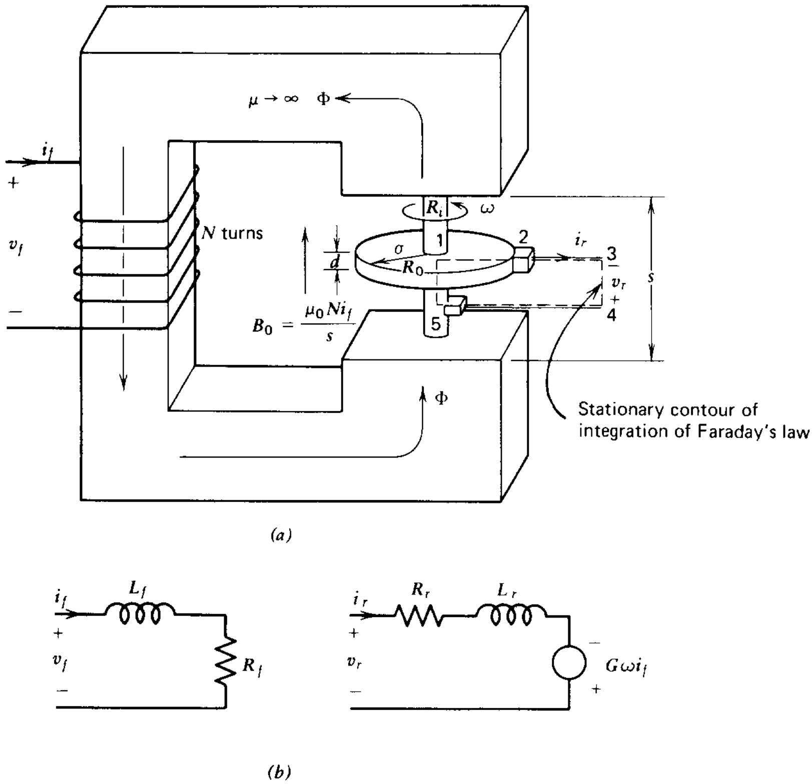



Figure 6-15 (*a*) A conducting disk rotating in an axial magnetic field is called a homopolar generator. (*b*) In addition to Ohmic and inductive voltages there is a speed voltage contribution proportional to the speed of the disk and the magnetic field.

Using (14) in (15) yields the terminal voltage as

$$\begin{aligned} v_r &= \int_{R_i}^{R_0} \left(\frac{i_r}{2\pi r\sigma d} - \omega r B_0\right) dr \\ &= \frac{i_r}{2\pi\sigma d} \ln \frac{R_0}{R_i} - \frac{\omega B_0}{2}(R_0^2 - R_i^2) \\ &= i_r R_r - G\omega i_f \end{aligned} \tag{16}$$

where $R_r$ is the internal rotor resistance of the disk and $G$ is called the speed coefficient:

$$R_r = \frac{\ln (R_0/R_i)}{2\pi\sigma d}, \qquad G = \frac{\mu_0 N}{2s}(R_0^2 - R_i^2) \tag{17}$$

We neglected the self-magnetic field due to the rotor current, assuming it to be much smaller than the applied field $B_0$, but it is represented in the equivalent rotor circuit in Figure 6-15*b* as the self-inductance $L_r$ in series with a resistor and a speed voltage source linearly dependent on the field current. The stationary field coil is represented by its self-inductance and resistance.

For a copper disk ($\sigma \approx 6\times 10^7$ siemen/m) of thickness 1 mm rotating at 3600 rpm ($\omega = 120\pi$ radian/sec) with outer and inner radii $R_0 = 10$ cm and $R_i = 1$ cm in a magnetic field of $B_0 = 1$ tesla, the open circuit voltage is

$$v_{oc} = -\frac{\omega B_0}{2}(R_0^2 - R_i^2) \approx -1.9\text{ V} \tag{18}$$

while the short circuit current is

$$i_{sc} = \frac{v_{oc}}{\ln(R_0/R_i)} 2\pi\sigma d \approx 3\times 10^5 \text{ amp} \tag{19}$$

Homopolar generators are typically high current, low voltage devices. The electromagnetic torque on the disk due to the Lorentz force is

$$\begin{aligned}\mathbf{T} &= \int_{\phi=0}^{2\pi}\int_{z=0}^{d}\int_{r=R_i}^{R_0} r\mathbf{i}_r \times (\mathbf{J}\times\mathbf{B}) r\, dr\, d\phi\, dz \\ &= -i_r B_0 \mathbf{i}_z \int_{R_i}^{R_0} r\, dr \\ &= -\frac{i_r B_0}{2}(R_0^2 - R_i^2)\mathbf{i}_z \\ &= -G i_f i_r \mathbf{i}_z \end{aligned} \tag{20}$$

The negative sign indicates that the Lorentz force acts on the disk in the direction opposite to the motion. An external torque equal in magnitude but opposite in direction to (20) is necessary to turn the shaft.

This device can also be operated as a motor if a rotor current into the disk ($i_r < 0$) is imposed. Then the electrical torque causes the disk to turn.

### (b) Self-Excited Generator

For generator operation it is necessary to turn the shaft and supply a field current to generate the magnetic field. However, if the field coil is connected to the rotor terminals, as in Figure 6-16*a*, the generator can supply its own field current. The equivalent circuit for self-excited operation is shown in Figure 6-16*b* where the series connection has $i_r = i_f$.

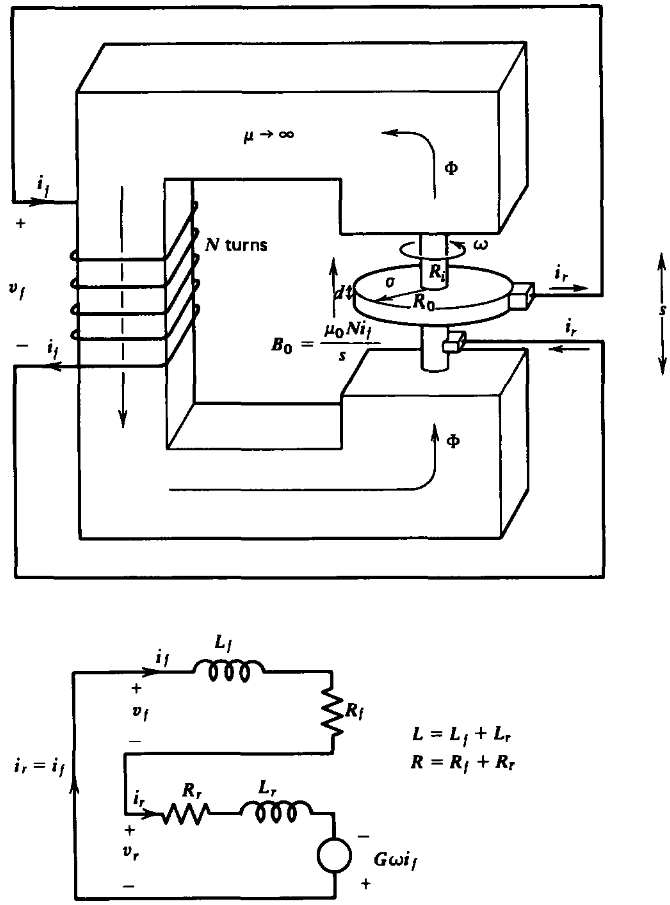



Figure 6-16 A homopolar generator can be self-excited where the generated rotor current is fed back to the field winding to generate its own magnetic field.

Kirchoff's voltage law around the loop is

$$L\frac{di}{dt} + i(R - G\omega) = 0, \qquad R = R_r + R_f, \qquad L = L_r + L_f \tag{21}$$

where $R$ and $L$ are the series resistance and inductance of the coil and disk. The solution to (21) is

$$i = I_0\, e^{-[(R - G\omega)/L]t} \tag{22}$$

where $I_0$ is the initial current at $t = 0$. If the exponential factor is positive

$$G\omega > R \tag{23}$$

the current grows with time no matter how small $I_0$ is. In practice, $I_0$ is generated by random fluctuations (noise) due to residual magnetism in the iron core. The exponential growth is limited by magnetic core saturation so that the current reaches a steady-state value. If the disk is rotating in the opposite direction ($\omega < 0$), the condition of (23) cannot be satisfied. It is then necessary for the field coil connection to be reversed so that $i_r = -i_f$. Such a dynamo model has been used as a model of the origin of the earth's magnetic field.

**(c) Self-Excited ac Operation**

Two such coupled generators can spontaneously generate two phase ac power if two independent field windings are connected, as in Figure 6-17. The field windings are connected so that if the flux through the two windings on one machine add, they subtract on the other machine. This accounts for the sign difference in the speed voltages in the equivalent circuits,

$$\begin{aligned} L\frac{di_1}{dt}+(R-G\omega)i_1+G\omega i_2=0 \\ L\frac{di_2}{dt}+(R-G\omega)i_2-G\omega i_1=0 \end{aligned} \tag{24}$$

where $L$ and $R$ are the total series inductance and resistance. The disks are each turned at the same angular speed $\omega$.

Since (24) are linear with constant coefficients, solutions are of the form

$$i_1=I_1\,e^{st}, \qquad i_2=I_2\,e^{st} \tag{25}$$

which when substituted back into (24) yields

$$\begin{aligned} (Ls+R-G\omega)I_1+G\omega I_2=0 \\ -G\omega I_1+(Ls+R-G\omega)I_2=0 \end{aligned} \tag{26}$$

For nontrivial solutions, the determinant of the coefficients of $I_1$ and $I_2$ must be zero,

$$(Ls+R-G\omega)^2=-(G\omega)^2 \tag{27}$$

which when solved for $s$ yields the complex conjugate natural frequencies,

$$\begin{aligned} s=-\frac{(R-G\omega)}{L}\pm j\frac{G\omega}{L} \\ I_1/I_2=\pm j \end{aligned} \tag{28}$$

where the currents are 90° out of phase. If the real part of $s$ is positive, the system is self-excited so that any perturbation

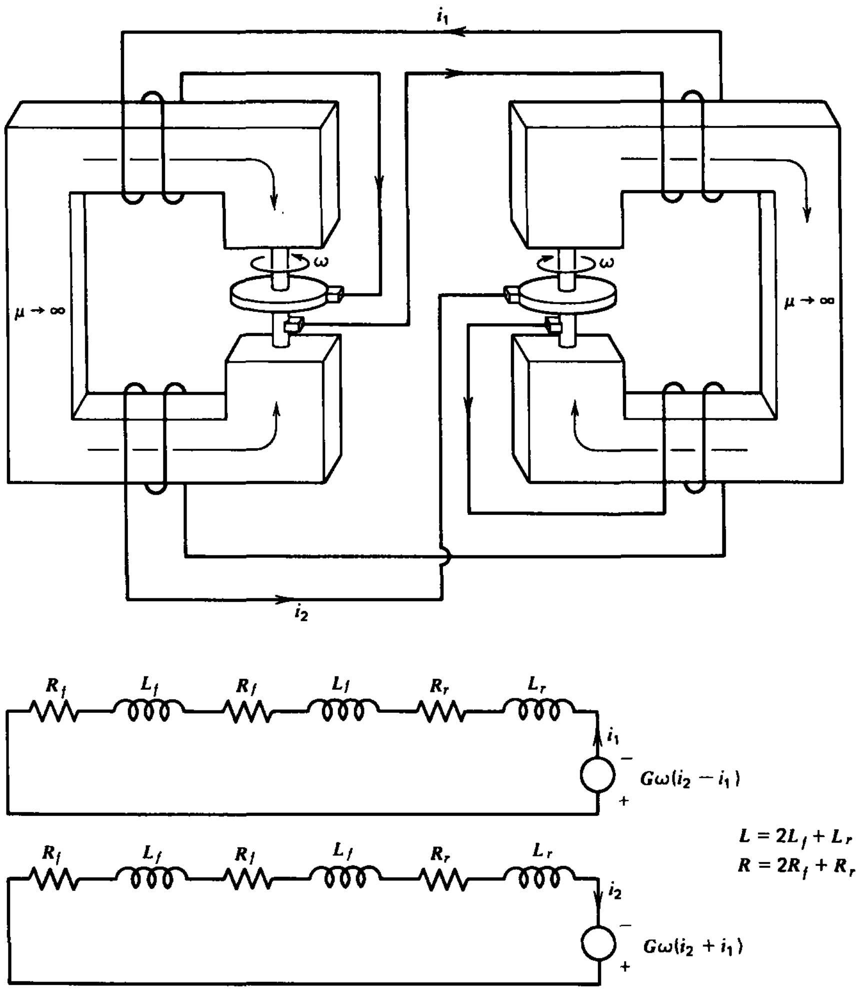



Figure 6-17 Cross-connecting two homopolar generators can result in self-excited two-phase alternating currents. Two independent field windings are required where on one machine the fluxes add while on the other they subtract.

grows at an exponential rate:

$$G\omega > R \tag{29}$$

The imaginary part of $s$ yields the oscillation frequency

$$\omega_0 = \operatorname{Im}(s) = G\omega/L \tag{30}$$

Again, core saturation limits the exponential growth so that two-phase power results. Such a model may help explain the periodic reversals in the earth's magnetic field every few hundred thousand years.

**(d) Periodic Motor Speed Reversals**

If the field winding of a motor is excited by a dc current, as in Figure 6-18, with the rotor terminals connected to a generator whose field and rotor terminals are in series, the circuit equation is

$$\frac{di}{dt}+\frac{(R-G_g\omega_g)}{L}i=\frac{G_m\omega_m}{L}I_f \tag{31}$$

where $L$ and $R$ are the total series inductances and resistances. The angular speed of the generator $\omega_g$ is externally

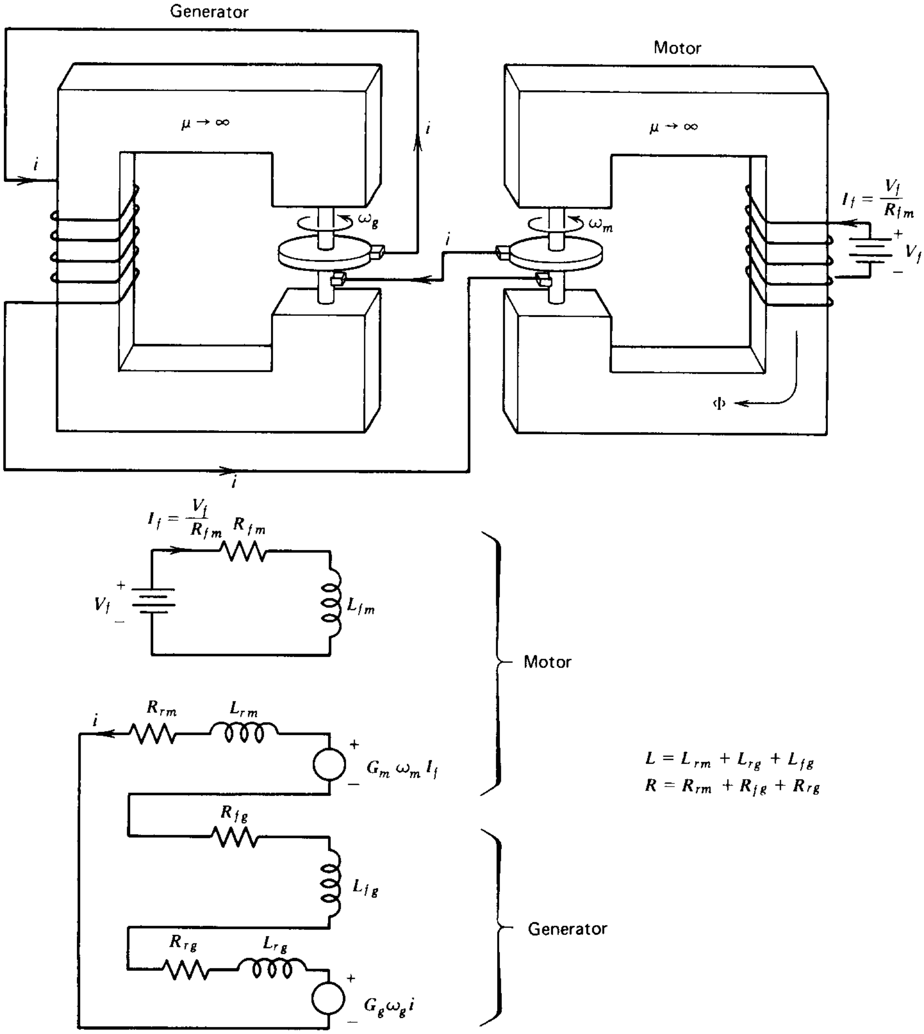



Figure 6-18 Cross connecting a homopolar generator and motor can result in spontaneous periodic speed reversals of the motor's shaft.

constrained to be a constant. The angular acceleration of the motor's shaft is equal to the torque of (20),

$$J\frac{d\omega_m}{dt} = -G_m I_f i \tag{32}$$

where $J$ is the moment of inertia of the shaft and $I_f = V_f/R_{fm}$ is the constant motor field current.

Solutions of these coupled, linear constant coefficient differential equations are of the form

$$\begin{aligned} i &= I\,e^{st} \\ \omega &= W\,e^{st} \end{aligned} \tag{33}$$

which when substituted back into (31) and (32) yield

$$\begin{aligned} I\left(s + \frac{R - G_g\omega_g}{L}\right) - W\left(\frac{G_m I_f}{L}\right) &= 0 \\ I\left(\frac{G_m I_f}{J}\right) + Ws &= 0 \end{aligned} \tag{34}$$

Again, for nontrivial solutions the determinant of coefficients of $I$ and $W$ must be zero,

$$s\left(s + \frac{R - G_g\omega_g}{L}\right) + \frac{(G_m I_f)^2}{JL} = 0 \tag{35}$$

which when solved for $s$ yields

$$s = -\frac{(R - G_g\omega_g)}{2L} \pm \left[\left(\frac{R - G_g\omega_g}{2L}\right)^2 - \frac{(G_m I_f)^2}{JL}\right]^{1/2} \tag{36}$$

For self-excitation the real part of $s$ must be positive,

$$G_g\omega_g > R \tag{37}$$

while oscillations will occur if $s$ has an imaginary part,

$$\frac{(G_m I_f)^2}{JL} > \left(\frac{R - G_g\omega_g}{2L}\right)^2 \tag{38}$$

Now, both the current and shaft's angular velocity spontaneously oscillate with time.

### 6-3-4 Basic Motors and Generators

#### (a) ac Machines

Alternating voltages are generated from a dc magnetic field by rotating a coil, as in Figure 6-19. An output voltage is measured via slip rings through carbon brushes. If the loop of area $A$ is vertical at $t = 0$ linking zero flux, the imposed flux

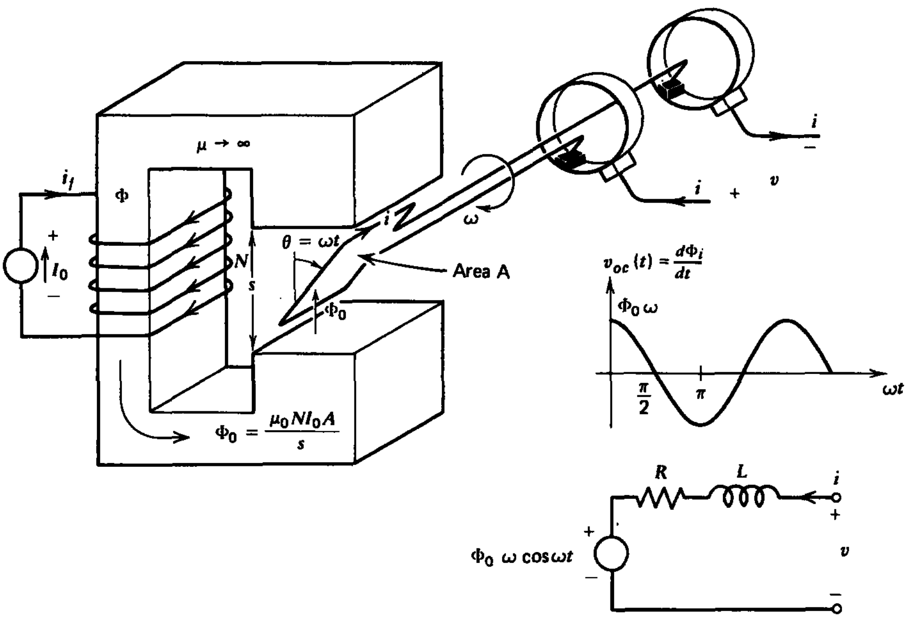



Figure 6-19 A coil rotated within a constant magnetic field generates a sinusoidal voltage.

through the loop at any time, varies sinusoidally with time due to the rotation as

$$\Phi_i = \Phi_0 \sin \omega t \tag{39}$$

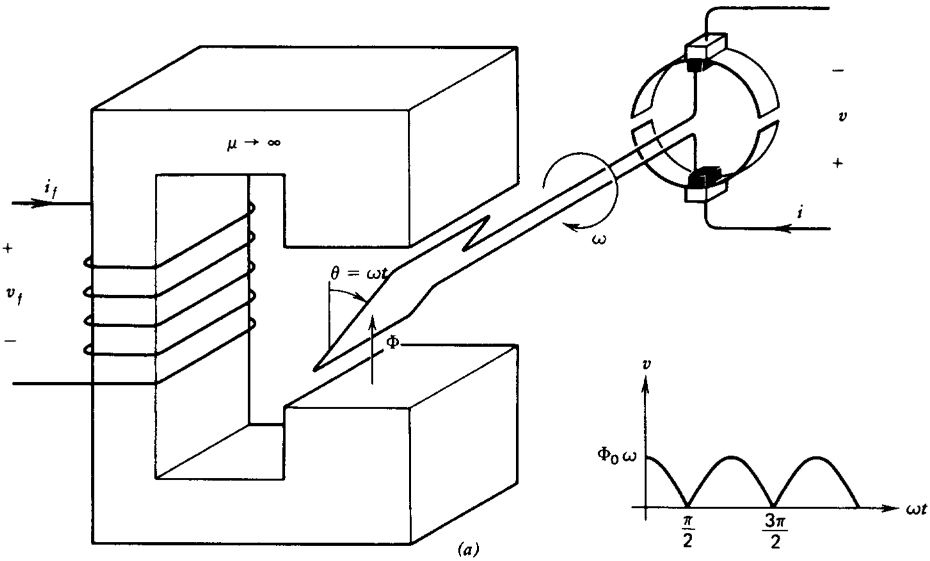
Faraday's law applied to a stationary contour instantaneously passing through the wire then gives the terminal voltage as

$$v = iR + \frac{d\Phi}{dt} = iR + L\frac{di}{dt} + \Phi_0\omega \cos \omega t \tag{40}$$

where $R$ and $L$ are the resistance and inductance of the wire. The total flux is equal to the imposed flux of (39) as well as self-flux (accounted for by $L$) generated by the current $i$. The equivalent circuit is then similar to that of the homopolar generator, but the speed voltage term is now sinusoidal in time.

### (b) dc Machines

DC machines have a similar configuration except that the slip ring is split into two sections, as in Figure 6-20$a$. Then whenever the output voltage tends to change sign, the terminals are also reversed yielding the waveform shown, which is of one polarity with periodic variations from zero to a peak value.



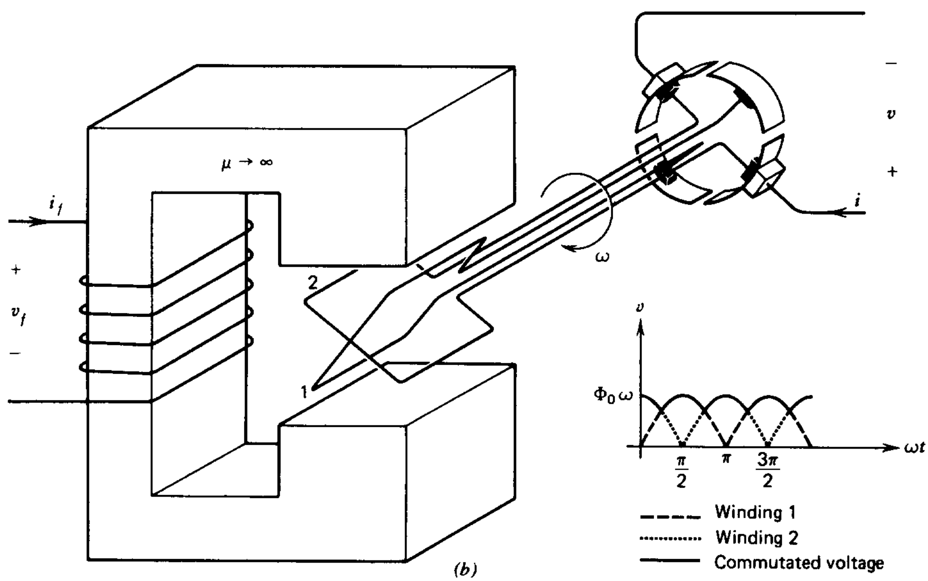




Figure 6-20 (*a*) If the slip rings are split so that when the voltage tends to change sign the terminals are also reversed, the resulting voltage is of one polarity. (*b*) The voltage waveform can be smoothed out by placing a second coil at right angles to the first and using a four-section commutator.

The voltage waveform can be smoothed out by using a four-section commutator and placing a second coil perpendicular to the first, as in Figure 6-20*b*. This second coil now generates its peak voltage when the first coil generates zero voltage. With more commutator sections and more coils, the dc voltage can be made as smooth as desired.

### 6-3-5 MHD Machines

Magnetohydrodynamic machines are based on the same principles as rotating machines, replacing the rigid rotor by a conducting fluid. For the linear machine in Figure 6-21, a fluid with Ohmic conductivity $\sigma$ flowing with velocity $v_y$ moves perpendicularly to an applied magnetic field $B_0\mathbf{i}_z$. The terminal voltage $V$ is related to the electric field and current as

$$\mathbf{E}=\mathbf{i}_x\frac{V}{s}, \qquad \mathbf{J}=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B})=\sigma\left(\frac{V}{s}+v_yB_0\right)\mathbf{i}_x=\frac{i}{Dd}\mathbf{i}_x \tag{41}$$

which can be rewritten as

$$V=iR-v_yB_0s \tag{42}$$

which has a similar equivalent circuit as for the homopolar generator.

The force on the channel is then

$$\begin{aligned}\mathbf{f}&=\int_V \mathbf{J}\times\mathbf{B}\,dV\\&=-iB_0s\mathbf{i}_y\end{aligned} \tag{43}$$

again opposite to the fluid motion.

### 6-3-6 Paradoxes

Faraday's law is prone to misuse, which has led to numerous paradoxes. The confusion arises because the same

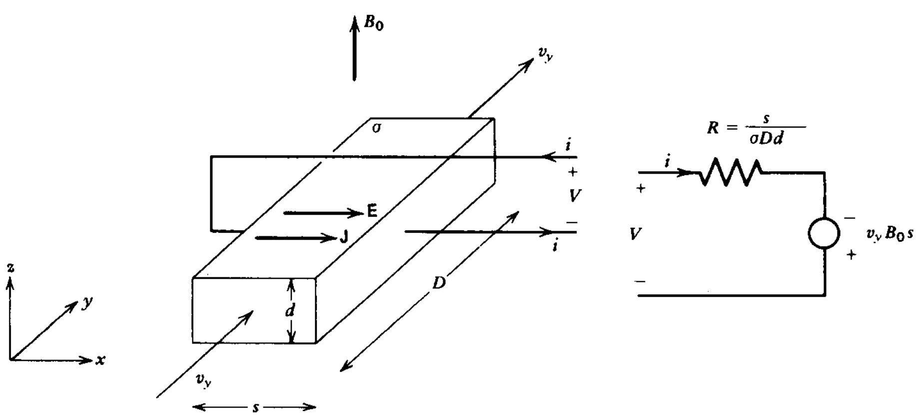


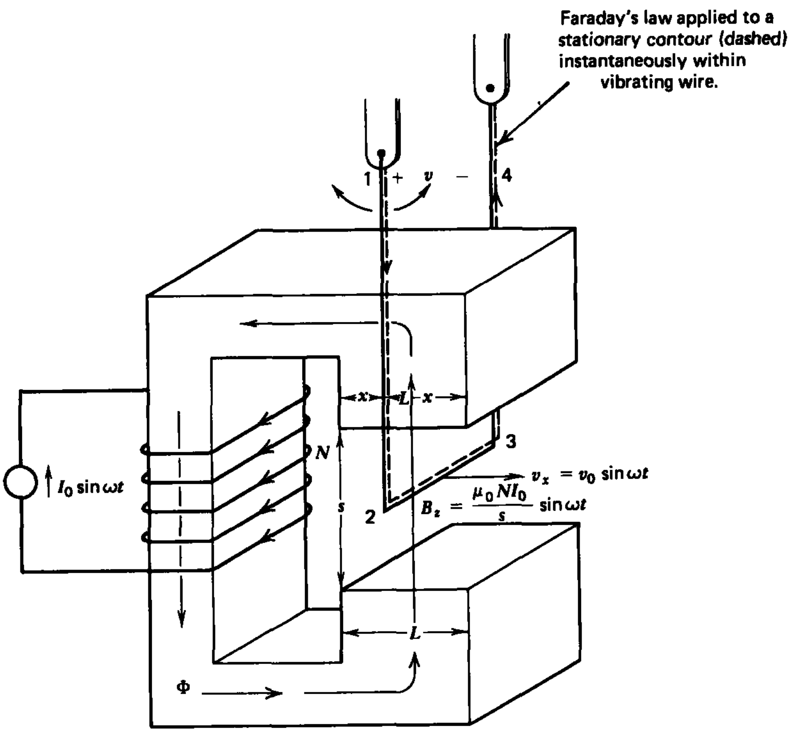

Figure 6-21 An MHD (magnetohydrodynamic) machine replaces a rotating conductor by a moving fluid.

contribution can arise from either the electromotive force side of the law, as a speed voltage when a conductor moves orthogonal to a magnetic field, or as a time rate of change of flux through the contour. This flux term itself has two contributions due to a time varying magnetic field or due to a contour that changes its shape, size, or orientation. With all these potential contributions it is often easy to miss a term or to double count.

### (a) A Commutatorless dc Machine*

Many persons have tried to make a commutatorless dc machine but to no avail. One novel unsuccessful attempt is illustrated in Figure 6-22, where a highly conducting wire is vibrated within the gap of a magnetic circuit with sinusoidal velocity:

$$v_x = v_0 \sin \omega t \tag{44}$$




Fcc 6-22 It is impossible to design a commutatorless dc machine. Although the speed voltage alone can have a dc average, it will be canceled by the transformer electromotive force due to the time rate of change of magnetic flux through the loop. The total terminal voltage will always have a zero time average.

* *H. Sohon, Electrical Essays for Recreation.* Electrical Engineering, *May (1946), p. 294.*

The sinusoidal current imposes the air gap flux density at the same frequency $\omega$:

$$B_z = B_0 \sin \omega t, \qquad B_0 = \mu_0 N I_0 / s \tag{45}$$

Applying Faraday's law to a stationary contour instantaneously within the open circuited wire yields

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \int_1^2 \cancelto{0}{\mathbf{E}'} \cdot d\mathbf{l} + \int_2^3 \underbrace{\mathbf{E} \cdot d\mathbf{l}}_{\mathbf{E} = -\mathbf{v} \times \mathbf{B}} + \int_3^4 \cancelto{0}{\mathbf{E}'} \cdot d\mathbf{l} + \int_4^1 \underbrace{\mathbf{E} \cdot d\mathbf{l}}_{-v}$$
$$= -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \tag{46}$$

where the electric field within the highly conducting wire as measured by an observer moving with the wire is zero. The electric field on the 2–3 leg within the air gap is given by (11), where $\mathbf{E}' = 0$, while the 4–1 leg defines the terminal voltage. If we erroneously argue that the flux term on the right-hand side is zero because the magnetic field $\mathbf{B}$ is perpendicular to $d\mathbf{S}$, the terminal voltage is

$$v = v_x B_z l = v_0 B_0 l \sin^2 \omega t \tag{47}$$

which has a dc time-average value. Unfortunately, this result is not complete because we forgot to include the flux that turns the corner in the magnetic core and passes perpendicularly through our contour. Only the flux to the right of the wire passes through our contour, which is the fraction $(L - x)/L$ of the total flux. Then the correct evaluation of (46) is

$$-v + v_x B_z l = +\frac{d}{dt}[(L - x) B_z l] \tag{48}$$

where $x$ is treated as a constant because the contour is stationary. The change in sign on the right-hand side arises because the flux passes through the contour in the direction opposite to its normal defined by the right-hand rule. The voltage is then

$$v = v_x B_z l - (L - x) l \frac{dB_z}{dt} \tag{49}$$

where the wire position is obtained by integrating (44),

$$x = \int v_x \, dt = -\frac{v_0}{\omega}(\cos \omega t - 1) + x_0 \tag{50}$$

and $x_0$ is the wire's position at $t=0$. Then (49) becomes

$$\begin{aligned} v &= l\frac{d}{dt}(xB_z) - Ll\frac{dB_z}{dt} \\ &= B_0 l v_0\left[\left(\frac{x_0\omega}{v_0}+1\right)\cos\omega t - \cos 2\omega t\right] - LlB_0\omega\cos\omega t \end{aligned} \tag{51}$$

which has a zero time average.

**(b) Changes in Magnetic Flux Due to Switching**

Changing the configuration of a circuit using a switch does not result in an electromotive force unless the magnetic flux itself changes.

In Figure 6-23*a*, the magnetic field through the loop is externally imposed and is independent of the switch position. Moving the switch does not induce an EMF because the magnetic flux through any surface remains unchanged.

In Figure 6-23*b*, a dc current source is connected to a circuit through a switch *S*. If the switch is instantaneously moved from contact 1 to contact 2, the magnetic field due to the source current *I* changes. The flux through any fixed area has thus changed resulting in an EMF.

**(c) Time Varying Number of Turns on a Coil***

If the number of turns on a coil is changing with time, as in Figure 6-24, the voltage is equal to the time rate of change of flux through the coil. Is the voltage then

$$v \stackrel{?}{=} N\frac{d\Phi}{dt} \tag{52}$$

or

$$v \stackrel{?}{=} \frac{d}{dt}(N\Phi) = N\frac{d\Phi}{dt} + \Phi\frac{dN}{dt} \tag{53}$$

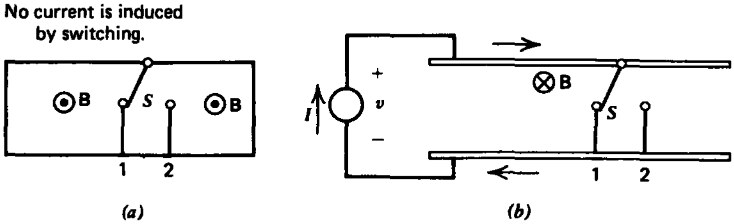



Figure 6-23 (*a*) Changes in a circuit through the use of a switch does not by itself generate an EMF. (*b*) However, an EMF can be generated if the switch changes the magnetic field.

* *L. V. Bewley.* Flux Linkages and Electromagnetic Induction. *Macmillan, New York, 1952.*

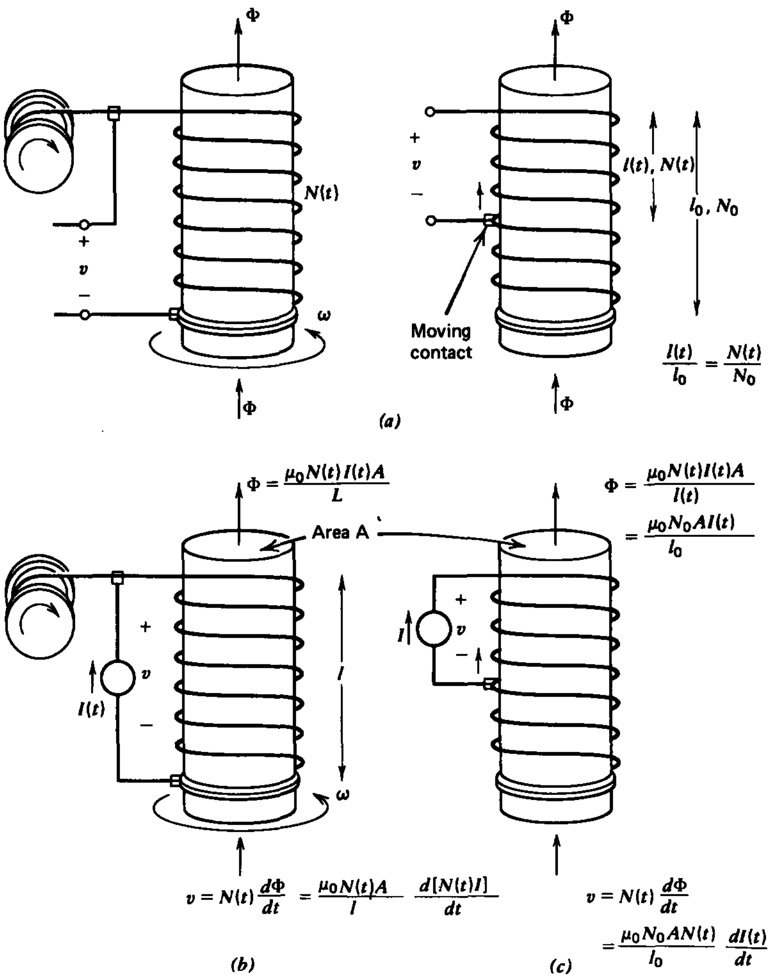



Figure 6-24 (*a*) If the number of turns on a coil is changing with time, the induced voltage is $v = N(t)\, d\Phi/dt$. A constant flux does not generate any voltage. (*b*) If the flux itself is proportional to the number of turns, a dc current can generate a voltage. (*c*) With the tap changing coil, the number of turns per unit length remains constant so that a dc current generates no voltage because the flux does not change with time.

For the first case a dc flux generates no voltage while the second does.

We use Faraday's law with a stationary contour instantaneously within the wire. Because the contour is stationary, its area of $NA$ is not changing with time and so can be taken outside the time derivative in the flux term of Faraday's law so that the voltage is given by (52) and (53) is wrong. Note that there is no speed voltage contribution in the electromotive force because the velocity of the wire is in the same direction as the contour ($\mathbf{v}\times\mathbf{B}\cdot\mathbf{dl} = 0$).

If the flux $\Phi$ itself depends on the number of turns, as in Figure 6-24*b*, there may be a contribution to the voltage even if the exciting current is dc. This is true for the turns being wound onto the cylinder in Figure 6-24*b*. For the tap changing configuration in Figure 6-24*c*, with uniformly wound turns, the ratio of turns to effective length is constant so that a dc current will still not generate a voltage.

## 6-4 MAGNETIC DIFFUSION INTO AN OHMIC CONDUCTOR*

If the current distribution is known, the magnetic field can be directly found from the Biot–Savart or Ampere's laws. However, when the magnetic field varies with time, the generated electric field within an Ohmic conductor induces further currents that also contribute to the magnetic field.

### 6-4-1 Resistor–Inductor Model

A thin conducting shell of radius $R_i$, thickness $\Delta$, and depth $l$ is placed within a larger conducting cylinder, as shown in Figure 6-25. A step current $I_0$ is applied at $t=0$ to the larger cylinder, generating a surface current $\mathbf{K}=(I_0/l)\mathbf{i}_\phi$. If the length $l$ is much greater than the outer radius $R_0$, the magnetic field is zero outside the cylinder and uniform inside for $R_i<r<R_0$. Then from the boundary condition on the discontinuity of tangential $\mathbf{H}$ given in Section 5-6-1, we have

$$\mathbf{H}_0=\frac{I_0}{l}\mathbf{i}_z, \qquad R_i<r<R_0 \tag{1}$$

The magnetic field is different inside the conducting shell because of the induced current, which from Lenz's law, flows in the opposite direction to the applied current. Because the shell is assumed to be very thin ($\Delta \ll R_i$), this induced current can be considered a surface current related to the volume current and electric field in the conductor as

$$K_\phi=J_\phi\Delta=(\sigma\Delta)E_\phi \tag{2}$$

The product $(\sigma\Delta)$ is called the surface conductivity. Then the magnetic fields on either side of the thin shell are also related by the boundary condition of Section 5-6-1:

$$H_i-H_0=K_\phi=(\sigma\Delta)E_\phi \tag{3}$$

* *Much of the treatment of this section is similar to that of H. H. Woodson and J. R. Melcher,* Electromechanical Dynamics, *Part II, Wiley, N.Y., 1968, Ch. 7.*

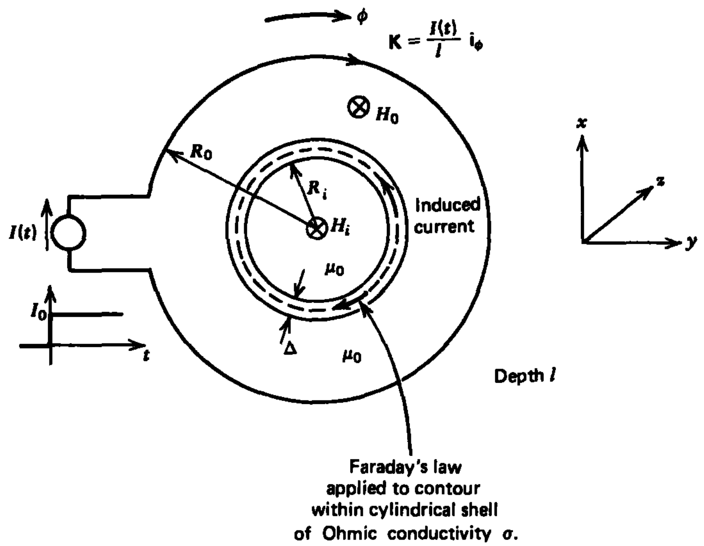



Figure 6-25 A step change in magnetic field causes the induced current within an Ohmic conductor to flow in the direction where its self-flux opposes the externally imposed flux. Ohmic dissipation causes the induced current to exponentially decay with time with a $L/R$ time constant.

Applying Faraday's law to a contour within the conducting shell yields

$$\oint_L \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} \Rightarrow E_\phi 2\pi R_i = -\mu_0 \pi R_i^2 \frac{dH_i}{dt} \qquad (4)$$

where only the magnetic flux due to $H_i$ passes through the contour. Then using (1)–(3) in (4) yields a single equation in $H_i$:

$$\frac{dH_i}{dt} + \frac{H_i}{\tau} = \frac{I(t)}{l\tau}, \qquad \tau = \frac{\mu_0 R_i \sigma \Delta}{2} \qquad (5)$$

where we recognize the time constant $\tau$ as just being the ratio of the shell's self-inductance to resistance:

$$L = \frac{\Phi}{K_\phi l} = \frac{\mu_0 \pi R_i^2}{l}, \qquad R = \frac{2\pi R_i}{\sigma l \Delta}, \qquad \tau = \frac{L}{R} = \frac{\mu_0 R_i \sigma \Delta}{2} \qquad (6)$$

The solution to (5) for a step current with zero initial magnetic field is

$$H_i = \frac{I_0}{l}(1 - e^{-t/\tau}) \qquad (7)$$

Initially, the magnetic field is excluded from inside the conducting shell by the induced current. However, Ohmic dissipation causes the induced current to decay with time so that the magnetic field may penetrate through the shell with characteristic time constant $\tau$.

### 6-4-2 The Magnetic Diffusion Equation

The transient solution for a thin conducting shell could be solved using the integral laws because the geometry constrained the induced current to flow azimuthally with no radial variations. If the current density is not directly known, it becomes necessary to self-consistently solve for the current density with the electric and magnetic fields:

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\quad\text{(Faraday's law)}\tag{8}$$

$$\nabla\times\mathbf{H}=\mathbf{J}_f\quad\text{(Ampere's law)}\tag{9}$$

$$\nabla\cdot\mathbf{B}=0\quad\text{(Gauss's law)}\tag{10}$$

For linear magnetic materials with constant permeability $\mu$ and constant Ohmic conductivity $\sigma$ moving with velocity $\mathbf{U}$, the constitutive laws are

$$\mathbf{B}=\mu\mathbf{H},\qquad \mathbf{J}_f=\sigma(\mathbf{E}+\mathbf{U}\times\mu\mathbf{H})\tag{11}$$

We can reduce (8)–(11) to a single equation in the magnetic field by taking the curl of (9), using (8) and (11) as

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{H})&=\nabla\times\mathbf{J}_f\\&=\sigma[\nabla\times\mathbf{E}+\mu\nabla\times(\mathbf{U}\times\mathbf{H})]\\&=\mu\sigma\left(-\frac{\partial\mathbf{H}}{\partial t}+\nabla\times(\mathbf{U}\times\mathbf{H})\right)\end{aligned}\tag{12}$$

The double cross product of $\mathbf{H}$ can be simplified using the vector identity

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{H})&=\nabla(\overset{0}{\nabla\cdot\mathbf{H}})-\nabla^2\mathbf{H}\\ \Rightarrow\frac{1}{\mu\sigma}\nabla^2\mathbf{H}&=\frac{\partial\mathbf{H}}{\partial t}-\nabla\times(\mathbf{U}\times\mathbf{H})\end{aligned}\tag{13}$$

where $\mathbf{H}$ has no divergence from (10). Remember that the Laplacian operator on the left-hand side of (13) also differentiates the directionally dependent unit vectors in cylindrical ($\mathbf{i}_r$ and $\mathbf{i}_\phi$) and spherical ($\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\phi$) coordinates.

### 6-4-3 Transient Solution with No Motion (U = 0)

A step current is turned on at $t=0$, in the parallel plate geometry shown in Figure 6-26. By the right-hand rule and with the neglect of fringing, the magnetic field is in the $z$ direction and only depends on the $x$ coordinate, $B_z(x, t)$, so that (13) reduces to

$$\frac{\partial^2 H_z}{\partial x^2}-\sigma\mu\frac{\partial H_z}{\partial t}=0 \tag{14}$$

which is similar in form to the diffusion equation of a distributed resistive-capacitive cable developed in Section 3-6-4.

In the dc steady state, the second term is zero so that the solution in each region is of the form

$$\frac{\partial^2 H_z}{\partial x^2}=0 \Rightarrow H_z=ax+b \tag{15}$$

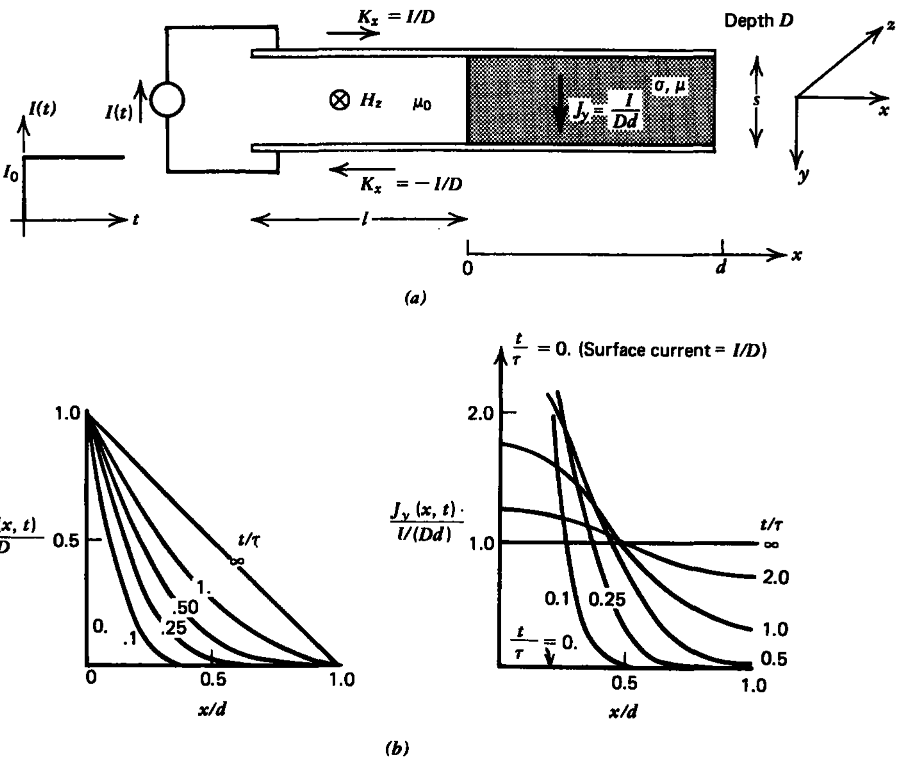



Figure 6-26 (*a*) A current source is instantaneously turned on at $t=0$. The resulting magnetic field within the Ohmic conductor remains continuous and is thus zero at $t=0$ requiring a surface current at $x=0$. (*b*) For later times the magnetic field and current diffuse into the conductor with longest time constant $\tau=\sigma\mu d^2/\pi^2$ towards a steady state of uniform current with a linear magnetic field.

where $a$ and $b$ are found from the boundary conditions. The current on the electrodes immediately spreads out to a uniform surface distribution $\pm(I/D)\mathbf{i}_x$ traveling from the upper to lower electrode uniformly through the Ohmic conductor. Then, the magnetic field is uniform in the free space region, decreasing linearly to zero within the Ohmic conductor being continuous across the interface at $x=0$:

$$\lim_{t\to\infty} H_z(x) = \begin{cases} \dfrac{I}{D}, & -l \le x \le 0 \\[2ex] \dfrac{I}{Dd}(d-x), & 0 \le x \le d \end{cases} \tag{16}$$

In the free space region where $\sigma=0$, the magnetic field remains constant for all time. Within the conducting slab, there is an initial charging transient as the magnetic field builds up to the linear steady-state distribution in (16). Because (14) is a linear equation, for the total solution of the magnetic field as a function of time and space, we use superposition and guess a solution that is the sum of the steady-state solution in (16) and a transient solution which dies off with time:

$$H_z(x,t) = \frac{I}{Dd}(d-x) + \hat{H}(x)\,e^{-\alpha t} \tag{17}$$

We follow the same procedures as for the lossy cable in Section 3-6-4. At this point we do not know the function $\hat{H}(x)$ or the parameter $\alpha$. Substituting the assumed solution of (17) back into (14) yields the ordinary differential equation

$$\frac{d^2\hat{H}(x)}{dx^2} + \sigma\mu\alpha\hat{H}(x) = 0 \tag{18}$$

which has the trigonometric solutions

$$\hat{H}(x) = A_1 \sin\sqrt{\sigma\mu\alpha}\,x + A_2\cos\sqrt{\sigma\mu\alpha}\,x \tag{19}$$

Since the time-independent part in (17) already meets the boundary conditions of

$$\begin{aligned} H_z(x=0) &= I/D \\ H_z(x=d) &= 0 \end{aligned} \tag{20}$$

the transient part of the solution must be zero at the ends

$$\begin{aligned} \hat{H}(x=0) &= 0 \Rightarrow A_2 = 0 \\ \hat{H}(x=d) &= 0 \Rightarrow A_1 \sin\sqrt{\sigma\mu\alpha}\,d = 0 \end{aligned} \tag{21}$$

which yields the allowed values of $\alpha$ as

$$\sqrt{\sigma\mu\alpha}\,d = n\pi \Rightarrow \alpha_n = \frac{1}{\mu\sigma}\left(\frac{n\pi}{d}\right)^2, \quad n=1,2,3,\ldots \tag{22}$$

Since there are an infinite number of allowed values of $\alpha$, the most general solution is the superposition of all allowed solutions:

$$H_z(x,t)=\frac{I}{Dd}(d-x)+\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{d} e^{-\alpha_n t} \tag{23}$$

This relation satisfies the boundary conditions but not the initial conditions at $t=0$ when the current is first turned on. Before the current takes its step at $t=0$, the magnetic field is zero in the slab. Right after the current is turned on, the magnetic field must remain zero. Faraday's law would otherwise make the electric field and thus the current density infinite within the slab, which is nonphysical. Thus we impose the initial condition

$$H_z(x,t=0)=0=\frac{I}{Dd}(d-x)+\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{d} \tag{24}$$

which will allow us to solve for the amplitudes $A_n$ by multiplying (24) through by $\sin(m\pi x/d)$ and then integrating over $x$ from 0 to $d$:

$$0=\frac{I}{Dd}\int_0^d (d-x)\sin\frac{m\pi x}{d}\,dx+\sum_{n=1}^{\infty} A_n \int_0^d \sin\frac{n\pi x}{d}\sin\frac{m\pi x}{d}\,dx \tag{25}$$

The first term on the right-hand side is easily integrable* while the product of sine terms integrates to zero unless $m=n$, yielding

$$A_m=-\frac{2I}{m\pi D} \tag{26}$$

The total solution is thus

$$H_z(x,t)=\frac{I}{D}\left(1-\frac{x}{d}-2\sum_{n=1}^{\infty}\frac{\sin(n\pi x/d)}{n\pi}e^{-n^2 t/\tau}\right) \tag{27}$$

where we define the fundamental continuum magnetic diffusion time constant $\tau$ as

$$\tau=\frac{1}{\alpha_1}=\frac{\mu\sigma d^2}{\pi^2} \tag{28}$$

analogous to the lumped parameter time constant of (5) and (6).

* $\int_0^d (d-x)\sin\frac{m\pi x}{d}\,dx=\frac{d^2}{m\pi}$

The magnetic field approaches the steady state in times long compared to $\tau$. For a perfect conductor ($\sigma \to \infty$), this time is infinite and the magnetic field is forever excluded from the slab. The current then flows only along the $x=0$ surface. However, even for copper ($\sigma \approx 6\times 10^7$ siemens/m) 10-cm thick, the time constant is $\tau \approx 80$ msec, which is fast for many applications. The current then diffuses into the conductor where the current density is easily obtained from Ampere's law as

$$\mathbf{J}_f = \nabla\times\mathbf{H} = -\frac{\partial H_z}{\partial x}\mathbf{i}_y$$
$$= \frac{I}{Dd}\left(1+2\sum_{n=1}^{\infty}\cos\frac{n\pi x}{d}e^{-n^2 t/\tau}\right)\mathbf{i}_y \tag{29}$$

The diffusion of the magnetic field and current density are plotted in Figure 6-26$b$ for various times

The force on the conducting slab is due to the Lorentz force tending to expand the loop and a magnetization force due to the difference of permeability of the slab and the surrounding free space as derived in Section 5-8-1:

$$\mathbf{F} = \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H} + \mu_0\mathbf{J}_f\times\mathbf{H}$$
$$= (\mu-\mu_0)(\mathbf{H}\cdot\nabla)\mathbf{H} + \mu_0\mathbf{J}_f\times\mathbf{H} \tag{30}$$

For our case with $\mathbf{H} = H_z(x)\mathbf{i}_z$, the magnetization force density has no contribution so that (30) reduces to

$$\mathbf{F} = \mu_0\mathbf{J}_f\times\mathbf{H}$$
$$= \mu_0(\nabla\times\mathbf{H})\times\mathbf{H}$$
$$= \mu_0(\mathbf{H}\cdot\nabla)\mathbf{H} - \nabla(\tfrac{1}{2}\mu_0\mathbf{H}\cdot\mathbf{H})$$
$$= -\frac{d}{dx}(\tfrac{1}{2}\mu_0 H_z^2)\mathbf{i}_x \tag{31}$$

Integrating (31) over the slab volume with the magnetic field independent of $y$ and $z$,

$$f_x = -\int_0^d sD\frac{d}{dx}(\tfrac{1}{2}\mu_0 H_z^2)\,dx$$
$$= -\tfrac{1}{2}\mu_0 H_z^2 sD\Big|_0^d$$
$$= \tfrac{1}{2}\frac{\mu_0 I^2 s}{D} \tag{32}$$

gives us a constant force with time that is independent of the permeability. Note that our approach of expressing the current density in terms of the magnetic field in (31) was easier than multiplying the infinite series of (27) and (29), as the result then only depended on the magnetic field at the boundaries that are known from the boundary conditions of (20). The resulting integration in (32) was easy because the force density in (31) was expressed as a pure derivative of $x$.

### 6-4-4 The Sinusoidal Steady State (Skin Depth)

We now place an infinitely thick conducting slab a distance $d$ above a sinusoidally varying current sheet $K_0 \cos \omega t \mathbf{i}_y$, which lies on top of a perfect conductor, as in Figure 6-27*a*. The

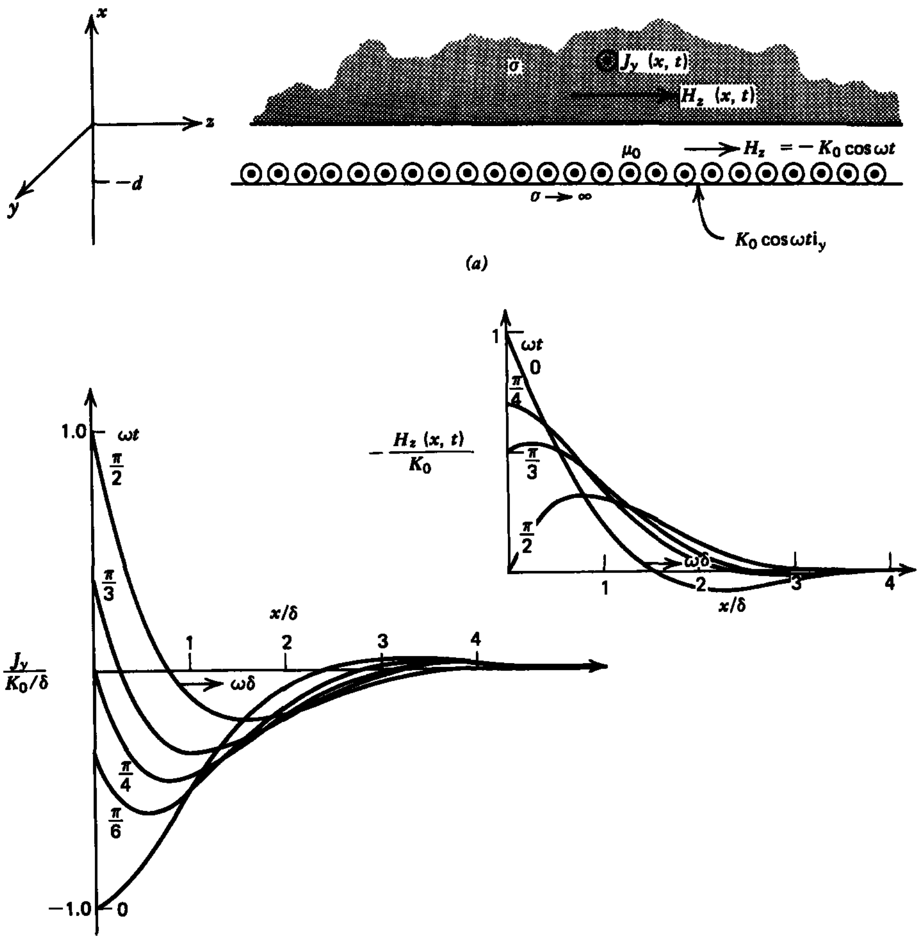



Figure 6-27 (*a*) A stationary conductor lies above a sinusoidal surface current placed upon a perfect conductor so that $\mathbf{H} = 0$ for $x < -d$. (*b*) The magnetic field and current density propagates and decays into the conductor with the same characteristic length given by the skin depth $\delta = \sqrt{2/(\omega\mu\sigma)}$. The phase speed of the wave is $\omega\delta$.

magnetic field within the conductor is then also sinusoidally varying with time:

$$H_z(x,t)=\text{Re}\,[\hat{H}_z(x)\,e^{j\omega t}] \tag{33}$$

Substituting (33) into (14) yields

$$\frac{d^2\hat{H}_z}{dx^2}-j\omega\mu\sigma\hat{H}_z=0 \tag{34}$$

with solution

$$\hat{H}_z(x)=A_1\,e^{(1+j)x/\delta}+A_2\,e^{-(1+j)x/\delta} \tag{35}$$

where the skin depth $\delta$ is defined as

$$\delta=\sqrt{2/(\omega\mu\sigma)} \tag{36}$$

Since the magnetic field must remain finite far from the current sheet, $A_1$ must be zero. The magnetic field is also continuous across the $x=0$ boundary because there is no surface current, so that the solution is

$$\begin{aligned}H_z(x,t)&=\text{Re}\,[-K_0\,e^{-(1+j)x/\delta}\,e^{j\omega t}]\\&=-K_0\cos(\omega t-x/\delta)\,e^{-x/\delta},\quad x\geq 0\end{aligned} \tag{37}$$

where the magnetic field in the gap is uniform, determined by the discontinuity in tangential **H** at $x=-d$ to be $H_z=-K_y$ for $-d<x\leq 0$ since within the perfect conductor $(x<-d)\mathbf{H}=0$. The magnetic field diffuses into the conductor as a strongly damped propagating wave with characteristic penetration depth $\delta$. The skin depth $\delta$ is also equal to the propagating wavelength, as drawn in Figure 6-27*b*. The current density within the conductor

$$\begin{aligned}\mathbf{J}_f=\nabla\times\mathbf{H}&=-\frac{\partial H_z}{\partial x}\mathbf{i}_y\\&=+\frac{K_0\,e^{-x/\delta}}{\delta}\left[\sin\left(\omega t-\frac{x}{\delta}\right)-\cos\left(\omega t-\frac{x}{\delta}\right)\right]\mathbf{i}_y\end{aligned} \tag{38}$$

is also drawn in Figure 6-27*b* at various times in the cycle, being confined near the interface to a depth on the order of $\delta$. For a perfect conductor, $\delta\to 0$, and the volume current becomes a surface current.

Seawater has a conductivity of $\approx 4$ siemens/m so that at a frequency of $f=1$ MHz $(\omega=2\pi f)$ the skin depth is $\delta\approx 0.25$ m. This is why radio communications to submarines are difficult. The conductivity of copper is $\sigma\approx 6\times 10^7$ siemens/m so that at 60 Hz the skin depth is $\delta\approx 8$ mm. Power cables with larger radii have most of the current confined near the surface so that the center core carries very little current. This reduces the cross-sectional area through which the current flows, raising the cable resistance leading to larger power dissipation.

Again, the magnetization force density has no contribution to the force density since $H_z$ only depends on $x$:

$$\begin{aligned}\mathbf{F} &= \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H}+\mu_0\mathbf{J}_f\times\mathbf{H}\\ &= \mu_0(\nabla\times\mathbf{H})\times\mathbf{H}\\ &= -\nabla(\tfrac{1}{2}\mu_0\mathbf{H}\cdot\mathbf{H})\end{aligned} \tag{39}$$

The total force per unit area on the slab obtained by integrating (39) over $x$ depends only on the magnetic field at $x=0$:

$$\begin{aligned}f_x &= -\int_0^\infty \frac{d}{dx}\left(\frac{\mu_0}{2}H_z^2\right)dx\\ &= -\tfrac{1}{2}\mu_0 H_z^2\big|_0^\infty\\ &= \tfrac{1}{2}\mu_0 K_0^2\cos^2\omega t\end{aligned} \tag{40}$$

because again $\mathbf{H}$ is independent of $y$ and $z$ and the $x$ component of the force density of (39) was written as a pure derivative with respect to $x$. Note that this approach was easier than integrating the cross product of (38) with (37).

This force can be used to levitate the conductor. Note that the region for $x>\delta$ is dead weight, as it contributes very little to the magnetic force.

### 6-4-5 Effects of Convection

A distributed dc surface current $-K_0\mathbf{i}_y$ at $x=0$ flows along parallel electrodes and returns via a conducting fluid moving to the right with constant velocity $v_0\mathbf{i}_x$, as shown in Figure 6-28*a*. The flow is not impeded by the current source at $x=0$. With the neglect of fringing, the magnetic field is purely $z$ directed and only depends on the $x$ coordinate, so that (13) in the dc steady state, with $\mathbf{U}=v_0\mathbf{i}_x$ being a constant, becomes*

$$\frac{d^2H_z}{dx^2}-\mu\sigma v_0\frac{dH_z}{dx}=0 \tag{41}$$

Solutions of the form

$$H_z(x)=A\,e^{px} \tag{42}$$

$$^*\,\nabla\times(\mathbf{U}\times\mathbf{H})=\mathbf{U}\,(\cancelto{0}{\nabla\cdot\mathbf{H}})-\mathbf{H}(\cancelto{0}{\nabla\cdot\mathbf{U}})+(\cancelto{0}{\mathbf{H}\cdot\nabla})\mathbf{U}-(\mathbf{U}\cdot\nabla)\mathbf{H}=-v_0\frac{d\mathbf{H}}{dx}$$

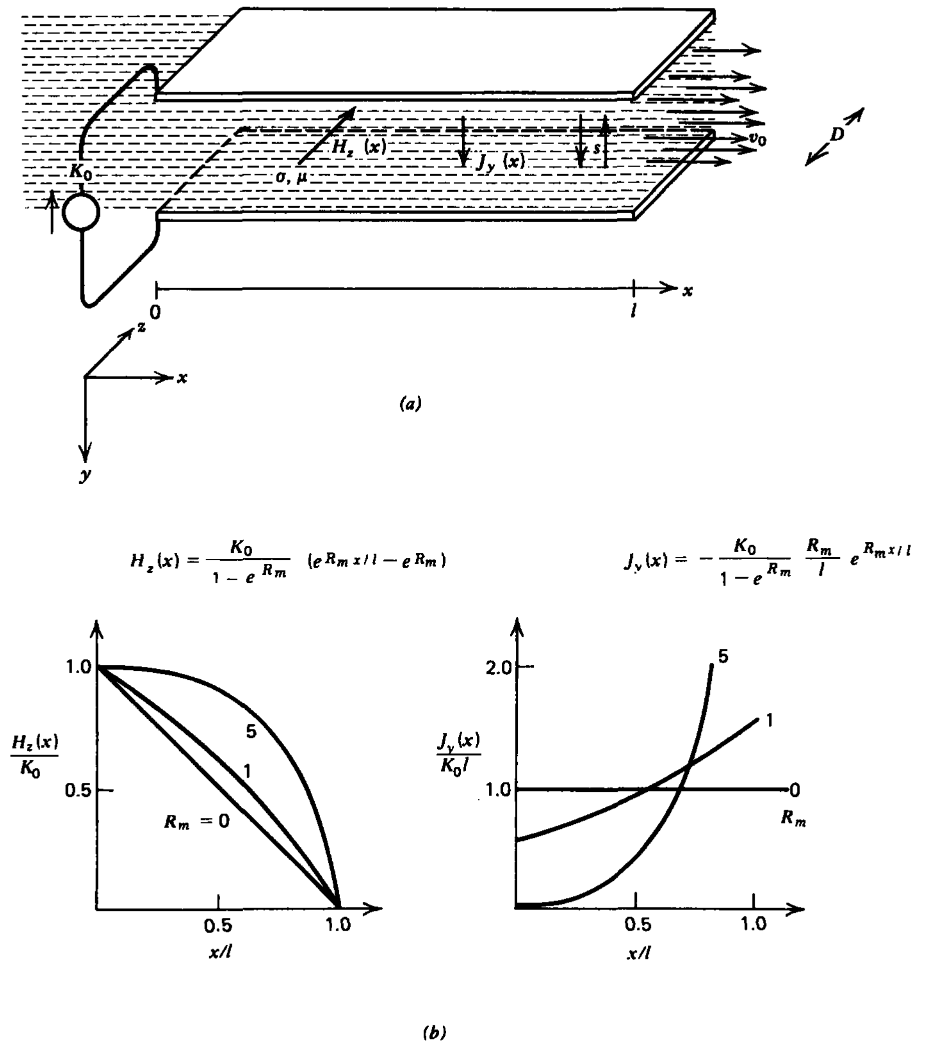



Figure 6-28 (*a*) A conducting material moving through a magnetic field tends to pull the magnetic field and current density with it. (*b*) The magnetic field and current density are greatly disturbed by the flow when the magnetic Reynolds number is large, $R_m = \sigma\mu Ul \gg 1$.

when substituted back into (41) yield two allowed values of $p$,

$$p^2 - \mu\sigma v_0 p = 0 \Rightarrow p = 0, \qquad p = \mu\sigma v_0 \tag{43}$$

Since (41) is linear, the most general solution is just the sum of the two allowed solutions,

$$H_z(x) = A_1 e^{R_m x/l} + A_2 \tag{44}$$

where the magnetic Reynold's number is defined as

$$R_m = \sigma\mu v_0 l = \frac{\sigma\mu l^2}{l/v_0} \tag{45}$$

and represents the ratio of a representative magnetic diffusion time given by (28) to a fluid transport time $(l/v_0)$. The boundary conditions are

$$H_z(x=0) = K_0, \qquad H_z(x=l) = 0 \tag{46}$$

so that the solution is

$$H_z(x) = \frac{K_0}{1-e^{R_m}}(e^{R_m x/l} - e^{R_m}) \tag{47}$$

The associated current distribution is then

$$\begin{aligned} \mathbf{J}_f = \nabla \times \mathbf{H} &= -\frac{\partial H_z}{\partial x}\mathbf{i}_y \\ &= -\frac{K_0}{1-e^{R_m}}\frac{R_m}{l}e^{R_m x/l}\mathbf{i}_y \end{aligned} \tag{48}$$

The field and current distributions plotted in Figure 6-28*b* for various $R_m$ show that the magnetic field and current are pulled along in the direction of flow. For small $R_m$ the magnetic field is hardly disturbed from the zero flow solution of a linear field and constant current distribution. For very large $R_m \gg 1$, the magnetic field approaches a uniform distribution while the current density approaches a surface current at $x = l$.

The force on the moving fluid is independent of the flow velocity:

$$\begin{aligned} \mathbf{f} &= \int_0^l \mathbf{J} \times \mu_0 \mathbf{H} s D\, dx \\ &= -\frac{K_0^2}{(1-e^{R_m})^2}\mu_0 \frac{R_m}{l} sD \int_0^l e^{R_m x/l}(e^{R_m x/l} - e^{R_m})\, dx\, \mathbf{i}_x \\ &= -\frac{K_0^2 \mu_0 s D}{(1-e^{R_m})^2} e^{R_m x/l}\left(\frac{e^{R_m x/l}}{2} - e^{R_m}\right)\Bigg|_0^l \mathbf{i}_x \\ &= \tfrac{1}{2}\mu_0 K_0^2 s D \mathbf{i}_x \end{aligned} \tag{49}$$

## 6-4-6 A Linear Induction Machine

The induced currents in a conductor due to a time varying magnetic field give rise to a force that can cause the conductor to move. This describes a motor. The inverse effect is when we cause a conductor to move through a time varying magnetic field generating a current, which describes a generator.

The linear induction machine shown in Figure 6-29*a* assumes a conductor moves to the right at constant velocity $U\mathbf{i}_z$. Directly below the conductor with no gap is a surface current placed on top of an infinitely permeable medium

$$\mathbf{K}(t) = -K_0 \cos(\omega t - kz)\mathbf{i}_y = \operatorname{Re}\left[-K_0\, e^{j(\omega t - kz)}\mathbf{i}_y\right] \qquad (50)$$

which is a traveling wave moving to the right at speed $\omega/k$. For $x > 0$, the magnetic field will then have $x$ and $z$ components of the form

$$\begin{aligned} H_z(x, z, t) &= \operatorname{Re}\left[\hat{H}_z(x)\, e^{j(\omega t - kz)}\right] \\ H_x(x, z, t) &= \operatorname{Re}\left[\hat{H}_x(x)\, e^{j(\omega t - kz)}\right] \end{aligned} \qquad (51)$$

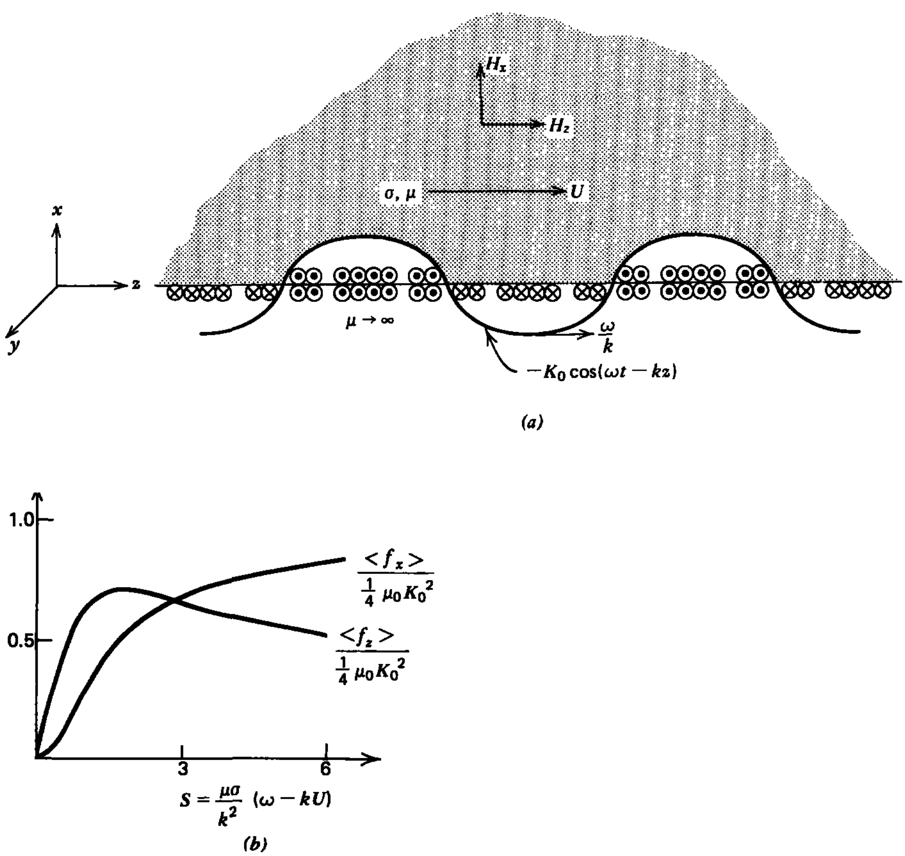



Figure 6-29 (*a*) A traveling wave of surface current induces currents in a conductor that is moving at a velocity $U$ different from the wave speed $\omega/k$. (*b*) The resulting forces can levitate and propel the conductor as a function of the slip $S$, which measures the difference in speeds of the conductor and traveling wave.

where (10) ($\nabla \cdot \mathbf{B} = 0$) requires these components to be related as

$$\frac{d\hat{H}_x}{dx} - jk\hat{H}_z = 0 \tag{52}$$

The $z$ component of the magnetic diffusion equation of (13) is

$$\frac{d^2\hat{H}_z}{dx^2} - k^2\hat{H}_z = j\mu\sigma(\omega - kU)\hat{H}_z \tag{53}$$

which can also be written as

$$\frac{d^2\hat{H}_z}{dx^2} - \gamma^2\hat{H}_z = 0 \tag{54}$$

where

$$\gamma^2 = k^2(1 + jS), \qquad S = \frac{\mu\sigma}{k^2}(\omega - kU) \tag{55}$$

and $S$ is known as the slip. Solutions of (54) are again exponential but complex because $\gamma$ is complex:

$$\hat{H}_z = A_1 e^{\gamma x} + A_2 e^{-\gamma x} \tag{56}$$

Because $\hat{H}_z$ must remain finite far from the current sheet, $A_1 = 0$, so that using (52) the magnetic field is of the form

$$\hat{\mathbf{H}} = K_0 e^{-\gamma x}\left(\mathbf{i}_z - \frac{jk}{\gamma}\mathbf{i}_x\right) \tag{57}$$

where we use the fact that the tangential component of $\mathbf{H}$ is discontinuous in the surface current, with $\mathbf{H} = 0$ for $x < 0$.

The current density in the conductor is

$$\begin{aligned}
\mathbf{J}_f = \nabla \times \mathbf{H} = \mathbf{i}_y\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) \Rightarrow \hat{J}_y &= -jk\hat{H}_x - \frac{d\hat{H}_z}{dx} \\
&= K_0 e^{-\gamma x}\frac{(\gamma^2 - k^2)}{\gamma} \\
&= \frac{K_0 k^2 jS e^{-\gamma x}}{\gamma}
\end{aligned} \tag{58}$$

If the conductor and current wave travel at the same speed ($\omega/k = U$), no current is induced as the slip is zero. Currents are only induced if the conductor and wave travel at different velocities. This is the principle of all induction machines.

The force per unit area on the conductor then has $x$ and $z$ components:

$$\mathbf{f} = \int_0^{\infty} \mathbf{J} \times \mu_0 \mathbf{H}\, dx$$
$$= \int_0^{\infty} \mu_0 J_y (H_z \mathbf{i}_x - H_x \mathbf{i}_z)\, dx \tag{59}$$

These integrations are straightforward but lengthy because first the instantaneous field and current density must be found from (51) by taking the real parts. More important is the time-average force per unit area over a period of excitation:

$$\langle \mathbf{f} \rangle = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \mathbf{f}\, dt \tag{60}$$

Since the real part of a complex quantity is equal to half the sum of the quantity and its complex conjugate,

$$\begin{aligned} A &= \mathrm{Re}\,[\hat{A}\, e^{j\omega t}] = \tfrac{1}{2}(\hat{A}\, e^{j\omega t} + \hat{A}^*\, e^{-j\omega t}) \\ B &= \mathrm{Re}\,[\hat{B}\, e^{j\omega t}] = \tfrac{1}{2}(\hat{B}\, e^{j\omega t} + \hat{B}^*\, e^{-j\omega t}) \end{aligned} \tag{61}$$

the time-average product of two quantities is

$$\frac{\omega}{2\pi} \int_0^{2\pi/\omega} AB\, dt = \frac{1}{4} \frac{\omega}{2\pi} \int_0^{2\pi/\omega} (\hat{A}\hat{B}\, e^{2j\omega t} + \hat{A}^*\hat{B} + \hat{A}\hat{B}^* + \hat{A}^*\hat{B}^*\, e^{-2j\omega t})\, dt$$
$$= \tfrac{1}{4}(\hat{A}^*\hat{B} + \hat{A}\hat{B}^*)$$
$$= \tfrac{1}{2}\,\mathrm{Re}\,(\hat{A}\hat{B}^*) \tag{62}$$

which is a formula often used for the time-average power in circuits where $A$ and $B$ are the voltage and current.

Then using (62) in (59), the $x$ component of the time-average force per unit area is

$$\langle f_x \rangle = \tfrac{1}{2}\,\mathrm{Re}\left(\int_0^{\infty} \mu_0 \hat{J}_y \hat{H}_z^*\, dx\right)$$
$$= \frac{\mu_0}{2} K_0^2 k^2 S\, \mathrm{Re}\left(\frac{j}{\gamma} \int_0^{\infty} e^{-(\gamma+\gamma^*)x}\, dx\right)$$
$$= \frac{\mu_0}{2} K_0^2 k^2 S\, \mathrm{Re}\left(\frac{j}{\gamma(\gamma+\gamma^*)}\right)$$
$$= \frac{1}{4} \frac{\mu_0 K_0^2 S^2}{[1+S^2+(1+S^2)^{1/2}]} = \tfrac{1}{4}\mu_0 K_0^2 \left(\frac{\sqrt{1+S^2}-1}{\sqrt{1+S^2}}\right) \tag{63}$$

where the last equalities were evaluated in terms of the slip $S$ from (55).

We similarly compute the time-average shear force per unit area as

$$
\begin{aligned}
<f_z> &= -\tfrac{1}{2}\,\mathrm{Re}\left(\int_0^\infty \mu_0 J_y H_x^*\,dx\right)\\
&= \frac{\mu_0}{2}\frac{K_0^2 k^3 S}{\gamma\gamma^*}\,\mathrm{Re}\left(\int_0^\infty e^{-(\gamma+\gamma^*)x}\,dx\right)\\
&= \frac{\mu_0}{2}\frac{k^3 K_0^2 S}{\gamma\gamma^*}\,\mathrm{Re}\left(\frac{1}{(\gamma+\gamma^*)}\right)\\
&= \frac{\mu_0 K_0^2 S}{4\sqrt{1+S^2}\,\mathrm{Re}\,(\sqrt{1+jS})}
\end{aligned}
\tag{64}
$$

When the wave speed exceeds the conductor's speed ($\omega/k > U$), the force is positive as $S>0$ so that the wave pulls the conductor along. When $S<0$, the slow wave tends to pull the conductor back as $<f_z><0$. The forces of (63) and (64), plotted in Figure 6-29*b*, can be used to simultaneously lift and propel a conducting material. There is no force when the wave and conductor travel at the same speed ($\omega/k = U$) as the slip is zero ($S=0$). For large $S$, the levitating force $<f_x>$ approaches the constant value $\frac{1}{4}\mu_0 K_0^2$ while the shear force approaches zero. There is an optimum value of $S$ that maximizes $<f_z>$. For smaller $S$, less current is induced while for larger $S$ the phase difference between the imposed and induced currents tend to decrease the time-average force.

### 6-4-7 Superconductors

In the limit of infinite Ohmic conductivity ($\sigma\to\infty$), the diffusion time constant of (28) becomes infinite while the skin depth of (36) becomes zero. The magnetic field cannot penetrate a perfect conductor and currents are completely confined to the surface.

However, in this limit the Ohmic conduction law is no longer valid and we should use the superconducting constitutive law developed in Section 3-2-2*d* for a single charge carrier:

$$\frac{\partial \mathbf{J}}{\partial t} = \omega_p^2 \varepsilon \mathbf{E} \tag{65}$$

Then for a stationary medium, following the same procedure as in (12) and (13) with the constitutive law of (65), (8)–(11) reduce to

$$\nabla^2 \frac{\partial \mathbf{H}}{\partial t} - \omega_p^2 \varepsilon\mu \frac{\partial \mathbf{H}}{\partial t} = 0 \Rightarrow \nabla^2(\mathbf{H}-\mathbf{H}_0) - \omega_p^2\varepsilon\mu(\mathbf{H}-\mathbf{H}_0) = 0 \tag{66}$$

where $\mathbf{H}_0$ is the instantaneous magnetic field at $t=0$. If the superconducting material has no initial magnetic field when an excitation is first turned on, then $\mathbf{H}_0=0$.

If the conducting slab in Figure 6-27$a$ becomes superconducting, (66) becomes

$$\frac{d^2H_z}{dx^2}-\frac{\omega_p^2}{c^2}H_z=0, \qquad c=\frac{1}{\sqrt{\varepsilon\mu}} \tag{67}$$

where $c$ is the speed of light in the medium.

The solution to (67) is

$$\begin{aligned} H_z &= A_1\,e^{\omega_p x/c}+A_2\,e^{-\omega_p x/c} \\ &= -K_0\cos\omega t\,e^{-\omega_p x/c} \end{aligned} \tag{68}$$

where we use the boundary condition of continuity of tangential $\mathbf{H}$ at $x=0$.

The current density is then

$$\begin{aligned} J_y &= -\frac{\partial H_z}{\partial x} \\ &= -\frac{K_0\omega_p}{c}\cos\omega t\,e^{-\omega_p x/c} \end{aligned} \tag{69}$$

For any frequency $\omega$, including dc ($\omega=0$), the field and current decay with characteristic length:

$$l_c=c/\omega_p \tag{70}$$

Since the plasma frequency $\omega_p$ is typically on the order of $10^{15}$ radian/sec, this characteristic length is very small, $l_c\approx 3\times10^8/10^{15}\approx 3\times10^{-7}$ m. Except for this thin sheath, the magnetic field is excluded from the superconductor while the volume current is confined to this region near the interface.

There is one experimental exception to the governing equation in (66), known as the Meissner effect. If an ordinary conductor is placed within a dc magnetic field $\mathbf{H}_0$ and then cooled through the transition temperature for superconductivity, the magnetic flux is pushed out except for a thin sheath of width given by (70). This is contrary to (66), which allows the time-independent solution $\mathbf{H}=\mathbf{H}_0$, where the magnetic field remains trapped within the superconductor. Although the reason is not well understood, superconductors behave as if $\mathbf{H}_0=0$ no matter what the initial value of magnetic field.

## 6-5 ENERGY STORED IN THE MAGNETIC FIELD

### 6-5-1 A Single Current Loop

The differential amount of work necessary to overcome the electric and magnetic forces on a charge $q$ moving an incremental distance **ds** at velocity **v** is

$$dW_q = -q(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\mathbf{ds} \tag{1}$$

**(a) Electrical Work**

If the charge moves solely under the action of the electrical and magnetic forces with no other forces of mechanical origin, the incremental displacement in a small time $dt$ is related to its velocity as

$$\mathbf{ds} = \mathbf{v}\,dt \tag{2}$$

Then the magnetic field cannot contribute to any work on the charge because the magnetic force is perpendicular to the charge's displacement:

$$dW_q = -q\mathbf{v}\cdot\mathbf{E}\,dt \tag{3}$$

and the work required is entirely due to the electric field. Within a charge neutral wire, the electric field is not due to Coulombic forces but rather arises from Faraday's law. The moving charge constitutes an incremental current element,

$$q\mathbf{v} = i\,\mathbf{dl} \Rightarrow dW_q = -i\mathbf{E}\cdot\mathbf{dl}\,dt \tag{4}$$

so that the total work necessary to move all the charges in the closed wire is just the sum of the work done on each current element,

$$\begin{aligned} dW &= \oint_L dW_q = -i\,dt \oint_L \mathbf{E}\cdot\mathbf{dl} \\ &= i\,dt\frac{d}{dt}\int_S \mathbf{B}\cdot\mathbf{dS} \\ &= i\,dt\frac{d\Phi}{dt} \\ &= i\,d\Phi \end{aligned} \tag{5}$$

which through Faraday's law is proportional to the change of flux through the current loop. This flux may be due to other currents and magnets (mutual flux) as well as the self-flux due to the current $i$. Note that the third relation in (5) is just equivalent to the circuit definition of electrical power delivered to the loop:

$$p = \frac{dW}{dt} = i\frac{d\Phi}{dt} = vi \tag{6}$$

All of this energy supplied to accelerate the charges in the wire is stored as no energy is dissipated in the lossless loop and no mechanical work is performed if the loop is held stationary.

**(b) Mechanical Work**

The magnetic field contributed no work in accelerating the charges. This is not true when the current-carrying wire is itself moved a small vector displacement **ds** requiring us to perform mechanical work,

$$\begin{aligned} dW &= -(i\,\mathbf{dl}\times\mathbf{B})\cdot\mathbf{ds} = i(\mathbf{B}\times\mathbf{dl})\cdot\mathbf{ds} \\ &= i\mathbf{B}\cdot(\mathbf{dl}\times\mathbf{ds}) \end{aligned} \qquad (7)$$

where we were able to interchange the dot and the cross using the scalar triple product identity proved in Problem 1-10*a*. We define $S_1$ as the area originally bounding the loop and $S_2$ as the bounding area after the loop has moved the distance **ds**, as shown in Figure 6-30. The incremental area $\mathbf{dS}_3$ is then the strip joining the two positions of the loop defined by the bracketed quantity in (7):

$$\mathbf{dS}_3 = \mathbf{dl}\times\mathbf{ds} \qquad (8)$$

The flux through each of the contours is

$$\Phi_1 = \int_{S_1} \mathbf{B}\cdot\mathbf{dS}, \qquad \Phi_2 = \int_{S_2} \mathbf{B}\cdot\mathbf{dS} \qquad (9)$$

where their difference is just the flux that passes outward through $\mathbf{dS}_3$:

$$d\Phi = \Phi_1 - \Phi_2 = \mathbf{B}\cdot\mathbf{dS}_3 \qquad (10)$$

The incremental mechanical work of (7) necessary to move the loop is then identical to (5):

$$dW = i\mathbf{B}\cdot\mathbf{dS}_3 = i\,d\Phi \qquad (11)$$

Here there was no change of electrical energy input, with the increase of stored energy due entirely to mechanical work in moving the current loop.

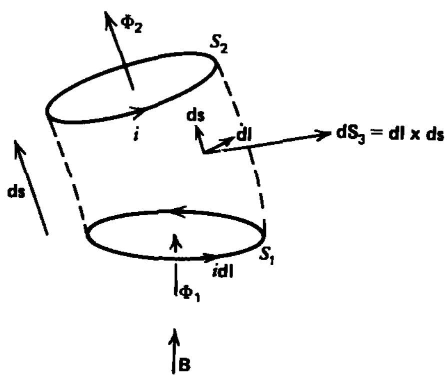



Figure 6-30 The mechanical work necessary to move a current-carrying loop is stored as potential energy in the magnetic field.

### 6-5-2 Energy and Inductance

If the loop is isolated and is within a linear permeable material, the flux is due entirely to the current, related through the self-inductance of the loop as

$$\Phi = Li \tag{12}$$

so that (5) or (11) can be integrated to find the total energy in a loop with final values of current $I$ and flux $\Phi$:

$$\begin{aligned} W &= \int_0^\Phi i\,d\Phi \\ &= \int_0^\Phi \frac{\Phi}{L}\,d\Phi \\ &= \frac{1}{2}\frac{\Phi^2}{L} = \frac{1}{2}LI^2 = \frac{1}{2}I\Phi \end{aligned} \tag{13}$$

### 6-5-3 Current Distributions

The results of (13) are only true for a single current loop. For many interacting current loops or for current distributions, it is convenient to write the flux in terms of the vector potential using Stokes' theorem:

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{dS} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{dS} = \oint_L \mathbf{A} \cdot \mathbf{dl} \tag{14}$$

Then each incremental-sized current element carrying a current $I$ with flux $d\Phi$ has stored energy given by (13):

$$dW = \tfrac{1}{2} I\,d\Phi = \tfrac{1}{2}\mathbf{I} \cdot \mathbf{A}\,dl \tag{15}$$

For $N$ current elements, (15) generalizes to

$$\begin{aligned} W &= \tfrac{1}{2}(\mathbf{I}_1 \cdot \mathbf{A}_1\,dl_1 + \mathbf{I}_2 \cdot \mathbf{A}_2\,dl_2 + \cdots + \mathbf{I}_N \cdot \mathbf{A}_N\,dl_N) \\ &= \tfrac{1}{2}\sum_{n=1}^{N} \mathbf{I}_n \cdot \mathbf{A}_n\,dl_n \end{aligned} \tag{16}$$

If the current is distributed over a line, surface, or volume, the summation is replaced by integration:

$$W = \begin{cases} \tfrac{1}{2}\int_L \mathbf{I}_f \cdot \mathbf{A}\,dl & \text{(line current)} \\ \tfrac{1}{2}\int_S \mathbf{K}_f \cdot \mathbf{A}\,dS & \text{(surface current)} \\ \tfrac{1}{2}\int_V \mathbf{J}_f \cdot \mathbf{A}\,dV & \text{(volume current)} \end{cases} \tag{17}$$

Remember that in (16) and (17) the currents and vector potentials are all evaluated at their final values as opposed to (11), where the current must be expressed as a function of flux.

### 6-5-4 Magnetic Energy Density

This stored energy can be thought of as being stored in the magnetic field. Assuming that we have a free volume distribution of current $\mathbf{J}_f$, we use (17) with Ampere's law to express $\mathbf{J}_f$ in terms of $\mathbf{H}$,

$$W=\frac{1}{2}\int_V \mathbf{J}_f\cdot\mathbf{A}\,dV=\frac{1}{2}\int_V (\nabla\times\mathbf{H})\cdot\mathbf{A}\,dV \tag{18}$$

where the volume V is just the volume occupied by the current. Larger volumes (including all space) can be used in (18), for the region outside the current has $\mathbf{J}_f=0$ so that no additional contributions arise.

Using the vector identity

$$\begin{aligned}\nabla\cdot(\mathbf{A}\times\mathbf{H})&=\mathbf{H}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{H})\\&=\mathbf{H}\cdot\mathbf{B}-\mathbf{A}\cdot(\nabla\times\mathbf{H})\end{aligned} \tag{19}$$

we rewrite (18) as

$$W=\frac{1}{2}\int_V [\mathbf{H}\cdot\mathbf{B}-\nabla\cdot(\mathbf{A}\times\mathbf{H})]\,dV \tag{20}$$

The second term on the right-hand side can be converted to a surface integral using the divergence theorem:

$$\int_V \nabla\cdot(\mathbf{A}\times\mathbf{H})\,dV=\oint_S (\mathbf{A}\times\mathbf{H})\cdot\mathbf{dS} \tag{21}$$

It now becomes convenient to let the volume extend over all space so that the surface is at infinity. If the current distribution does not extend to infinity the vector potential dies off at least as $1/r$ and the magnetic field as $1/r^2$. Then, even though the area increases as $r^2$, the surface integral in (21) decreases at least as $1/r$ and thus is zero when $S$ is at infinity. Then (20) becomes simply

$$W=\frac{1}{2}\int_V \mathbf{H}\cdot\mathbf{B}\,dV=\frac{1}{2}\int_V \mu H^2\,dV=\frac{1}{2}\int_V \frac{B^2}{\mu}\,dV \tag{22}$$

where the volume V now extends over all space. The magnetic energy density is thus

$$w=\frac{1}{2}\mathbf{H}\cdot\mathbf{B}=\frac{1}{2}\mu H^2=\frac{1}{2}\frac{B^2}{\mu} \tag{23}$$

These results are only true for linear materials where $\mu$ does not depend on the magnetic field, although it can depend on position.

For a single coil, the total energy in (22) must be identical to (13), which gives us an alternate method to calculating the self-inductance from the magnetic field.

## 6-5-5 The Coaxial Cable

**(a) External Inductance**

A typical cable geometry consists of two perfectly conducting cylindrical shells of radii $a$ and $b$ and length $l$, as shown in Figure 6-31. An imposed current $I$ flows axially as a surface current in opposite directions on each cylinder. We neglect fringing field effects near the ends so that the magnetic field is the same as if the cylinder were infinitely long. Using Ampere's law we find that

$$H_\phi = \frac{I}{2\pi r}, \quad a < r < b \tag{24}$$

The total magnetic flux between the two conductors is

$$\begin{aligned} \Phi &= \int_a^b \mu_0 H_\phi l \, dr \\ &= \frac{\mu_0 I l}{2\pi} \ln \frac{b}{a} \end{aligned} \tag{25}$$

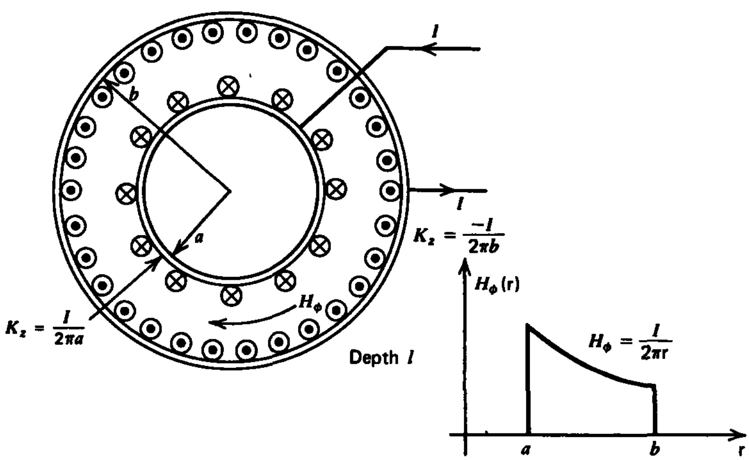



Figure 6-31 The magnetic field between two current-carrying cylindrical shells forming a coaxial cable is confined to the region between cylinders.

giving the self-inductance as

$$L = \frac{\Phi}{I} = \frac{\mu_0 l}{2\pi} \ln \frac{b}{a} \tag{26}$$

The same result can just as easily be found by computing the energy stored in the magnetic field

$$W = \tfrac{1}{2} L I^2 = \tfrac{1}{2}\mu_0 \int_a^b H_\phi^2 2\pi r l \, dr$$

$$= \frac{\mu_0 l I^2}{4\pi} \ln \frac{b}{a} \Rightarrow L = \frac{2W}{I^2} = \frac{\mu_0 l \ln (b/a)}{2\pi} \tag{27}$$

**(b) Internal Inductance**

If the inner cylinder is now solid, as in Figure 6-32, the current at low enough frequencies where the skin depth is much larger than the radius, is uniformly distributed with density

$$J_z = \frac{I}{\pi a^2} \tag{28}$$

so that a linearly increasing magnetic field is present within the inner cylinder while the outside magnetic field is

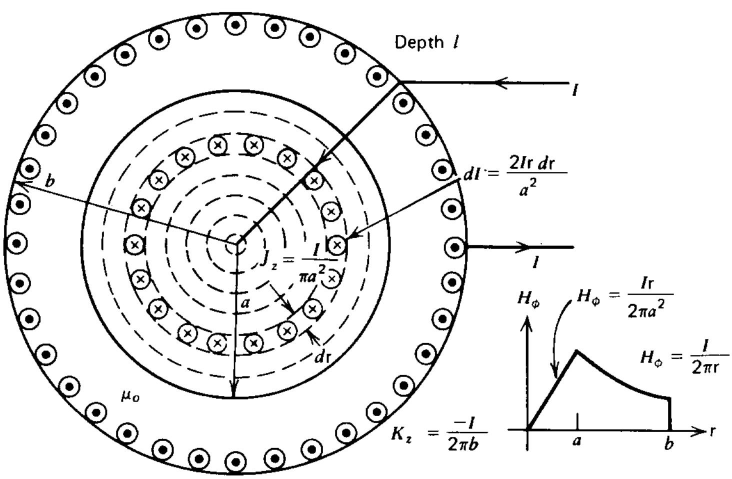



Figure 6-32 At low frequencies the current in a coaxial cable is uniformly distributed over the solid center conductor so that the internal magnetic field increases linearly with radius. The external magnetic field remains unchanged. The inner cylinder can be thought of as many incremental cylindrical shells of thickness $dr$ carrying a fraction of the total current. Each shell links its own self-flux as well as the mutual flux of the other shells of smaller radius. The additional flux within the current-carrying conductor results in the internal inductance of the cable.

unchanged from (24):

$$H_\phi = \begin{cases} \dfrac{I\mathrm{r}}{2\pi a^2}, & 0<\mathrm{r}<a \\ \dfrac{I}{2\pi \mathrm{r}}, & a<\mathrm{r}<b \end{cases} \tag{29}$$

The self-inductance cannot be found using the flux per unit current definition for a current loop since the current is not restricted to a thin filament. The inner cylinder can be thought of as many incremental cylindrical shells, as in Figure 6-32, each linking its own self-flux as well as the mutual flux of the other shells of smaller radius. Note that each shell is at a different voltage due to the differences in enclosed flux, although the terminal wires that are in a region where the magnetic field is negligible have a well-defined unique voltage difference.

The easiest way to compute the self-inductance as seen by the terminal wires is to use the energy definition of (22):

$$\begin{aligned} W &= \tfrac{1}{2}\mu_0 \int_0^b H_\phi^2 2\pi l \mathrm{r}\, d\mathrm{r} \\ &= \pi l \mu_0 \left[\int_0^a \left(\frac{I\mathrm{r}}{2\pi a^2}\right)^2 \mathrm{r}\, d\mathrm{r} + \int_a^b \left(\frac{I}{2\pi \mathrm{r}}\right)^2 \mathrm{r}\, d\mathrm{r}\right] \\ &= \frac{\mu_0 l I^2}{4\pi}\left(\frac{1}{4} + \ln\frac{b}{a}\right) \end{aligned} \tag{30}$$

which gives the self-inductance as

$$L = \frac{2W}{I^2} = \frac{\mu_0 l}{2\pi}\left(\frac{1}{4} + \ln\frac{b}{a}\right) \tag{31}$$

The additional contribution of $\mu_0 l/8\pi$ is called the internal inductance and is due to the flux within the current-carrying conductor.

### 6-5-6 Self-Inductance, Capacitance, and Resistance

We can often save ourselves further calculations for the external self-inductance if we already know the capacitance or resistance for the same two-dimensional geometry composed of highly conducting electrodes with no internal inductance contribution. For the arbitrary geometry shown in Figure 6-33 of depth $d$, the capacitance, resistance, and inductance are defined as the ratios of line and surface integrals:

$$C = \frac{\varepsilon d \oint_S \mathbf{E} \cdot \mathbf{n}_s \, ds}{\int_L \mathbf{E} \cdot \mathbf{dl}}$$

$$R = \frac{\int_L \mathbf{E} \cdot \mathbf{dl}}{\sigma d \oint_S \mathbf{E} \cdot \mathbf{n}_s \, ds} \tag{32}$$

$$L = \frac{\mu d \int_L \mathbf{H} \cdot \mathbf{n}_l \, dl}{\oint_S \mathbf{H} \cdot \mathbf{ds}}$$

Because the homogeneous region between electrodes is charge and current free, both the electric and magnetic fields can be derived from a scalar potential that satisfies Laplace's equation. However, the electric field must be incident normally onto the electrodes while the magnetic field is incident tangentially so that **E** and **H** are perpendicular everywhere, each being along the potential lines of the other. This is accounted for in (32) and Figure 6-33 by having $\mathbf{n}_s \, ds$ perpendicular to **ds** and $\mathbf{n}_l \, dl$ perpendicular to **dl**. Then since $C$, $R$, and $L$ are independent of the field strengths, we can take **E** and **H** to both have unit magnitude so that in the products of $LC$ and $L/R$ the line and surface integrals cancel:

$$\begin{aligned} LC &= \varepsilon\mu d^2 = d^2/c^2, \qquad c = 1/\sqrt{\varepsilon\mu} \\ L/R &= \mu\sigma d^2, \qquad RC = \varepsilon/\sigma \end{aligned} \tag{33}$$

These products are then independent of the electrode geometry and depend only on the material parameters and the depth of the electrodes.

We recognize the $L/R$ ratio to be proportional to the magnetic diffusion time of Section 6-4-3 while $RC$ is just the charge relaxation time of Section 3-6-1. In Chapter 8 we see that the $\sqrt{LC}$ product is just equal to the time it takes an electromagnetic wave to propagate a distance $d$ at the speed of light $c$ in the medium.

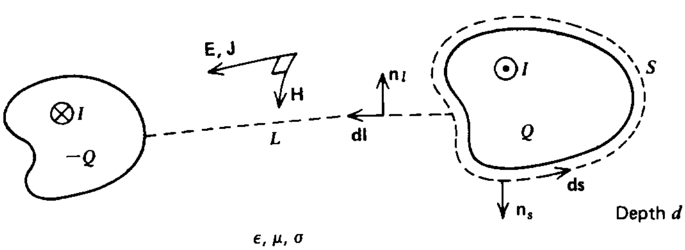



Figure 6-33 The electric and magnetic fields in the two-dimensional homogeneous charge and current-free region between hollow electrodes can be derived from a scalar potential that obeys Laplace's equation. The electric field lines are along the magnetic potential lines and vice versa so **E** and **H** are perpendicular. The inductance-capacitance product is then a constant.

## 6-6 THE ENERGY METHOD FOR FORCES

### 6-6-1 The Principle of Virtual Work

In Section 6-5-1 we calculated the energy stored in a current-carrying loop by two methods. First we calculated the electric energy input to a loop with no mechanical work done. We then obtained the same answer by computing the mechanical work necessary to move a current-carrying loop in an external field with no further electrical inputs. In the most general case, an input of electrical energy can result in stored energy $dW$ and mechanical work by the action of a force $f_x$ causing a small displacement $dx$:

$$i\,d\Phi = dW + f_x\,dx \tag{1}$$

If we knew the total energy stored in the magnetic field as a function of flux and position, the force is simply found as

$$f_x = -\left.\frac{\partial W}{\partial x}\right|_{\Phi} \tag{2}$$

We can easily compute the stored energy by realizing that no matter by what process or order the system is assembled, if the final position $x$ and flux $\Phi$ are the same, the energy is the same. Since the energy stored is independent of the order that we apply mechanical and electrical inputs, we choose to mechanically assemble a system first to its final position $x$ with no electrical excitations so that $\Phi = 0$. This takes no work as with zero flux there is no force of electrical origin. Once the system is mechanically assembled so that its position remains constant, we apply the electrical excitation to bring the system to its final flux value. The electrical energy required is

$$W = \int_{x\ \mathrm{const}} i\,d\Phi \tag{3}$$

For linear materials, the flux and current are linearly related through the inductance that can now be a function of $x$ because the inductance depends on the geometry:

$$i = \Phi/L(x) \tag{4}$$

Using (4) in (3) allows us to take the inductance outside the integral because $x$ is held constant so that the inductance is also constant:

$$\begin{aligned} W &= \frac{1}{L(x)}\int_0^{\Phi} \Phi\,d\Phi \\ &= \frac{\Phi^2}{2L(x)} = \tfrac{1}{2}L(x)i^2 \end{aligned} \tag{5}$$

The stored energy is the same as found in Section 6-5-2 even when mechanical work is included and the inductance varies with position.

To find the force on the moveable member, we use (2) with the energy expression in (5), which depends only on flux and position:

$$\begin{aligned} f_x &= -\frac{\partial W}{\partial x}\bigg|_{\Phi} \\ &= -\frac{\Phi^2}{2}\frac{d[1/L(x)]}{dx} \\ &= \frac{1}{2}\frac{\Phi^2}{L^2(x)}\frac{dL(x)}{dx} \\ &= \frac{1}{2}i^2\frac{dL(x)}{dx} \end{aligned} \tag{6}$$

### 6-6-2 Circuit Viewpoint

This result can also be obtained using a circuit description with the linear flux-current relation of (4):

$$\begin{aligned} v &= \frac{d\Phi}{dt} \\ &= L(x)\frac{di}{dt} + i\frac{dL(x)}{dt} \\ &= L(x)\frac{di}{dt} + i\frac{dL(x)}{dx}\frac{dx}{dt} \end{aligned} \tag{7}$$

The last term, proportional to the speed of the moveable member, just adds to the usual inductive voltage term. If the geometry is fixed and does not change with time, there is no electromechanical coupling term.

The power delivered to the system is

$$p = vi = i\frac{d}{dt}[L(x)i] \tag{8}$$

which can be expanded as

$$p = \frac{d}{dt}(\tfrac{1}{2}L(x)i^2) + \tfrac{1}{2}i^2\frac{dL(x)}{dx}\frac{dx}{dt} \tag{9}$$

This is in the form

$$p = \frac{dW}{dt} + f_x\frac{dx}{dt}, \quad \begin{cases} W = \frac{1}{2}L(x)i^2 \\ f_x = \frac{1}{2}i^2\dfrac{dL(x)}{dx} \end{cases} \tag{10}$$

which states that the power delivered to the inductor is equal to the sum of the time rate of energy stored and mechanical power performed on the inductor. This agrees with the energy method approach. If the inductance does not change with time because the geometry is fixed, all the input power is stored as potential energy W.

**Example 6-2 MAGNETIC FIELDS AND FORCES**

**(a) Relay**

Find the force on the moveable slug in the magnetic circuit shown in Figure 6-34.

**SOLUTION**

---

It is necessary to find the inductance of the system as a function of the slug's position so that we can use (6). Because of the infinitely permeable core and slug, the **H** field is non-zero only in the air gap of length $x$. We use Ampere's law to obtain

$$H = NI/x$$

The flux through the gap

$$\Phi = \mu_0 NIA/x$$

is equal to the flux through each turn of the coil yielding the inductance as

$$L(x) = \frac{N\Phi}{I} = \frac{\mu_0 N^2 A}{x}$$

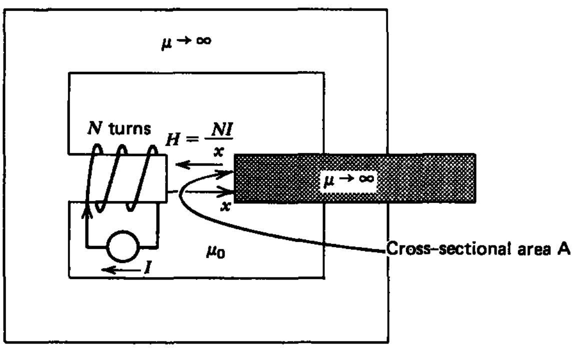



Figure 6-34 The magnetic field exerts a force on the moveable member in the relay pulling it into the magnetic circuit.

The force is then

$$f_x = \tfrac{1}{2} I^2 \frac{dL(x)}{dx}$$

$$= -\frac{\mu_o N^2 A I^2}{2x^2}$$

The minus sign means that the force is opposite to the direction of increasing $x$, so that the moveable piece is attracted to the coil.

**(b) One Turn Loop**

Find the force on the moveable upper plate in the one turn loop shown in Figure 6-35.

**SOLUTION**

---

The current distributes itself uniformly as a surface current $K = I/D$ on the moveable plate. If we neglect nonuniform field effects near the corners, the **H** field being tangent to the conductors just equals $K$:

$$H_z = I/D$$

The total flux linked by the current source is then

$$\Phi = \mu_0 H_z x l$$

$$= \frac{\mu_0 x l}{D} I$$

which gives the inductance as

$$L(x) = \frac{\Phi}{I} = \frac{\mu_0 x l}{D}$$

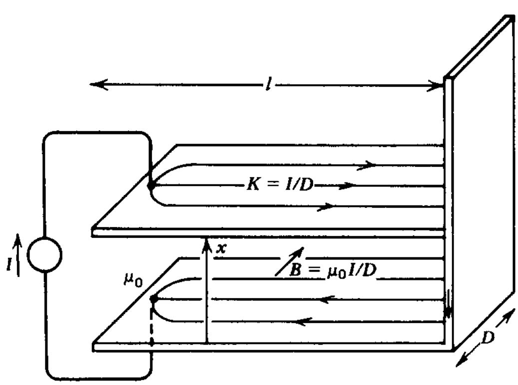



Figure 6-35 The magnetic force on a current-carrying loop tends to expand the loop.

The force is then constant

$$f_x = \tfrac{1}{2} I^2 \frac{dL(x)}{dx}$$
$$= \frac{1}{2} \frac{\mu_0 l I^2}{D}$$

### 6-6-3 Magnetization Force

A material with permeability $\mu$ is partially inserted into the magnetic circuit shown in Figure 6-36. With no free current in the moveable material, the $x$-directed force density from Section 5-8-1 is

$$\begin{aligned} F_x &= \mu_0(\mathbf{M} \cdot \nabla) H_x \\ &= (\mu - \mu_0)(\mathbf{H} \cdot \nabla) H_x \\ &= (\mu - \mu_0)\left(H_x \frac{\partial H_x}{\partial x} + H_y \frac{\partial H_x}{\partial y}\right) \end{aligned} \tag{11}$$

where we neglect variations with $z$. This force arises in the fringing field because within the gap the magnetic field is essentially uniform:

$$H_y = NI/s \tag{12}$$

Because the magnetic field in the permeable block is curl free,

$$\nabla \times \mathbf{H} = 0 \Rightarrow \frac{\partial H_x}{\partial y} = \frac{\partial H_y}{\partial x} \tag{13}$$

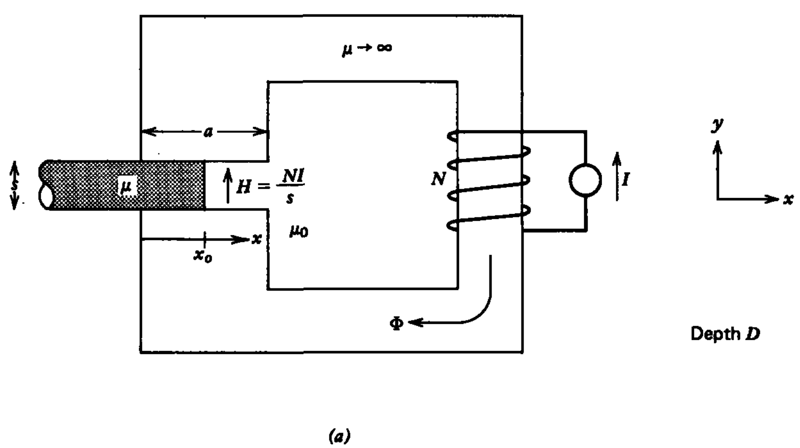



Figure 6-36 · A permeable material tends to be pulled into regions of higher magnetic field.

(11) can be rewritten as

$$F_x = \frac{(\mu - \mu_0)}{2}\frac{\partial}{\partial x}(H_x^2 + H_y^2) \tag{14}$$

The total force is then

$$\begin{aligned} f_x &= sD\int_{-\infty}^{x_0} F_x\,dx \\ &= \frac{(\mu-\mu_0)}{2} sD(H_x^2 + H_y^2)\Big|_{-\infty}^{x_0} \\ &= \frac{(\mu-\mu_0)}{2}\frac{N^2I^2D}{s} \end{aligned} \tag{15}$$

where the fields at $x = -\infty$ are zero and the field at $x = x_0$ is given by (12). High permeability material is attracted to regions of stronger magnetic field. It is this force that causes iron materials to be attracted towards a magnet. Diamagnetic materials ($\mu < \mu_0$) will be repelled.

This same result can more easily be obtained using (6) where the flux through the gap is

$$\Phi = HD[\mu x + \mu_0(a - x)] = \frac{NID}{s}[(\mu - \mu_0)x + a\mu_0] \tag{16}$$

so that the inductance is

$$L = \frac{N\Phi}{I} = \frac{N^2D}{s}[(\mu - \mu_0)x + a\mu_0] \tag{17}$$

Then the force obtained using (6) agrees with (15)

$$\begin{aligned} f_x &= \tfrac{1}{2}I^2\frac{dL(x)}{dx} \\ &= \frac{(\mu-\mu_0)}{2s}N^2I^2D \end{aligned} \tag{18}$$

## PROBLEMS

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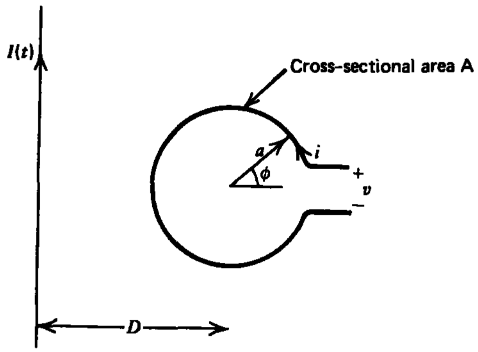
Section 6-1

1. A circular loop of radius $a$ with Ohmic conductivity $\sigma$ and cross-sectional area $A$ has its center a small distance $D$ away from an infinitely long time varying current.




(a) Find the mutual inductance $M$ and resistance $R$ of the loop. **Hint:**

$$\int \frac{dx}{a+b\cos x}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\frac{\sqrt{a^2-b^2}\tan(x/2)}{a+b}\right],$$

$$\int \frac{r\,dr}{\sqrt{D^2-r^2}}=-\sqrt{D^2-r^2}$$

(b) This loop is stationary and has a self-inductance $L$. What is the time dependence of the induced short circuit current when the line current is instantaneously stepped on to a dc level $I$ at $t=0$?

(c) Repeat (b) when the line current has been on a long time and is suddenly turned off at $t=T$.

(d) If the loop has no resistance and is moving with radial velocity $v_r=dr/dt$, what is the short circuit current and open circuit voltage for a dc line current?

(e) What is the force on the loop when it carries a current $i$? **Hint:**

$$\int \frac{\cos\phi\,d\phi}{D+a\cos\phi}=-\frac{1}{a}\sin^{-1}[\cos\phi]$$

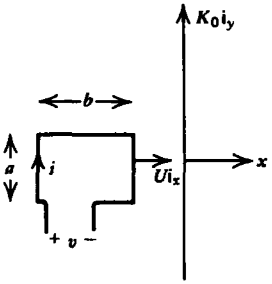
$$+\frac{D}{a\sqrt{D^2-a^2}}\sin^{-1}\left(\frac{a+D\cos\phi}{D+a\cos\phi}\right)$$

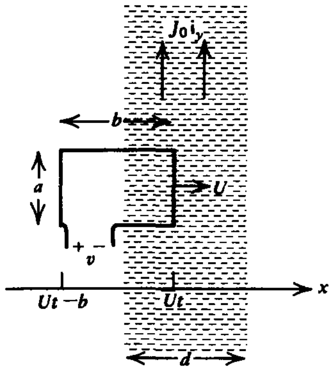
2. A rectangular loop at the far left travels with constant velocity $U\mathbf{i}_x$ towards and through a dc surface current sheet $K_0\mathbf{i}_y$ at $x=0$. The right-hand edge of the loop first reaches the current sheet at $t=0$.

(a) What is the loop's open circuit voltage as a function of time?

(b) What is the short circuit current if the loop has self-inductance $L$ and resistance $R$?

(c) Find the open circuit voltage if the surface current is replaced by a fluid with uniformly distributed volume current. The current is undisturbed as the loop passes through.

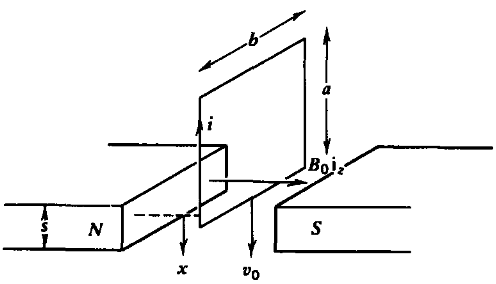



(a) (c)

Specifically consider the case when $d>b$ and then sketch the results when $d=b$ and $d<b$. The right edge of the current loop reaches the volume current at $t=0$.

3. A short circuited rectangular loop of mass $m$ and self-inductance $L$ is dropped with initial velocity $v_0\mathbf{i}_x$ between the pole faces of a magnet that has a concentrated uniform magnetic field $B_0\mathbf{i}_z$. Neglect gravity.




(a) What is the imposed flux through the loop as a function of the loop's position $x$ $(0<x<s)$ within the magnet?

(b) If the wire has conductivity $\sigma$ and cross-sectional area $A$, what equation relates the induced current $i$ in the loop and the loop's velocity?

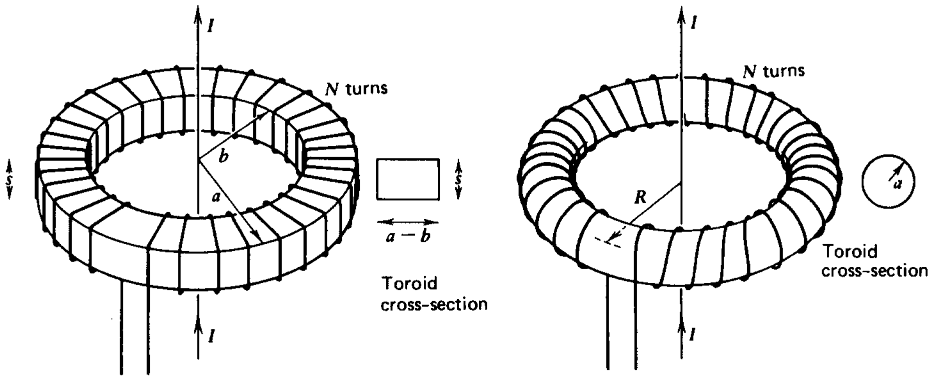
(c) What is the force on the loop in terms of $i$? Obtain a single equation for the loop's velocity. (**Hint:** Let $\omega_0^2 = B_0^2b^2/mL$, $\alpha = R/L$.)

(d) How does the loop's velocity and induced current vary with time?

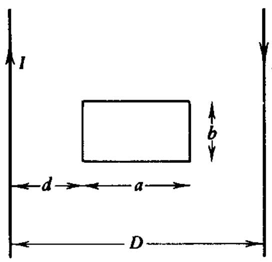
(e) If $\sigma \to \infty$, what minimum initial velocity is necessary for the loop to pass through the magnetic field?

4. Find the mutual inductance between the following currents:

(a) Toroidal coil of rectangular or circular cross section




(a)




(b)

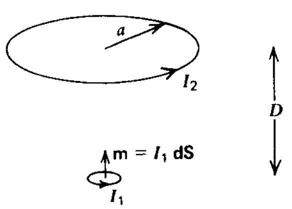
coaxially centered about an infinitely long line current. **Hint:**

$$\int \frac{dx}{a+b\cos x}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left\{\frac{\sqrt{a^2-b^2}\tan(x/2)}{a+b}\right\},$$

$$\int \frac{\mathrm{r}\,dr}{\sqrt{R^2-\mathrm{r}^2}}=-\sqrt{R^2-\mathrm{r}^2}$$

(b) A very long rectangular current loop, considered as two infinitely long parallel line currents, a distance $D$ apart, carrying the same current $I$ in opposite directions near a small rectangular loop of width $a$, which is a distance $d$ away from the left line current. Consider the cases $d+a<D$, $d<D<d+a$, and $d>D$.

5. A circular loop of radius $a$ is a distance $D$ above a point magnetic dipole of area $dS$ carrying a current $I_1$.



(a) What is the vector potential due to the dipole at all points on the circular loop? (**Hint:** See Section 5-5-1.)

(b) How much flux of the dipole passes through the circular loop?

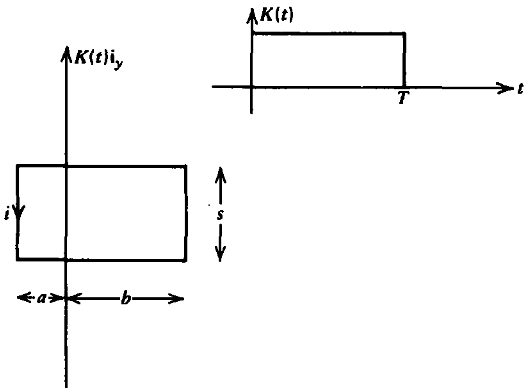
(c) What is the mutual inductance between the dipole and the loop?

(d) If the loop carries a current $I_2$, what is the magnetic field due to $I_2$ at the position of the point dipole? (**Hint:** See Section 5-2-4*a*.)

(e) How much flux due to $I_2$ passes through the magnetic dipole?

(f) What is the mutual inductance? Does your result agree with (c)?

6. A small rectangular loop with self-inductance $L$, Ohmic conductivity $\sigma$, and cross-sectional area $A$ straddles a current sheet.



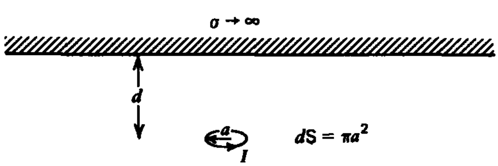

(a) The current sheet is instantaneously turned on to a dc level $K_0\mathbf{i}_y$ at $t=0$. What is the induced loop current?

(b) After a long time $T$ the sheet current is instantaneously set to zero. What is the induced loop current?

(c) What is the induced loop current if the current sheet varies sinusoidally with time as $K_0 \cos \omega t\,\mathbf{i}_y$.

7. A point magnetic dipole with area $dS$ lies a distance $d$ below a perfectly conducting plane of infinite extent. The dipole current $I$ is instantaneously turned on at $t=0$.

(a) Using the method of images, find the magnetic field everywhere along the conducting plane. (**Hint:** $\mathbf{i}_r \cdot \mathbf{i}_r = \sin\theta$,



$\mathbf{i}_\theta \cdot \mathbf{i}_r = \cos\theta$.)

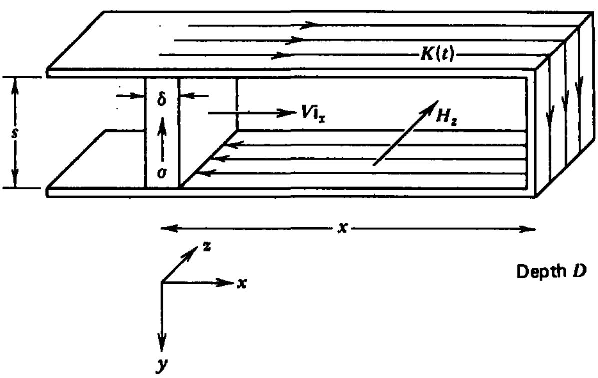
(b) What is the surface current distribution?

(c) What is the force on the plane? **Hint:**

$$\int \frac{r^3\,dr}{(r^2+d^2)^5} = -\frac{(r^2+d^2/4)}{6(r^2+d^2)^4}$$

(d) If the plane has a mass $M$ in the gravity field $g$, what current $I$ is necessary to just lift the conductor? Evaluate for $M = 10^{-3}$ kg, $d = 10^{-2}$ m, and $a = 10^{-3}$ m.

8. A thin block with Ohmic conductivity $\sigma$ and thickness $\delta$ moves with constant velocity $V\mathbf{i}_x$ between short circuited perfectly conducting parallel plates. An initial surface current $K_0$ is imposed at $t = 0$ when $x = x_0$, but the source is then removed.




(a) The surface current on the plates $K(t)$ will vary with time. What is the magnetic field in terms of $K(t)$? Neglect fringing effects.

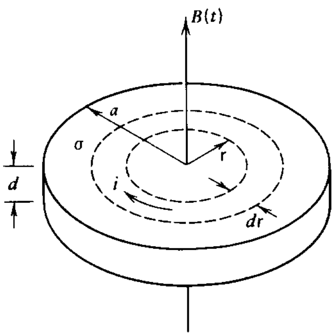
(b) Because the moving block is so thin, the current is uniformly distributed over the thickness $\delta$. Using Faraday's law, find $K(t)$ as a function of time.

(c) What value of velocity will just keep the magnetic field constant with time until the moving block reaches the end?

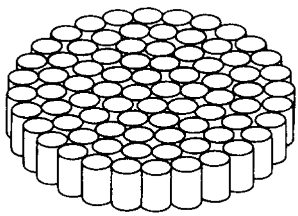
(d) What happens to the magnetic field for larger and smaller velocities?

9. A thin circular disk of radius $a$, thickness $d$, and conductivity $\sigma$ is placed in a uniform time varying magnetic field $B(t)$.

(a) Neglecting the magnetic field of the eddy currents, what is the current induced in a thin circular filament at radius r of thickness $dr$.




(a)



(d)

(b) What power is dissipated in this incremental current loop?

(c) How much power is dissipated in the whole disk?

(d) If the disk is instead cut up into $N$ smaller circular disks with negligible wastage, what is the approximate radius of each smaller disk?

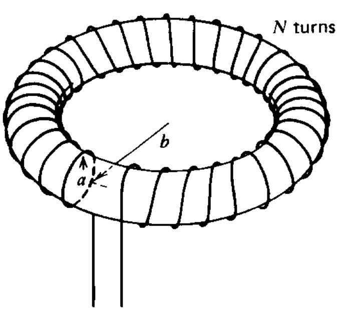
(e) If these $N$ smaller disks are laminated together to form a thin disk of closely packed cylindrical wires, what is the power dissipated?

Section 6-2

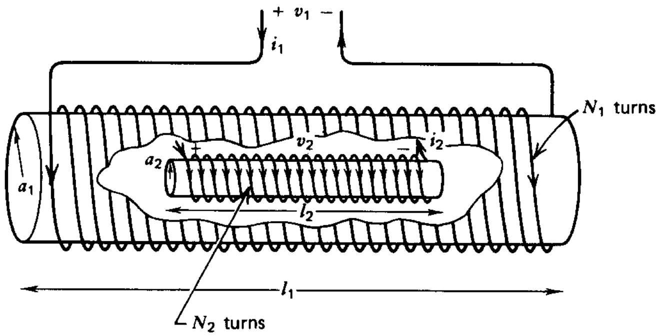
10. Find the self-inductance of an $N$ turn toroidal coil of circular cross-sectional radius $a$ and mean radius $b$. **Hint:**

$$\int \frac{d\theta}{b+\mathrm{r}\cos\theta}=\frac{2}{\sqrt{b^2-\mathrm{r}^2}}\tan^{-1}\frac{\sqrt{b^2-\mathrm{r}^2}\tan(\theta/2)}{b+\mathrm{r}}$$

$$\int \frac{\mathrm{r}\,d\mathrm{r}}{\sqrt{b^2-\mathrm{r}^2}}=-\sqrt{b^2-\mathrm{r}^2}$$




11. A large solenoidal coil of long length $l_1$, radius $a_1$, and number of turns $N_1$ coaxially surrounds a smaller coil of long length $l_2$, radius $a_2$, and turns $N_2$.




(a) Neglecting fringing field effects find the self-inductances and mutual inductances of each coil. (**Hint:** Assume the magnetic field is essentially uniform within the cylinders.)

(b) What is the voltage across each coil in terms of $i_1$ and $i_2$?

(c) If the coils are connected in series so that $i_1 = i_2$ with the fluxes of each coil in the same direction, what is the total self-inductance?

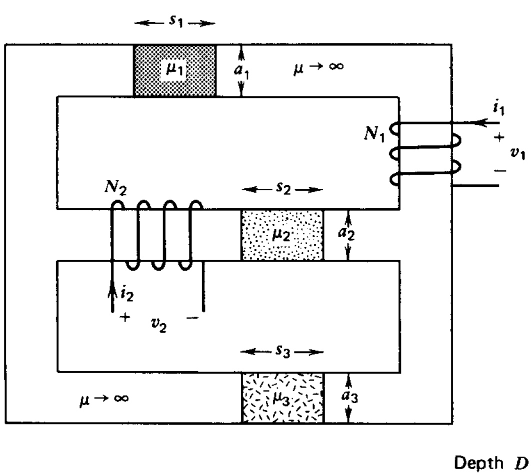
(d) Repeat (c) if the series connection is reversed so that $i_1 = -i_2$ and the fluxes due to each coil are in opposite directions.

(e) What is the total self-inductance if the coils are connected in parallel so that $v_1 = v_2$ or $v_1 = -v_2$?

12. The iron core shown with infinite permeability has three gaps filled with different permeable materials.

(a) What is the equivalent magnetic circuit?

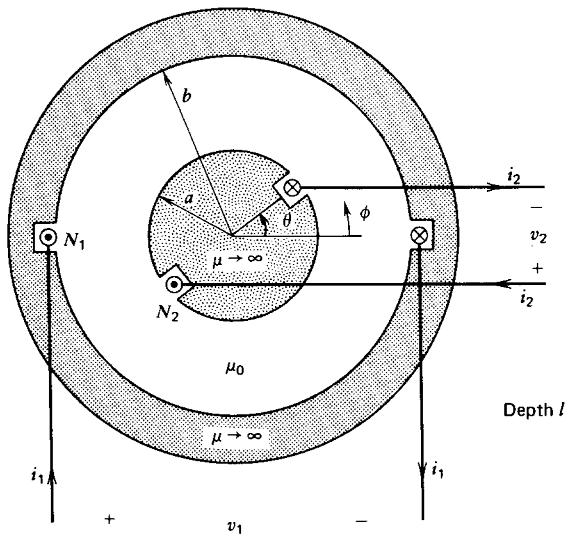
(b) Find the magnetic flux everywhere in terms of the gap reluctances.



(c) What is the total magnetic flux through each winding?

(d) What is the self-inductance and mutual inductance of each winding?

13. A cylindrical shell of infinite permeability, length $l$ and inner radius $b$ coaxially surrounds a solid cylinder also with infinite permeability and length $l$ but with smaller radius $a$ so that there is a small gap $g = b - a$. An $N_1$ turn coil carrying a current $I_1$ is placed within two slots on the inner surface of the outer cylinder.




(a) What is the magnetic field everywhere? Neglect all radial variations in the narrow air gap. (**Hint:** Separately consider $0 < \phi < \pi$ and $\pi < \phi < 2\pi$.)

(b) What is the self-inductance of the coil?

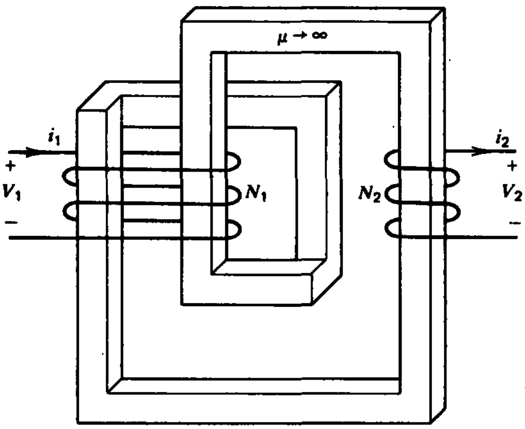
(c) A second coil with $N_2$ turns carrying a current $I_2$ is placed in slots on the inner cylinder that is free to rotate. When the rotor is at angle $\theta$, what is the total magnetic field due to currents $I_1$ and $I_2$? (**Hint:** Separately consider $0 < \phi < \theta$, $\theta < \phi < \pi$, $\pi < \phi < \pi + \theta$, and $\pi + \theta < \phi < 2\pi$.)

(d) What is the self-inductance and mutual inductance of coil 2 as a function of $\theta$?

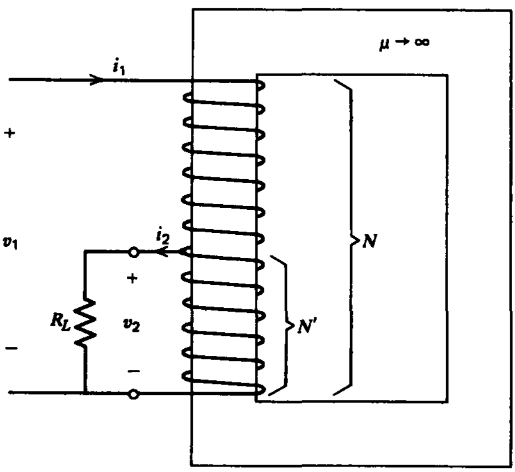
(e) What is the torque on the rotor coil?

14. (a) What is the ratio of terminal voltages and currents for the odd twisted ideal transformer shown?

(b) A resistor $R_L$ is placed across the secondary winding ($v_2$, $i_2$). What is the impedance as seen by the primary winding?




15. An $N$ turn coil is wound onto an infinitely permeable magnetic core. An autotransformer is formed by connecting a tap of $N'$ turns.

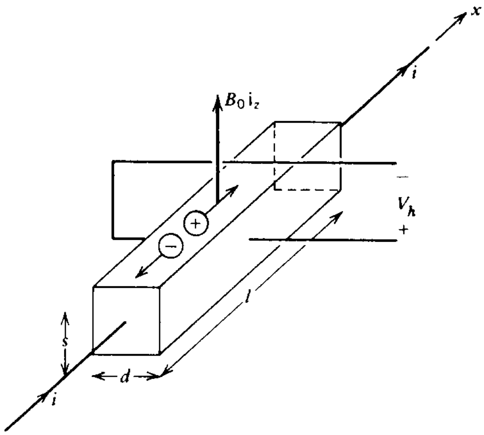



(a) What are the terminal voltage ($v_2/v_1$) and current ($i_2/i_1$) ratios?

(b) A load resistor $R_L$ is connected across the terminals of the tap. What is the impedance as seen by the input terminals?

Section 6-3

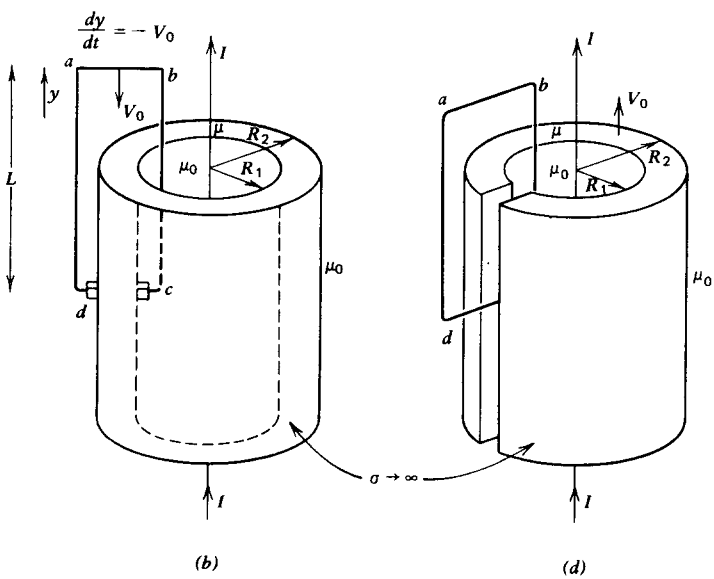
16. A conducting material with current density $J_x\mathbf{i}_x$ has two species of charge carriers with respective mobilities $\mu_+$ and $\mu_-$ and number densities $n_+$ and $n_-$. A magnetic field $B_0\mathbf{i}_z$ is imposed perpendicular to the current flow.




(a) What is the open circuit Hall voltage? (**Hint:** The transverse current of each carrier must be zero.)

(b) What is the short circuit Hall current?

17. A highly conducting hollow iron cylinder with permeability $\mu$ and inner and outer radii $R_1$ and $R_2$ is concentric to an infinitely long dc line current (adapted from L. V. Bewley, *Flux Linkages and Electromagnetic Induction.* Macmillan, New York, 1952, pp. 71–77).



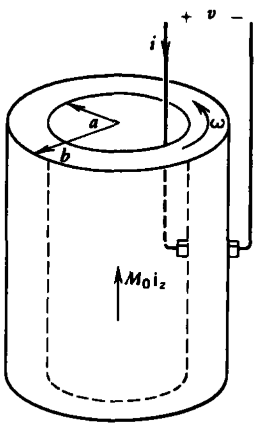

(a) What is the magnetic flux density everywhere? Find the electromotive force (EMF) of the loop for each of the following cases.

(b) A highly conducting circuit *abcd* is moving downward with constant velocity $V_0$ making contact with the surfaces of the cylinders via sliding brushes. The circuit is completed from *c* to *d* via the iron cylinder.

(c) Now the circuit remains stationary and the iron cylinder moves upwards at velocity $V_0$.

(d) Now a thin axial slot is cut in the cylinder so that it can slip by the complete circuit *abcd*, which remains stationary as the cylinder moves upwards at speed $V_0$. The brushes are removed and a highly conducting wire completes the *c–d* path.

18. A very long permanently magnetized cylinder $M_0\mathbf{i}_z$ rotates on a shaft at constant angular speed $\omega$. The inner and outer surfaces at r = *a* and r = *b* are perfectly conducting, so that brushes can make electrical contact.




(a) If the cylinder is assumed very long compared to its radius, what are the approximate values of **B** and **H** in the magnet?

(b) What is the open circuit voltage?

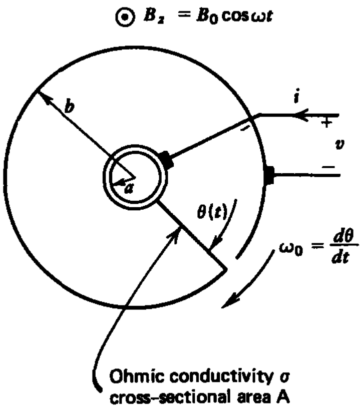
(c) If the magnet has an Ohmic conductivity $\sigma$, what is the equivalent circuit of this generator?

(d) What torque is required to turn the magnet if the terminals are short circuited?

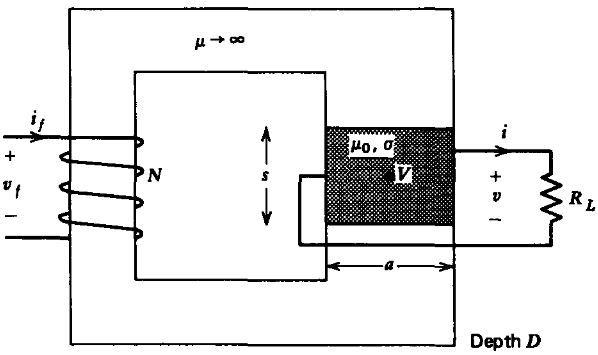
19. A single spoke wheel has a perfectly conducting cut rim. The spoke has Ohmic conductivity $\sigma$ and cross-sectional area $A$. The wheel rotates at constant angular speed $\omega_0$ in a sinusoidally varying magnetic field $B_z = B_0 \cos \omega t$.

(a) What is the open circuit voltage and short circuit current?

(b) What is the equivalent circuit?




20. An MHD machine is placed within a magnetic circuit.




(a) A constant dc current $i_f = I_0$ is applied to the $N$ turn coil. How much power is delivered to the load resistor $R_L$?

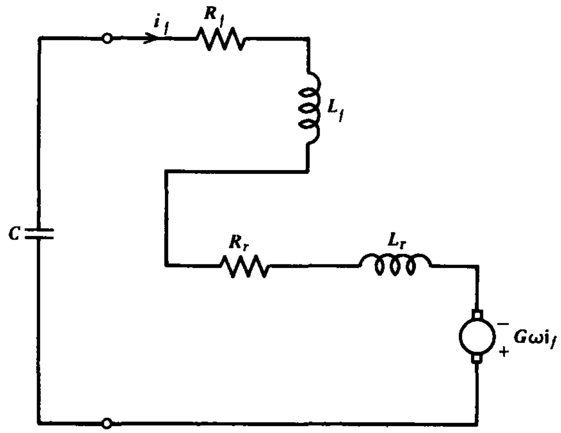
(b) The MHD machine and load resistor $R_L$ are now connected in series with the $N$ turn coil that has a resistance $R_f$. No current is applied. For what minimum flow speed can the MHD machine be self-excited?

21. The field winding of a homopolar generator is connected in series with the rotor terminals through a capacitor $C$. The rotor is turned at constant speed $\omega$.

(a) For what minimum value of rotor speed is the system self-excited?

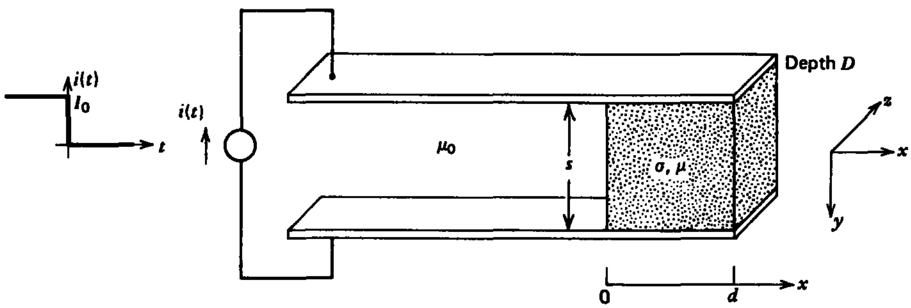
(b) For the self-excited condition of (a) what range of values of $C$ will result in dc self-excitation or in ac self-excitation?

(c) What is the frequency for ac self-excitation?




Section 6-4

22. An Ohmic block separates two perfectly conducting parallel plates. A dc current that has been applied for a long time is instantaneously turned off at $t=0$.




(a) What are the initial and final magnetic field distributions? What are the boundary conditions?

(b) What are the transient magnetic field and current distributions?

(c) What is the force on the block as a function of time?

23. A block of Ohmic material is placed within a magnetic circuit. A step current $I_0$ is applied at $t=0$.

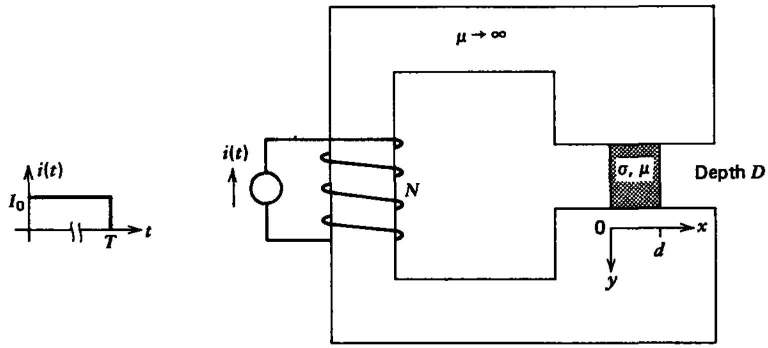
(a) What is the dc steady-state solution for the magnetic field distribution?

(b) What are the boundary and initial conditions for the magnetic field in the conducting block?

(c) What are the transient field and current distributions?

(d) What is the time dependence of the force on the conductor?

(e) The current has been on a long time so that the system is in the dc steady state found in (a) when at $t=T$ the current

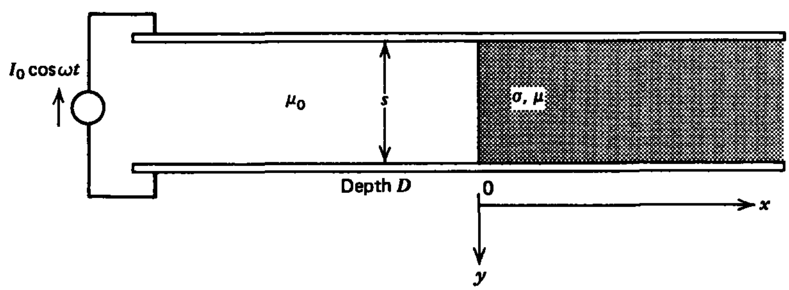



is instantaneously turned off. What are the transient field and current distributions in the conductor?

(f) If the applied coil current varies sinusoidally with time as $i(t) = I_0 \cos \omega t$, what are the sinusoidal steady-state field and current distributions? (**Hint:** Leave your answer in terms of complex amplitudes.)

(g) What is the force on the block?

24. A semi-infinite conducting block is placed between parallel perfect conductors. A sinusoidal current source is applied.




(a) What are the magnetic field and current distributions within the conducting block?

(b) What is the total force on the block?

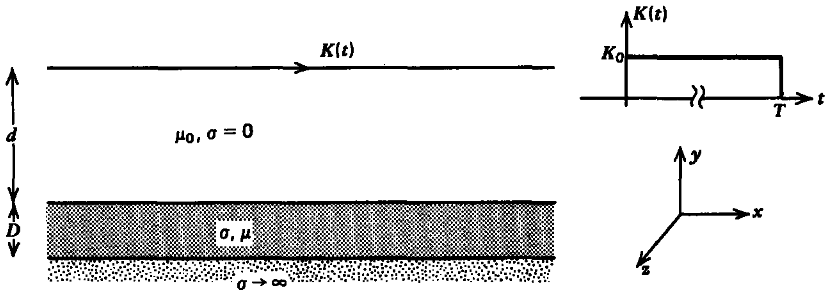
(c) Repeat (a) and (b) if the block has length $d$.

25. A current sheet that is turned on at $t = 0$ lies a distance $d$ above a conductor of thickness $D$ and conductivity $\sigma$. The conductor lies on top of a perfectly conducting plane.

(a) What are the initial and steady-state solutions? What are the boundary conditions?

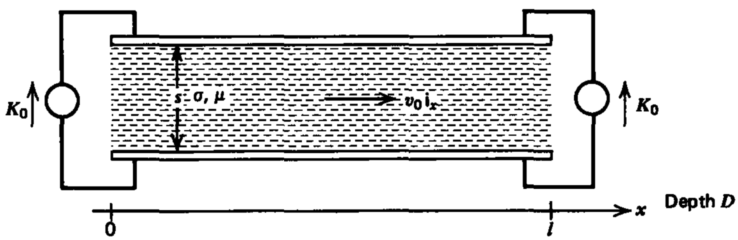
(b) What are the transient magnetic field and current distributions?

(c) After a long time $T$, so that the system has reached the dc steady state, the surface current is set to zero. What are the subsequent field and current distributions?



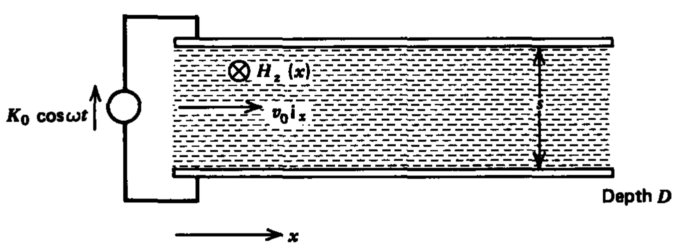

(d) What are the field and current distributions if the current sheet varies as $K_0 \cos \omega t$?

26. Distributed dc currents at $x = 0$ and $x = l$ flow through a conducting fluid moving with constant velocity $v_0\mathbf{i}_x$.




(a) What are the magnetic field and current distributions?
(b) What is the force on the fluid?

27. A sinusoidal surface current at $x = 0$ flows along parallel electrodes and returns through a conducting fluid moving to the right with constant velocity $v_0\mathbf{i}_x$. The flow is not impeded by the current source. The system extends to $x = \infty$.



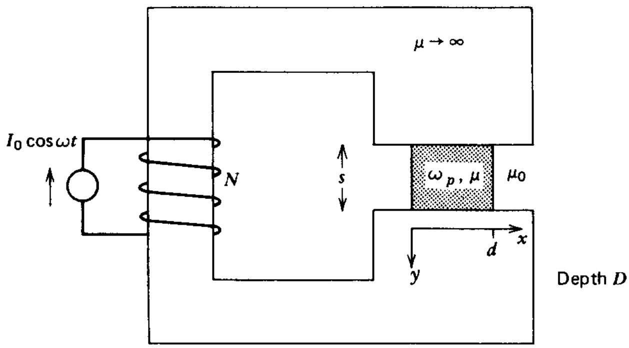

(a) What are the magnetic field and current density distributions?
(b) What is the time-average force on the fluid?

28. The surface current for the linear induction machine treated in Section 6-4-6 is now put a distance $s$ below the conductor.

(a) What are the magnetic field and current distributions in each region of space? (**Hint:** Check your answer with Section 6-4-6 when $s = 0$.)

(b) Repeat (a) if $s$ is set to zero but the conductor has a finite thickness $d$.

29. A superconducting block with plasma frequency $\omega_p$ is placed within a magnetic circuit with exciting current $I_0 \cos \omega t$.

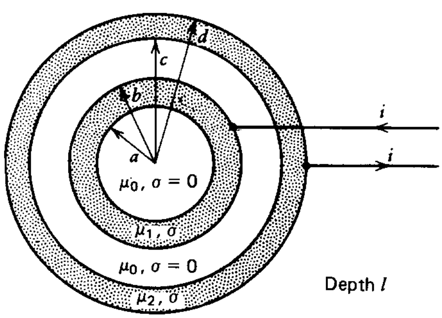



(a) What are the magnetic field and current distributions in the superconductor?

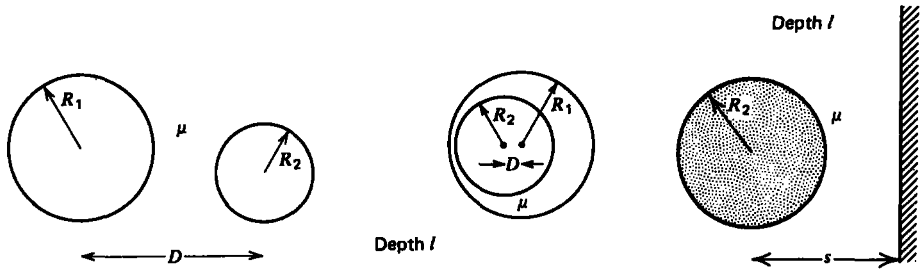
(b) What is the force on the block?

Section 6.5

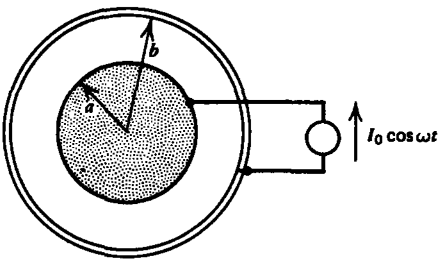
30. Find the magnetic energy stored and the self-inductance for the geometry below where the current in each shell is uniformly distributed.




31. Find the external self-inductance of the two wire lines shown. (**Hint:** See Section 2-6-4$c$.)




32. A coaxial cable with solid inner conductor is excited by a sinusoidally varying current $I_0 \cos \omega t$ at high enough frequency so that the skin depth is small compared to the radius $a$. The current is now nonuniformly distributed over the inner conductor.




(a) Assuming that $\mathbf{H} = H_\phi(\mathrm{r})\mathbf{i}_\phi$, what is the governing equation for $H_\phi(\mathrm{r})$ within the inner cylinder. (**Hint:** $\nabla^2\mathbf{H} = \nabla(\cancelto{0}{\nabla\cdot\mathbf{H}}) - \nabla\times(\nabla\times\mathbf{H})$.)

(b) Solve (a) for solutions of the form

$$H_\phi(\mathrm{r}) = \mathrm{Re}\,[\hat{H}_\phi(\mathrm{r})\, e^{j\omega t}]$$

**Hint:** Bessel's equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$$

with solutions

$$y = A_1 J_p(x) + A_2 Y_p(x)$$

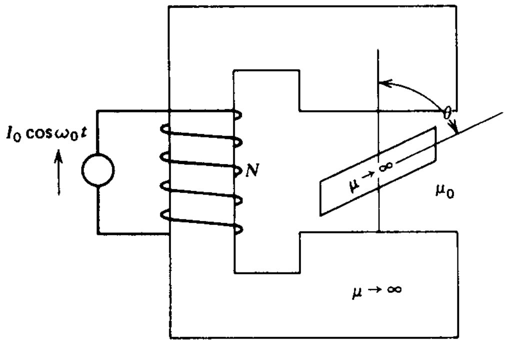
where $Y_p$ is singular at $x = 0$.

(c) What is the current distribution? **Hint:**

$$\frac{d}{dx}[J_1(x)] + \frac{1}{x}J_1(x) = J_0(x)$$

Section 6-6

33. A reluctance motor is made by placing a high permeability material, which is free to rotate, in the air gap of a magnetic circuit excited by a sinusoidal current $I_0 \cos \omega_0 t$.




The inductance of the circuit varies as

$$L(\theta) = L_0 + L_1 \cos 2\theta$$

where the maximum inductance $L_0 + L_1$ occurs when $\theta = 0$ or $\theta = \pi$ and the minimum inductance $L_0 - L_1$ occurs when $\theta = \pm\pi/2$.

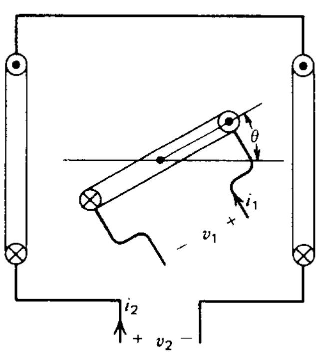
(a) What is the torque on the slab as a function of the angle $\theta$?

(b) The rotor is rotating at constant speed $\omega$, where $\theta = \omega t + \delta$ and $\delta$ is the angle of the rotor at $t = 0$. At what value of $\omega$ does the torque have a nonzero time average. The reluctance motor is then a synchronous machine. **Hint:**

$$\cos^2 \omega_0 t \sin 2\theta = \tfrac{1}{2}[\sin 2\theta + \cos 2\omega_0 t \sin 2\theta]$$
$$= \tfrac{1}{2}\{\sin 2\theta + \tfrac{1}{2}[\sin 2(\omega_0 t + \theta) + \sin 2(\theta - \omega_0 t)]\}$$

(c) What is the maximum torque that can be delivered by the shaft and at what angle $\delta$ does it occur?

34. A system of two coupled coils have the following flux-current relations:




(c)

$$\Phi_1 = L_1(\theta)i_1 + M(\theta)i_2$$

$$\Phi_2 = M(\theta)i_1 + L_2(\theta)i_2$$

(a) What is the power $p$ delivered to the coils?
(b) Write this power in the form

$$p = \frac{dW}{dt} + T\frac{d\theta}{dt}$$

What are $W$ and $T$?

(c) A small coil is free to rotate in the uniform magnetic field produced by another coil. The flux-current relation is

$$\Phi_1 = L_1 i_1 + M_0 i_2 \sin\theta$$

$$\Phi_2 = M_0 i_1 \sin\theta + L_2 i_2$$

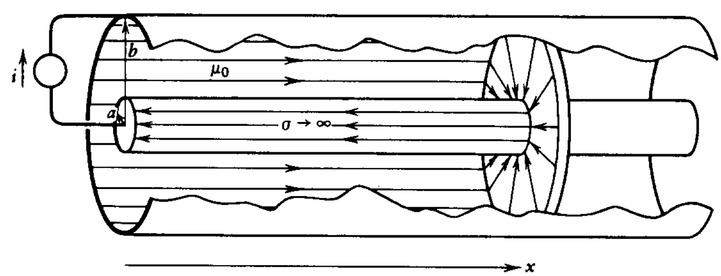
The coils are excited by dc currents $I_1$ and $I_2$. What is the torque on the small coil?

(d) If the small coil has conductivity $\sigma$, cross-sectional area $A$, total length $l$, and is short circuited, what differential equation must the current $i_1$ obey if $\theta$ is a function of time? A dc current $I_2$ is imposed in coil 2.

(e) The small coil has moment of inertia $J$. Consider only small motions around $\theta = 0$ so that $\cos\theta \approx 1$. With the torque and current equations linearized, try exponential solutions of the form $e^{st}$ and solve for the natural frequencies.

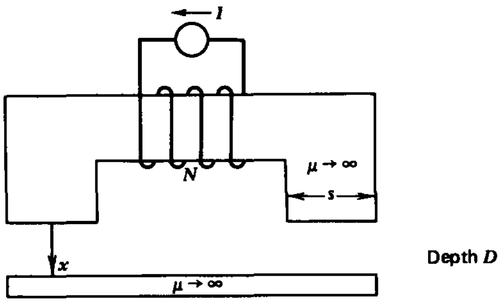
(f) The coil is released from rest at $\theta = \theta_0$. What is $\theta(t)$ and $i_1(t)$? Under what conditions are the solutions oscillatory? Damped?

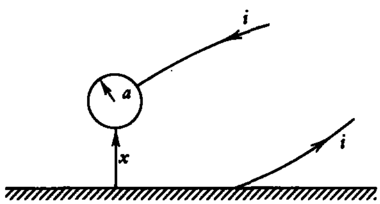
35. A coaxial cable has its short circuited end free to move.



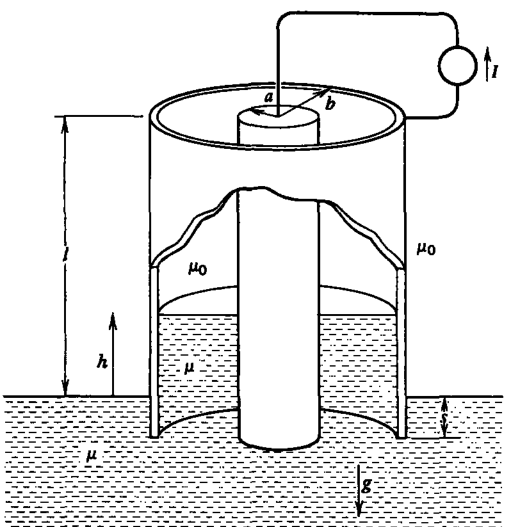

(a) What is the inductance of the cable as a function of $x$?
(b) What is the force on the end?

36. For the following geometries, find:
(a) The inductance;
(b) The force on the moveable member.




37. A coaxial cylinder is dipped into a magnetizable fluid with permeability $\mu$ and mass density $\rho_m$. How high $h$ does the fluid rise within the cylinder?



# chapter 7

*electrodynamics—*
*fields and waves*

The electromagnetic field laws, derived thus far from the empirically determined Coulomb–Lorentz forces, are correct on the time scales of our own physical experiences. However, just as Newton's force law must be corrected for material speeds approaching that of light, the field laws must be corrected when fast time variations are on the order of the time it takes light to travel over the length of a system. Unlike the abstractness of relativistic mechanics, the complete electrodynamic equations describe a familiar phenomenon—propagation of electromagnetic waves. Throughout the rest of this text, we will examine when appropriate the low-frequency limits to justify the past quasi-static assumptions.

## 7-1 MAXWELL'S EQUATIONS

### 7-1-1 Displacement Current Correction to Ampere's Law

In the historical development of electromagnetic field theory through the nineteenth century, charge and its electric field were studied separately from currents and their magnetic fields. Until Faraday showed that a time varying magnetic field generates an electric field, it was thought that the electric and magnetic fields were distinct and uncoupled. Faraday believed in the duality that a time varying electric field should also generate a magnetic field, but he was not able to prove this supposition.

It remained for James Clerk Maxwell to show that Faraday's hypothesis was correct and that without this correction Ampere's law and conservation of charge were inconsistent:

$$\nabla \times \mathbf{H} = \mathbf{J}_f \Rightarrow \nabla \cdot \mathbf{J}_f = 0 \tag{1}$$

$$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0 \tag{2}$$

for if we take the divergence of Ampere's law in (1), the current density must have zero divergence because the divergence of the curl of a vector is always zero. This result contradicts (2) if a time varying charge is present. Maxwell realized that adding the displacement current on the right-hand side of Ampere's law would satisfy charge conservation, because of Gauss's law relating $\mathbf{D}$ to $\rho_f$ ($\nabla \cdot \mathbf{D} = \rho_f$).

This simple correction has far-reaching consequences, because we will be able to show the existence of electromagnetic waves that travel at the speed of light $c$, thus proving that light is an electromagnetic wave. Because of the significance of Maxwell's correction, the complete set of coupled electromagnetic field laws are called Maxwell's equations:

Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \oint_L \mathbf{E} \cdot \mathbf{dl} = -\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{dS} \tag{3}$$

Ampere's law with Maxwell's displacement current correction

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \Rightarrow \oint_L \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J}_f \cdot \mathbf{dS} + \frac{d}{dt}\int_S \mathbf{D} \cdot \mathbf{dS} \tag{4}$$

Gauss's laws

$$\nabla \cdot \mathbf{D} = \rho_f \Rightarrow \oint_S \mathbf{D} \cdot \mathbf{dS} = \int_V \rho_f \, dV \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \oint_S \mathbf{B} \cdot \mathbf{dS} = 0 \tag{6}$$

Conservation of charge

$$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0 \Rightarrow \oint_S \mathbf{J}_f \cdot \mathbf{dS} + \frac{d}{dt}\int_V \rho_f \, dV = 0 \tag{7}$$

As we have justified, (7) is derived from the divergence of (4) using (5).

Note that (6) is not independent of (3) for if we take the divergence of Faraday's law, $\nabla \cdot \mathbf{B}$ could at most be a time-independent function. Since we assume that at some point in time $\mathbf{B} = 0$, this function must be zero.

The symmetry in Maxwell's equations would be complete if a magnetic charge density appeared on the right-hand side of Gauss's law in (6) with an associated magnetic current due to the flow of magnetic charge appearing on the right-hand side of (3). Thus far, no one has found a magnetic charge or current, although many people are actively looking. Throughout this text we accept (3)–(7) keeping in mind that if magnetic charge is discovered, we must modify (3) and (6) and add an equation like (7) for conservation of magnetic charge.

### 7-1-2 Circuit Theory as a Quasi-static Approximation

Circuit theory assumes that the electric and magnetic fields are highly localized within the circuit elements. Although the displacement current is dominant within a capacitor, it is negligible outside so that Ampere's law can neglect time variations of **D** making the current divergence-free. Then we obtain Kirchoff's current law that the algebraic sum of all currents flowing into (or out of) a node is zero:

$$\nabla \cdot \mathbf{J}=0 \Rightarrow \oint_S \mathbf{J} \cdot d\mathbf{S}=0 \Rightarrow \Sigma i_k=0 \tag{8}$$

Similarly, time varying magnetic flux that is dominant within inductors and transformers is assumed negligible outside so that the electric field is curl free. We then have Kirchoff's voltage law that the algebraic sum of voltage drops (or rises) around any closed loop in a circuit is zero:

$$\nabla \times \mathbf{E}=0 \Rightarrow \mathbf{E}=-\nabla V \Rightarrow \oint_L \mathbf{E} \cdot d\mathbf{l}=0 \Rightarrow \Sigma v_k=0 \tag{9}$$

## 7-2 CONSERVATION OF ENERGY

### 7-2-1 Poynting's Theorem

We expand the vector quantity

$$\begin{aligned}\nabla \cdot (\mathbf{E}\times\mathbf{H})&=\mathbf{H}\cdot(\nabla\times\mathbf{E})-\mathbf{E}\cdot(\nabla\times\mathbf{H})\\ &=-\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}-\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}-\mathbf{E}\cdot\mathbf{J}_f\end{aligned} \tag{1}$$

where we change the curl terms using Faraday's and Ampere's laws.

For linear homogeneous media, including free space, the constitutive laws are

$$\mathbf{D}=\varepsilon\mathbf{E}, \qquad \mathbf{B}=\mu\mathbf{H} \tag{2}$$

so that (1) can be rewritten as

$$\nabla \cdot (\mathbf{E}\times\mathbf{H})+\frac{\partial}{\partial t}\left(\tfrac{1}{2}\varepsilon E^2+\tfrac{1}{2}\mu H^2\right)=-\mathbf{E}\cdot\mathbf{J}_f \tag{3}$$

which is known as Poynting's theorem. We integrate (3) over a closed volume, using the divergence theorem to convert the first term to a surface integral:

$$\underbrace{\oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S}}_{\int_V \nabla\cdot(\mathbf{E}\times\mathbf{H})\,dV}+\frac{d}{dt}\int_V (\tfrac{1}{2}\varepsilon E^2+\tfrac{1}{2}\mu H^2)\,dV=-\int_V \mathbf{E}\cdot\mathbf{J}_f\,dV \quad (4)$$

We recognize the time derivative in (4) as operating on the electric and magnetic energy densities, which suggests the interpretation of (4) as

$$P_{\text{out}}+\frac{dW}{dt}=-P_d \quad (5)$$

where $P_{\text{out}}$ is the total electromagnetic power flowing out of the volume with density

$$\boldsymbol{S}=\mathbf{E}\times\mathbf{H}\ \text{watts/m}^2\ [\text{kg-s}^{-3}] \quad (6)$$

where $\boldsymbol{S}$ is called the Poynting vector, $W$ is the electromagnetic stored energy, and $P_d$ is the power dissipated or generated:

$$P_{\text{out}}=\oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{S}=\oint_S \boldsymbol{S}\cdot d\mathbf{S}$$

$$W=\int_V [\tfrac{1}{2}\varepsilon E^2+\tfrac{1}{2}\mu H^2]\,dV \quad (7)$$

$$P_d=\oint_V \mathbf{E}\cdot\mathbf{J}_f\,dV$$

If $\mathbf{E}$ and $\mathbf{J}_f$ are in the same direction as in an Ohmic conductor ($\mathbf{E}\cdot\mathbf{J}_f=\sigma E^2$), then $P_d$ is positive, representing power dissipation since the right-hand side of (5) is negative. A source that supplies power to the volume has $\mathbf{E}$ and $\mathbf{J}_f$ in opposite directions so that $P_d$ is negative.

### 7-2-2 A Lossy Capacitor

Poynting's theorem offers a different and to some a paradoxical explanation of power flow to circuit elements. Consider the cylindrical lossy capacitor excited by a time varying voltage source in Figure 7-1. The terminal current has both Ohmic and displacement current contributions:

$$i=\frac{\varepsilon A}{l}\frac{dv}{dt}+\frac{\sigma A v}{l}=C\frac{dv}{dt}+\frac{v}{R},\qquad C=\frac{\varepsilon A}{l},\qquad R=\frac{l}{\sigma A} \quad (8)$$

From a circuit theory point of view we would say that the power flows from the terminal wires, being dissipated in the

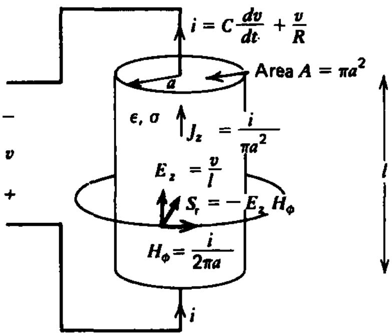



Figure 7-1 The power delivered to a lossy cylindrical capacitor $vi$ is partly dissipated by the Ohmic conduction and partly stored in the electric field. This power can also be thought to flow-in radially from the surrounding electric and magnetic fields via the Poynting vector $S = \mathbf{E} \times \mathbf{H}$.

resistance and stored as electrical energy in the capacitor:

$$P = vi = \frac{v^2}{R} + \frac{d}{dt}\left(\tfrac{1}{2}Cv^2\right) \tag{9}$$

We obtain the same results from a field's viewpoint using Poynting's theorem. Neglecting fringing, the electric field is simply

$$E_z = v/l \tag{10}$$

while the magnetic field at the outside surface of the resistor is generated by the conduction and displacement currents:

$$\oint_L \mathbf{H} \cdot \mathbf{dl} = \int_S \left(J_z + \varepsilon \frac{\partial E_z}{\partial t}\right) dS \Rightarrow H_\phi 2\pi a = \frac{\sigma A v}{l} + \frac{\varepsilon}{l} A \frac{dv}{dt} = i \tag{11}$$

where we recognize the right-hand side as the terminal current in (8),

$$H_\phi = i/(2\pi a) \tag{12}$$

The power flow through the surface at $r = a$ surrounding the resistor is then radially inward,

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot \mathbf{dS} = -\int_S \frac{v}{l} \frac{i}{2\pi a} a \, d\phi \, dz = -vi \tag{13}$$

and equals the familiar circuit power formula. The minus sign arises because the left-hand side of (13) is the power out of the volume as the surface area element **dS** points radially outwards. From the field point of view, power flows into the lossy capacitor from the electric and magnetic fields outside the resistor via the Poynting vector. Whether the power is thought to flow along the terminal wires or from the surrounding fields is a matter of convenience as the results are identical. The presence of the electric and magnetic fields are directly due to the voltage and current. It is impossible to have the fields without the related circuit variables.

### 7-2-3 Power in Electric Circuits

We saw in (13) that the flux of $\mathbf{S}$ entering the surface surrounding a circuit element just equals $vi$. We can show this for the general network with $N$ terminals in Figure 7-2 using the quasi-static field laws that describe networks outside the circuit elements:

$$\begin{aligned} \nabla\times\mathbf{E}&=0\Rightarrow\mathbf{E}=-\nabla V\\ \nabla\times\mathbf{H}&=\mathbf{J}_f\Rightarrow\nabla\cdot\mathbf{J}_f=0 \end{aligned} \tag{14}$$

We then can rewrite the electromagnetic power into a surface as

$$\begin{aligned} P_{\text{in}}&=-\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{S}\\ &=-\int_V \nabla\cdot(\mathbf{E}\times\mathbf{H})\,dV\\ &=\int_V \nabla\cdot(\nabla V\times\mathbf{H})\,dV \end{aligned} \tag{15}$$

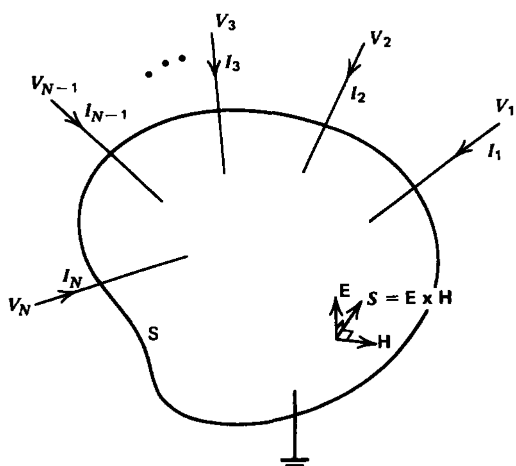



Figure 7-2 The circuit power into an $N$ terminal network $\sum_{k=1}^{N} V_kI_k$ equals the electromagnetic power flow into the surface surrounding the network, $-\oint_S \mathbf{E}\times\mathbf{H}\cdot d\mathbf{S}$.

where the minus is introduced because we want the power in and we use the divergence theorem to convert the surface integral to a volume integral. We expand the divergence term as

$$\begin{aligned}\nabla \cdot (\nabla V \times \mathbf{H}) &= \mathbf{H} \cdot (\cancelto{0}{\nabla \times \nabla V}) - \nabla V \cdot (\nabla \times \mathbf{H}) \\ &= -\mathbf{J}_f \cdot \nabla V = -\nabla \cdot (\mathbf{J}_f V)\end{aligned} \tag{16}$$

where we use (14).

Substituting (16) into (15) yields

$$\begin{aligned}P_{\text{in}} &= -\int_V \nabla \cdot (\mathbf{J}_f V)\, dV \\ &= -\oint_S \mathbf{J}_f V \cdot \mathbf{dS}\end{aligned} \tag{17}$$

where we again use the divergence theorem. On the surface S, the potential just equals the voltages on each terminal wire allowing $V$ to be brought outside the surface integral:

$$\begin{aligned}P_{\text{in}} &= \sum_{k=1}^{N} -V_k \oint_S \mathbf{J}_f \cdot \mathbf{dS} \\ &= \sum_{k=1}^{N} \mathrm{V}_k I_k\end{aligned} \tag{18}$$

where we recognize the remaining surface integral as just being the negative (remember **dS** points outward) of each terminal current flowing into the volume. This formula is usually given as a postulate along with Kirchoff's laws in most circuit theory courses. Their correctness follows from the quasi-static field laws that are only an approximation to more general phenomena which we continue to explore.

### 7-2-4 The Complex Poynting's Theorem

For many situations the electric and magnetic fields vary sinusoidally with time:

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= \mathrm{Re}\,[\hat{\mathbf{E}}(\mathbf{r})\, e^{j\omega t}] \\ \mathbf{H}(\mathbf{r}, t) &= \mathrm{Re}\,[\hat{\mathbf{H}}(\mathbf{r})\, e^{j\omega t}]\end{aligned} \tag{19}$$

where the caret is used to indicate a complex amplitude that can vary with position **r**. The instantaneous power density is obtained by taking the cross product of **E** and **H**. However, it is often useful to calculate the time-average power density $\langle S \rangle$, where we can avoid the lengthy algebraic and trigonometric manipulations in expanding the real parts in (19).

A simple rule for the time average of products is obtained by realizing that the real part of a complex number is equal to one half the sum of the complex number and its conjugate (denoted by a superscript asterisk). The power density is then

$$\begin{aligned}\boldsymbol{S}(\mathbf{r},t)&=\mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)\\&=\tfrac{1}{4}[\hat{\mathbf{E}}(\mathbf{r})\,e^{j\omega t}+\hat{\mathbf{E}}^*(\mathbf{r})\,e^{-j\omega t}]\times[\hat{\mathbf{H}}(\mathbf{r})\,e^{j\omega t}+\hat{\mathbf{H}}^*(\mathbf{r})\,e^{-j\omega t}]\\&=\tfrac{1}{4}[\hat{\mathbf{E}}(\mathbf{r})\times\hat{\mathbf{H}}(\mathbf{r})\,e^{2j\omega t}+\hat{\mathbf{E}}^*(\mathbf{r})\times\hat{\mathbf{H}}(\mathbf{r})+\hat{\mathbf{E}}(\mathbf{r})\times\hat{\mathbf{H}}^*(\mathbf{r})\\&\quad+\hat{\mathbf{E}}^*(\mathbf{r})\times\hat{\mathbf{H}}^*(\mathbf{r})\,e^{-2j\omega t}]\end{aligned}\tag{20}$$

The time average of (20) is then

$$\begin{aligned}<\boldsymbol{S}>&=\tfrac{1}{4}[\hat{\mathbf{E}}^*(\mathbf{r})\times\hat{\mathbf{H}}(\mathbf{r})+\hat{\mathbf{E}}(\mathbf{r})\times\hat{\mathbf{H}}^*(\mathbf{r})]\\&=\tfrac{1}{2}\,\mathrm{Re}\,[\hat{\mathbf{E}}(\mathbf{r})\times\hat{\mathbf{H}}^*(\mathbf{r})]\\&=\tfrac{1}{2}\,\mathrm{Re}\,[\hat{\mathbf{E}}^*(\mathbf{r})\times\hat{\mathbf{H}}(\mathbf{r})]\end{aligned}\tag{21}$$

as the complex exponential terms $e^{\pm 2j\omega t}$ average to zero over a period $T=2\pi/\omega$ and we again realized that the first bracketed term on the right-hand side of (21) was the sum of a complex function and its conjugate.

Motivated by (21) we define the complex Poynting vector as

$$\hat{\boldsymbol{S}}=\tfrac{1}{2}\hat{\mathbf{E}}(\mathbf{r})\times\hat{\mathbf{H}}^*(\mathbf{r})\tag{22}$$

whose real part is just the time-average power density.

We can now derive a complex form of Poynting's theorem by rewriting Maxwell's equations for sinusoidal time variations as

$$\begin{aligned}\nabla\times\hat{\mathbf{E}}(\mathbf{r})&=-j\omega\mu\,\hat{\mathbf{H}}(\mathbf{r})\\\nabla\times\hat{\mathbf{H}}(\mathbf{r})&=\hat{\mathbf{J}}_f(\mathbf{r})+j\omega\varepsilon\,\hat{\mathbf{E}}(\mathbf{r})\\\nabla\cdot\hat{\mathbf{E}}(\mathbf{r})&=\hat{\rho}_f(\mathbf{r})/\varepsilon\\\nabla\cdot\hat{\mathbf{B}}(\mathbf{r})&=0\end{aligned}\tag{23}$$

and expanding the product

$$\begin{aligned}\nabla\cdot\hat{\boldsymbol{S}}=\nabla\cdot[\tfrac{1}{2}\hat{\mathbf{E}}(\mathbf{r})\times\hat{\mathbf{H}}^*(\mathbf{r})]&=\tfrac{1}{2}[\hat{\mathbf{H}}^*(\mathbf{r})\cdot\nabla\times\hat{\mathbf{E}}(\mathbf{r})-\hat{\mathbf{E}}(\mathbf{r})\cdot\nabla\times\hat{\mathbf{H}}^*(\mathbf{r})]\\&=\tfrac{1}{2}[-j\omega\mu\,|\hat{\mathbf{H}}(\mathbf{r})|^2+j\omega\varepsilon\,|\hat{\mathbf{E}}(\mathbf{r})|^2]-\tfrac{1}{2}\hat{\mathbf{E}}(\mathbf{r})\cdot\hat{\mathbf{J}}_f^*(\mathbf{r})\end{aligned}\tag{24}$$

which can be rewritten as

$$\nabla\cdot\hat{\boldsymbol{S}}+2j\omega[<w_m>-<w_e>]=-\hat{P}_d\tag{25}$$

where

$$\begin{aligned}<w_m>&=\tfrac{1}{4}\mu\,|\hat{\mathbf{H}}(\mathbf{r})|^2\\<w_e>&=\tfrac{1}{4}\varepsilon\,|\hat{\mathbf{E}}(\mathbf{r})|^2\\\hat{P}_d&=\tfrac{1}{2}\hat{\mathbf{E}}(\mathbf{r})\cdot\hat{\mathbf{J}}_f^*(\mathbf{r})\end{aligned}\tag{26}$$

We note that $<w_m>$ and $<w_e>$ are the time-average magnetic and electric energy densities and that the complex Poynting's theorem depends on their difference rather than their sum.

## 7-3 TRANSVERSE ELECTROMAGNETIC WAVES

### 7-3-1 Plane Waves

Let us try to find solutions to Maxwell's equations that only depend on the $z$ coordinate and time in linear media with permittivity $\varepsilon$ and permeability $\mu$. In regions where there are no sources so that $\rho_f = 0$, $\mathbf{J}_f = 0$, Maxwell's equations then reduce to

$$-\frac{\partial E_y}{\partial z}\mathbf{i}_x + \frac{\partial E_x}{\partial z}\mathbf{i}_y = -\mu\frac{\partial \mathbf{H}}{\partial t} \tag{1}$$

$$-\frac{\partial H_y}{\partial z}\mathbf{i}_x + \frac{\partial H_x}{\partial z}\mathbf{i}_y = \varepsilon\frac{\partial \mathbf{E}}{\partial t} \tag{2}$$

$$\varepsilon\frac{\partial E_z}{\partial z} = 0 \tag{3}$$

$$\mu\frac{\partial H_z}{\partial z} = 0 \tag{4}$$

These relations tell us that at best $E_z$ and $H_z$ are constant in time and space. Because they are uncoupled, in the absence of sources we take them to be zero. By separating vector components in (1) and (2) we see that $E_x$ is coupled to $H_y$ and $E_y$ is coupled to $H_x$:

$$\begin{aligned} \frac{\partial E_x}{\partial z} &= -\mu\frac{\partial H_y}{\partial t}, \qquad & \frac{\partial E_y}{\partial z} &= \mu\frac{\partial H_x}{\partial t} \\ \frac{\partial H_y}{\partial z} &= -\varepsilon\frac{\partial E_x}{\partial t}, \qquad & \frac{\partial H_x}{\partial z} &= \varepsilon\frac{\partial E_y}{\partial t} \end{aligned} \tag{5}$$

forming two sets of independent equations. Each solution has perpendicular electric and magnetic fields. The power flow $\boldsymbol{S} = \mathbf{E}\times\mathbf{H}$ for each solution is $z$ directed also being perpendicular to $\mathbf{E}$ and $\mathbf{H}$. Since the fields and power flow are mutually perpendicular, such solutions are called transverse electromagnetic waves (TEM). They are waves because if we take $\partial/\partial z$ of the upper equations and $\partial/\partial t$ of the lower equations and solve for the electric fields, we obtain one-dimensional wave equations:

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}, \qquad \frac{\partial^2 E_y}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} \tag{6}$$

where $c$ is the speed of the wave,

$$c=\frac{1}{\sqrt{\varepsilon\mu}}=\frac{1}{\sqrt{\varepsilon_0\mu_0}\sqrt{\varepsilon_r\mu_r}}\approx\frac{3\times 10^8}{\sqrt{\varepsilon_r\mu_r}}\text{ m/sec} \tag{7}$$

In free space, where $\varepsilon_r = 1$ and $\mu_r = 1$, this quantity equals the speed of light in vacuum which demonstrated that light is a transverse electromagnetic wave. If we similarly take $\partial/\partial t$ of the upper and $\partial/\partial z$ of the lower equations in (5), we obtain wave equations in the magnetic fields:

$$\frac{\partial^2 H_y}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 H_y}{\partial t^2},\qquad \frac{\partial^2 H_x}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 H_x}{\partial t^2} \tag{8}$$

## 7-3-2 The Wave Equation

### (a) Solutions

These equations arise in many physical systems, so their solutions are well known. Working with the $E_x$ and $H_y$ equations, the solutions are

$$\begin{aligned} E_x(z,t)&=E_+(t-z/c)+E_-(t+z/c)\\ H_y(z,t)&=H_+(t-z/c)+H_-(t+z/c)\end{aligned} \tag{9}$$

where the functions $E_+$, $E_-$, $H_+$, and $H_-$ depend on initial conditions in time and boundary conditions in space. These solutions can be easily verified by defining the arguments $\alpha$ and $\beta$ with their resulting partial derivatives as

$$\begin{aligned} \alpha&=t-\frac{z}{c}\Rightarrow\frac{\partial\alpha}{\partial t}=1,\qquad \frac{\partial\alpha}{\partial z}=-\frac{1}{c}\\ \beta&=t+\frac{z}{c}\Rightarrow\frac{\partial\beta}{\partial t}=1,\qquad \frac{\partial\beta}{\partial z}=\frac{1}{c}\end{aligned} \tag{10}$$

and realizing that the first partial derivatives of $E_x(z, t)$ are

$$\begin{aligned} \frac{\partial E_x}{\partial t}&=\frac{dE_+}{d\alpha}\frac{\partial\alpha}{\partial t}+\frac{dE_-}{d\beta}\frac{\partial\beta}{\partial t}\\ &=\frac{dE_+}{d\alpha}+\frac{dE_-}{d\beta}\\ \frac{\partial E_x}{\partial z}&=\frac{dE_+}{d\alpha}\frac{\partial\alpha}{\partial z}+\frac{dE_-}{d\beta}\frac{\partial\beta}{\partial z}\\ &=\frac{1}{c}\left(-\frac{dE_+}{d\alpha}+\frac{dE_-}{d\beta}\right)\end{aligned} \tag{11}$$

The second derivatives are then

$$
\begin{aligned}
\frac{\partial^2 E_x}{\partial t^2} &= \frac{d^2E_+}{d\alpha^2}\frac{\partial\alpha}{\partial t} + \frac{d^2E_-}{d\beta^2}\frac{\partial\beta}{\partial t} \\
&= \frac{d^2E_+}{d\alpha^2} + \frac{d^2E_-}{d\beta^2} \\
\frac{\partial^2 E_x}{\partial z^2} &= \frac{1}{c}\left(-\frac{d^2E_+}{d\alpha^2}\frac{\partial\alpha}{\partial z} + \frac{d^2E_-}{d\beta^2}\frac{\partial\beta}{\partial z}\right) \\
&= \frac{1}{c^2}\left(\frac{d^2E_+}{d\alpha^2} + \frac{d^2E_-}{d\beta^2}\right) = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}
\end{aligned}
\tag{12}
$$

which satisfies the wave equation of (6). Similar operations apply for $H_y$, $E_y$, and $H_x$.

In (9), the pair $H_+$ and $E_+$ as well as the pair $H_-$ and $E_-$ are not independent, as can be seen by substituting the solutions of (9) back into (5) and using (11):

$$\frac{\partial E_x}{\partial z} = -\mu\frac{\partial H_y}{\partial t} \Rightarrow \frac{1}{c}\left(-\frac{dE_+}{d\alpha} + \frac{dE_-}{d\beta}\right) = -\mu\left(\frac{dH_+}{d\alpha} + \frac{dH_-}{d\beta}\right) \tag{13}$$

The functions of $\alpha$ and $\beta$ must separately be equal,

$$\frac{d}{d\alpha}(E_+ - \mu c H_+) = 0, \qquad \frac{d}{d\beta}(E_- + \mu c H_-) = 0 \tag{14}$$

which requires that

$$E_+ = \mu c H_+ = \sqrt{\frac{\mu}{\varepsilon}}H_+, \qquad E_- = -\mu c H_- = -\sqrt{\frac{\mu}{\varepsilon}}H_- \tag{15}$$

where we use (7). Since $\sqrt{\mu/\varepsilon}$ has units of Ohms, this quantity is known as the wave impedance $\eta$,

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \approx 120\pi\sqrt{\frac{\mu_r}{\varepsilon_r}} \tag{16}$$

and has value $120\pi \approx 377$ ohm in free space ($\mu_r = 1$, $\varepsilon_r = 1$).

The power flux density in TEM waves is

$$
\begin{aligned}
S = \mathbf{E}\times\mathbf{H} &= [E_+(t - z/c) + E_-(t + z/c)]\mathbf{i}_x \\
&\qquad \times [H_+(t - z/c) + H_-(t + z/c)]\mathbf{i}_y \\
&= (E_+H_+ + E_-H_- + E_-H_+ + E_+H_-)\mathbf{i}_z
\end{aligned}
\tag{17}
$$

Using (15) and (16) this result can be written as

$$S_z = \frac{1}{\eta}(E_+^2 - E_-^2) \tag{18}$$

where the last two cross terms in (17) cancel because of the minus sign relating $E_-$ to $H_-$ in (15). For TEM waves the total power flux density is due to the difference in power densities between the squares of the positively $z$-directed and negatively $z$-directed waves.

**(b) Properties**

The solutions of (9) are propagating waves at speed $c$. To see this, let us examine $E_+(t-z/c)$ and consider the case where at $z=0$, $E_+(t)$ is the staircase pulse shown in Figure 7-3*a*. In Figure 7-3*b* we replace the argument $t$ by $t-z/c$. As long as the function $E_+$ is plotted versus its argument, no matter what its argument is, the plot remains unchanged. However, in Figure 7-3*c* the function $E_+(t-z/c)$ is plotted versus $t$ resulting in the pulse being translated in time by an amount $z/c$. To help in plotting this translated function, we use the following logic:

(i) The pulse jumps to amplitude $E_0$ when the argument is zero. When the argument is $t-z/c$, this occurs for $t=z/c$.
(ii) The pulse jumps to amplitude $2E_0$ when the argument is $T$. When the argument is $t-z/c$, this occurs for $t=T+z/c$.
(iii) The pulse returns to zero when the argument is $2T$. For the argument $t-z/c$, we have $t=2T+z/c$.

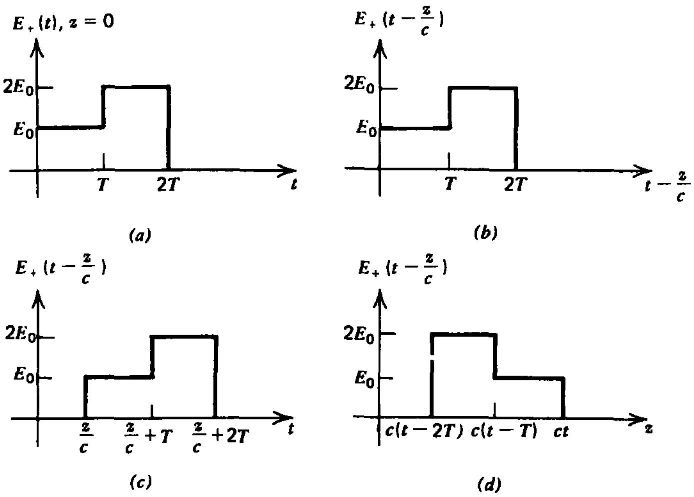



Figure 7-3 (*a*) $E_+(t)$ at $z=0$ is a staircase pulse. (*b*) $E_+(\phi)$ always has the same shape as (*a*) when plotted versus $\phi$, no matter what $\phi$ is. Here $\phi=t-z/c$. (*c*) When plotted versus $t$, the pulse is translated in time where $z$ must be positive to keep $t$ positive. (*d*) When plotted versus $z$, it is translated and inverted. The pulse propagates at speed $c$ in the positive $z$ direction.

Note that $z$ can only be positive as causality imposes the condition that time can only be increasing. The response at any positive position $z$ to an initial $E_+$ pulse imposed at $z=0$ has the same shape in time but occurs at a time $z/c$ later. The pulse travels the distance $z$ at the speed $c$. This is why the function $E_+(t-z/c)$ is called a positively traveling wave.

In Figure 7-3$d$ we plot the same function versus $z$. Its appearance is inverted as that part of the pulse generated first (step of amplitude $E_0$) will reach any positive position $z$ first. The second step of amplitude $2E_0$ has not traveled as far since it was generated a time $T$ later. To help in plotting, we use the same criterion on the argument as used in the plot versus time, only we solve for $z$. The important rule we use is that as long as the argument of a function remains constant, the value of the function is unchanged, no matter how the individual terms in the argument change.

Thus, as long as

$$t - z/c = \text{const} \tag{19}$$

$E_+(t-z/c)$ is unchanged. As time increases, so must $z$ to satisfy (19) at the rate

$$t - \frac{z}{c} = \text{const} \Rightarrow \frac{dz}{dt} = c \tag{20}$$

to keep the $E_+$ function constant.

For similar reasons $E_-(t+z/c)$ represents a traveling wave at the speed $c$ in the negative $z$ direction as an observer must move to keep the argument $t+z/c$ constant at speed:

$$t + \frac{z}{c} = \text{const} \Rightarrow \frac{dz}{dt} = -c \tag{21}$$

as demonstrated for the same staircase pulse in Figure 7-4. Note in Figure 7-4$d$ that the pulse is not inverted when plotted versus $z$ as it was for the positively traveling wave, because that part of the pulse generated first (step of amplitude $E_0$) reaches the maximum distance but in the negative $z$ direction. These differences between the positively and negatively traveling waves are functionally due to the difference in signs in the arguments $(t-z/c)$ and $(t+z/c)$.

### 7-3-3 Sources of Plane Waves

These solutions are called plane waves because at any constant $z$ plane the fields are constant and do not vary with the $x$ and $y$ coordinates.

The idealized source of a plane wave is a time varying current sheet of infinite extent that we take to be $x$ directed,

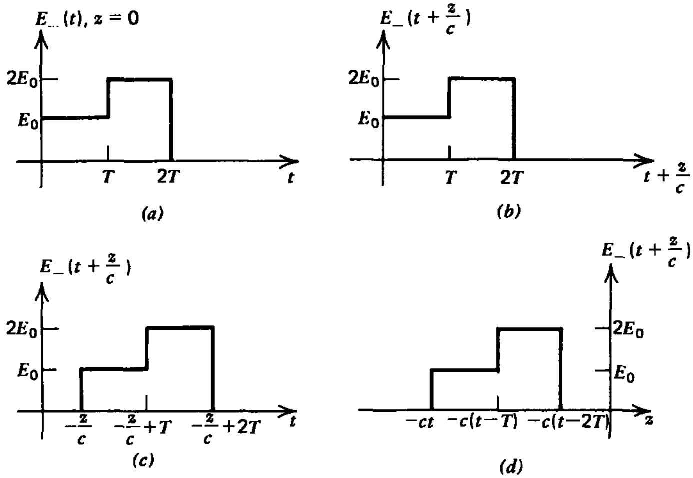



Figure 7-4 (*a*) $E_-(t)$ at $z=0$ is a staircase pulse. (*b*) $E_-(\phi)$ always has the same form of (a) when plotted versus $\phi$. Here $\phi = t + z/c$. (*c*) When plotted versus $t$, the pulse is translated in time where $z$ must be negative to keep $t$ positive. (*d*) When plotted versus $z$, it is translated but not inverted.

as shown in Figure 7-5. From the boundary condition on the discontinuity of tangential **H**, we find that the $x$-directed current sheet gives rise to a $y$-directed magnetic field:

$$H_y(z=0_+) - H_y(z=0_-) = -K_x(t) \tag{22}$$

In general, a uniform current sheet gives rise to a magnetic field perpendicular to the direction of current flow but in the plane of the sheet. Thus to generate an $x$-directed magnetic field, a $y$-directed surface current is required.

Since there are no other sources, the waves must travel away from the sheet so that the solutions on each side of the sheet are of the form

$$H_y(z,t) = \begin{cases} H_+(t-z/c) \\ H_-(t+z/c) \end{cases} \qquad E_x(z,t) = \begin{cases} \eta H_+(t-z/c), & z>0 \\ -\eta H_-(t+z/c), & z<0 \end{cases} \tag{23}$$

For $z>0$, the waves propagate only in the positive $z$ direction. In the absence of any other sources or boundaries, there can be no negatively traveling waves in this region. Similarly for $z<0$, we only have waves propagating in the $-z$ direction. In addition to the boundary condition of (22), the tangential component of **E** must be continuous across the sheet at $z=0$

$$\left.\begin{array}{r} H_+(t) - H_-(t) = -K_x(t) \\ \eta[H_+(t) + H_-(t)] = 0 \end{array}\right\} \Rightarrow H_+(t) = -H_-(t) = \frac{-K_x(t)}{2} \tag{24}$$

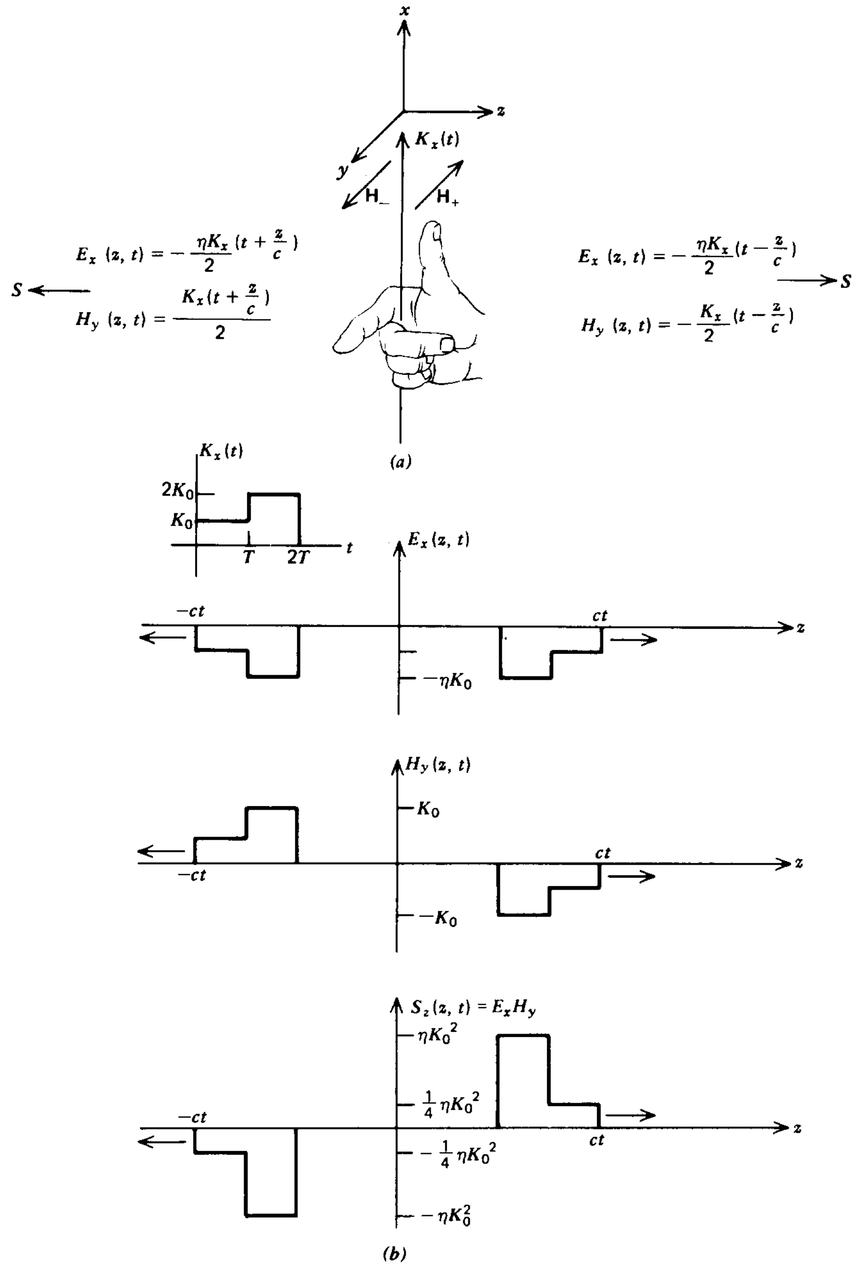



Figure 7-5 (*a*) A linearly polarized plane wave is generated by an infinite current sheet. The electric field is in the direction opposite to the current on either side of the sheet. The magnetic field is perpendicular to the current but in the plane of the current sheet and in opposite directions as given by the right-hand rule on either side of the sheet. The power flow $\boldsymbol{S}$ is thus perpendicular to the current and to the sheet. (*b*) The field solutions for $t > 2T$ if the current source is a staircase pulse in time.

so that the electric and magnetic fields have the same shape as the current. Because the time and space shape of the fields remains unchanged as the waves propagate, linear dielectric media are said to be nondispersive.

Note that the electric field at $z=0$ is in the opposite direction as the current, so the power per unit area delivered by the current sheet,

$$-\mathbf{E}(z=0,t)\cdot\mathbf{K}_x(t)=\frac{\eta K_x^2(t)}{2} \tag{25}$$

is equally carried away by the Poynting vector on each side of the sheet:

$$\mathbf{S}(z \doteq 0)=\mathbf{E}\times\mathbf{H}=\begin{cases}\dfrac{\eta K_x^2(t)}{4}\mathbf{i}_z, & z>0\\[2ex] \dfrac{-\eta K_x^2(t)}{4}\mathbf{i}_z, & z<0\end{cases} \tag{26}$$

### 7-3-4 A Brief Introduction to the Theory of Relativity

Maxwell's equations show that electromagnetic waves propagate at the speed $c_0=1/\sqrt{\varepsilon_0\mu_0}$ in vacuum. Our natural intuition would tell us that if we moved at a speed $v$ we would measure a wave speed of $c_0-v$ when moving in the same direction as the wave, and a speed $c_0+v$ when moving in the opposite direction. However, our intuition would be wrong, for nowhere in the free space, source-free Maxwell's equations does the speed of the observer appear. Maxwell's equations predict that the speed of electromagnetic waves is $c_0$ for all observers no matter their relative velocity. This assumption is a fundamental postulate of the theory of relativity and has been verified by all experiments. The most notable experiment was performed by A. A. Michelson and E. W. Morley in the late nineteenth century, where they showed that the speed of light reflected between mirrors is the same whether it propagated in the direction parallel or perpendicular to the velocity of the earth. This postulate required a revision of the usual notions of time and distance.

If the surface current sheet of Section 7-3-3 is first turned on at $t=0$, the position of the wave front on either side of the sheet at time $t$ later obeys the equality

$$z^2-c_0^2t^2=0 \tag{27}$$

Similarly, an observer in a coordinate system moving with constant velocity $u\mathbf{i}_z$ which is aligned with the current sheet at

$t=0$ finds the wavefront position to obey the equality

$$z'^2-c_0^2t'^2=0 \tag{28}$$

The two coordinate systems must be related by a linear transformation of the form

$$z'=a_1z+a_2t, \qquad t'=b_1z+b_2t \tag{29}$$

The position of the origin of the moving frame ($z'=0$) as measured in the stationary frame is $z=vt$, as shown in Figure 7-6, so that $a_1$ and $a_2$ are related as

$$0=a_1vt+a_2t \Rightarrow a_1v+a_2=0 \tag{30}$$

We can also equate the two equalities of (27) and (28),

$$z^2-c_0^2t^2=z'^2-c_0^2t'^2=(a_1z+a_2t)^2-c_0^2(b_1z+b_2t)^2 \tag{31}$$

so that combining terms yields

$$z^2(1-a_1^2+c_0^2b_1^2)-c_0^2t^2\left(1+\frac{a_2^2}{c_0^2}-b_2^2\right)-2(a_1a_2-c_0^2b_1b_2)zt=0 \tag{32}$$

Since (32) must be true for all $z$ and $t$, each of the coefficients must be zero, which with (30) gives solutions

$$\begin{aligned} a_1&=\frac{1}{\sqrt{1-(v/c_0)^2}}, & b_1&=\frac{-v/c_0^2}{\sqrt{1-(v/c_0)^2}} \\ a_2&=\frac{-v}{\sqrt{1-(v/c_0)^2}}, & b_2&=\frac{1}{\sqrt{1-(v/c_0)^2}} \end{aligned} \tag{33}$$

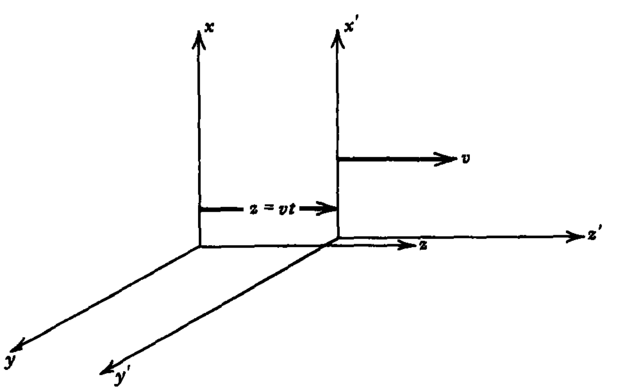



Figure 7-6 The primed coordinate system moves at constant velocity $v\mathbf{i}_z$ with respect to a stationary coordinate system. The free space speed of an electromagnetic wave is $c_0$ as measured by observers in either coordinate system no matter the velocity $v$.

The transformations of (29) are then

$$z' = \frac{z - vt}{\sqrt{1-(v/c_0)^2}}, \qquad t' = \frac{t - vz/c_0^2}{\sqrt{1-(v/c_0)^2}} \tag{34}$$

and are known as the Lorentz transformations. Measured lengths and time intervals are different for observers moving at different speeds. If the velocity $v$ is much less than the speed of light, (34) reduces to the Galilean transformations,

$$\lim_{v/c \ll 1} z' \approx z - vt, \qquad t' \approx t \tag{35}$$

which describe our usual experiences at nonrelativistic speeds.

The coordinates perpendicular to the motion are unaffected by the relative velocity between reference frames

$$x' = x, \qquad y' = y \tag{36}$$

Continued development of the theory of relativity is beyond the scope of this text and is worth a course unto itself. Applying the Lorentz transformation to Newton's law and Maxwell's equations yield new results that at first appearance seem contrary to our experiences because we live in a world where most material velocities are much less than $c_0$. However, continued experiments on such disparate time and space scales as between atomic physics and astronomics verify the predictions of relativity theory, in part spawned by Maxwell's equations.

## 7-4 SINUSOIDAL TIME VARIATIONS

### 7-4-1 Frequency and Wavenumber

If the current sheet of Section 7-3-3 varies sinusoidally with time as $\text{Re}\,(K_0 e^{j\omega t})$, the wave solutions require the fields to vary as $e^{j\omega(t-z/c)}$ and $e^{j\omega(t+z/c)}$:

$$H_y(z,t) = \begin{cases} \text{Re}\left(-\dfrac{K_0}{2} e^{j\omega(t-z/c)}\right), & z>0 \\ \text{Re}\left(+\dfrac{K_0}{2} e^{j\omega(t+z/c)}\right), & z<0 \end{cases}$$
$$E_x(z,t) = \begin{cases} \text{Re}\left(-\dfrac{\eta K_0}{2} e^{j\omega(t-z/c)}\right), & z>0 \\ \text{Re}\left(-\dfrac{\eta K_0}{2} e^{j\omega(t+z/c)}\right), & z<0 \end{cases} \tag{1}$$

At a fixed time the fields then also vary sinusoidally with position so that it is convenient to define the wavenumber as

$$k=\frac{2\pi}{\lambda}=\frac{\omega}{c}=\omega\sqrt{\mu\varepsilon} \tag{2}$$

where $\lambda$ is the fundamental spatial period of the wave. At a fixed position the waveform is also periodic in time with period $T$:

$$T=\frac{1}{f}=\frac{2\pi}{\omega} \tag{3}$$

where $f$ is the frequency of the source. Using (3) with (2) gives us the familiar frequency-wavelength formula:

$$\omega=kc\Rightarrow f\lambda=c \tag{4}$$

Throughout the electromagnetic spectrum, summarized in Figure 7-7, time varying phenomena differ only in the scaling of time and size. No matter the frequency or wavelength, although easily encompassing 20 orders of magnitude, electromagnetic phenomena are all described by Maxwell's equations. Note that visible light only takes up a tiny fraction of the spectrum.

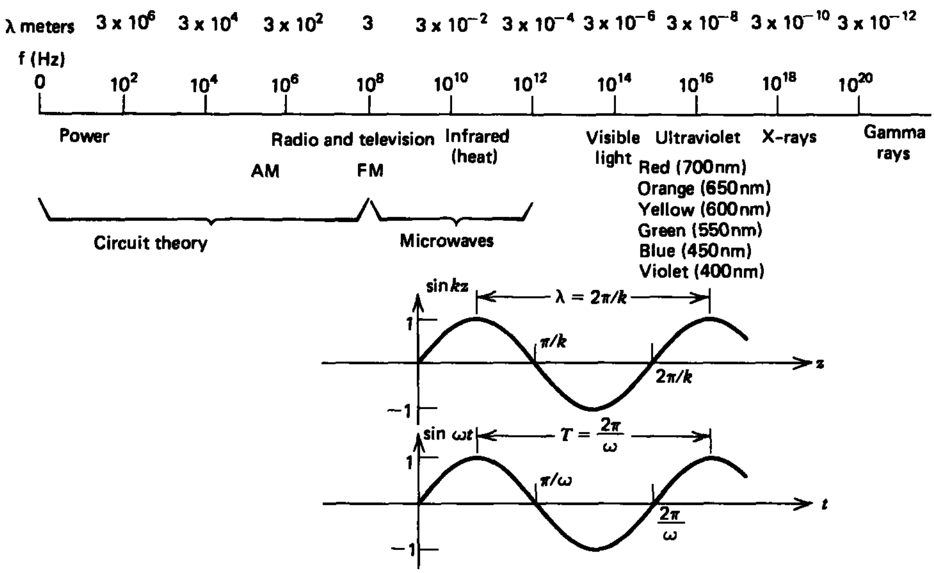



Figure 7-7 Time varying electromagnetic phenomena differ only in the scaling of time (frequency) and size (wavelength). In linear dielectric media the frequency and wavelength are related as $f\lambda=c$ ($\omega=kc$), where $c=1/\sqrt{\varepsilon\mu}$ is the speed of light in the medium.

For a single sinusoidally varying plane wave, the time-average electric and magnetic energy densities are equal because the electric and magnetic field amplitudes are related through the wave impedance $\eta$:

$$<w_m> = <w_e> = \tfrac{1}{4}\mu|\mathbf{H}|^2 = \tfrac{1}{4}\varepsilon|\mathbf{E}|^2 = \tfrac{1}{16}\mu K_0^2 \tag{5}$$

From the complex Poynting theorem derived in Section 7-2-4, we then see that in a lossless region with no sources for $|z|>0$ that $\hat{P}_d = 0$ so that the complex Poynting vector has zero divergence. With only one-dimensional variations with $z$, this requires the time-average power density to be a constant throughout space on each side of the current sheet:

$$\begin{aligned} <\boldsymbol{S}> &= \tfrac{1}{2}\operatorname{Re}[\hat{\mathbf{E}}(\mathbf{r}) \times \hat{\mathbf{H}}^*(\mathbf{r})] \\ &= \begin{cases} \tfrac{1}{8}\eta K_0^2 \mathbf{i}_z, & z>0 \\ -\tfrac{1}{8}\eta K_0^2 \mathbf{i}_z, & z<0 \end{cases} \end{aligned} \tag{6}$$

The discontinuity in $<\boldsymbol{S}>$ at $z=0$ is due to the power output of the source.

### 7-4-2 Doppler Frequency Shifts

If the sinusoidally varying current sheet $\operatorname{Re}(K_0 e^{j\omega t})$ moves with constant velocity $v\mathbf{i}_z$, as in Figure 7-8, the boundary conditions are no longer at $z=0$ but at $z=vt$. The general form of field solutions are then:

$$\begin{aligned} H_y(z,t) &= \begin{cases} \operatorname{Re}(\hat{H}_+ e^{j\omega_+(t-z/c)}), & z>vt \\ \operatorname{Re}(\hat{H}_- e^{j\omega_-(t+z/c)}), & z<vt \end{cases} \\ E_x(z,t) &= \begin{cases} \operatorname{Re}(\eta\hat{H}_+ e^{j\omega_+(t-z/c)}), & z>vt \\ \operatorname{Re}(-\eta\hat{H}_- e^{j\omega_-(t+z/c)}), & z<vt \end{cases} \end{aligned} \tag{7}$$

where the frequencies of the fields $\omega_+$ and $\omega_-$ on each side of the sheet will be different from each other as well as differing from the frequency of the current source $\omega$. We assume $v/c \ll 1$ so that we can neglect relativistic effects discussed in Section 7-3-4. The boundary conditions

$$\begin{aligned} E_{x_+}(z=vt) = E_{x_-}(z=vt) &\Rightarrow \hat{H}_+ e^{j\omega_+ t(1-v/c)} = -\hat{H}_- e^{j\omega_- t(1+v/c)} \\ H_{y_+}(z=vt) - H_{y_-}(z=vt) &= -K_x \\ &\Rightarrow \hat{H}_+ e^{j\omega_+ t(1-v/c)} - \hat{H}_- e^{j\omega_- t(1+v/c)} = -K_0 e^{j\omega t} \end{aligned} \tag{8}$$

must be satisfied for all values of $t$ so that the exponential time factors in (8) must all be equal, which gives the shifted

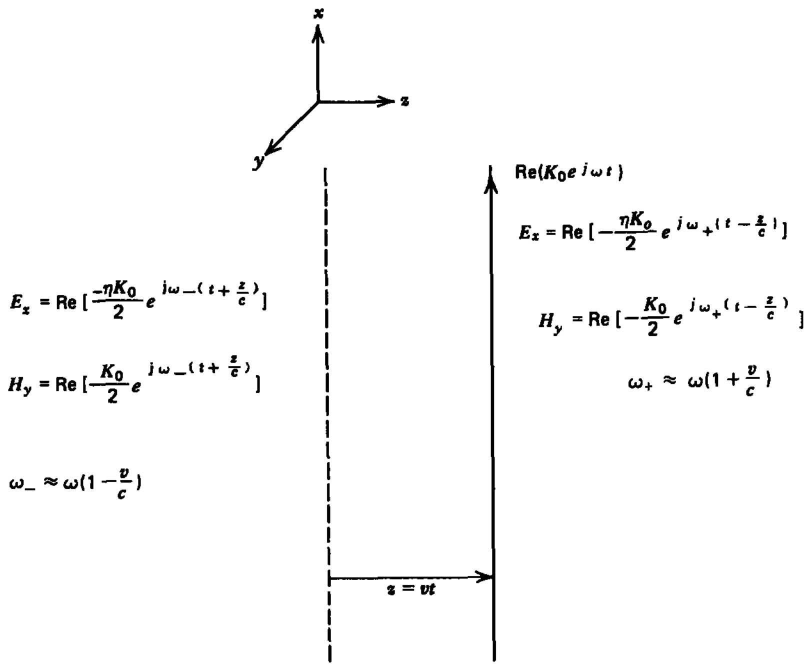



Figure 7-8 When a source of electromagnetic waves moves towards an observer, the frequency is raised while it is lowered when it moves away from an observer.

frequencies on each side of the sheet as

$$\begin{aligned}\omega_+ &= \frac{\omega}{1-v/c} \approx \omega\left(1+\frac{v}{c}\right),\\ \omega_- &= \frac{\omega}{1+v/c} \approx \omega\left(1-\frac{v}{c}\right) \Rightarrow \hat{H}_+ = -\hat{H}_- = -\frac{K_0}{2}\end{aligned} \tag{9}$$

where $v/c \ll 1$. When the source is moving towards an observer, the frequency is raised while it is lowered when it moves away. Such frequency changes due to the motion of a source or observer are called Doppler shifts and are used to measure the velocities of moving bodies in radar systems. For $v/c \ll 1$, the frequency shifts are a small percentage of the driving frequency, but in absolute terms can be large enough to be easily measured. At a velocity $v = 300$ m/sec with a driving frequency of $f = 10^{10}$ Hz, the frequency is raised and lowered on each side of the sheet by $\Delta f = \pm f(v/c) = \pm 10^4$ Hz.

### 7-4-3 Ohmic Losses

Thus far we have only considered lossless materials. If the medium also has an Ohmic conductivity $\sigma$, the electric field will cause a current flow that must be included in Ampere's law:

$$\frac{\partial E_x}{\partial z}=-\mu\frac{\partial H_y}{\partial t}$$
$$\frac{\partial H_y}{\partial z}=-J_x-\varepsilon\frac{\partial E_x}{\partial t}=-\sigma E_x-\varepsilon\frac{\partial E_x}{\partial t} \tag{10}$$

where for conciseness we only consider the $x$-directed electric field solution as the same results hold for the $E_y$, $H_x$ solution. Our wave solutions of Section 7-3-2 no longer hold with this additional term, but because Maxwell's equations are linear with constant coefficients, for sinusoidal time variations the solutions in space must also be exponential functions, which we write as

$$E_x(z,t)=\text{Re}\,(\hat{E}_0\,e^{j(\omega t-kz)})$$
$$H_y(z,t)=\text{Re}\,(\hat{H}_0\,e^{j(\omega t-kz)}) \tag{11}$$

where $\hat{E}_0$ and $\hat{H}_0$ are complex amplitudes and the wavenumber $k$ is no longer simply related to $\omega$ as in (4) but is found by substituting (11) back into (10):

$$-jk\hat{E}_0=-j\omega\mu\hat{H}_0$$
$$-jk\hat{H}_0=-j\omega\varepsilon(1+\sigma/j\omega\varepsilon)\hat{E}_0 \tag{12}$$

This last relation was written in a way that shows that the conductivity enters in the same way as the permittivity so that we can define a complex permittivity $\hat{\varepsilon}$ as

$$\hat{\varepsilon}=\varepsilon(1+\sigma/j\omega\varepsilon) \tag{13}$$

Then the solutions to (12) are

$$\frac{\hat{E}_0}{\hat{H}_0}=\frac{\omega\mu}{k}=\frac{k}{\omega\hat{\varepsilon}}\Rightarrow k^2=\omega^2\mu\hat{\varepsilon}=\omega^2\mu\varepsilon\left(1+\frac{\sigma}{j\omega\varepsilon}\right) \tag{14}$$

which is similar in form to (2) with a complex permittivity.

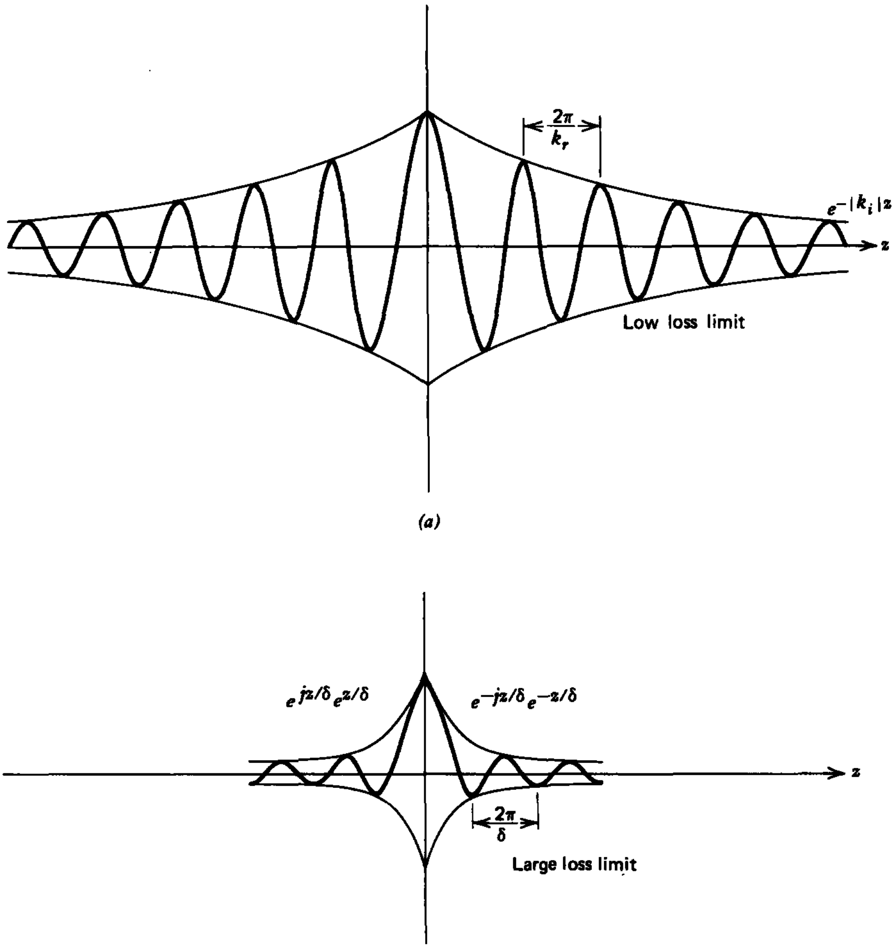
There are two interesting limits of (14):

**(a) Low Loss Limit**

If the conductivity is small so that $\sigma/\omega\varepsilon \ll 1$, then the solution of (14) reduces to

$$\lim_{\sigma/\omega\varepsilon\ll 1} k=\pm\omega\sqrt{\mu\varepsilon}\left(1+\frac{\sigma}{2j\omega\varepsilon}\right)=\pm\left(\frac{\omega}{c}-\frac{j\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}\right) \tag{15}$$

where $c$ is the speed of the light in the medium if there were no losses, $c=1/\sqrt{\mu\varepsilon}$. Because of the spatial exponential dependence in (11), the real part of $k$ is the same as for the lossless case and represents the sinusoidal spatial distribution of the fields. The imaginary part of $k$ represents the exponential decay of the fields due to the Ohmic losses with exponential decay length $\frac{1}{2}\sigma\eta$, where $\eta=\sqrt{\mu/\varepsilon}$ is the wave impedance. Note that for waves traveling in the positive $z$ direction we take the upper positive sign in (15) using the lower negative sign for negatively traveling waves so that the solutions all decay and do not grow for distances far from the source. This solution is only valid for small $\sigma$ so that the wave is only slightly damped as it propagates, as illustrated in Figure 7-9$a$.




(a)

(b)

Figure 7-9 ($a$) In a slightly lossy dielectric, the fields decay away from a source at a slow rate while the wavelength is essentially unchanged. ($b$) In the large loss limit the spatial decay rate is equal to the skin depth. The wavelength also equals the skin depth.

**(b) Large Loss Limit**

In the other extreme of a highly conducting material so that $\sigma/\omega\varepsilon \gg 1$, (14) reduces to

$$\lim_{\sigma/\omega\varepsilon \gg 1} k^2 \approx -j\omega\mu\sigma \Rightarrow k = \pm\frac{(1-j)}{\delta}, \qquad \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{16}$$

where $\delta$ is just the skin depth found in Section 6-4-3 for magneto-quasi-static fields within a conductor. The skin-depth term also arises for electrodynamic fields because the large loss limit has negligible displacement current compared to the conduction currents.

Because the real and imaginary part of $k$ have equal magnitudes, the spatial decay rate is large so that within a few oscillation intervals the fields are negligibly small, as illustrated in Figure 7-9*b*. For a metal like copper with $\mu = \mu_0 = 4\pi \times 10^{-7}$ henry/m and $\sigma \approx 6 \times 10^7$ siemens/m at a frequency of 1 MHz, the skin depth is $\delta \approx 6.5 \times 10^{-5}$ m.

### 7-4-4 High-Frequency Wave Propagation in Media

Ohm's law is only valid for frequencies much below the collision frequencies of the charge carriers, which is typically on the order of $10^{13}$ Hz. In this low-frequency regime the inertia of the particles is negligible. For frequencies much higher than the collision frequency the inertia dominates and the current constitutive law for a single species of charge carrier $q$ with mass $m$ and number density $n$ is as found in Section 3-2-2*d*:

$$\partial \mathbf{J}_f/\partial t = \omega_p^2 \varepsilon \mathbf{E} \tag{17}$$

where $\omega_p = \sqrt{q^2 n/m\varepsilon}$ is the plasma frequency. This constitutive law is accurate for radio waves propagating in the ionosphere, for light waves propagating in many dielectrics, and is also valid for superconductors where the collision frequency is zero.

Using (17) rather than Ohm's law in (10) for sinusoidal time and space variations as given in (11), Maxwell's equations are

$$\begin{aligned} \frac{\partial E_x}{\partial z} &= -\mu\frac{\partial H_y}{\partial t} \Rightarrow -jk\hat{E}_0 = -j\omega\mu\hat{H}_0 \\ \frac{\partial H_y}{\partial z} &= -J_x - \varepsilon\frac{\partial E_x}{\partial t} \Rightarrow -jk\hat{H}_0 = -j\omega\varepsilon\left(1 - \frac{\omega_p^2}{\omega^2}\right)\hat{E}_0 \end{aligned} \tag{18}$$

The effective permittivity is now frequency dependent:

$$\hat{\varepsilon} = \varepsilon(1 - \omega_p^2/\omega^2) \tag{19}$$

The solutions to (18) are

$$\frac{\hat{E}_0}{\hat{H}_0}=\frac{\omega\mu}{k}=\frac{k}{\omega\hat{\varepsilon}}\Rightarrow k^2=\omega^2\mu\hat{\varepsilon}=\frac{\omega^2-\omega_p^2}{c^2} \tag{20}$$

For $\omega>\omega_p$, $k$ is real and we have pure propagation where the wavenumber depends on the frequency. For $\omega<\omega_p$, $k$ is imaginary representing pure exponential decay.

Poynting's theorem for this medium is

$$\begin{aligned}\nabla\cdot\boldsymbol{S}+\frac{\partial}{\partial t}\left(\tfrac{1}{2}\varepsilon|\mathbf{E}|^2+\tfrac{1}{2}\mu|\mathbf{H}|^2\right)&=-\mathbf{E}\cdot\mathbf{J}_f=-\frac{1}{\omega_p^2\varepsilon}\mathbf{J}_f\cdot\frac{\partial\mathbf{J}_f}{\partial t}\\&=-\frac{\partial}{\partial t}\left(\frac{1}{\omega_p^2\varepsilon}\frac{|\mathbf{J}_f|^2}{2}\right)\end{aligned} \tag{21}$$

Because this system is lossless, the right-hand side of (21) can be brought to the left-hand side and lumped with the energy densities:

$$\nabla\cdot\boldsymbol{S}+\frac{\partial}{\partial t}\left[\tfrac{1}{2}\varepsilon|\mathbf{E}|^2+\tfrac{1}{2}\mu|\mathbf{H}|^2+\frac{1}{2}\frac{1}{\omega_p^2\varepsilon}|\mathbf{J}_f|^2\right]=0 \tag{22}$$

This new energy term just represents the kinetic energy density of the charge carriers since their velocity is related to the current density as

$$\mathbf{J}_f=qn\mathbf{v}\Rightarrow\frac{1}{2}\frac{1}{\omega_p^2\varepsilon}|\mathbf{J}_f|^2=\tfrac{1}{2}mn|\mathbf{v}|^2 \tag{23}$$

### 7-4-5 Dispersive Media

When the wavenumber is not proportional to the frequency of the wave, the medium is said to be dispersive. A nonsinusoidal time signal (such as a square wave) will change shape and become distorted as the wave propagates because each Fourier component of the signal travels at a different speed.

To be specific, consider a stationary current sheet source at $z=0$ composed of two signals with slightly different frequencies:

$$\begin{aligned}K(t)&=K_0[\cos(\omega_0+\Delta\omega)t+\cos(\omega_0-\Delta\omega)t]\\&=2K_0\cos\Delta\omega t\cos\omega_0 t\end{aligned} \tag{24}$$

With $\Delta\omega\ll\omega$ the fast oscillations at frequency $\omega_0$ are modulated by the slow envelope function at frequency $\Delta\omega$. In a linear dielectric medium this wave packet would propagate away from the current sheet at the speed of light, $c=1/\sqrt{\varepsilon\mu}$).

If the medium is dispersive with the wavenumber $k(\omega)$ being a function of $\omega$, each frequency component in (24) travels at a slightly different speed. Since each frequency is very close to $\omega_0$ we expand $k(\omega)$ as

$$\begin{aligned} k(\omega_0+\Delta\omega) &\approx k(\omega_0)+\frac{dk}{d\omega}\bigg|_{\omega_0}\Delta\omega \\ k(\omega_0-\Delta\omega) &\approx k(\omega_0)-\frac{dk}{d\omega}\bigg|_{\omega_0}\Delta\omega \end{aligned} \tag{25}$$

where for propagation $k(\omega_0)$ must be real.

The fields for waves propagating in the $+z$ direction are then of the following form:

$$\begin{aligned} E_x(z,t) &= \mathrm{Re}\,\hat{E}_0\left(\exp\left\{j\left[(\omega_0+\Delta\omega)t-\left(k(\omega_0)+\frac{dk}{d\omega}\bigg|_{\omega_0}\Delta\omega\right)z\right]\right\}\right. \\ &\quad \left.+\exp\left\{j\left[(\omega_0-\Delta\omega)t-\left(k(\omega_0)-\frac{dk}{d\omega}\bigg|_{\omega_0}\Delta\omega\right)z\right]\right\}\right) \\ &= \mathrm{Re}\left(\hat{E}_0\exp\{j[\omega_0 t-k(\omega_0)z]\}\left\{\exp\left[j\,\Delta\omega\left(t-\frac{dk}{d\omega}\bigg|_{\omega_0}z\right)\right]\right.\right. \\ &\quad \left.\left.+\exp\left[-j\,\Delta\omega\left(t-\frac{dk}{d\omega}\bigg|_{\omega_0}z\right)\right]\right\}\right) \\ &= 2E_0\cos(\omega_0 t-k(\omega_0)z)\cos\Delta\omega\left(t-\frac{dk}{d\omega}\bigg|_{\omega_0}z\right) \end{aligned} \tag{26}$$

where without loss of generality we assume in the last relation that $\hat{E}_0=E_0$ is real. This result is plotted in Figure 7-10 as a function of $z$ for fixed time. The fast waves with argument $\omega_0 t-k(\omega_0)z$ travel at the phase speed $v_p=\omega_0/k(\omega_0)$ through the modulating envelope with argument $\Delta\omega(t-dk/d\omega|_{\omega_0}z)$. This envelope itself travels at the slow speed

$$t-\frac{dk}{d\omega}\bigg|_{\omega_0}z=\text{const}\Rightarrow\frac{dz}{dt}=v_g=\frac{d\omega}{dk}\bigg|_{\omega_0} \tag{27}$$

known as the group velocity, for it is the velocity at which a packet of waves within a narrow frequency band around $\omega_0$ will travel.

For linear media the group and phase velocities are equal:

$$\begin{aligned} \omega = kc &\Rightarrow v_p=\frac{\omega}{k}=c \\ v_g &= \frac{d\omega}{dk}=c \end{aligned} \tag{28}$$

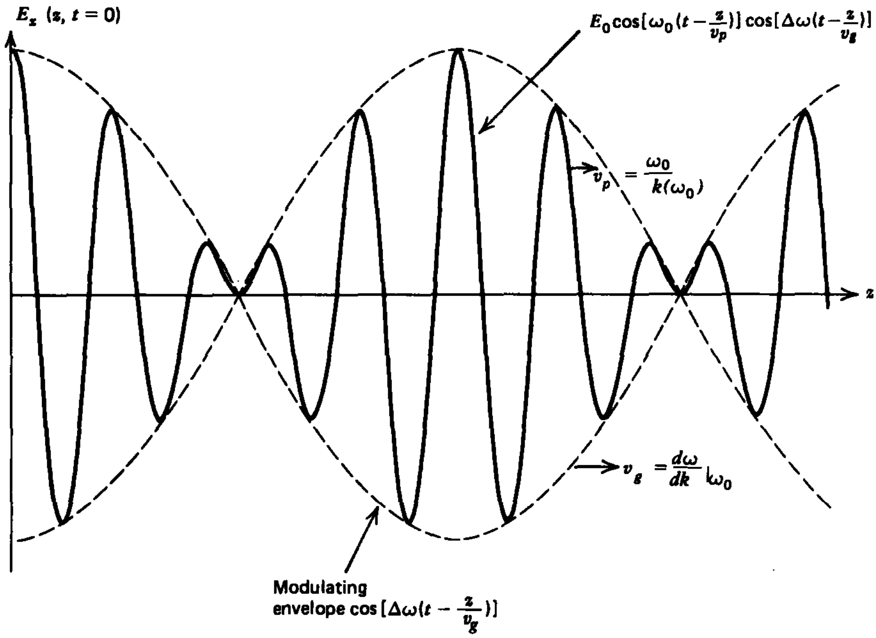



Figure 7-10 In a dispersive medium the shape of the waves becomes distorted so the velocity of a wave is not uniquely defined. For a group of signals within a narrow frequency band the modulating envelope travels at the group velocity $v_g$. The signal within the envelope propagates through at the phase velocity $v_p$.

while from Section 7-4-4 in the high-frequency limit for conductors, we see that

$$\begin{aligned} \omega^2 = k^2c^2 + \omega_p^2 \Rightarrow v_p &= \frac{\omega}{k} \\ v_g &= \frac{d\omega}{dk} = \frac{k}{\omega}c^2 \end{aligned} \tag{29}$$

where the velocities only make sense when $k$ is real so that $\omega > \omega_p$. Note that in this limit

$$v_g v_p = c^2 \tag{30}$$

Group velocity only has meaning in a dispersive medium when the signals of interest are clustered over a narrow frequency range so that the slope defined by (27) is approximately constant and real.

### 7-4-6 Polarization

The two independent sets of solutions of Section 7-3-1 both have their power flow $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ in the $z$ direction. One solution is said to have its electric field polarized in the $x$ direction while the second has its electric field polarized in the $y$ direction. Each solution alone is said to be linearly polarized because the electric field always points in the same direction for all time. If both field solutions are present, the direction of net electric field varies with time. In particular, let us say that the $x$ and $y$ components of electric field at any value of $z$ differ in phase by angle $\phi$:

$$\mathbf{E} = \text{Re}\,[E_{x_0}\mathbf{i}_x + E_{y_0}\,e^{j\phi}\mathbf{i}_y]\,e^{j\omega t} = E_{x_0}\cos\omega t\,\mathbf{i}_x + E_{y_0}\cos(\omega t+\phi)\mathbf{i}_y \tag{31}$$

We can eliminate time as a parameter, realizing from (31) that

$$\cos\omega t = E_x/E_{x_0}$$
$$\sin\omega t = \frac{\cos\omega t\cos\phi - E_y/E_{y_0}}{\sin\phi} = \frac{(E_x/E_{x_0})\cos\phi - E_y/E_{y_0}}{\sin\phi} \tag{32}$$

and using the identity that

$$\sin^2\omega t + \cos^2\omega t$$
$$= 1 = \left(\frac{E_x}{E_{x_0}}\right)^2 + \frac{(E_x/E_{x_0})^2\cos^2\phi + (E_y/E_{y_0})^2 - (2E_xE_y/E_{x_0}E_{y_0})\cos\phi}{\sin^2\phi} \tag{33}$$

to give us the equation of an ellipse relating $E_x$ to $E_y$:

$$\left(\frac{E_x}{E_{x_0}}\right)^2 + \left(\frac{E_y}{E_{y_0}}\right)^2 - \frac{2E_xE_y}{E_{x_0}E_{y_0}}\cos\phi = \sin^2\phi \tag{34}$$

as plotted in Figure 7-11*a*. As time increases the electric field vector traces out an ellipse each period so this general case of the superposition of two linear polarizations with arbitrary phase $\phi$ is known as elliptical polarization. There are two important special cases:

**(a) Linear Polarization**

If $E_x$ and $E_y$ are in phase so that $\phi = 0$, (34) reduces to

$$\left(\frac{E_x}{E_{x_0}} - \frac{E_y}{E_{y_0}}\right)^2 = 0 \Rightarrow \tan\theta = \frac{E_y}{E_x} = \frac{E_{y_0}}{E_{x_0}} \tag{35}$$

The electric field at all times is at a constant angle $\theta$ to the $x$ axis. The electric field amplitude oscillates with time along this line, as in Figure 7-11*b*. Because its direction is always along the same line, the electric field is linearly polarized.

**(b) Circular Polarization**

If both components have equal amplitudes but are 90° out of phase,

$$E_{x_0} = E_{y_0} \equiv E_0, \qquad \phi = \pm\pi/2 \tag{36}$$

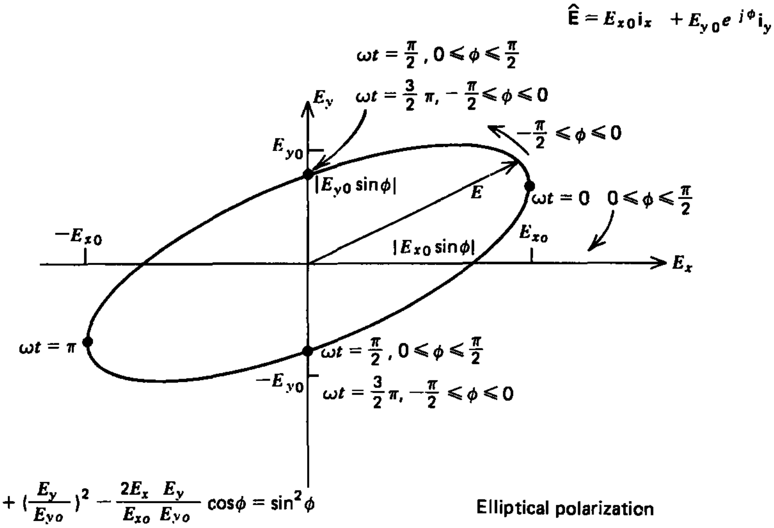



Figure 7-11 (*a*) Two perpendicular field components with phase difference $\phi$ have the tip of the net electric field vector tracing out an ellipse each period. (*b*) If both field components are in phase, the ellipse reduces to a straight line. (*c*) If the field components have the same magnitude but are 90° out of phase, the ellipse becomes a circle. The polarization is left circularly polarized to $z$-directed power flow if the electric field rotates clockwise and is (*d*) right circularly polarized if it rotates counterclockwise.

(34) reduces to the equation of a circle:

$$E_x^2 + E_y^2 = E_0^2 \tag{37}$$

The tip of the electric field vector traces out a circle as time evolves over a period, as in Figure 7-11*c*. For the upper (+) sign for $\phi$ in (36), the electric field rotates clockwise while the negative sign has the electric field rotating counterclockwise. These cases are, respectively, called left and right circular polarization for waves propagating in the $+z$ direction as found by placing the thumb of either hand in the direction of power flow. The fingers on the left hand curl in the direction of the rotating field for left circular polarization, while the fingers of the right hand curl in the direction of the rotating field for right circular polarization. Left and right circular polarizations reverse for waves traveling in the $-z$ direction.

### 7-4-7 Wave Propagation in Anisotropic Media

Many properties of plane waves have particular applications to optics. Because visible light has a wavelength on the order of 500 nm, even a pencil beam of light 1 mm wide is 2000 wavelengths wide and thus approximates a plane wave.

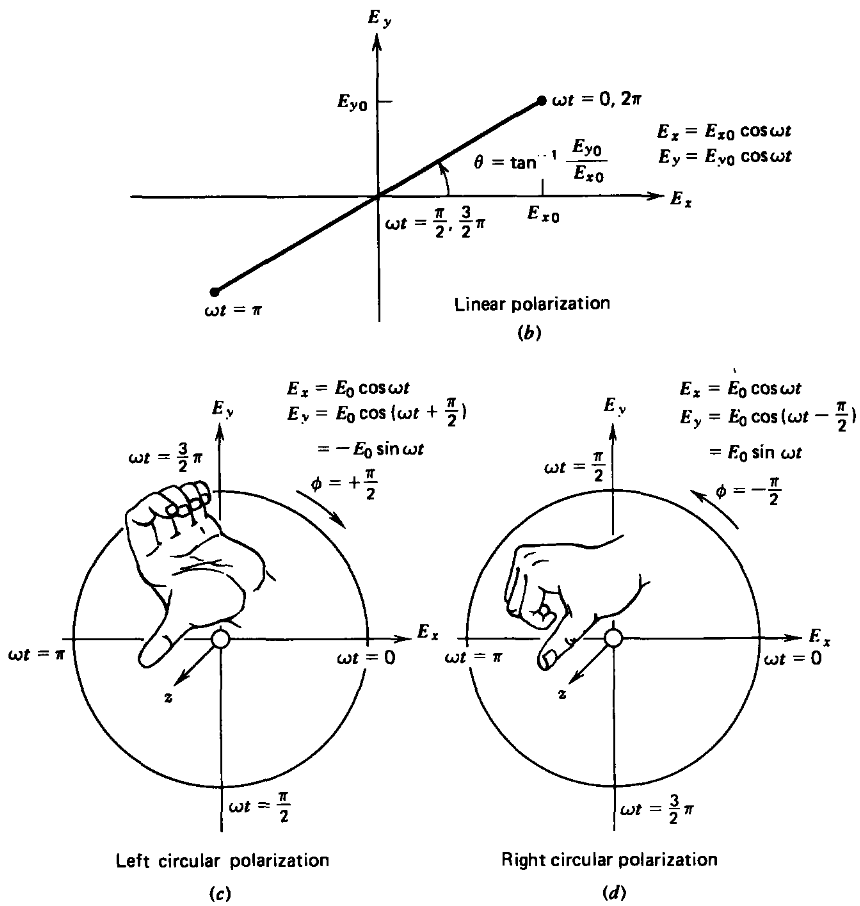



Figure 7-11

**(a) Polarizers**

Light is produced by oscillating molecules whether in a light bulb or by the sun. This natural light is usually unpolarized as each molecule oscillates in time and direction independent of its neighbors so that even though the power flow may be in a single direction the electric field phase changes randomly with time and the source is said to be incoherent. Lasers, an acronym for "light amplification by stimulated emission of radiation," emits coherent light by having all the oscillating molecules emit in time phase.

A polarizer will only pass those electric field components aligned with the polarizer's transmission axis so that the transmitted light is linearly polarized. Polarizers are made of such crystals as tourmaline, which exhibit dichroism—the selective absorption of the polarization along a crystal axis.

The polarization perpendicular to this axis is transmitted.

Because tourmaline polarizers are expensive, fragile, and of small size, improved low cost and sturdy sheet polarizers were developed by embedding long needlelike crystals or chainlike molecules in a plastic sheet. The electric field component in the long direction of the molecules or crystals is strongly absorbed while the perpendicular component of the electric field is passed.

For an electric field of magnitude $E_0$ at angle $\phi$ to the transmission axis of a polarizer, the magnitude of the transmitted field is

$$E_t = E_0 \cos \phi \tag{38}$$

so that the time-average power flux density is

$$\begin{aligned} <S> &= \left|\tfrac{1}{2}\operatorname{Re}[\hat{\mathbf{E}}(\mathbf{r})\times\hat{\mathbf{H}}^*(\mathbf{r})]\right| \\ &= \frac{1}{2}\frac{E_0^2}{\eta}\cos^2\phi \end{aligned} \tag{39}$$

which is known as the law of Malus.

**(b) Double Refraction (Birefringence)**

If a second polarizer, now called the analyzer, is placed parallel to the first but with its transmission axis at right angles, as in Figure 7-12, no light is transmitted. The combination is called a polariscope. However, if an anisotropic crystal is inserted between the polarizer and analyzer, light is transmitted through the analyzer. In these doubly refracting crystals, light polarized along the optic axis travels at speed $c_\parallel$ while light polarized perpendicular to the axis travels at a slightly different speed $c_\perp$. The crystal is said to be birefringent. If linearly polarized light is incident at 45° to the axis,

$$\mathbf{E}(z=0, t) = E_0(\mathbf{i}_x + \mathbf{i}_y)\operatorname{Re}(e^{j\omega t}) \tag{40}$$

the components of electric field along and perpendicular to the axis travel at different speeds:

$$\begin{aligned} E_x(z, t) &= E_0 \operatorname{Re}(e^{j(\omega t - k_\parallel z)}), \qquad k_\parallel = \omega/c_\parallel \\ E_y(z, t) &= E_0 \operatorname{Re}(e^{j(\omega t - k_\perp z)}), \qquad k_\perp = \omega/c_\perp \end{aligned} \tag{41}$$

After exiting the crystal at $z = l$, the total electric field is

$$\begin{aligned} \mathbf{E}(z=l, t) &= E_0 \operatorname{Re}[e^{j\omega t}(e^{-jk_\parallel l}\mathbf{i}_x + e^{-jk_\perp l}\mathbf{i}_y)] \\ &= E_0 \operatorname{Re}[e^{j(\omega t - k_\parallel l)}(\mathbf{i}_x + e^{j(k_\parallel - k_\perp)l}\mathbf{i}_y)] \end{aligned} \tag{42}$$

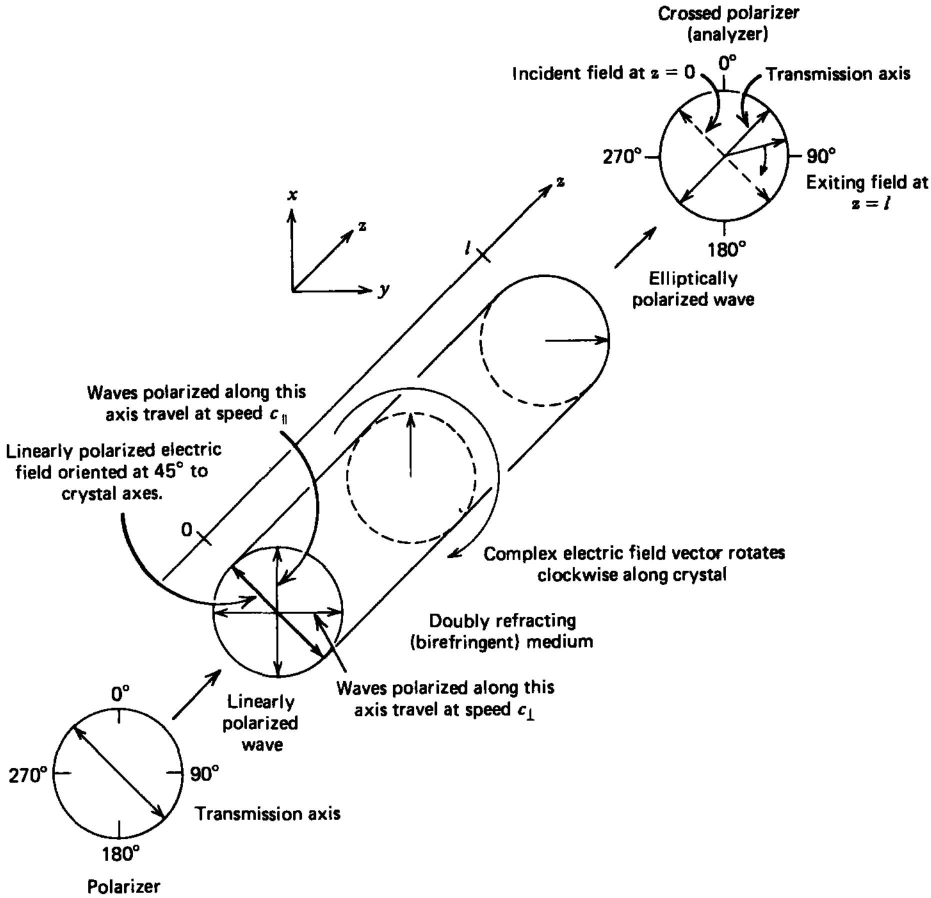



Figure 7-12 When a linearly polarized wave passes through a doubly refracting (birefringent) medium at an angle to the crystal axes, the transmitted light is elliptically polarized.

which is of the form of (31) for an elliptically polarized wave where the phase difference is

$$\phi = (k_\parallel - k_\perp)l = \omega l\left(\frac{1}{c_\parallel} - \frac{1}{c_\perp}\right) \tag{43}$$

When $\phi$ is an integer multiple of $2\pi$, the light exiting the crystal is the same as if the crystal were not there so that it is not transmitted through the analyzer. If $\phi$ is an odd integer multiple of $\pi$, the exiting light is also linearly polarized but perpendicularly to the incident light so that it is polarized in the same direction as the transmission axis of the analyzer, and thus is transmitted. Such elements are called half-wave plates at the frequency of operation. When $\phi$ is an odd integer multiple of $\pi/2$, the exiting light is circularly polarized and the crystal serves as a quarter-wave plate. However, only that polarization of light along the transmission axis of the analyzer is transmitted.

Double refraction occurs naturally in many crystals due to their anisotropic molecular structure. Many plastics and glasses that are generally isotropic have induced birefringence when mechanically stressed. When placed within a polariscope the photoelastic stress patterns can be seen. Some liquids, notably nitrobenzene, also become birefringent when stressed by large electric fields. This phenomena is called the Kerr effect. Electro-optical measurements allow electric field mapping in the dielectric between high voltage stressed electrodes, useful in the study of high voltage conduction and breakdown phenomena. The Kerr effect is also used as a light switch in high-speed shutters. A parallel plate capacitor is placed within a polariscope so that in the absence of voltage no light is transmitted. When the voltage is increased the light is transmitted, being a maximum when $\phi = \pi$. (See problem 17.)

## 7-5 NORMAL INCIDENCE ONTO A PERFECT CONDUCTOR

A uniform plane wave with $x$-directed electric field is normally incident upon a perfectly conducting plane at $z = 0$, as shown in Figure 7-13. The presence of the boundary gives rise to a reflected wave that propagates in the $-z$ direction. There are no fields within the perfect conductor. The known incident fields traveling in the $+z$ direction can be written as

$$\begin{aligned} \mathbf{E}_i(z,t) &= \mathrm{Re}\,(\hat{E}_i\, e^{j(\omega t - kz)}\mathbf{i}_x) \\ \mathbf{H}_i(z,t) &= \mathrm{Re}\left(\frac{\hat{E}_i}{\eta_0}\, e^{j(\omega t - kz)}\mathbf{i}_y\right) \end{aligned} \tag{1}$$

while the reflected fields propagating in the $-z$ direction are similarly

$$\begin{aligned} \mathbf{E}_r(z,t) &= \mathrm{Re}\,(\hat{E}_r\, e^{j(\omega t + kz)}\mathbf{i}_x) \\ \mathbf{H}_r(z,t) &= \mathrm{Re}\left(\frac{-\hat{E}_r}{\eta_0}\, e^{j(\omega t + kz)}\mathbf{i}_y\right) \end{aligned} \tag{2}$$

where in the lossless free space

$$\eta_0 = \sqrt{\mu_0/\varepsilon_0}, \qquad k = \omega\sqrt{\varepsilon_0\mu_0} \tag{3}$$

Note the minus sign difference in the spatial exponential phase factors of (1) and (2) as the waves are traveling in opposite directions. The amplitude of incident and reflected magnetic fields are given by the ratio of electric field amplitude to the wave impedance, as derived in Eq. (15) of Section

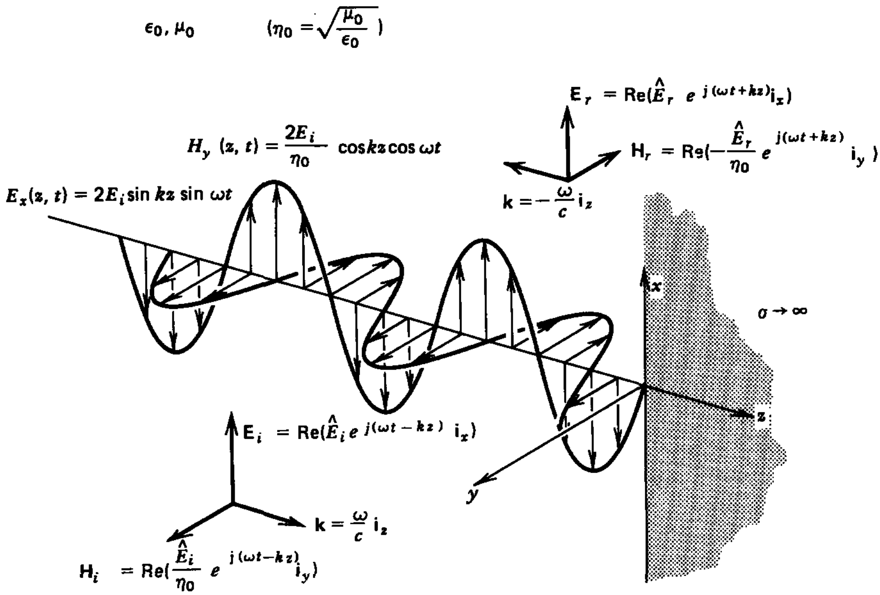



Figure 7-13 A uniform plane wave normally incident upon a perfect conductor has zero electric field at the conducting surface thus requiring a reflected wave. The source of this reflected wave is the surface current at $z=0$, which equals the magnetic field there. The total electric and magnetic fields are 90° out of phase in time and space.

7-3-2. The negative sign in front of the reflected magnetic field for the wave in the $-z$ direction arises because the power flow $\boldsymbol{S}_r = \mathbf{E}_r \times \mathbf{H}_r$ in the reflected wave must also be in the $-z$ direction.

The total electric and magnetic fields are just the sum of the incident and reflected fields. The only unknown parameter $E_r$ can be evaluated from the boundary condition at $z=0$ where the tangential component of $\mathbf{E}$ must be continuous and thus zero along the perfect conductor:

$$\hat{E}_i + \hat{E}_r = 0 \Rightarrow \hat{E}_r = -\hat{E}_i \tag{4}$$

The total fields are then the sum of the incident and reflected fields

$$\begin{aligned}
\mathbf{E}_x(z, t) &= \mathbf{E}_i(z, t) + \mathbf{E}_r(z, t) \\
&= \mathrm{Re}\,[\hat{E}_i(e^{-jkz} - e^{+jkz})\, e^{j\omega t}] \\
&= 2E_i \sin kz \sin \omega t \\
\mathbf{H}_y(z, t) &= \mathbf{H}_i(z, t) + \mathbf{H}_r(z, t) \\
&= \mathrm{Re}\left(\frac{\hat{E}_i}{\eta_0}(e^{-jkz} + e^{+jkz})\, e^{j\omega t}\right) \\
&= \frac{2E_i}{\eta_0} \cos kz \cos \omega t
\end{aligned} \tag{5}$$

where we take $\hat{E}_i = E_i$ to be real. The electric and magnetic fields are 90° out of phase with each other both in time and space. We note that the two oppositely traveling wave solutions combined for a standing wave solution. The total solution does not propagate but is a standing sinusoidal solution in space whose amplitude varies sinusoidally in time.

A surface current flows on the perfect conductor at $z=0$ due to the discontinuity in tangential component of $\mathbf{H}$,

$$K_x = H_y(z=0) = \frac{2E_i}{\eta_0}\cos\omega t \tag{6}$$

giving rise to a force per unit area on the conductor,

$$\mathbf{F} = \tfrac{1}{2}\mathbf{K}\times\mu_0\mathbf{H} = \tfrac{1}{2}\mu_0 H_y^2(z=0)\mathbf{i}_z = 2\varepsilon_0 E_i^2 \cos^2\omega t\,\mathbf{i}_z \tag{7}$$

known as the radiation pressure. The factor of $\frac{1}{2}$ arises in (7) because the force on a surface current is proportional to the average value of magnetic field on each side of the interface, here being zero for $z=0_+$.

## 7-6 NORMAL INCIDENCE ONTO A DIELECTRIC

### 7-6-1 Lossless Dielectric

We replace the perfect conductor with a lossless dielectric of permittivity $\varepsilon_2$ and permeability $\mu_2$, as in Figure 7-14, with a uniform plane wave normally incident from a medium with permittivity $\varepsilon_1$ and permeability $\mu_1$. In addition to the incident and reflected fields for $z<0$, there are transmitted fields which propagate in the $+z$ direction within the medium for $z>0$:

$$\left.\begin{aligned}
\mathbf{E}_i(z,t) &= \operatorname{Re}[\hat{E}_i e^{j(\omega t - k_1 z)}\mathbf{i}_x], & k_1 &= \omega\sqrt{\varepsilon_1\mu_1}\\
\mathbf{H}_i(z,t) &= \operatorname{Re}\left[\frac{\hat{E}_i}{\eta_1}e^{j(\omega t - k_1 z)}\mathbf{i}_y\right], & \eta_1 &= \sqrt{\frac{\mu_1}{\varepsilon_1}}\\
\mathbf{E}_r(z,t) &= \operatorname{Re}[\hat{E}_r e^{j(\omega t + k_1 z)}\mathbf{i}_x]\\
\mathbf{H}_r(z,t) &= \operatorname{Re}\left[-\frac{\hat{E}_r}{\eta_1}e^{j(\omega t + k_1 z)}\mathbf{i}_y\right]
\end{aligned}\right\}\quad z<0$$

$$\left.\begin{aligned}
\mathbf{E}_t(z,t) &= \operatorname{Re}[\hat{E}_t e^{j(\omega t - k_2 z)}\mathbf{i}_x], & k_2 &= \omega\sqrt{\varepsilon_2\mu_2}\\
\mathbf{H}_t(z,t) &= \operatorname{Re}\left[\frac{\hat{E}_t}{\eta_2}e^{j(\omega t - k_2 z)}\mathbf{i}_y\right], & \eta_2 &= \sqrt{\frac{\mu_2}{\varepsilon_2}}.
\end{aligned}\right\}\quad z>0 \tag{1}$$

It is necessary in (1) to use the appropriate wavenumber and impedance within each region. There is no wave traveling in the $-z$ direction in the second region as we assume no boundaries or sources for $z>0$.

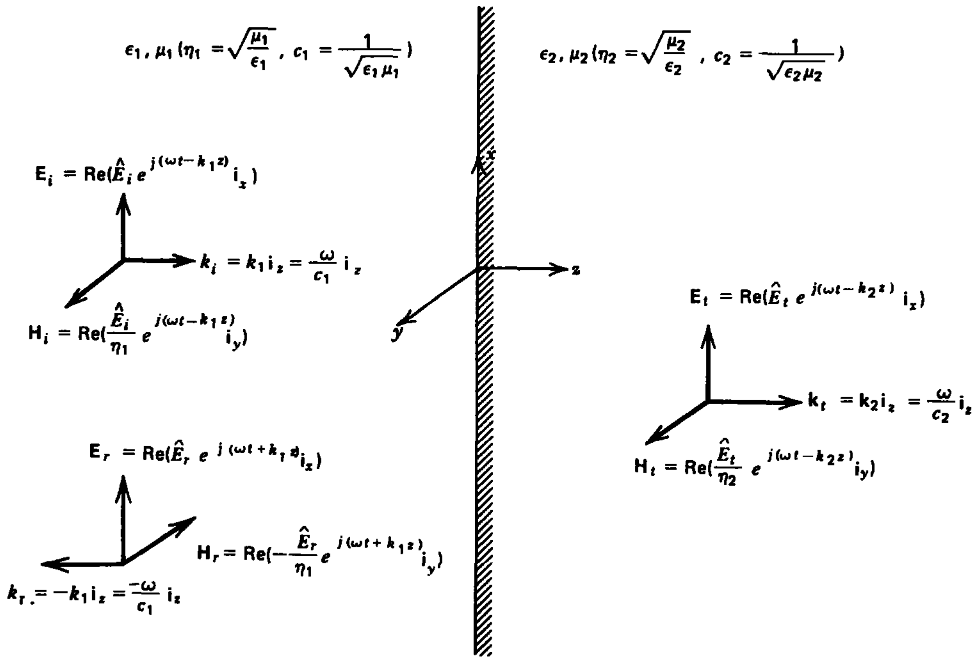



Figure 7-14 A uniform plane wave normally incident upon a dielectric interface separating two different materials has part of its power reflected and part transmitted.

The unknown quantities $\hat{E}_r$ and $\hat{E}_t$ can be found from the boundary conditions of continuity of tangential **E** and **H** at $z=0$,

$$\begin{gathered}\hat{E}_i+\hat{E}_r=\hat{E}_t\\ \frac{\hat{E}_i-\hat{E}_r}{\eta_1}=\frac{\hat{E}_t}{\eta_2}\end{gathered}\tag{2}$$

from which we find the reflection $R$ and transmission $T$ field coefficients as

$$\begin{aligned}R&=\frac{\hat{E}_r}{\hat{E}_i}=\frac{\eta_2-\eta_1}{\eta_2+\eta_1}\\ T&=\frac{\hat{E}_t}{\hat{E}_i}=\frac{2\eta_2}{\eta_2+\eta_1}\end{aligned}\tag{3}$$

where from (2)

$$1+R=T\tag{4}$$

If both mediums have the same wave impedance, $\eta_1=\eta_2$, there is no reflected wave.

### 7-6-2 Time-Average Power Flow

The time-average power flow in the region $z<0$ is

$$\begin{aligned}<S_{zi}> &= \tfrac{1}{2}\,\mathrm{Re}\,[\hat{E}_x(z)\hat{H}_y^*(z)]\\ &= \frac{1}{2\eta_1}\,\mathrm{Re}\,[\hat{E}_i e^{-jk_1z}+\hat{E}_r e^{+jk_1z}][\hat{E}_i^* e^{+jk_1z}-\hat{E}_r^* e^{-jk_1z}]\\ &= \frac{1}{2\eta_1}[|\hat{E}_i|^2-|\hat{E}_r|^2]\\ &\quad +\frac{1}{2\eta_1}\underbrace{\mathrm{Re}\,[\hat{E}_r\hat{E}_i^* e^{+2jk_1z}-\hat{E}_r^*\hat{E}_i e^{-2jk_1z}]}_{0}\end{aligned} \tag{5}$$

The last term on the right-hand side of (5) is zero as it is the difference between a number and its complex conjugate, which is pure imaginary and equals $2j$ times its imaginary part. Being pure imaginary, its real part is zero. Thus the time-average power flow just equals the difference in the power flows in the incident and reflected waves as found more generally in Section 7-3-2. The coupling terms between oppositely traveling waves have no time-average yielding the simple superposition of time-average powers:

$$\begin{aligned}<S_{zi}> &= \frac{1}{2\eta_1}[|\hat{E}_i|^2-|\hat{E}_r|^2]\\ &= \frac{|\hat{E}_i|^2}{2\eta_1}[1-R^2]\end{aligned} \tag{6}$$

This net time-average power flows into the dielectric medium, as it also equals the transmitted power;

$$<S_{zt}> = \frac{1}{2\eta_2}|\hat{E}_t|^2 = \frac{|\hat{E}_i|^2T^2}{2\eta_2} = \frac{|\hat{E}_i|^2}{2\eta_1}[1-R^2] \tag{7}$$

### 7-6-3 Lossy Dielectric

If medium 2 is lossy with Ohmic conductivity $\sigma$, the solutions of (3) are still correct if we replace the permittivity $\varepsilon_2$ by the complex permittivity $\hat{\varepsilon}_2$,

$$\hat{\varepsilon}_2 = \varepsilon_2\left(1+\frac{\sigma}{j\omega\varepsilon_2}\right) \tag{8}$$

so that the wave impedance in region 2 is complex:

$$\eta_2 = \sqrt{\mu_2/\hat{\varepsilon}_2} \tag{9}$$

We can easily explore the effect of losses in the low and large loss limits.

**(a) Low Losses**

If the Ohmic conductivity is small, we can neglect it in all terms except in the wavenumber $k_2$:

$$\lim_{\sigma/\omega\varepsilon_2 \ll 1} k_2 \approx \omega\sqrt{\varepsilon_2\mu_2} - \frac{j}{2}\sigma\sqrt{\frac{\mu_2}{\varepsilon_2}} \tag{10}$$

The imaginary part of $k_2$ gives rise to a small rate of exponential decay in medium 2 as the wave propagates away from the $z=0$ boundary.

**(b) Large Losses**

For large conductivities so that the displacement current is negligible in medium 2, the wavenumber and impedance in region 2 are complex:

$$\lim_{\sigma/\omega\varepsilon_2 \gg 1} \begin{cases} k_2 = \dfrac{1-j}{\delta}, & \delta = \sqrt{\dfrac{2}{\omega\mu_2\sigma}} \\ \eta_2 = \sqrt{\dfrac{j\omega\mu_2}{\sigma}} = \dfrac{1+j}{\sigma\delta} \end{cases} \tag{11}$$

The fields decay within a characteristic distance equal to the skin depth $\delta$. This is why communications to submerged submarines are difficult. For seawater, $\mu_2 = \mu_0 = 4\pi \times 10^{-7}$ henry/m and $\sigma \approx 4$ siemens/m so that for 1 MHz signals, $\delta \approx 0.25$ m. However, at 100 Hz the skin depth increases to 25 meters. If a submarine is within this distance from the surface, it can receive the signals. However, it is difficult to transmit these low frequencies because of the large free space wavelength, $\lambda \approx 3 \times 10^6$ m. Note that as the conductivity approaches infinity,

$$\lim_{\sigma\to\infty} \begin{cases} k_2 = \infty \\ \eta_2 = 0 \end{cases} \Rightarrow \begin{cases} R = -1 \\ T = 0 \end{cases} \tag{12}$$

so that the field solution approaches that of normal incidence upon a perfect conductor found in Section 7-5.

## EXAMPLE 7-1 DIELECTRIC COATING

A thin lossless dielectric with permittivity $\varepsilon$ and permeability $\mu$ is coated onto the interface between two infinite half-spaces of lossless media with respective properties $(\varepsilon_1, \mu_1)$ and $(\varepsilon_2, \mu_2)$, as shown in Figure 7-15. What coating parameters $\varepsilon$ and $\mu$ and thickness $d$ will allow all the time-average power

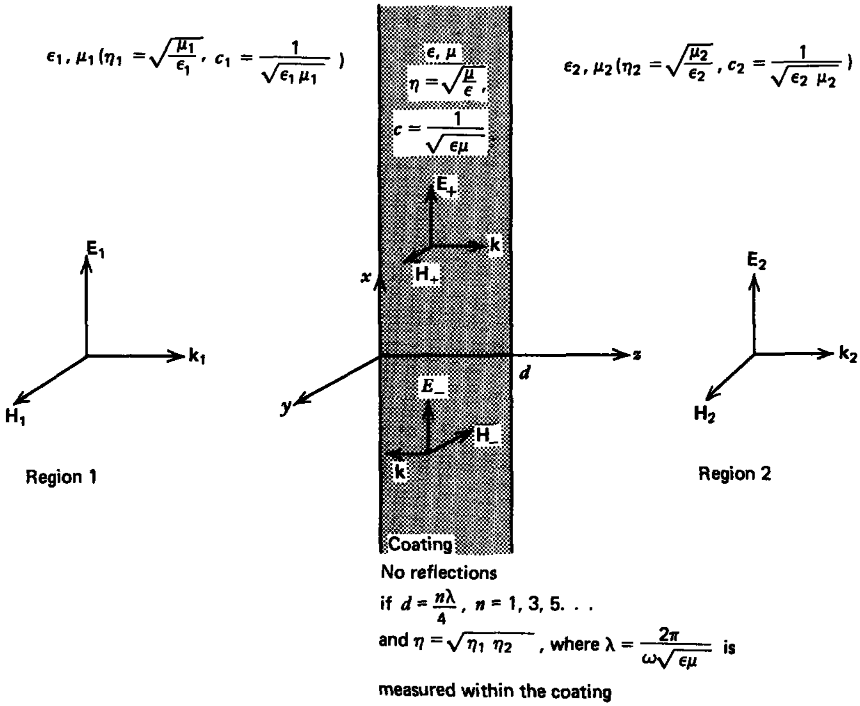



Figure 7-15 A suitable dielectric coating applied on the interface of discontinuity between differing media can eliminate reflections at a given frequency.

from region 1 to be transmitted through the coating to region 2? Such coatings are applied to optical components such as lenses to minimize unwanted reflections and to maximize the transmitted light intensity.

## SOLUTION

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For all the incident power to be transmitted into region 2, there can be no reflected field in region 1, although we do have oppositely traveling waves in the coating due to the reflection at the second interface. Region 2 only has positively $z$-directed power flow. The fields in each region are thus of the following form:

*Region 1*

$$\mathbf{E}_1 = \mathrm{Re}\,[\hat{E}_1\, e^{j(\omega t - k_1 z)}\mathbf{i}_x], \qquad k_1 = \omega/c_1 = \omega\sqrt{\varepsilon_1\mu_1}$$

$$\mathbf{H}_1 = \mathrm{Re}\left[\frac{\hat{E}_1}{\eta_1}\, e^{j(\omega t - k_1 z)}\mathbf{i}_y\right], \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

*Coating*

$$\mathbf{E}_+ = \mathrm{Re}\,[\hat{E}_+\, e^{j(\omega t - kz)}\mathbf{i}_x], \qquad k = \omega/c = \omega\sqrt{\varepsilon\mu}$$

$$\mathbf{H}_+ = \mathrm{Re}\left[\frac{\hat{E}_+}{\eta}\, e^{j(\omega t - kz)}\mathbf{i}_y\right], \qquad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\mathbf{E}_- = \mathrm{Re}\,[\hat{E}_-\, e^{j(\omega t + kz)}\mathbf{i}_x]$$

$$\mathbf{H}_- = \mathrm{Re}\left[-\frac{\hat{E}_-}{\eta}\, e^{j(\omega t + kz)}\mathbf{i}_y\right]$$

*Region 2*

$$\mathbf{E}_2 = \mathrm{Re}\,[\hat{E}_2\, e^{j(\omega t - k_2 z)}\mathbf{i}_x], \qquad k_2 = \omega/c_2 = \omega\sqrt{\varepsilon_2\mu_2}$$

$$\mathbf{H}_2 = \mathrm{Re}\left[\frac{\hat{E}_2}{\eta_2}\, e^{j(\omega t - k_2 z)}\mathbf{i}_y\right], \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

Continuity of tangential **E** and **H** at $z = 0$ and $z = d$ requires

$$\hat{E}_1 = \hat{E}_+ + \hat{E}_-, \qquad \frac{\hat{E}_1}{\eta_1} = \frac{\hat{E}_+ - \hat{E}_-}{\eta}$$

$$\hat{E}_+\, e^{-jkd} + \hat{E}_-\, e^{+jkd} = \hat{E}_2\, e^{-jk_2 d}$$

$$\frac{\hat{E}_+\, e^{-jkd} - \hat{E}_-\, e^{+jkd}}{\eta} = \frac{\hat{E}_2\, e^{-jk_2 d}}{\eta_2}$$

Each of these amplitudes in terms of $\hat{E}_1$ is then

$$\hat{E}_+ = \frac{1}{2}\left(1 + \frac{\eta}{\eta_1}\right)\hat{E}_1$$

$$\hat{E}_- = \frac{1}{2}\left(1 - \frac{\eta}{\eta_1}\right)\hat{E}_1$$

$$\hat{E}_2 = e^{jk_2 d}[\hat{E}_+\, e^{-jkd} + \hat{E}_-\, e^{+jkd}]$$

$$= \frac{\eta_2}{\eta}\, e^{jk_2 d}[\hat{E}_+\, e^{-jkd} - \hat{E}_-\, e^{+jkd}]$$

Solving this last relation self-consistently requires that

$$\hat{E}_+\, e^{-jkd}\left(1 - \frac{\eta_2}{\eta}\right) + \hat{E}_-\, e^{jkd}\left(1 + \frac{\eta_2}{\eta}\right) = 0$$

Writing $\hat{E}_+$ and $\hat{E}_-$ in terms of $\hat{E}_1$ yields

$$\left(1 + \frac{\eta}{\eta_1}\right)\left(1 - \frac{\eta_2}{\eta}\right) + e^{2jkd}\left(1 + \frac{\eta_2}{\eta}\right)\left(1 - \frac{\eta}{\eta_1}\right) = 0$$

Since this relation is complex, the real and imaginary parts must separately be satisfied. For the imaginary part to be zero requires that the coating thickness $d$ be an integral number of quarter wavelengths as measured within the coating,

$$2kd = n\pi \Rightarrow d = n\lambda/4, \qquad n = 1, 2, 3, \ldots$$

The real part then requires

$$\left(1+\frac{\eta}{\eta_1}\right)\left(1-\frac{\eta_2}{\eta}\right) \pm \left(1+\frac{\eta_2}{\eta}\right)\left(1-\frac{\eta}{\eta_1}\right) = 0 \begin{cases} n \text{ even} \\ n \text{ odd} \end{cases}$$

For the upper sign where $d$ is a multiple of half-wavelengths the only solution is

$$\eta_2 = \eta_1 \quad (d = n\lambda/4, \quad n = 2, 4, 6, \ldots)$$

which requires that media 1 and 2 be the same so that the coating serves no purpose. If regions 1 and 2 have differing wave impedances, we must use the lower sign where $d$ is an odd integer number of quarter wavelengths so that

$$\eta^2 = \eta_1\eta_2 \Rightarrow \eta = \sqrt{\eta_1\eta_2} \quad (d = n\lambda/4, \quad n = 1, 3, 5, \ldots)$$

Thus, if the coating is a quarter wavelength thick as measured within the coating, or any odd integer multiple of this thickness with its wave impedance equal to the geometrical average of the impedances in each adjacent region, all the time-average power flow in region 1 passes through the coating into region 2:

$$\begin{aligned} \langle S_z \rangle &= \frac{1}{2}\frac{|\hat{E}_1|^2}{\eta_1} = \frac{1}{2}\frac{|\hat{E}_2|^2}{\eta_2} \\ &= \tfrac{1}{2}\operatorname{Re}\left[(\hat{E}_+ e^{-jkz} + \hat{E}_- e^{+jkz})\frac{(\hat{E}_+^* e^{+jkz} - \hat{E}_-^* e^{-jkz})}{\eta}\right] \\ &= \frac{1}{2\eta}(|\hat{E}_+|^2 - |\hat{E}_-|^2) \end{aligned}$$

Note that for a given coating thickness $d$, there is no reflection only at select frequencies corresponding to wavelengths $d = n\lambda/4$, $n = 1, 3, 5, \ldots$. For a narrow band of wavelengths about these select wavelengths, reflections are small. The magnetic permeability of coatings and of the glass used in optical components are usually that of free space while the permittivities differ. The permittivity of the coating $\varepsilon$ is then picked so that

$$\varepsilon = \sqrt{\varepsilon_2\varepsilon_0}$$

and with a thickness corresponding to the central range of the wavelengths of interest (often in the visible).

## 7-7 UNIFORM AND NONUNIFORM PLANE WAVES

Our analysis thus far has been limited to waves propagating in the $z$ direction normally incident upon plane interfaces. Although our examples had the electric field polarized in the $x$ direction, the solution procedure is the same for the $y$-directed electric field polarization as both polarizations are parallel to the interfaces of discontinuity.

### 7-7-1 Propagation at an Arbitrary Angle

We now consider a uniform plane wave with power flow at an angle $\theta$ to the $z$ axis, as shown in Figure 7-16. The electric field is assumed to be $y$ directed, but the magnetic field that is perpendicular to both $\mathbf{E}$ and $\boldsymbol{S}$ now has components in the $x$ and $z$ directions.

The direction of the power flow, which we can call $z'$, is related to the Cartesian coordinates as

$$z' = x \sin\theta + z \cos\theta \tag{1}$$

so that the phase factor $kz'$ can be written as

$$kz' = k_x x + k_z z, \qquad \begin{aligned} k_x &= k \sin\theta \\ k_z &= k \cos\theta \end{aligned} \tag{2}$$

where the wavenumber magnitude is

$$k = \omega\sqrt{\varepsilon\mu} \tag{3}$$

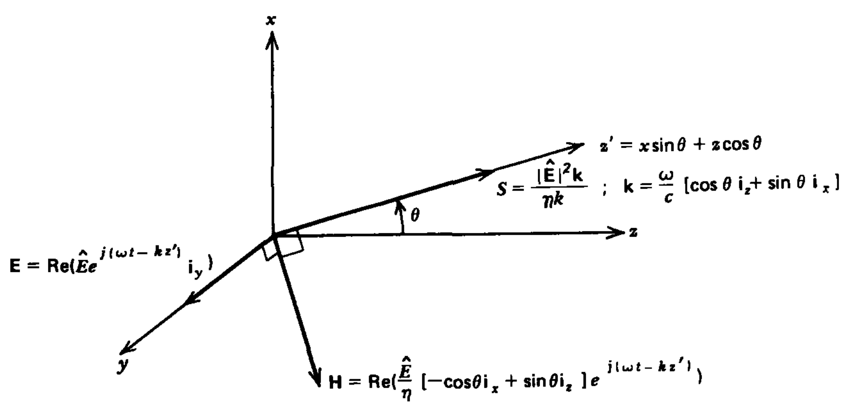



Figure 7-16 The spatial dependence of a uniform plane wave at an arbitrary angle $\theta$ can be expressed in terms of a vector wavenumber $\mathbf{k}$ as $\mathbf{e}^{-j\mathbf{k}\cdot\mathbf{r}}$, where $\mathbf{k}$ is in the direction of power flow and has magnitude $\omega/c$.

This allows us to write the fields as

$$\begin{aligned}\mathbf{E} &= \text{Re}\,[\hat{E}\, e^{j(\omega t - k_x x - k_z z)}\mathbf{i}_y]\\ \mathbf{H} &= \text{Re}\left[\frac{\hat{E}}{\eta}(-\cos\theta \mathbf{i}_x + \sin\theta \mathbf{i}_z)\, e^{j(\omega t - k_x x - k_z z)}\right]\end{aligned} \tag{4}$$

We note that the spatial dependence of the fields can be written as $e^{-j\mathbf{k}\cdot\mathbf{r}}$, where the wavenumber is treated as a vector:

$$\mathbf{k} = k_x \mathbf{i}_x + k_y \mathbf{i}_y + k_z \mathbf{i}_z \tag{5}$$

with

$$\mathbf{r} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z \tag{6}$$

so that

$$\mathbf{k}\cdot\mathbf{r} = k_x x + k_y y + k_z z \tag{7}$$

The magnitude of $\mathbf{k}$ is as given in (3) and its direction is the same as the power flow $\boldsymbol{S}$:

$$\begin{aligned}\boldsymbol{S} = \mathbf{E}\times\mathbf{H} &= \frac{|\hat{E}|^2}{\eta}(\cos\theta \mathbf{i}_z + \sin\theta \mathbf{i}_x)\cos^2(\omega t - \mathbf{k}\cdot\mathbf{r})\\ &= \frac{|\hat{E}|^2\mathbf{k}}{\omega\mu}\cos^2(\omega t - \mathbf{k}\cdot\mathbf{r})\end{aligned} \tag{8}$$

where without loss of generality we picked the phase of $\hat{E}$ to be zero so that it is real.

### 7-7-2 The Complex Propagation Constant

Let us generalize further by considering fields of the form

$$\begin{aligned}\mathbf{E} &= \text{Re}\,[\hat{\mathbf{E}}\, e^{j\omega t}\, e^{-\boldsymbol{\gamma}\cdot\mathbf{r}}] = \text{Re}\,[\hat{\mathbf{E}}\, e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}\, e^{-\boldsymbol{\alpha}\cdot\mathbf{r}}]\\ \mathbf{H} &= \text{Re}\,[\hat{\mathbf{H}}\, e^{j\omega t}\, e^{-\boldsymbol{\gamma}\cdot\mathbf{r}}] = \text{Re}\,[\hat{\mathbf{H}}\, e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}\, e^{-\boldsymbol{\alpha}\cdot\mathbf{r}}]\end{aligned} \tag{9}$$

where $\boldsymbol{\gamma}$ is the complex propagation vector and $\mathbf{r}$ is the position vector of (6):

$$\begin{aligned}\boldsymbol{\gamma} &= \boldsymbol{\alpha} + j\mathbf{k} = \gamma_x \mathbf{i}_x + \gamma_y \mathbf{i}_y + \gamma_z \mathbf{i}_z\\ \boldsymbol{\gamma}\cdot\mathbf{r} &= \gamma_x x + \gamma_y y + \gamma_z z\end{aligned} \tag{10}$$

We have previously considered uniform plane waves in lossless media where the wavenumber $\mathbf{k}$ is pure real and $z$ directed with $\alpha = 0$ so that $\boldsymbol{\gamma}$ is pure imaginary. The parameter $\boldsymbol{\alpha}$ represents the decay rate of the fields even though the medium is lossless. If $\boldsymbol{\alpha}$ is nonzero, the solutions are called nonuniform plane waves. We saw this decay in our quasi-static solutions of Laplace's equation where solutions had oscillations in one direction but decay in the perpendicular direction. We would expect this evanescence to remain at low frequencies.

The value of the assumed form of solutions in (9) is that the del ($\nabla$) operator in Maxwell's equations can be replaced by the vector operator $-\boldsymbol{\gamma}$:

$$\begin{aligned}\nabla &= \frac{\partial}{\partial x}\mathbf{i}_x + \frac{\partial}{\partial y}\mathbf{i}_y + \frac{\partial}{\partial z}\mathbf{i}_z \\ &= -\gamma_x \mathbf{i}_x - \gamma_y \mathbf{i}_y - \gamma_z \mathbf{i}_z \\ &= -\boldsymbol{\gamma} \end{aligned} \tag{11}$$

This is true because any spatial derivatives only operate on the exponential term in (9). Then the source free Maxwell's equations can be written in terms of the complex amplitudes as

$$\begin{aligned}-\boldsymbol{\gamma}\times\hat{\mathbf{E}} &= -j\omega\mu\hat{\mathbf{H}} \\ -\boldsymbol{\gamma}\times\hat{\mathbf{H}} &= j\omega\varepsilon\hat{\mathbf{E}} \\ -\boldsymbol{\gamma}\cdot\varepsilon\hat{\mathbf{E}} &= 0 \\ -\boldsymbol{\gamma}\cdot\mu\hat{\mathbf{H}} &= 0\end{aligned} \tag{12}$$

The last two relations tell us that $\boldsymbol{\gamma}$ is perpendicular to both **E** and **H**. If we take $\boldsymbol{\gamma}\times$ the top equation and use the second equation, we have

$$\begin{aligned}-\boldsymbol{\gamma}\times(\boldsymbol{\gamma}\times\hat{\mathbf{E}}) &= -j\omega\mu(\boldsymbol{\gamma}\times\hat{\mathbf{H}}) = -j\omega\mu(-j\omega\varepsilon\hat{\mathbf{E}}) \\ &= -\omega^2\mu\varepsilon\hat{\mathbf{E}}\end{aligned} \tag{13}$$

The double cross product can be expanded as

$$\begin{aligned}-\boldsymbol{\gamma}\times(\boldsymbol{\gamma}\times\hat{\mathbf{E}}) &= -\boldsymbol{\gamma}(\boldsymbol{\gamma}\cdot\hat{\mathbf{E}}) + (\boldsymbol{\gamma}\cdot\boldsymbol{\gamma})\hat{\mathbf{E}} \\ &= (\boldsymbol{\gamma}\cdot\boldsymbol{\gamma})\hat{\mathbf{E}} = -\omega^2\mu\varepsilon\hat{\mathbf{E}}\end{aligned} \tag{14}$$

The $\boldsymbol{\gamma}\cdot\hat{\mathbf{E}}$ term is zero from the third relation in (12). The dispersion relation is then

$$\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = (\alpha^2 - k^2 + 2j\boldsymbol{\alpha}\cdot\mathbf{k}) = -\omega^2\mu\varepsilon \tag{15}$$

For solution, the real and imaginary parts of (15) must be separately equal:

$$\begin{aligned}\alpha^2 - k^2 &= -\omega^2\mu\varepsilon \\ \boldsymbol{\alpha}\cdot\mathbf{k} &= 0\end{aligned} \tag{16}$$

When $\boldsymbol{\alpha}=0$, (16) reduces to the familiar frequency-wavenumber relation of Section 7-3-4.

The last relation now tells us that evanescence (decay) in space as represented by $\boldsymbol{\alpha}$ is allowed by Maxwell's equations, but must be perpendicular to propagation represented by **k**.

We can compute the time-average power flow for fields of the form of (9) using (12) in terms of either $\hat{\mathbf{E}}$ or $\hat{\mathbf{H}}$ as follows:

$$\begin{aligned}
<S> &= \tfrac{1}{2}\operatorname{Re}(\hat{\mathbf{E}}\times\hat{\mathbf{H}}^*),\\
&= -\tfrac{1}{2}\operatorname{Re}\left(\hat{\mathbf{E}}\times\frac{(\boldsymbol{\gamma}^*\times\hat{\mathbf{E}}^*)}{j\omega\mu}\right),\\
&= -\tfrac{1}{2}\operatorname{Re}\left(\frac{\boldsymbol{\gamma}^*|\hat{\mathbf{E}}|^2-\hat{\mathbf{E}}^*(\boldsymbol{\gamma}^*\cdot\hat{\mathbf{E}})}{j\omega\mu}\right),\\
&= \frac{1}{2}\frac{\mathbf{k}}{\omega\mu}|\hat{\mathbf{E}}|^2+\tfrac{1}{2}\operatorname{Re}\left(\frac{\hat{\mathbf{E}}^*(\boldsymbol{\gamma}^*\cdot\hat{\mathbf{E}})}{j\omega\mu}\right),\\
<S> &= \tfrac{1}{2}\operatorname{Re}(\hat{\mathbf{E}}\times\hat{\mathbf{H}}^*)\\
&= -\tfrac{1}{2}\operatorname{Re}\left(\frac{(\boldsymbol{\gamma}\times\hat{\mathbf{H}})}{j\omega\varepsilon}\times\hat{\mathbf{H}}^*\right)\\
&= \tfrac{1}{2}\operatorname{Re}\left(\frac{\boldsymbol{\gamma}|\hat{\mathbf{H}}|^2-\hat{\mathbf{H}}(\boldsymbol{\gamma}\cdot\hat{\mathbf{H}}^*)}{j\omega\varepsilon}\right)\\
&= \frac{1}{2}\frac{\mathbf{k}}{\omega\varepsilon}|\hat{\mathbf{H}}|^2-\tfrac{1}{2}\operatorname{Re}\left(\frac{\hat{\mathbf{H}}(\boldsymbol{\gamma}\cdot\hat{\mathbf{H}}^*)}{j\omega\varepsilon}\right)
\end{aligned} \tag{17}$$

Although both final expressions in (17) are equivalent, it is convenient to write the power flow in terms of either $\hat{\mathbf{E}}$ or $\hat{\mathbf{H}}$. When $\hat{\mathbf{E}}$ is perpendicular to both the real vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, defined in (10) and (16), the dot product $\boldsymbol{\gamma}^*\cdot\hat{\mathbf{E}}$ is zero. Such a mode is called transverse electric (TE), and we see in (17) that the time-average power flow is still in the direction of the wavenumber $\mathbf{k}$. Similarly, when $\mathbf{H}$ is perpendicular to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the dot product $\boldsymbol{\gamma}\cdot\hat{\mathbf{H}}^*$ is zero and we have a transverse magnetic (TM) mode. Again, the time-average power flow in (17) is in the direction of $\mathbf{k}$. The magnitude of $\mathbf{k}$ is related to $\omega$ in (16).

Note that our discussion has been limited to lossless systems. We can include Ohmic losses if we replace $\varepsilon$ by the complex permittivity $\hat{\varepsilon}$ of Section 7-4-3 in (15) and (17). Then, there is always decay ($\alpha\neq0$) because of Ohmic dissipation (see Problem 22).

### 7-7-3 Nonuniform Plane Waves

We can examine nonuniform plane wave solutions with nonzero $\boldsymbol{\alpha}$ by considering a current sheet in the $z=0$ plane, which is a traveling wave in the $x$ direction:

$$K_x(z=0)=K_0\cos(\omega t-k_x x)=\operatorname{Re}(K_0\,e^{j(\omega t-k_x x)}) \tag{18}$$

The $x$-directed surface current gives rise to a $y$-directed magnetic field. Because the system does not depend on the $y$ coordinate, solutions are thus of the following form:

$$H_y = \begin{cases} \operatorname{Re}(\hat{H}_1 e^{j\omega t} e^{-\boldsymbol{\gamma}_1\cdot\mathbf{r}}), & z>0 \\ \operatorname{Re}(\hat{H}_2 e^{j\omega t} e^{-\boldsymbol{\gamma}_2\cdot\mathbf{r}}), & z<0 \end{cases}$$
$$\mathbf{E} = \begin{cases} \operatorname{Re}\left[-\dfrac{\boldsymbol{\gamma}_1\times\hat{H}_1}{j\omega\varepsilon}\mathbf{i}_y e^{j\omega t} e^{-\boldsymbol{\gamma}_1\cdot\mathbf{r}}\right], & z>0 \\ \operatorname{Re}\left[-\dfrac{\boldsymbol{\gamma}_2\times\hat{H}_2}{j\omega\varepsilon}\mathbf{i}_y e^{j\omega t} e^{-\boldsymbol{\gamma}_2\cdot\mathbf{r}}\right], & z<0 \end{cases} \tag{19}$$

where $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are the complex propagation vectors on each side of the current sheet:

$$\begin{aligned} \boldsymbol{\gamma}_1 &= \gamma_{1x}\mathbf{i}_x + \gamma_{1z}\mathbf{i}_z \\ \boldsymbol{\gamma}_2 &= \gamma_{2x}\mathbf{i}_x + \gamma_{2z}\mathbf{i}_z \end{aligned} \tag{20}$$

The boundary condition of the discontinuity of tangential $\mathbf{H}$ at $z=0$ equaling the surface current yields

$$-\hat{H}_1 e^{-\gamma_{1x}x} + \hat{H}_2 e^{-\gamma_{2x}x} = K_0 e^{-jk_x x} \tag{21}$$

which tells us that the $x$ components of the complex propagation vectors equal the trigonometric spatial dependence of the surface current:

$$\gamma_{1x} = \gamma_{2x} = jk_x \tag{22}$$

The $z$ components of $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are then determined from (15) as

$$\gamma_x^2 + \gamma_z^2 = -\omega^2\varepsilon\mu \Rightarrow \gamma_z = \pm(k_x^2 - \omega^2\varepsilon\mu)^{1/2} \tag{23}$$

If $k_x^2 < \omega^2\varepsilon\mu$, $\gamma_z$ is pure imaginary representing propagation and we have uniform plane waves. If $k_x^2 > \omega^2\varepsilon\mu$, then $\gamma_z$ is pure real representing evanescence in the $z$ direction so that we generate nonuniform plane waves. When $\omega = 0$, (23) corresponds to Laplacian solutions that oscillate in the $x$ direction but decay in the $z$ direction.

The $z$ component of $\boldsymbol{\gamma}$ is of opposite sign in each region,

$$\gamma_{1z} = -\gamma_{2z} = +(k_x^2 - \omega^2\varepsilon\mu)^{1/2} \tag{24}$$

as the waves propagate or decay away from the sheet. Continuity of the tangential component of $\mathbf{E}$ requires

$$\gamma_{1z}\hat{H}_1 = \gamma_{2z}\hat{H}_2 \Rightarrow \hat{H}_2 = -\hat{H}_1 = K_0/2 \tag{25}$$

If $k_x = 0$, we re-obtain the solution of Section 7-4-1. Increasing $k_x$ generates propagating waves with power flow in the $k_x\mathbf{i}_x \pm k_z\mathbf{i}_z$ directions. At $k_x^2 = \omega^2\varepsilon\mu$, $k_z = 0$ so that the power flow is purely $x$ directed with no spatial dependence on $z$. Further increasing $k_x$ converts $k_z$ to $\alpha_z$ as $\gamma_z$ becomes real and the fields decay with $z$.

## 7-8 OBLIQUE INCIDENCE ONTO A PERFECT CONDUCTOR

### 7-8-1 E Field Parallel to the Interface

In Figure 7-17*a* we show a uniform plane wave incident upon a perfect conductor with power flow at an angle $\theta_i$ to the normal. The electric field is parallel to the surface with the magnetic field having both $x$ and $z$ components:

$$\mathbf{E}_i = \text{Re}\,[\hat{E}_i e^{j(\omega t - k_{xi}x - k_{zi}z)}\mathbf{i}_y]$$
$$\mathbf{H}_i = \text{Re}\left[\frac{\hat{E}_i}{\eta}(-\cos\theta_i \mathbf{i}_x + \sin\theta_i \mathbf{i}_z)\, e^{j(\omega t - k_{xi}x - k_{zi}z)}\right] \quad (1)$$

where

$$\left.\begin{aligned} k_{xi} &= k\ \sin\theta_i \\ k_{zi} &= k\ \cos\theta_i \end{aligned}\right\} \quad k = \omega\sqrt{\varepsilon\mu}, \quad \eta = \sqrt{\frac{\mu}{\epsilon}} \quad (2)$$

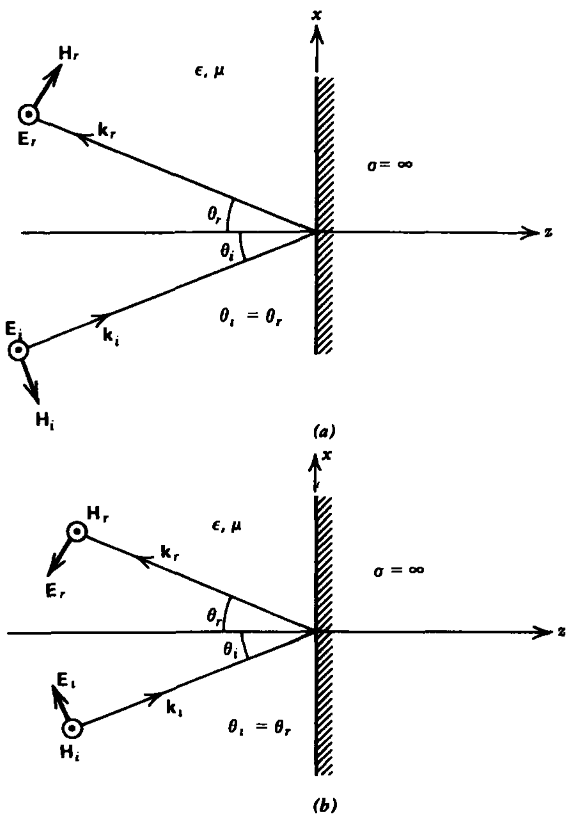



Figure 7-17 A uniform plane wave obliquely incident upon a perfect conductor has its angle of incidence equal to the angle of reflection. (*a*) Electric field polarized parallel to the interface. (*b*) Magnetic field parallel to the interface.

There are no transmitted fields within the perfect conductor, but there is a reflected field with power flow at angle $\theta_r$ from the interface normal. The reflected electric field is also in the $y$ direction so the magnetic field, which must be perpendicular to both $\mathbf{E}$ and $\mathbf{S}=\mathbf{E}\times\mathbf{H}$, is in the direction shown in Figure 7-17$a$:

$$\begin{aligned}\mathbf{E}_r &= \text{Re}\,[\hat{E}_r\,e^{j(\omega t-k_{xr}x+k_{zr}z)}\mathbf{i}_y]\\ \mathbf{H}_r &= \text{Re}\left[\frac{\hat{E}_r}{\eta}(\cos\theta_r\mathbf{i}_x+\sin\theta_r\mathbf{i}_z)\,e^{j(\omega t-k_{xr}x+k_{zr}z)}\right]\end{aligned}\tag{3}$$

where the reflected wavenumbers are

$$\begin{aligned}k_{xr}&=k\sin\theta_r\\ k_{zr}&=k\cos\theta_r\end{aligned}\tag{4}$$

At this point we do not know the angle of reflection $\theta_r$ or the reflected amplitude $\hat{E}_r$. They will be determined from the boundary conditions at $z=0$ of continuity of tangential $\mathbf{E}$ and normal $\mathbf{B}$. Because there are no fields within the perfect conductor these boundary conditions at $z=0$ are

$$\begin{aligned}\hat{E}_i\,e^{-jk_{xi}x}+\hat{E}_r\,e^{-jk_{xr}x}&=0\\ \frac{\mu}{\eta}(\hat{E}_i\sin\theta_i\,e^{-jk_{xi}x}+\hat{E}_r\sin\theta_r\,e^{-jk_{xr}x})&=0\end{aligned}\tag{5}$$

These conditions must be true for every value of $x$ along $z=0$ so that the phase factors given in (2) and (4) must be equal,

$$k_{xi}=k_{xr}\Rightarrow\theta_i=\theta_r\equiv\theta\tag{6}$$

giving the well-known rule ***that the angle of incidence equals the angle of reflection.*** The reflected field amplitude is then

$$\hat{E}_r=-\hat{E}_i\tag{7}$$

with the boundary conditions in (5) being redundant as they both yield (7). The total fields are then:

$$\begin{aligned}E_y &= \text{Re}\,[\hat{E}_i(e^{-jk_zz}-e^{+jk_zz})\,e^{j(\omega t-k_xx)}]\\ &=2E_i\sin k_zz\sin(\omega t-k_xx)\\ \mathbf{H} &= \text{Re}\left[\frac{\hat{E}_i}{\eta}[\cos\theta(-e^{-jk_zz}-e^{+jk_zz})\mathbf{i}_x+\sin\theta(e^{-jk_zz}\right.\\ &\qquad\left.-e^{+jk_zz})\mathbf{i}_z]\,e^{j(\omega t-k_xx)}\right]\\ &=\frac{2E_i}{\eta}[-\cos\theta\cos k_zz\cos(\omega t-k_xx)\mathbf{i}_x\\ &\qquad+\sin\theta\sin k_zz\sin(\omega t-k_xx)\mathbf{i}_z]\end{aligned}\tag{8}$$

where without loss of generality we take $\hat{E}_i$ to be real.

We drop the $i$ and $r$ subscripts on the wavenumbers and angles because they are equal. The fields travel in the $x$ direction parallel to the interface, but are stationary in the $z$ direction. Note that another perfectly conducting plane can be placed at distances $d$ to the left of the interface at

$$k_z d = n\pi \tag{9}$$

where the electric field is already zero without disturbing the solutions of (8). The boundary conditions at the second conductor are automatically satisfied. Such a structure is called a waveguide and is discussed in Section 8-6.

Because the tangential component of **H** is discontinuous at $z=0$, a traveling wave surface current flows along the interface,

$$K_y = -H_x(z=0) = \frac{2E_i}{\eta}\cos\theta\cos(\omega t - k_x x) \tag{10}$$

From (8) we compute the time-average power flow as

$$\begin{aligned} <S> &= \tfrac{1}{2}\operatorname{Re}[\hat{\mathbf{E}}(x,z)\times\hat{\mathbf{H}}^*(x,z)] \\ &= \frac{2E_i^2}{\eta}\sin\theta\sin^2 k_z z\,\mathbf{i}_x \end{aligned} \tag{11}$$

We see that the only nonzero power flow is in the direction parallel to the interfacial boundary and it varies as a function of $z$.

### 7-8-2 H Field Parallel to the Interface

If the **H** field is parallel to the conducting boundary, as in Figure 7-17*b*, the incident and reflected fields are as follows:

$$\begin{aligned} \mathbf{E}_i &= \operatorname{Re}[\hat{E}_i(\cos\theta_i\mathbf{i}_x - \sin\theta_i\mathbf{i}_z)\,e^{j(\omega t - k_{xi}x - k_{zi}z)}] \\ \mathbf{H}_i &= \operatorname{Re}\left[\frac{\hat{E}_i}{\eta}e^{j(\omega t - k_{xi}x - k_{zi}z)}\mathbf{i}_y\right] \\ \mathbf{E}_r &= \operatorname{Re}[\hat{E}_r(-\cos\theta_r\mathbf{i}_x - \sin\theta_r\mathbf{i}_z)\,e^{j(\omega t - k_{xr}x + k_{zr}z)}] \\ \mathbf{H}_r &= \operatorname{Re}\left[\frac{\hat{E}_r}{\eta}e^{j(\omega t - k_{xr}x + k_{zr}z)}\mathbf{i}_y\right] \end{aligned} \tag{12}$$

The tangential component of **E** is continuous and thus zero at $z=0$:

$$\hat{E}_i\cos\theta_i\,e^{-jk_{xi}x} - \hat{E}_r\cos\theta_r\,e^{-jk_{xr}x} = 0 \tag{13}$$

There is no normal component of **B**. This boundary condition must be satisfied for all values of $x$ so again the angle of incidence must equal the angle of reflection ($\theta_i = \theta_r$) so that

$$\hat{E}_i = \hat{E}_r \tag{14}$$

The total **E** and **H** fields can be obtained from (12) by adding the incident and reflected fields and taking the real part;

$$\begin{aligned}
\mathbf{E} &= \operatorname{Re}\{\hat{E}_i\,[\cos\theta(e^{-jk_z z} - e^{+jk_z z})\mathbf{i}_x \\
&\quad - \sin\theta(e^{-jk_z z} + e^{+jk_z z})\mathbf{i}_z]\,e^{j(\omega t - k_x x)}\} \\
&= 2E_i\{\cos\theta\sin k_z z\sin(\omega t - k_x x)\mathbf{i}_x \\
&\quad - \sin\theta\cos k_z z\cos(\omega t - k_x x)\mathbf{i}_z\} \\
\mathbf{H} &= \operatorname{Re}\left[\frac{\hat{E}_i}{\eta}(e^{-jk_z z} + e^{+jk_z z})\,e^{j(\omega t - k_x x)}\mathbf{i}_y\right] \\
&= \frac{2E_i}{\eta}\cos k_z z\cos(\omega t - k_x x)\mathbf{i}_y
\end{aligned} \tag{15}$$

The surface current on the conducting surface at $z = 0$ is given by the tangential component of **H**

$$K_x(z=0) = H_y(z=0) = \frac{2E_i}{\eta}\cos(\omega t - k_x x) \tag{16}$$

while the surface charge at $z = 0$ is proportional to the normal component of electric field,

$$\sigma_f(z=0) = -\varepsilon E_z(z=0) = 2\varepsilon E_i\sin\theta\cos(\omega t - k_x x) \tag{17}$$

Note that (16) and (17) satisfy conservation of current on the conducting surface,

$$\nabla_\Sigma\cdot\mathbf{K} + \frac{\partial\sigma_f}{\partial t} = 0 \Rightarrow \frac{\partial K_x}{\partial x} + \frac{\partial\sigma_f}{\partial t} = 0 \tag{18}$$

where

$$\nabla_\Sigma = \frac{\partial}{\partial x}\mathbf{i}_x + \frac{\partial}{\partial y}\mathbf{i}_y$$

is the surface divergence operator. The time-average power flow for this polarization is also $x$ directed:

$$\begin{aligned}
\langle S\rangle &= \tfrac{1}{2}\operatorname{Re}(\hat{\mathbf{E}}\times\hat{\mathbf{H}}^*) \\
&= \frac{2E_i^2}{\eta}\sin\theta\cos^2 k_z z\,\mathbf{i}_x
\end{aligned} \tag{19}$$

## 7-9 OBLIQUE INCIDENCE ONTO A DIELECTRIC

### 7-9-1 E Parallel to the Interface

A plane wave incident upon a dielectric interface, as in Figure 7-18*a*, now has transmitted fields as well as reflected fields. For the electric field polarized parallel to the interface, the fields in each region can be expressed as

$$
\begin{aligned}
\mathbf{E}_i &= \mathrm{Re}\,[\hat{E}_i e^{j(\omega t - k_{xi}x - k_{zi}z)}\mathbf{i}_y]\\
\mathbf{H}_i &= \mathrm{Re}\left[\frac{\hat{E}_i}{\eta_1}(-\cos\theta_i\mathbf{i}_x + \sin\theta_i\mathbf{i}_z)\,e^{j(\omega t - k_{xi}x - k_{zi}z)}\right]\\
\mathbf{E}_r &= \mathrm{Re}\,[\hat{E}_r e^{j(\omega t - k_{xr}x + k_{zr}z)}\mathbf{i}_y]\\
\mathbf{H}_r &= \mathrm{Re}\left[\frac{\hat{E}_r}{\eta_1}(\cos\theta_r\mathbf{i}_x + \sin\theta_r\mathbf{i}_z)\,e^{j(\omega t - k_{xr}x + k_{zr}z)}\right]\\
\mathbf{E}_t &= \mathrm{Re}\,[\hat{E}_t e^{j(\omega t - k_{xt}x - k_{zt}z)}\mathbf{i}_y]\\
\mathbf{H}_t &= \mathrm{Re}\left[\frac{\hat{E}_t}{\eta_2}(-\cos\theta_t\mathbf{i}_x + \sin\theta_t\mathbf{i}_z)\,e^{j(\omega t - k_{xt}x - k_{zt}z)}\right]
\end{aligned}
\qquad (1)
$$

where $\theta_i$, $\theta_r$, and $\theta_t$ are the angles from the normal of the incident, reflected, and transmitted power flows. The wavenumbers in each region are

$$
\begin{aligned}
k_{xi} &= k_1\sin\theta_i, \qquad k_{xr} = k_1\sin\theta_r, \qquad k_{xt} = k_2\sin\theta_t\\
k_{zi} &= k_1\cos\theta_i, \qquad k_{zr} = k_1\cos\theta_r, \qquad k_{zt} = k_2\cos\theta_t
\end{aligned}
\qquad (2)
$$

where the wavenumber magnitudes, wave speeds, and wave impedances are

$$
\begin{aligned}
k_1 &= \frac{\omega}{c_1}, \qquad k_2 = \frac{\omega}{c_2}, \qquad c_1 = \frac{1}{\sqrt{\varepsilon_1\mu_1}}\\
\eta_1 &= \sqrt{\frac{\mu_1}{\varepsilon_1}}, \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}, \qquad c_2 = \frac{1}{\sqrt{\varepsilon_2\mu_2}}
\end{aligned}
\qquad (3)
$$

The unknown angles and amplitudes in (1) are found from the boundary conditions of continuity of tangential **E** and **H** at the $z = 0$ interface.

$$
\begin{aligned}
&\hat{E}_i e^{-jk_{xi}x} + \hat{E}_r e^{-jk_{xr}x} = \hat{E}_t e^{-jk_{xt}x}\\
&\frac{-\hat{E}_i\cos\theta_i e^{-jk_{xi}x} + \hat{E}_r\cos\theta_r e^{-jk_{xr}x}}{\eta_1} = -\frac{\hat{E}_t\cos\theta_t e^{-jk_{xt}x}}{\eta_2}
\end{aligned}
\qquad (4)
$$

These boundary conditions must be satisfied point by point for all $x$. This requires that the exponential factors also be

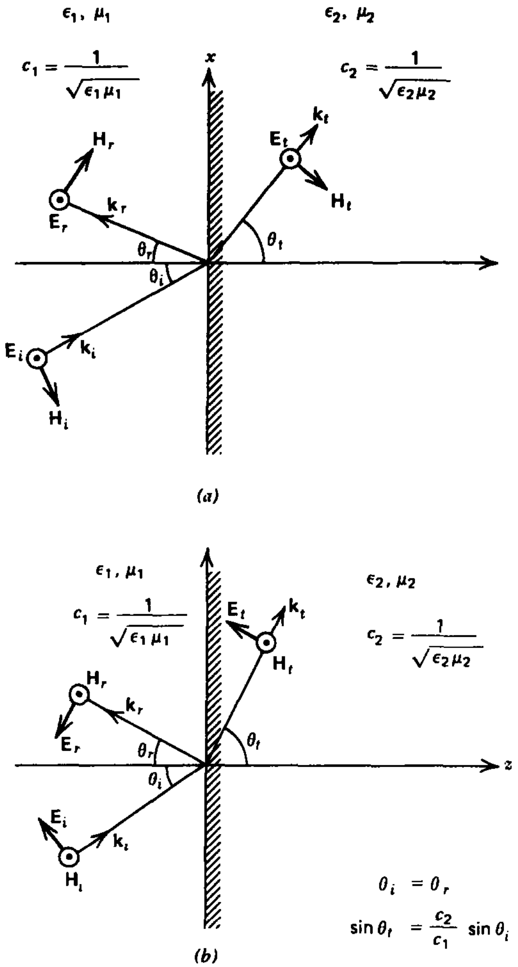



Figure 7-18 A uniform plane wave obliquely incident upon a dielectric interface also has its angle of incidence equal to the angle of reflection while the transmitted angle is given by Snell's law. (*a*) Electric field polarized parallel to the interface. (*b*) Magnetic field parallel to the interface.

equal so that the $x$ components of all wavenumbers must be equal,

$$k_{xi} = k_{xr} = k_{xt} \Rightarrow k_1 \sin\theta_i = k_1 \sin\theta_r = k_2 \sin\theta_t \tag{5}$$

which relates the angles as

$$\theta_r = \theta_i \tag{6}$$

$$\sin\theta_t = (c_2/c_1)\sin\theta_i \tag{7}$$

As before, the angle of incidence equals the angle of reflection. The transmission angle obeys a more complicated relation called Snell's law relating the sines of the angles. The angle from the normal is largest in that region which has the faster speed of electromagnetic waves.

In optics, the ratio of the speed of light in vacuum, $c_0 = 1/\sqrt{\varepsilon_0\mu_0}$, to the speed of light in the medium is defined as the index of refraction,

$$n_1 = c_0/c_1, \qquad n_2 = c_0/c_2 \tag{8}$$

which is never less than unity. Then Snell's law is written as

$$\sin\theta_t = (n_1/n_2)\sin\theta_i \tag{9}$$

With the angles related as in (6), the reflected and transmitted field amplitudes can be expressed in the same way as for normal incidence (see Section 7-6-1) if we replace the wave impedances by $\eta \to \eta/\cos\theta$ to yield

$$\begin{aligned} R &= \frac{\hat{E}_r}{\hat{E}_i} = \frac{\dfrac{\eta_2}{\cos\theta_t} - \dfrac{\eta_1}{\cos\theta_i}}{\dfrac{\eta_2}{\cos\theta_t} + \dfrac{\eta_1}{\cos\theta_i}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \\ T &= \frac{\hat{E}_t}{\hat{E}_i} = \frac{2\eta_2}{\cos\theta_t\left(\dfrac{\eta_2}{\cos\theta_t} + \dfrac{\eta_1}{\cos\theta_i}\right)} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \end{aligned} \tag{10}$$

In (4) we did not consider the boundary condition of continuity of normal $\mathbf{B}$ at $z = 0$. This boundary condition is redundant as it is the same condition as the upper equation in (4):

$$\frac{\mu_1}{\eta_1}(\hat{E}_i + \hat{E}_r)\sin\theta_i = \frac{\mu_2}{\eta_2}\hat{E}_t\sin\theta_t \Rightarrow (\hat{E}_i + \hat{E}_r) = \hat{E}_t \tag{11}$$

where we use the relation between angles in (6). Since

$$\frac{\mu_1}{\eta_1} = \sqrt{\mu_1\varepsilon_1} = \frac{1}{c_1}, \qquad \frac{\mu_2}{\eta_2} = \sqrt{\mu_2\varepsilon_2} = \frac{1}{c_2} \tag{12}$$

the trigonometric terms in (11) cancel due to Snell's law. There is no normal component of $\mathbf{D}$ so it is automatically continuous across the interface.

### 7-9-2 Brewster's Angle of No Reflection

We see from (10) that at a certain angle of incidence, there is no reflected field as $R = 0$. This angle is called Brewster's angle:

$$R = 0 \Rightarrow \eta_2\cos\theta_i = \eta_1\cos\theta_t \tag{13}$$

By squaring (13), replacing the cosine terms with sine terms ($\cos^2\theta = 1-\sin^2\theta$), and using Snell's law of (6), the Brewster angle $\theta_B$ is found as

$$\sin^2\theta_B = \frac{1-\varepsilon_2\mu_1/(\varepsilon_1\mu_2)}{1-(\mu_1/\mu_2)^2} \tag{14}$$

There is not always a real solution to (14) as it depends on the material constants. The common dielectric case, where $\mu_1 = \mu_2 \equiv \mu$ but $\varepsilon_1 \neq \varepsilon_2$, does not have a solution as the right-hand side of (14) becomes infinite. Real solutions to (14) require the right-hand side to be between zero and one. A Brewster's angle does exist for the uncommon situation where $\varepsilon_1 = \varepsilon_2$ and $\mu_1 \neq \mu_2$:

$$\sin^2\theta_B = \frac{1}{1+\mu_1/\mu_2} \Rightarrow \tan\theta_B = \sqrt{\frac{\mu_2}{\mu_1}} \tag{15}$$

At this Brewster's angle, the reflected and transmitted power flows are at right angles ($\theta_B + \theta_t = \pi/2$) as can be seen by using (6), (13), and (14):

$$\begin{aligned}\cos(\theta_B+\theta_t) &= \cos\theta_B\cos\theta_t - \sin\theta_B\sin\theta_t\\ &= \cos^2\theta_B\sqrt{\frac{\mu_2}{\mu_1}} - \sin^2\theta_B\sqrt{\frac{\mu_1}{\mu_2}}\\ &= \sqrt{\frac{\mu_2}{\mu_1}} - \sin^2\theta_B\left(\sqrt{\frac{\mu_1}{\mu_2}}+\sqrt{\frac{\mu_2}{\mu_1}}\right) = 0\end{aligned} \tag{16}$$

### 7-9-3 Critical Angle of Transmission

Snell's law in (6) shows us that if $c_2 > c_1$, large angles of incident angle $\theta_i$ could result in $\sin\theta_t$ being greater than unity. There is no real angle $\theta_t$ that satisfies this condition. The critical incident angle $\theta_c$ is defined as that value of $\theta_i$ that makes $\theta_t = \pi/2$,

$$\sin\theta_c = c_1/c_2 \tag{17}$$

which has a real solution only if $c_1 < c_2$. At the critical angle, the wavenumber $k_{zt}$ is zero. Lesser incident angles have real values of $k_{zt}$. For larger incident angles there is no real angle $\theta_t$ that satisfies (6). Snell's law must always be obeyed in order to satisfy the boundary conditions at $z=0$ for all $x$. What happens is that $\theta_t$ becomes a complex number that satisfies (6). Although $\sin\theta_t$ is still real, $\cos\theta_t$ is imaginary when $\sin\theta_t$ exceeds unity:

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} \tag{18}$$

This then makes $k_{zt}$ imaginary, which we can write as

$$k_{zt} = k_2 \cos\theta_t = -j\alpha \tag{19}$$

The negative sign of the square root is taken so that waves now decay with $z$:

$$\begin{aligned} \mathbf{E}_t &= \text{Re}\,[\hat{E}_t\, e^{j(\omega t - k_{xt}x)}\, e^{-\alpha z}\,\mathbf{i}_y] \\ \mathbf{H}_t &= \text{Re}\left[\frac{\hat{E}_t}{\eta_2}(-\cos\theta_t\mathbf{i}_x + \sin\theta_t\mathbf{i}_z)\, e^{j(\omega t - k_{xt}x)}\, e^{-\alpha z}\right] \end{aligned} \tag{20}$$

The solutions are now nonuniform plane waves, as discussed in Section 7-7.

Complex angles of transmission are a valid mathematical concept. What has happened is that in (1) we wrote our assumed solutions for the transmitted fields in terms of pure propagating waves. Maxwell's equations for an incident angle greater than the critical angle require spatially decaying waves with $z$ in region 2 so that the mathematics forced $k_{zt}$ to be imaginary.

There is no power dissipation since the $z$-directed time-average power flow is zero,

$$\begin{aligned} \langle S_z \rangle &= -\tfrac{1}{2}\,\text{Re}\,[E_y H_x^*] \\ &= -\tfrac{1}{2}\,\text{Re}\left[\frac{\hat{E}_t \hat{E}_t^*}{\eta_2}(-\cos\theta_t)^*\, e^{-2\alpha z}\right] = 0 \end{aligned} \tag{21}$$

because $\cos\theta_t$ is pure imaginary so that the bracketed term in (21) is pure imaginary. The incident $z$-directed time-average power is totally reflected. Even though the time-averaged $z$-directed transmitted power is zero, there are nonzero but exponentially decaying fields in region 2.

### 7-9-4 H Field Parallel to the Boundary

For this polarization, illustrated in Figure 7-18$b$, the fields are

$$\begin{aligned} \mathbf{E}_i &= \text{Re}\,[\hat{E}_i\,(\cos\theta_i\mathbf{i}_x - \sin\theta_i\mathbf{i}_z)\, e^{j(\omega t - k_{xi}x - k_{zi}z)}] \\ \mathbf{H}_i &= \text{Re}\left[\frac{\hat{E}_i}{\eta_1}\, e^{j(\omega t - k_{xi}x - k_{zi}z)}\,\mathbf{i}_y\right] \\ \mathbf{E}_r &= \text{Re}\,[\hat{E}_r\,(-\cos\theta_r\mathbf{i}_x - \sin\theta_r\mathbf{i}_z)\, e^{j(\omega t - k_{xr}x + k_{zr}z)}] \\ \mathbf{H}_r &= \text{Re}\left[\frac{\hat{E}_r}{\eta_1}\, e^{j(\omega t - k_{xr}x + k_{zr}z)}\,\mathbf{i}_y\right] \\ \mathbf{E}_t &= \text{Re}\,[\hat{E}_t\,(\cos\theta_t\mathbf{i}_x - \sin\theta_t\mathbf{i}_z)\, e^{j(\omega t - k_{xt}x - k_{zt}z)}] \\ \mathbf{H}_t &= \text{Re}\left[\frac{\hat{E}_t}{\eta_2}\, e^{j(\omega t - k_{xt}x - k_{zt}z)}\,\mathbf{i}_y\right] \end{aligned} \tag{22}$$

where the wavenumbers and impedances are the same as in (2) and (3).

Continuity of tangential **E** and **H** at $z=0$ requires

$$\begin{gathered}\hat{E}_i \cos\theta_i\, e^{-jk_{xi}x} - \hat{E}_r \cos\theta_r\, e^{-jk_{xr}x} = \hat{E}_t \cos\theta_t\, e^{-jk_{xt}x}\\ \frac{\hat{E}_i\, e^{-jk_{xi}x} + \hat{E}_r\, e^{-jk_{xr}x}}{\eta_1} = \frac{\hat{E}_t\, e^{-jk_{xt}x}}{\eta_2}\end{gathered} \tag{23}$$

Again the phase factors must be equal so that (5) and (6) are again true. Snell's law and the angle of incidence equalling the angle of reflection are independent of polarization.

We solve (23) for the field reflection and transmission coefficients as

$$R = \frac{\hat{E}_r}{\hat{E}_i} = \frac{\eta_1 \cos\theta_i - \eta_2 \cos\theta_t}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \tag{24}$$

$$T = \frac{\hat{E}_t}{\hat{E}_i} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \tag{25}$$

Now we note that the boundary condition of continuity of normal **D** at $z=0$ is redundant to the lower relation in (23),

$$\varepsilon_1 \hat{E}_1 \sin\theta_i + \varepsilon_1 \hat{E}_r \sin\theta_r = \varepsilon_2 \hat{E}_t \sin\theta_t \tag{26}$$

using Snell's law to relate the angles.

For this polarization the condition for no reflected waves is

$$R = 0 \Rightarrow \eta_2 \cos\theta_t = \eta_1 \cos\theta_i \tag{27}$$

which from Snell's law gives the Brewster angle:

$$\sin^2\theta_B = \frac{1 - \varepsilon_1\mu_2/(\varepsilon_2\mu_1)}{1 - (\varepsilon_1/\varepsilon_2)^2} \tag{28}$$

There is now a solution for the usual case where $\mu_1 = \mu_2$ but $\varepsilon_1 \neq \varepsilon_2$:

$$\sin^2\theta_B = \frac{1}{1 + \varepsilon_1/\varepsilon_2} \Rightarrow \tan\theta_B = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \tag{29}$$

At this Brewster's angle the reflected and transmitted power flows are at right angles $(\theta_B + \theta_t) = \pi/2$ as can be seen by using (6), (27), and (29)

$$\begin{aligned}\cos(\theta_B + \theta_t) &= \cos\theta_B \cos\theta_t - \sin\theta_B \sin\theta_t\\ &= \cos^2\theta_B \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} - \sin^2\theta_B \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\\ &= \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} - \sin^2\theta_B\left(\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right) = 0\end{aligned} \tag{30}$$

Because Snell's law is independent of polarization, the critical angle of (17) is the same for both polarizations. Note that the Brewster's angle for either polarization, if it exists, is always less than the critical angle of (17), as can be particularly seen when $\mu_1 = \mu_2$ for the magnetic field polarized parallel to the interface or when $\varepsilon_1 = \varepsilon_2$ for the electric field polarized parallel to the interface, as then

$$\frac{1}{\sin^2 \theta_B} = \frac{1}{\sin^2 \theta_c} + 1 \tag{31}$$

## 7-10 APPLICATIONS TO OPTICS

Reflection and refraction of electromagnetic waves obliquely incident upon the interface between dissimilar linear lossless media are governed by the two rules illustrated in Figure 7-19:

(i) The angle of incidence equals the angle of reflection.
(ii) Waves incident from a medium of high light velocity (low index of refraction) to one of low velocity (high index of refraction) are bent towards the normal. If the wave is incident from a low velocity (high index) to high velocity (low index) medium, the light is bent away from the normal. The incident and refracted angles are related by Snell's law.

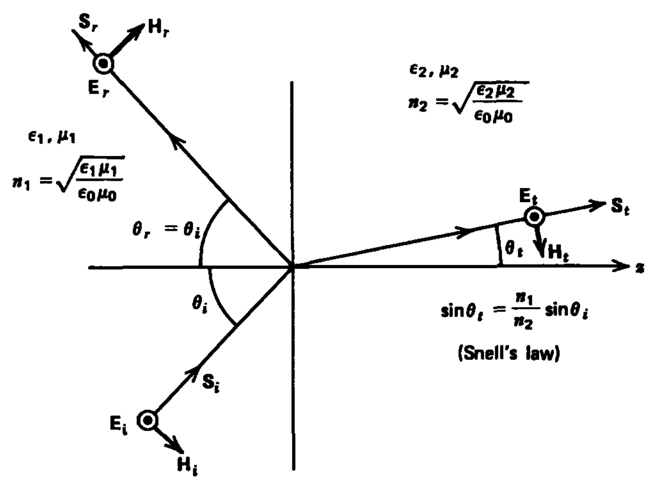



Figure 7-19 A summary of reflection and refraction phenomena across the interface separating two linear media. When $\theta_i = \theta_B$ (Brewster's angle), there is no reflected ray. When $\theta_i > \theta_c$ (critical angle), the transmitted fields decay with $z$.

Most optical materials, like glass, have a permeability of free space $\mu_0$. Therefore, a Brewster's angle of no reflection only exists if the **H** field is parallel to the boundary.

At the critical angle, which can only exist if light travels from a high index of refraction material (low light velocity) to one of low index (high light velocity), there is a transmitted field that decays with distance as a nonuniform plane wave. However, there is no time-average power carried by this evanescent wave so that all the time-average power is reflected. This section briefly describes various applications of these special angles and the rules governing reflection and refraction.

### 7-10-1 Reflections from a Mirror

A person has their eyes at height $h$ above their feet and a height $\Delta h$ below the top of their head, as in Figure 7-20. A mirror in front extends a distance $\Delta y$ above the eyes and a distance $y$ below. How large must $y$ and $\Delta y$ be so that the person sees their entire image? The light reflected off the person into the mirror must be reflected again into the person's eyes. Since the angle of incidence equals the angle of reflection, Figure 7-20 shows that $\Delta y = \Delta h/2$ and $y = h/2$.

### 7-10-2 Lateral Displacement of a Light Ray

A light ray is incident from free space upon a transparent medium with index of refraction $n$ at angle $\theta_i$, as shown in Figure 7-21. The angle of the transmitted light is given by Snell's law:

$$\sin \theta_t = (1/n) \sin \theta_i \tag{1}$$

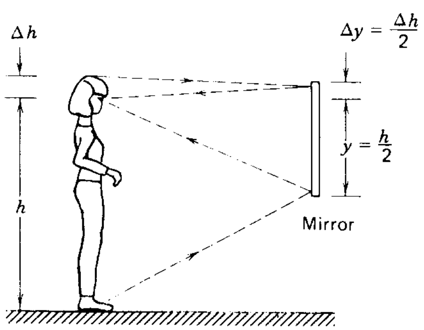



Figure 7-20 Because the angle of incidence equals the angle of reflection, a person can see their entire image if the mirror extends half the distance of extent above and below the eyes.

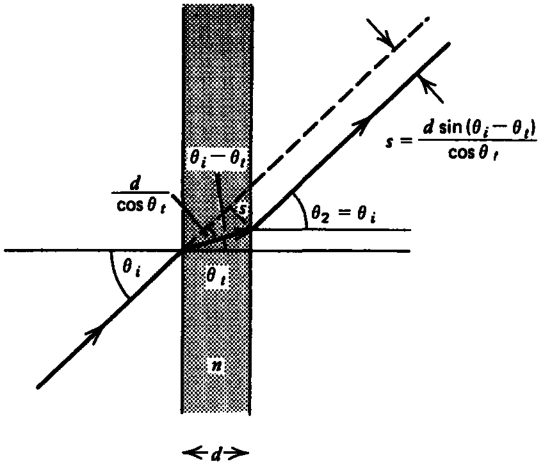



Figure 7-21 A light ray incident upon a glass plate exits the plate into the original medium parallel to its original trajectory but laterally displaced.

When this light hits the second interface, the angle $\theta_t$ is now the incident angle so that the transmitted angle $\theta_2$ is again given by Snell's law:

$$\sin\theta_2 = n \sin\theta_t = \sin\theta_i \tag{2}$$

so that the light exits at the original incident angle $\theta_i$. However, it is now shifted by the amount:

$$s = \frac{d \sin(\theta_i - \theta_t)}{\cos\theta_t} \tag{3}$$

If the plate is glass with refractive index $n = 1.5$ and thickness $d = 1$ mm with incident angle $\theta_i = 30°$, the angle $\theta_t$ in the glass is

$$\sin\theta_t = 0.33 \Rightarrow \theta_t = 19.5° \tag{4}$$

so that the lateral displacement is $s = 0.19$ mm.

### 7-10-3 Polarization By Reflection

Unpolarized light is incident upon the piece of glass in Section 7-10-2 with index of refraction $n = 1.5$. Unpolarized light has both **E** and **H** parallel to the interface. We assume that the permeability of the glass equals that of free space and that the light is incident at the Brewster's angle $\theta_B$ for light polarized with **H** parallel to the interface. The incident and transmitted angles are then

$$\begin{aligned}\tan \theta_B &= \sqrt{\varepsilon/\varepsilon_0} = n \Rightarrow \theta_B = 56.3^\circ \\ \tan \theta_t &= \sqrt{\varepsilon_0/\varepsilon} = 1/n \Rightarrow \theta_t = 33.7^\circ\end{aligned} \tag{5}$$

The Brewster's angle is also called the polarizing angle because it can be used to separate the two orthogonal polarizations. The polarization, whose **H** field is parallel to the interface, is entirely transmitted at the first interface with no reflection. The other polarization with electric field parallel to the interface is partially transmitted and reflected. At the second (glass-free space) interface the light is incident at angle $\theta_t$. From (5) we see that this angle is the Brewster's angle with **H** parallel to the interface for light incident from the glass side onto the glass-free space interface. Then again, the **H** parallel to the interface polarization is entirely transmitted while the **E** parallel to the interface polarization is partially reflected and partially transmitted. Thus, the reflected wave is entirely polarized with electric field parallel to the interface. The transmitted waves, although composed of both polarizations, have the larger amplitude with **H**

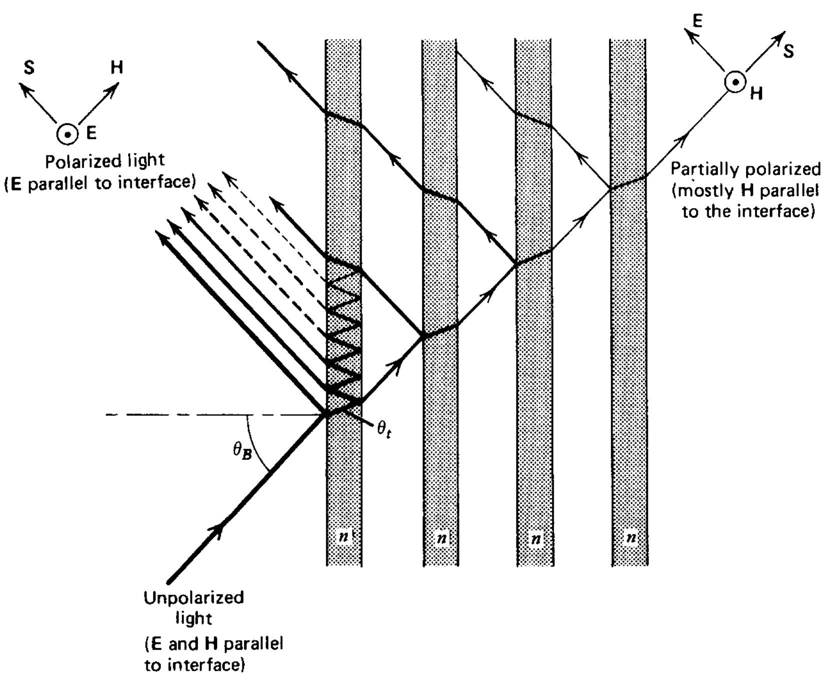



Figure 7-22 Unpolarized light incident upon glass with $\mu = \mu_0$ can be polarized by reflection if it is incident at the Brewster's angle for the polarization with **H** parallel to the interface. The transmitted light becomes more polarized with **H** parallel to the interface by adding more parallel glass plates.

parallel to the interface because it was entirely transmitted with no reflection at both interfaces.

By passing the transmitted light through another parallel piece of glass, the polarization with electric field parallel to the interface becomes further diminished because it is partially reflected, while the other polarization is completely transmitted. With more glass elements, as in Figure 7-22, the transmitted light can be made essentially completely polarized with **H** field parallel to the interface.

### 7-10-4 Light Propagation In Water

**(a) Submerged Source**

A light source is a distance $d$ below the surface of water with refractive index $n = 1.33$, as in Figure 7-23. The rays emanate from the source as a cone. Those rays at an angle from the normal greater than the critical angle,

$$\sin \theta_c = 1/n \Rightarrow \theta_c = 48.8^\circ \tag{6}$$

are not transmitted into the air but undergo total internal reflection. A circle of light with diameter

$$D = 2d \tan \theta_c \approx 2.28d \tag{7}$$

then forms on the water's surface due to the exiting light.

**(b) Fish Below a Boat**

A fish swims below a circular boat of diameter $D$, as in Figure 7-24. As we try to view the fish from the air above, the incident light ray is bent towards the normal. The region below the boat that we view from above is demarcated by the light rays at grazing incidence to the surface ($\theta_i = \pi/2$) just entering the water ($n = 1.33$) at the sides of the boat. The transmitted angle of these light rays is given from Snell's law as

$$\sin \theta_t = \frac{\sin \theta_i}{n} = \frac{1}{n} \Rightarrow \theta_t = 48.8^\circ \tag{8}$$

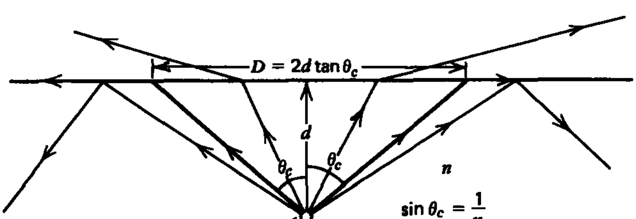



Figure 7-23 Light rays emanating from a source within a high index of refraction medium are totally internally reflected from the surface for angles greater than the critical angle. Lesser angles of incidence are transmitted.

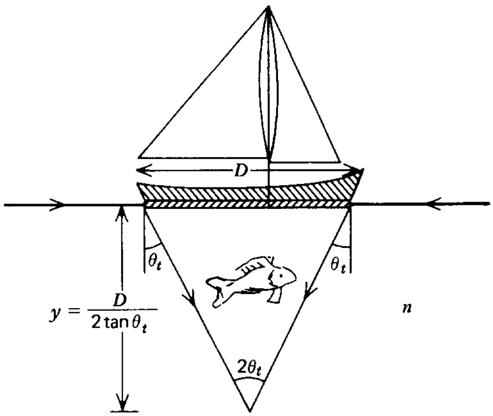



Figure 7-24 A fish cannot be seen from above if it swims below a circular boat within the cone bounded by light rays at grazing incidence entering the water at the side of the boat.

These rays from all sides of the boat intersect at the point a distance $y$ below the boat, where

$$\tan\theta_t = \frac{D}{2y} \Rightarrow y = \frac{D}{2\tan\theta_t} \approx 0.44D \tag{9}$$

If the fish swims within the cone, with vertex at the point $y$ below the boat, it cannot be viewed from above.

### 7-10-5 Totally Reflecting Prisms

The glass isoceles right triangle in Figure 7-25 has an index of refraction of $n = 1.5$ so that the critical angle for total

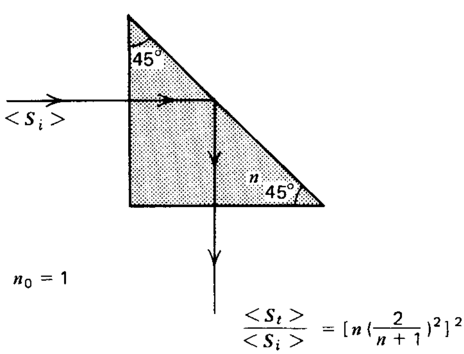



Figure 7-25 A totally reflecting prism. The index of refraction $n$ must exceed $\sqrt{2}$ so that the light incident on the hypotenuse at 45° exceeds the critical angle.

internal reflection is

$$\sin\theta_c = \frac{1}{n} = \frac{1}{1.5} \Rightarrow \theta_c = 41.8° \tag{10}$$

The light is normally incident on the vertical face of the prism. The transmission coefficient is then given in Section 7-6-1 as

$$T_1 = \frac{\hat{E}_t}{\hat{E}_i} = \frac{2\eta}{\eta+\eta_0} = \frac{2/n}{1+1/n} = \frac{2}{n+1} = 0.8 \tag{11}$$

where because the permeability of the prism equals that of free space $n = \sqrt{\varepsilon/\varepsilon_0}$ while $\eta/\eta_0 = \sqrt{\varepsilon_0/\varepsilon} = 1/n$. The transmitted light is then incident upon the hypotenuse of the prism at an angle of 45°, which exceeds the critical angle so that no power is transmitted and the light is totally reflected being turned through a right angle. The light is then normally incident upon the horizontal face with transmission coefficient:

$$T_2 = \frac{\hat{E}_2}{0.8\hat{E}_i} = \frac{2\eta_0}{\eta+\eta_0} = \frac{2}{1/n+1} = \frac{2n}{n+1} = 1.2 \tag{12}$$

The resulting electric field amplitude is then

$$\hat{E}_2 = T_1 T_2 \hat{E}_i = 0.96\hat{E}_i \tag{13}$$

The ratio of transmitted to incident power density is

$$\frac{\langle S\rangle}{\langle S_i\rangle} = \frac{\frac{1}{2}|\hat{E}_2|^2/\eta_0}{\frac{1}{2}|\hat{E}_i|^2/\eta_0} = \frac{|\hat{E}_2|^2}{|\hat{E}_i|^2} = \left(\frac{24}{25}\right)^2 \approx 0.92 \tag{14}$$

This ratio can be increased to unity by applying a quarter-wavelength-thick dielectric coating with index of refraction $n_{\text{coating}} = \sqrt{n}$, as developed in Example 7-1. This is not usually done because the ratio in (14) is already large without the expense of a coating.

## 7-10-6 Fiber Optics

### (a) Straight Light Pipe

Long thin fibers of transparent material can guide light along a straight path if the light within the pipe is incident upon the wall at an angle greater than the critical angle ($\sin\theta_c = 1/n$):

$$\sin\theta_2 = \cos\theta_t \geq \sin\theta_c \tag{15}$$

The light rays are then totally internally reflected being confined to the pipe until they exit, as in Figure 7-26. The

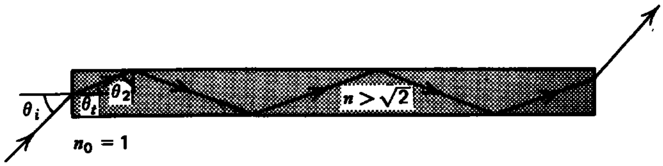



Figure 7-26 The index of refraction of a straight light pipe must be greater than $\sqrt{2}$ for total internal reflections of incident light at any angle.

incident angle is related to the transmitted angle from Snell's law,

$$\sin\theta_t = (1/n)\sin\theta_i \tag{16}$$

so that (15) becomes

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-(1/n^2)\sin^2\theta_i} \geq 1/n \tag{17}$$

which when solved for $n$ yields

$$n^2 \geq 1 + \sin^2\theta_i \tag{18}$$

If this condition is met for grazing incidence ($\theta_i = \pi/2$), all incident light will be passed by the pipe, which requires that

$$n^2 \geq 2 \Rightarrow n \geq \sqrt{2} \tag{19}$$

Most types of glass have $n \approx 1.5$ so that this condition is easily met.

**(b) Bent Fibers**

Light can also be guided along a tortuous path if the fiber is bent, as in the semi-circular pipe shown in Figure 7-27. The minimum angle to the radial normal for the incident light shown is at the point $A$. This angle in terms of the radius of the bend and the light pipe width must exceed the critical angle

$$\sin\theta_A = \frac{R}{R+d} \geq \sin\theta_c \tag{20}$$

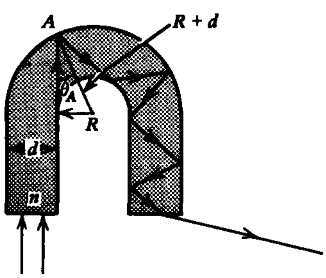



Figure 7-27 Light can be guided along a circularly bent fiber if $R/d > 1/(n-1)$ as then there is always total internal reflection each time the light is incident on the walls.

so that

$$\frac{R/d}{R/d+1} \geq \frac{1}{n} \tag{21}$$

which when solved for $R/d$ requires

$$\frac{R}{d} \geq \frac{1}{n-1} \tag{22}$$

## PROBLEMS

Section 7-1

1. For the following electric fields in a linear media of permittivity $\varepsilon$ and permeability $\mu$ find the charge density, magnetic field, and current density.

(a) $\mathbf{E} = E_0(x\mathbf{i}_x + y\mathbf{i}_y) \sin \omega t$

(b) $\mathbf{E} = E_0(y\mathbf{i}_x - x\mathbf{i}_y) \cos \omega t$

(c) $\mathbf{E} = \text{Re}\,[E_0\, e^{j(\omega t - k_x x - k_z z)}\mathbf{i}_y]$. How must $k_x$, $k_z$, and $\omega$ be related so that $\mathbf{J} = 0$?

2. An Ohmic conductor of arbitrary shape has an initial charge distribution $\rho_0(\mathbf{r})$ at $t = 0$.

(a) What is the charge distribution for all time?

(b) The initial charge distribution is uniform and is confined between parallel plate electrodes of spacing $d$. What are the electric and magnetic fields when the electrodes are opened or short circuited?

(c) Repeat (b) for coaxial cylindrical electrodes of inner radius $a$ and outer radius $b$.

(d) When does a time varying electric field not generate a magnetic field?

3. (a) For linear media of permittivity $\varepsilon$ and permeability $\mu$, use the magnetic vector potential $\mathbf{A}$ to rewrite Faraday's law as the curl of a function.

(b) Can a scalar potential function $V$ be defined? What is the electric field in terms of $V$ and $\mathbf{A}$? The choice of $V$ is not unique so pick $V$ so that under static conditions $\mathbf{E} = -\nabla V$.

(c) Use the results of (a) and (b) in Ampere's law with Maxwell's displacement current correction to obtain a single equation in $\mathbf{A}$ and $V$. (**Hint:** $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$.)

(d) Since we are free to specify $\nabla \cdot \mathbf{A}$, what value should we pick to make (c) an equation just in $\mathbf{A}$? This is called setting the gauge.

(e) Use the results of (a)–(d) in Gauss's law for $\mathbf{D}$ to obtain a single equation in $V$.

(f) Consider a sinusoidally varying point charge at $r=0$, $\hat{Q}e^{j\omega t}$. Solve (e) for $r>0$.

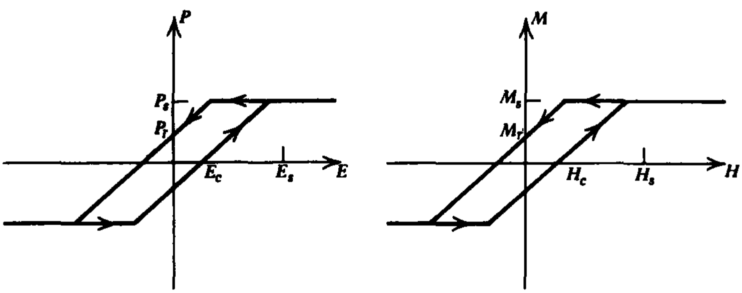
**Hint:**

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=\frac{\partial^2}{\partial r^2}(rV)$$

Define a new variable $(rV)$. By symmetry, $V$ only depends on $r$ and waves can only propagate away from the charge and not towards it. As $r\to 0$, the potential approaches the quasi-static Coulomb potential.

Section 7-2

4. Poynting's theorem must be modified if we have a hysteretic material with a nonlinear and double-valued relationship between the polarization **P** and electric field **E** and the magnetization **M** and magnetic field **H**.




(a) For these nonlinear constitutive laws put Poynting's theorem in the form

$$\nabla\cdot\boldsymbol{S}+\frac{\partial w}{\partial t}=-P_d-P_P-P_M$$

where $P_P$ and $P_M$ are the power densities necessary to polarize and magnetize the material.

(b) Sinusoidal electric and magnetic fields $\mathbf{E}=\mathbf{E}_s\cos\omega t$ and $\mathbf{H}=\mathbf{H}_s\cos\omega t$ are applied. How much energy density is dissipated per cycle?

5. An electromagnetic field is present within a superconductor with constituent relation

$$\frac{\partial\mathbf{J}_f}{\partial t}=\omega_p^2\varepsilon\mathbf{E}$$

(a) Show that Poynting's theorem can be written in the form

$$\nabla\cdot\boldsymbol{S}+\frac{\partial w}{\partial t}=0$$

What is $w$?

(b) What is the velocity of the charge carriers each with charge $q$ in terms of the current density $\mathbf{J}_f$? The number density of charge carriers is $n$.

(c) What kind of energy does the superconductor add?

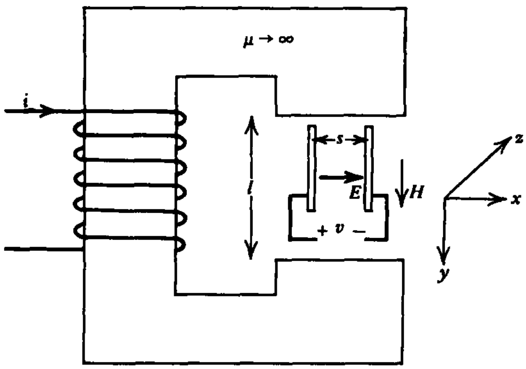
(d) Rewrite Maxwell's equations with this constitutive law for fields that vary sinusoidally with time.

(e) Derive the complex Poynting theorem in the form

$$\nabla \cdot [\tfrac{1}{2}\hat{\mathbf{E}}(\mathbf{r}) \times \hat{\mathbf{H}}^*(\mathbf{r})] + 2j\omega \langle w \rangle = 0$$

What is $\langle w \rangle$?

6. A paradoxical case of Poynting's theorem occurs when a static electric field is applied perpendicularly to a static magnetic field, as in the case of a pair of electrodes placed within a magnetic circuit.




(a) What are $\mathbf{E}$, $\mathbf{H}$, and $\boldsymbol{S}$?

(b) What is the energy density stored in the system?

(c) Verify Poynting's theorem.

7. The complex electric field amplitude has real and imaginary parts

$$\hat{\mathbf{E}}(\mathbf{r}) = \mathbf{E}_r + j\mathbf{E}_i$$

Under what conditions are the following scalar and vector products zero:

(a) $\hat{\mathbf{E}} \cdot \hat{\mathbf{E}} \overset{?}{=} 0$

(b) $\hat{\mathbf{E}} \cdot \hat{\mathbf{E}}^* \overset{?}{=} 0$

(c) $\hat{\mathbf{E}} \times \hat{\mathbf{E}} \overset{?}{=} 0$

(d) $\hat{\mathbf{E}} \times \hat{\mathbf{E}}^* \overset{?}{=} 0$

Section 7.3

8. Consider a lossy medium of permittivity $\varepsilon$, permeability $\mu$, and Ohmic conductivity $\sigma$.

(a) Write down the field equations for an $x$-directed electric field.

(b) Obtain a single equation in $E_x$.

(c) If the fields vary sinusoidally with time,

$$E_x = \mathrm{Re}\,[\hat{E}_x(z)\,e^{j\omega t}]$$

what are the spatial dependences of the fields?

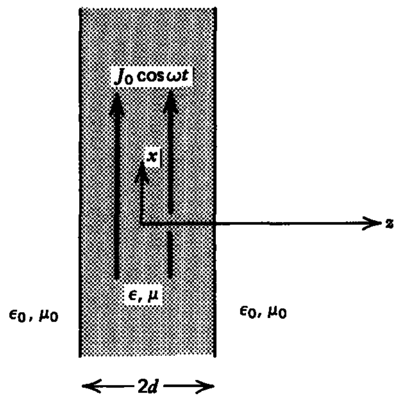
(d) Specialize (c) to the (i) low loss limit ($\sigma/\omega\varepsilon \ll 1$) and (ii) large loss limit ($\sigma/\omega\varepsilon \gg 1$).

(e) Repeat (a)–(c) if the medium is a plasma with constitutive law

$$\frac{\partial \mathbf{J}}{\partial t} = \omega_p^2 \varepsilon \mathbf{E}$$

(f) A current sheet $K_0 \cos \omega t\,\mathbf{i}_x$ is placed at $z = 0$. Find the electric and magnetic fields if the sheet is placed within an Ohmic conductor or within a plasma.

9. A uniformly distributed volume current of thickness $2d$, $J_0 \cos \omega t\,\mathbf{i}_x$, is a source of plane waves.




(a) From Maxwell's equations obtain a single differential equation relating $E_x$ to $J_x$.

(b) Find the electric and magnetic fields within and outside the current distribution.

(c) How much time-average power per unit area is delivered by the current?

(d) How does this generated power compare to the electromagnetic time-average power per unit area leaving the volume current at $z = \pm d$?

10. A TEM wave ($E_x$, $H_y$) propagates in a medium whose permittivity and permeability are functions of $z$, $\varepsilon(z)$, and $\mu(z)$.

(a) Write down Maxwell's equations and obtain single partial differential equations in $E_x$ and $H_y$.

(b) Consider the idealized case where $\varepsilon(z) = \varepsilon\, e^{\alpha|z|}$ and $\mu(z) = \mu\, e^{-\alpha|z|}$. A current sheet $K_0\, e^{j\omega t}\mathbf{i}_x$ is at $z = 0$. What are the resulting electric and magnetic fields on each side of the sheet?

(c) For what values of $\alpha$ are the solutions spatially evanescent or oscillatory?

11. We wish to compare various measurements between two observers, the second moving at a constant velocity $v\mathbf{i}_z$ with respect to the first.

(a) The first observer measures simultaneous events at two positions $z_1$ and $z_2$ so that $t_1 = t_2$. What is the time interval between the two events $t_1' - t_2'$ as measured by the second observer?

(b) The first observer measures a time interval $\Delta t = t_1 - t_2$ between two events at the same position $z$. What is the time interval as measured by the second observer?

(c) The first observer measures the length of a stick as $L = z_2 - z_1$. What is the length of the stick as measured by the second observer?

12. A stationary observer measures the velocity of a particle as $\mathbf{u} = u_x\mathbf{i}_x + u_y\mathbf{i}_y + u_z\mathbf{i}_z$.

(a) What velocity, $\mathbf{u}' = u_x'\mathbf{i}_x + u_y'\mathbf{i}_y + u_z'\mathbf{i}_z$, does another observer moving at constant speed $v\mathbf{i}_z$ measure?

(b) Find $\mathbf{u}'$ for the following values of $\mathbf{u}$ where $c_0$ is the free space speed of light:

(i) $\mathbf{u} = c_0\mathbf{i}_x$
(ii) $\mathbf{u} = c_0\mathbf{i}_y$
(iii) $\mathbf{u} = c_0\mathbf{i}_z$
(iv) $\mathbf{u} = (c_0/\sqrt{3})[\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z]$

(c) Do the results of (a) and (b) agree with the postulate that the speed of light for all observers is $c_0$?

Section 7.4

13. An electric field is of the form

$$\mathbf{E} = 100\, e^{j(2\pi \times 10^6 t - 2\pi \times 10^{-2} z)}\mathbf{i}_x \text{ volts/m}$$

(a) What is the frequency, wavelength, and speed of light in the medium?

(b) If the medium has permeability $\mu_0 = 4\pi \times 10^{-7}$ henry/m, what is the permittivity $\varepsilon$, wave impedance $\eta$, and the magnetic field?

(c) How much time-average power per unit area is carried by the wave?

14. The electric field of an elliptically polarized plane wave in a medium with wave impedance $\eta$ is

$$\mathbf{E} = \text{Re}\,(E_{x0}\mathbf{i}_x + E_{y0}\, e^{j\phi}\mathbf{i}_y)\, e^{j(\omega t - kz)}$$

where $E_{x0}$ and $E_{y0}$ are real.

(a) What is the magnetic field?

(b) What is the instantaneous and time-average power flux densities?

15. In Section 3-1-4 we found that the force on one of the charges $Q$ of a spherical atomic electric dipole of radius $R_0$ is

$$\mathbf{F}=Q\left[\mathbf{E}-\frac{Q\mathbf{d}}{4\pi\varepsilon_0 R_0^3}\right]$$

where $\mathbf{d}$ is the dipole spacing.

(a) Write Newton's law for this moveable charge with mass $M$ assuming that the electric field varies sinusoidally with time as $E_0 \cos \omega t$ and solve for $\mathbf{d}$. (**Hint:** Let $\omega_0^2 = Q^2/(M4\pi\varepsilon_0 R_0^3)$.)

(b) What is the polarization $\mathbf{P}$ as a function of $\mathbf{E}$ if there are $N$ dipoles per unit volume? What is the frequency dependent permittivity function $\varepsilon(\omega)$, where

$$\mathbf{D}(\mathbf{r})=\varepsilon(\omega)\mathbf{E}(\mathbf{r})$$

This model is often appropriate for light propagating in dielectric media.

(c) Use the results of (b) in Maxwell's equations to find the relation between the wavenumber $k$ and frequency $\omega$.

(d) For what frequency ranges do we have propagation or evanescence?

(e) What are the phase and group velocities of the waves?

(f) Derive the complex Poynting's theorem for this dispersive dielectric.

16. High-frequency wave propagation in the ionosphere is partially described by the development in Section 7-4-4 except that we must include the earth's dc magnetic field, which we take to be $H_0\mathbf{i}_z$.

(a) The charge carriers have charge $q$ and mass $m$. Write the three components of Newton's force law neglecting collisions but including inertia and the Coulomb–Lorentz force law. Neglect the magnetic field amplitudes of the propagating waves compared to $H_0$ in the Lorentz force law.

(b) Solve for each component of the current density $\mathbf{J}$ in terms of the charge velocity components assuming that the propagating waves vary sinusoidally with time as $e^{j\omega t}$. **Hint:** Define

$$\omega_p^2=\frac{q^2 n}{m\varepsilon},\qquad \omega_0=\frac{q\mu_0 H_0}{m}$$

(c) Use the results of (b) in Maxwell's equations for fields of the form $e^{j(\omega t-kz)}$ to solve for the wavenumber $k$ in terms of $\omega$.

(d) At what frequencies is the wavenumber zero or infinite? Over what frequency range do we have evanescence or propagation?

(e) For each of the two modes found in (c), what is the polarization of the electric field?

(f) What is the phase velocity of each wave? Since each mode travels at a different speed, the atmosphere acts like an anisotropic birefringent crystal. A linearly polarized wave $E_0 e^{j(\omega t - k_0 z)}\mathbf{i}_x$ is incident upon such a medium. Write this field as the sum of right and left circularly polarized waves.

**Hint:**

$$E_0\mathbf{i}_x = \frac{E_0}{2}(\mathbf{i}_x + j\mathbf{i}_y) + \frac{E_0}{2}(\mathbf{i}_x - j\mathbf{i}_y)$$

(g) If the transmitted field at $z = 0$ just inside the medium has amplitude $E_t e^{j\omega t}\mathbf{i}_x$, what are the electric and magnetic fields throughout the medium?

17. Nitrobenzene with $\mu = \mu_0$ and $\varepsilon = 35\varepsilon_0$ is placed between parallel plate electrodes of spacing $s$ and length $l$ stressed by a dc voltage $V_0$. Measurements have shown that light polarized parallel to the dc electric field travels at the speed $c_\|$, while light polarized perpendicular to the dc electric field travels slightly faster at the speed $c_\perp$, being related to the dc electric field $E_0$ and free space light wavelength as

$$\frac{1}{c_\|} - \frac{1}{c_\perp} = \lambda B E_0^2$$

where $B$ is called the Kerr constant which for nitrobenzene is $B \approx 4.3 \times 10^{-12}\ \text{sec/V}^2$ at $\lambda = 500$ nm.

(a) Linearly polarized light with free space wavelength $\lambda =$ 500 nm is incident at 45° to the dc electric field. After exiting the Kerr cell, what is the phase difference between the field components of the light parallel and perpendicular to the dc electric field?

(b) What are all the values of electric field strengths that allow the Kerr cell to act as a quarter- or half-wave plate?

(c) The Kerr cell is placed between crossed polarizers (polariscope). What values of electric field allow maximum light transmission? No light transmission?

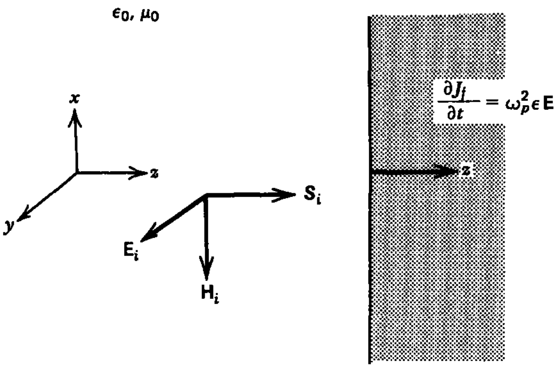
Section 7.5

18. A uniform plane wave with $y$-directed electric field is normally incident upon a plasma medium at $z = 0$ with constitutive law $\partial \mathbf{J}_f/\partial t = \omega_p^2 \varepsilon \mathbf{E}$. The fields vary sinusoidally in time as $e^{j\omega t}$.

(a) What is the general form of the incident, reflected, and transmitted fields?

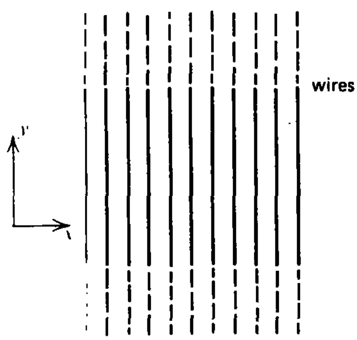
(b) Applying the boundary conditions, find the field amplitudes.

(c) What is the time-average electromagnetic power density in each region for $\omega > \omega_p$ and for $\omega < \omega_p$?




19. A polarizing filter to microwaves is essentially formed by many highly conducting parallel wires whose spacing is much smaller than a wavelength. That polarization whose electric field is transverse to the wires passes through. The incident electric field is

$$\mathbf{E} = E_x \cos(\omega t - kz)\mathbf{i}_x + E_y \sin(\omega t - kz)\mathbf{i}_y$$




(a) What is the incident magnetic field and incident power density?

(b) What are the transmitted fields and power density?

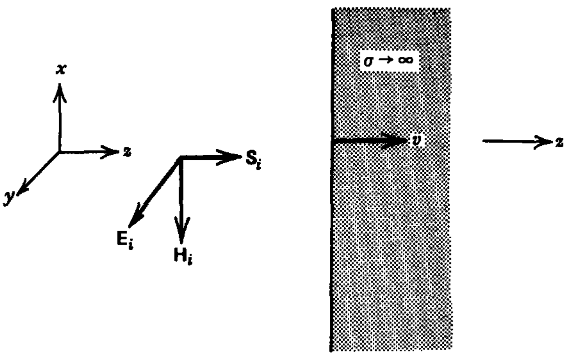
(c) Another set of polarizing wires are placed parallel but a distance $d$ and orientated at an angle $\phi$ to the first. What are the transmitted fields?

20. A uniform plane wave with $y$-directed electric field $E_y = E_0 \cos \omega(t - z/c)$ is normally incident upon a perfectly conducting plane that is moving with constant velocity $v\mathbf{i}_z$, where $v \ll c$.

(a) What are the total electric and magnetic fields in each region?

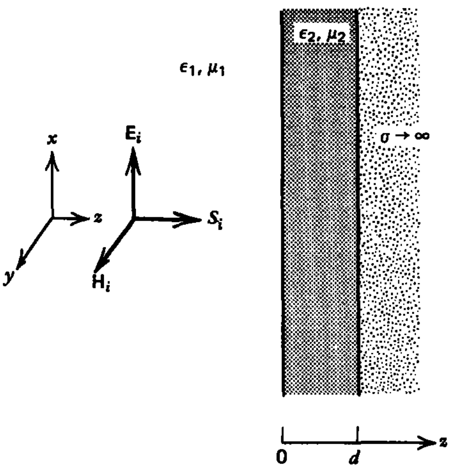
(b) What is the frequency of the reflected wave?

(c) What is the power flow density? Why can't we use the complex Poynting vector to find the time-average power?




Section 7.6

21. A dielectric ($\varepsilon_2$, $\mu_2$) of thickness $d$ coats a perfect conductor. A uniform plane wave is normally incident onto the coating from the surrounding medium with properties ($\varepsilon_1$, $\mu_1$).




(a) What is the general form of the fields in the two dielectric media? (**Hint:** Why can the transmitted electric field be written as $\mathbf{E}_t = \text{Re}\,[\hat{E}_t \sin k_2(z-d)\, e^{j\omega t}\mathbf{i}_x]$?)

(b) Applying the boundary conditions, what are the field amplitudes?

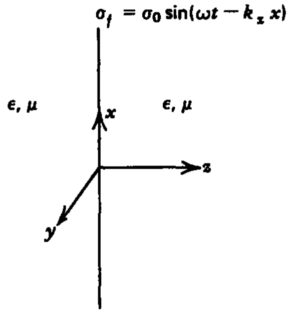
(c) What is the time-average power flow in each region?

(d) What is the time-average radiation pressure on the conductor?

Section 7.7

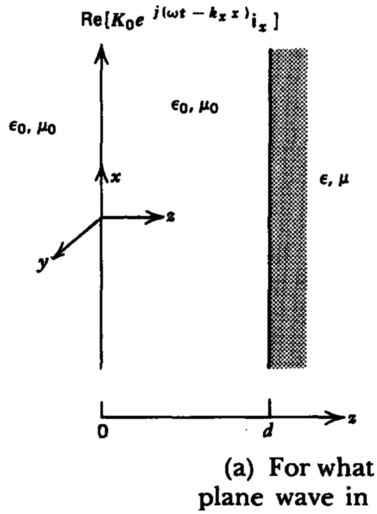
22. An electric field of the form $\text{Re}\,(\hat{E}\, e^{j\omega t}\, e^{-\boldsymbol{\gamma}\cdot\mathbf{r}})$ propagates in a lossy conductor with permittivity $\varepsilon$, permeability $\mu$, and conductivity $\sigma$. If $\boldsymbol{\gamma} = \boldsymbol{\alpha} + j\mathbf{k}$, what equalities must $\boldsymbol{\alpha}$ and $\mathbf{k}$ obey?

23. A sheet of surface charge with charge density $\sigma_0 \sin(\omega t - k_x x)$ is placed at $z = 0$ within a linear medium with properties $(\varepsilon, \mu)$.



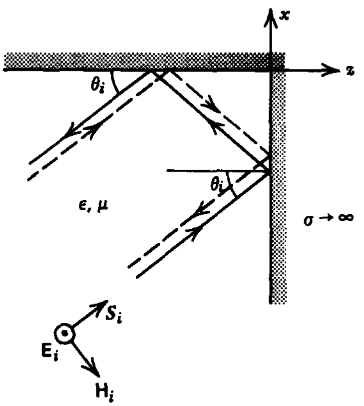

(a) What are the electric and magnetic fields?
(b) What surface current flows on the sheet?

24. A current sheet of the form $\text{Re}\,(K_0\, e^{j(\omega t - k_x x)}\mathbf{i}_x)$ is located in free space at $z = 0$. A dielectric medium $(\varepsilon, \mu)$ of semi-infinite extent is placed at $z = d$.




(a) For what range of frequency can we have a nonuniform plane wave in free space and a uniform plane wave in the dielectric? Nonuniform plane wave in each region? Uniform plane wave in each region?
(b) What are the electric and magnetic fields everywhere?
(c) What is the time-average $z$-directed power flow density in each region if we have a nonuniform plane wave in free space but a uniform plane wave in the dielectric?

Section 7.8
25. A uniform plane wave $\text{Re}\,(E_0\, e^{j(\omega t - k_x x - k_z z)}\mathbf{i}_y)$ is obliquely incident upon a right-angled perfectly conducting corner. The wave is incident at angle $\theta_i$ to the $z = 0$ wall.




(a) Try a solution composed of the incident and reflected waves off each surface of the conductor. What is the general form of solution? (**Hint:** There are four different waves.)

(b) Applying the boundary conditions, what are the electric and magnetic fields?

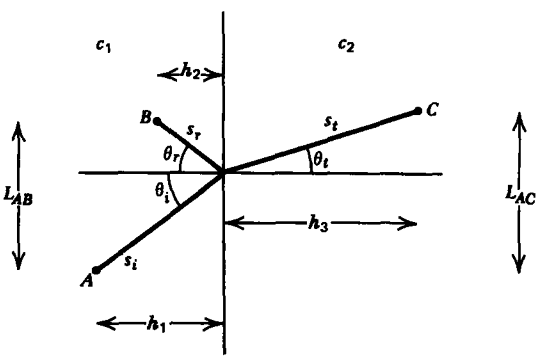
(c) What are the surface charge and current distributions on the conducting walls?

(d) What is the force per unit area on each wall?

(e) What is the power flow density?

Section 7.9

26. Fermat's principle of least time states that light, when reflected or refracted off an interface, will pick the path of least time to propagate between two points.




(a) A beam of light from point $A$ is incident upon a dielectric interface at angle $\theta_i$ from the normal and is reflected through the point $B$ at angle $\theta_r$. In terms of $\theta_i$, $\theta_r$, $h_1$ and $h_2$, and the speed of light $c_1$, how long does it take light to travel from $A$ to $B$ along this path? What other relation is there between $\theta_i$, $\theta_r$, $L_{AB}$, $h_1$ and $h_2$?

(b) Find the angle $\theta_i$ that satisfies Fermat's principle. What is $\theta_r$?

(c) In terms of $\theta_i$, $\theta_t$, $h_1$, $h_3$, and the light speeds $c_1$ and $c_2$ in each medium, how long does it take light to travel from $A$ to $C$?

(d) Find the relationship between $\theta_i$ and $\theta_t$ that satisfies Fermat's principle.

27. In many cases the permeability of dielectric media equals that of free space. In this limit show that the reflection and transmission coefficients for waves obliquely incident upon dielectric media are: **E** parallel to the interface

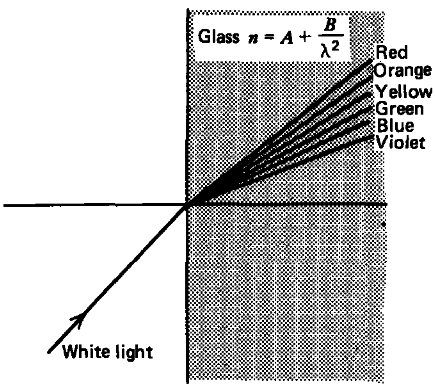
$$R = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}, \qquad T = \frac{2\cos\theta_i \sin\theta_t}{\sin(\theta_i + \theta_t)}$$

**H** parallel to the interface

$$R = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}, \qquad T = \frac{2\cos\theta_i \sin\theta_t}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)}$$

28. White light is composed of the entire visible spectrum. The index of refraction $n$ for most materials is a weak function of wavelength $\lambda$, often described by Cauchy's equation

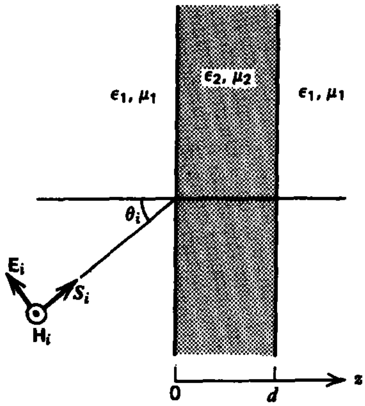
$$n = A + B/\lambda^2$$




A beam of white light is incident at 30° to a piece of glass with $A = 1.5$ and $B = 5 \times 10^{-15}$ m$^2$. What are the transmitted angles for the colors violet (400 nm), blue (450 nm), green (550 nm), yellow (600 nm), orange (650 nm), and red (700 nm)? This separation of colors is called dispersion.

29. A dielectric slab of thickness $d$ with speed of light $c_2$ is placed within another dielectric medium of infinite extent with speed of light $c_1$, where $c_1 < c_2$. An electromagnetic wave with **H** parallel to the interface is incident onto the slab at angle $\theta_i$.

(a) Find the electric and magnetic fields in each region. (**Hint:** Use Cramer's rule to find the four unknown field amplitudes in terms of $E_i$.)

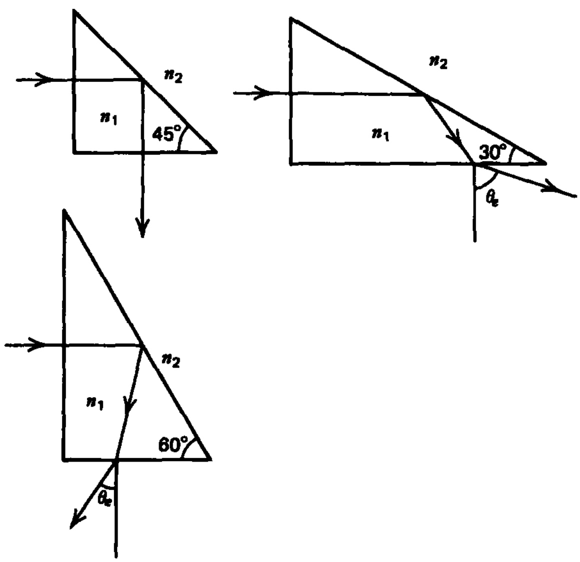



(b) For what range of incident angle do we have uniform or nonuniform plane waves through the middle region?

(c) What is the transmitted time-average power density with uniform or nonuniform plane waves through the middle region. How can we have power flow through the middle region with nonuniform plane waves?

Section 7.10

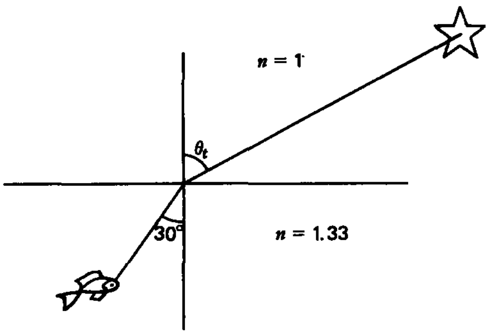
30. Consider the various prisms shown.



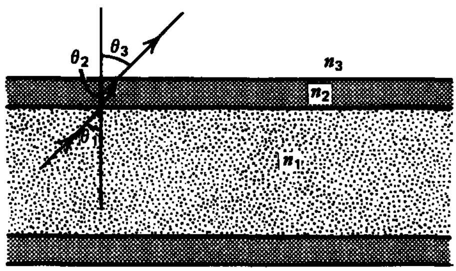

(a) What is the minimum index of refraction $n_1$ necessary for no time-average power to be transmitted across the hypotenuse when the prisms are in free space, $n_2 = 1$, or water, $n_2 = 1.33$?

(b) At these values of refractive index, what are the exiting angles $\theta_e$?

31. A fish below the surface of water with index of refraction $n = 1.33$ sees a star that he measures to be at 30° from the normal. What is the star's actual angle from the normal?




32. A straight light pipe with refractive index $n_1$ has a dielectric coating with index $n_2$ added for protection. The light pipe is usually within free space so that $n_3$ is typically unity.




(a) Light within the pipe is incident upon the first interface at angle $\theta_1$. What are the angles $\theta_2$ and $\theta_3$?

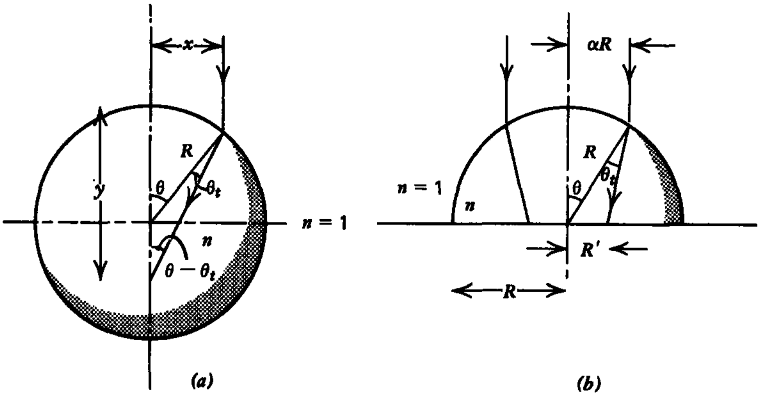
(b) What value of $\theta_1$ will make $\theta_3$ just equal the critical angle for total internal reflection at the second interface?

(c) How does this value differ from the critical angle if the coating was not present so that $n_1$ was directly in contact with $n_3$?

(d) If we require that total reflection occur at the first interface, what is the allowed range of incident angle $\theta_1$. Must the coating have a larger or smaller index of refraction than the light pipe?

33. A spherical piece of glass of radius $R$ has refractive index $n$.

(a) A vertical light ray is incident at the distance $x$ ($x < R$) from the vertical diameter. At what distance $y$ from the top of the sphere will the light ray intersect the vertical diameter? For what range of $n$ and $x$ will the refracted light intersect the vertical diameter within the sphere?




(b) A vertical light beam of radius $\alpha R$ $(\alpha < 1)$ is incident upon a hemisphere of this glass that rests on a table top. What is the radius $R'$ of the light on the table?

# chapter 8

## *guided electromagnetic waves*

The uniform plane wave solutions developed in Chapter 7 cannot in actuality exist throughout all space, as an infinite amount of energy would be required from the sources. However, TEM waves can also propagate in the region of finite volume between electrodes. Such electrode structures, known as transmission lines, are used for electromagnetic energy flow from power (60 Hz) to microwave frequencies, as delay lines due to the finite speed $c$ of electromagnetic waves, and in pulse forming networks due to reflections at the end of the line. Because of the electrode boundaries, more general wave solutions are also permitted where the electric and magnetic fields are no longer perpendicular. These new solutions also allow electromagnetic power flow in closed single conductor structures known as waveguides.

## 8-1 THE TRANSMISSION LINE EQUATIONS

### 8-1-1 The Parallel Plate Transmission Line

The general properties of transmission lines are illustrated in Figure 8-1 by the parallel plate electrodes a small distance $d$ apart enclosing linear media with permittivity $\varepsilon$ and permeability $\mu$. Because this spacing $d$ is much less than the width $w$ or length $l$, we neglect fringing field effects and assume that the fields only depend on the $z$ coordinate.

The perfectly conducting electrodes impose the boundary conditions:

(i) The tangential component of **E** is zero.
(ii) The normal component of **B** (and thus **H** in the linear media) is zero.

With these constraints and the neglect of fringing near the electrode edges, the fields cannot depend on $x$ or $y$ and thus are of the following form:

$$\begin{aligned} \mathbf{E} &= E_x(z, t)\mathbf{i}_x \\ \mathbf{H} &= H_y(z, t)\mathbf{i}_y \end{aligned} \tag{1}$$

which when substituted into Maxwell's equations yield

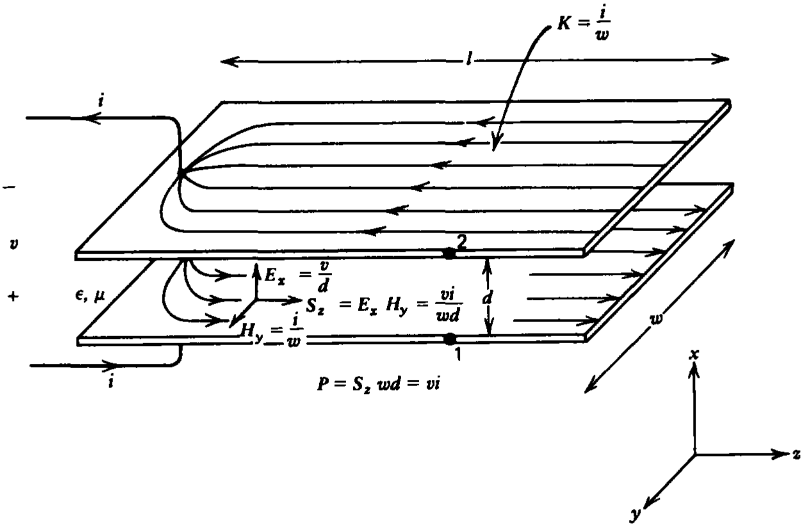



Figure 8-1 The simplest transmission line consists of two parallel perfectly conducting plates a small distance $d$ apart.

$$\begin{aligned}
\nabla \times \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} \Rightarrow \frac{\partial E_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \\
\nabla \times \mathbf{H} &= \varepsilon \frac{\partial \mathbf{E}}{\partial t} \Rightarrow \frac{\partial H_y}{\partial z} = -\varepsilon \frac{\partial E_x}{\partial t}
\end{aligned} \tag{2}$$

We recognize these equations as the same ones developed for plane waves in Section 7-3-1. The wave solutions found there are also valid here. However, now it is more convenient to introduce the circuit variables of voltage and current along the transmission line, which will depend on $z$ and $t$.

Kirchoff's voltage and current laws will not hold along the transmission line as the electric field in (2) has nonzero curl and the current along the electrodes will have a divergence due to the time varying surface charge distribution, $\sigma_f = \pm\varepsilon E_x(z, t)$. Because $\mathbf{E}$ has a curl, the voltage difference measured between any two points is not unique, as illustrated in Figure 8-2, where we see time varying magnetic flux passing through the contour $L_1$. However, no magnetic flux passes through the path $L_2$, where the potential difference is measured between the two electrodes at the same value of $z$, as the magnetic flux is parallel to the surface. Thus, the voltage can be uniquely defined between the two electrodes at the same value of $z$:

$$v(z, t) = \int_{\substack{1 \\ z=\text{const}}}^{2} \mathbf{E} \cdot d\mathbf{l} = E_x(z, t)d \tag{3}$$

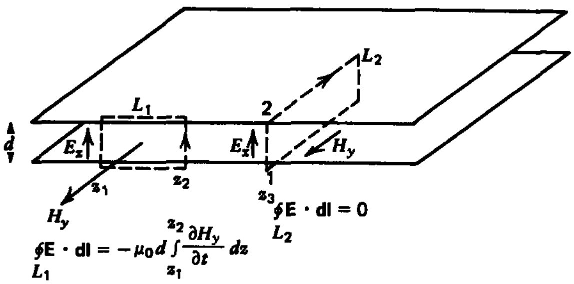



Figure 8-2 The potential difference measured between any two arbitrary points at different positions $z_1$ and $z_2$ on the transmission line is not unique—the line integral $L_1$ of the electric field is nonzero since the contour has magnetic flux passing through it. If the contour $L_2$ lies within a plane of constant $z$ such as at $z_3$, no magnetic flux passes through it so that the voltage difference between the two electrodes at the same value of $z$ is unique.

Similarly, the tangential component of **H** is discontinuous at each plate by a surface current $\pm\mathbf{K}$. Thus, the total current $i(z, t)$ flowing in the $z$ direction on the lower plate is

$$i(z,t) = K_z w = H_y w \tag{4}$$

Substituting (3) and (4) back into (2) results in the transmission line equations:

$$\begin{aligned}\frac{\partial v}{\partial z} &= -L\frac{\partial i}{\partial t}\\ \frac{\partial i}{\partial z} &= -C\frac{\partial v}{\partial t}\end{aligned} \tag{5}$$

where $L$ and $C$ are the inductance and capacitance per unit length of the parallel plate structure:

$$L = \frac{\mu d}{w}\ \text{henry/m}, \qquad C = \frac{\varepsilon w}{d}\ \text{farad/m} \tag{6}$$

If both quantities are multiplied by the length of the line $l$, we obtain the inductance of a single turn plane loop if the line were short circuited, and the capacitance of a parallel plate capacitor if the line were open circuited.

It is no accident that the $LC$ product

$$LC = \varepsilon\mu = 1/c^2 \tag{7}$$

is related to the speed of light in the medium.

### 8-1-2 General Transmission Line Structures

The transmission line equations of (5) are valid for any two-conductor structure of arbitrary shape in the transverse

$xy$ plane but whose cross-sectional area does not change along its axis in the $z$ direction. $L$ and $C$ are the inductance and capacitance per unit length as would be calculated in the quasi-static limits. Various simple types of transmission lines are shown in Figure 8-3. Note that, in general, the field equations of (2) must be extended to allow for $x$ and $y$ components but still no $z$ components:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_T(x, y, z, t) = E_x \mathbf{i}_x + E_y \mathbf{i}_y, \qquad E_z = 0 \\ \mathbf{H} &= \mathbf{H}_T(x, y, z, t) = H_x \mathbf{i}_x + H_y \mathbf{i}_y, \qquad H_z = 0 \end{aligned} \tag{8}$$

We use the subscript $T$ in (8) to remind ourselves that the fields lie purely in the transverse $xy$ plane. We can then also distinguish between spatial derivatives along the $z$ axis $(\partial/\partial z)$ from those in the transverse plane $(\partial/\partial x, \partial/\partial y)$:

$$\nabla = \underbrace{\nabla_T}_{\mathbf{i}_x \frac{\partial}{\partial x} + \mathbf{i}_y \frac{\partial}{\partial y}} + \mathbf{i}_z \frac{\partial}{\partial z} \tag{9}$$

We may then write Maxwell's equations as

$$\begin{aligned} \nabla_T \times \mathbf{E}_T + \frac{\partial}{\partial z}(\mathbf{i}_z \times \mathbf{E}_T) &= -\mu \frac{\partial \mathbf{H}_T}{\partial t} \\ \nabla_T \times \mathbf{H}_T + \frac{\partial}{\partial z}(\mathbf{i}_z \times \mathbf{H}_T) &= \varepsilon \frac{\partial \mathbf{E}_T}{\partial t} \\ \nabla_T \cdot \mathbf{E}_T &= 0 \\ \nabla_T \cdot \mathbf{H}_T &= 0 \end{aligned} \tag{10}$$

The following vector properties for the terms in (10) apply:

(i) $\nabla_T \times \mathbf{H}_T$ and $\nabla_T \times \mathbf{E}_T$ lie purely in the $z$ direction.
(ii) $\mathbf{i}_z \times \mathbf{E}_T$ and $\mathbf{i}_z \times \mathbf{H}_T$ lie purely in the $xy$ plane.

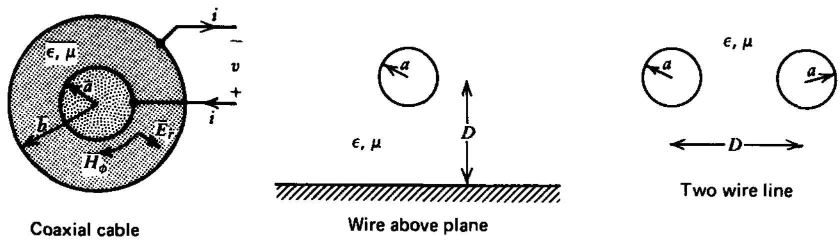



Figure 8-3 Various types of simple transmission lines.

Thus, the equations in (10) may be separated by equating vector components:

$$\begin{aligned} \nabla_T \times \mathbf{E}_T = 0, \qquad \nabla_T \times \mathbf{H}_T = 0 \\ \nabla_T \cdot \mathbf{E}_T = 0, \qquad \nabla_T \cdot \mathbf{H}_T = 0 \end{aligned} \tag{11}$$

$$\begin{aligned} &\frac{\partial}{\partial z}(\mathbf{i}_z \times \mathbf{E}_T) = -\mu \frac{\partial \mathbf{H}_T}{\partial t} \Rightarrow \frac{\partial \mathbf{E}_T}{\partial z} = \mu \frac{\partial}{\partial t}(\mathbf{i}_z \times \mathbf{H}_T) \\ &\frac{\partial}{\partial z}(\mathbf{i}_z \times \mathbf{H}_T) = \varepsilon \frac{\partial \mathbf{E}_T}{\partial t} \end{aligned} \tag{12}$$

where the Faraday's law equalities are obtained by crossing with $\mathbf{i}_z$ and expanding the double cross product

$$\mathbf{i}_z \times (\mathbf{i}_z \times \mathbf{E}_T) = \mathbf{i}_z (\cancelto{0}{\mathbf{i}_z \cdot \mathbf{E}_T}) - \mathbf{E}_T(\mathbf{i}_z \cdot \mathbf{i}_z) = -\mathbf{E}_T \tag{13}$$

and remembering that $\mathbf{i}_z \cdot \mathbf{E}_T = 0$.

The set of equations in (11) tell us that the field dependences on the transverse coordinates are the same as if the system were static and source free. Thus, all the tools developed for solving static field solutions, including the two-dimensional Laplace's equations and the method of images, can be used to solve for $\mathbf{E}_T$ and $\mathbf{H}_T$ in the transverse $xy$ plane.

We need to relate the fields to the voltage and current defined as a function of $z$ and $t$ for the transmission line of arbitrary shape shown in Figure 8-4 as

$$\begin{aligned} v(z, t) &= \int_{1 \atop z=\text{const}}^{2} \mathbf{E}_T \cdot \mathbf{dl} \\ i(z, t) &= \oint_{\substack{\text{contour } L \\ \text{at constant } z \\ \text{enclosing the} \\ \text{inner conductor}}} \mathbf{H}_T \cdot \mathbf{ds} \end{aligned} \tag{14}$$

The related quantities of charge per unit length $q$ and flux per unit length $\lambda$ along the transmission line are

$$\begin{aligned} q(z, t) &= \varepsilon \oint_{L \atop z=\text{const}} \mathbf{E}_T \cdot \mathbf{n}\, ds \\ \lambda(z, t) &= \mu \int_{1 \atop z=\text{const}}^{2} \mathbf{H}_T \cdot (\mathbf{i}_z \times \mathbf{dl}) \end{aligned} \tag{15}$$

The capacitance and inductance per unit length are then defined as the ratios:

$$\begin{aligned} C &= \frac{q(z, t)}{v(z, t)} = \left. \frac{\varepsilon \oint_L \mathbf{E}_T \cdot \mathbf{ds}}{\int_1^2 \mathbf{E}_T \cdot \mathbf{dl}} \right|_{z=\text{const}} \\ L &= \frac{\lambda(z, t)}{i(z, t)} = \left. \frac{\mu \int_1^2 \mathbf{H}_T \cdot (\mathbf{i}_z \times \mathbf{dl})}{\oint_L \mathbf{H}_T \cdot \mathbf{ds}} \right|_{z=\text{const}} \end{aligned} \tag{16}$$

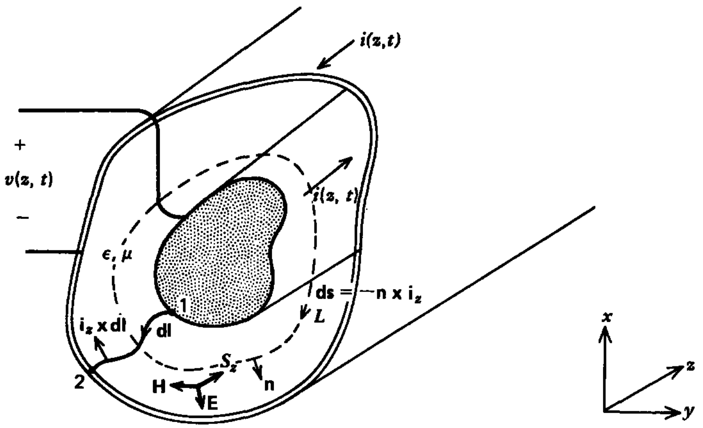



Figure 8-4 A general transmission line has two perfect conductors whose cross-sectional area does not change in the direction along its $z$ axis, but whose shape in the transverse $xy$ plane is arbitrary. The electric and magnetic fields are perpendicular, lie in the transverse $xy$ plane, and have the same dependence on $x$ and $y$ as if the fields were static.

which are constants as the geometry of the transmission line does not vary with $z$. Even though the fields change with $z$, the ratios in (16) do not depend on the field amplitudes.

To obtain the general transmission line equations, we dot the upper equation in (12) with $\mathbf{dl}$, which can be brought inside the derivatives since $\mathbf{dl}$ only varies with $x$ and $y$ and not $z$ or $t$. We then integrate the resulting equation over a line at constant $z$ joining the two electrodes:

$$\frac{\partial}{\partial z}\left(\int_1^2 \mathbf{E}_T \cdot \mathbf{dl}\right) = \frac{\partial}{\partial t}\left(\mu \int_1^2 (\mathbf{i}_z \times \mathbf{H}_T) \cdot \mathbf{dl}\right)$$
$$= -\frac{\partial}{\partial t}\left(\mu \int_1^2 \mathbf{H}_T \cdot (\mathbf{i}_z \times \mathbf{dl})\right) \tag{17}$$

where the last equality is obtained using the scalar triple product allowing the interchange of the dot and the cross:

$$(\mathbf{i}_z \times \mathbf{H}_T) \cdot \mathbf{dl} = -(\mathbf{H}_T \times \mathbf{i}_z) \cdot \mathbf{dl} = -\mathbf{H}_T \cdot (\mathbf{i}_z \times \mathbf{dl}) \tag{18}$$

We recognize the left-hand side of (17) as the $z$ derivative of the voltage defined in (14), while the right-hand side is the negative time derivative of the flux per unit length defined in (15):

$$\frac{\partial v}{\partial z} = -\frac{\partial \lambda}{\partial t} = -L\frac{\partial i}{\partial t} \tag{19}$$

We could also have derived this last relation by dotting the upper equation in (12) with the normal $\mathbf{n}$ to the inner conductor and then integrating over the contour $L$ surrounding the inner conductor:

$$\frac{\partial}{\partial z}\left(\oint_L \mathbf{n}\cdot\mathbf{E}_T\,ds\right)=\frac{\partial}{\partial t}\left(\mu\oint_L \mathbf{n}\cdot(\mathbf{i}_z\times\mathbf{H}_T)\,ds\right)=-\frac{\partial}{\partial t}\left(\mu\oint_L \mathbf{H}_T\cdot\mathbf{ds}\right) \tag{20}$$

where the last equality was again obtained by interchanging the dot and the cross in the scalar triple product identity:

$$\mathbf{n}\cdot(\mathbf{i}_z\times\mathbf{H}_T)=(\mathbf{n}\times\mathbf{i}_z)\cdot\mathbf{H}_T=-\mathbf{H}_T\cdot\mathbf{ds} \tag{21}$$

The left-hand side of (20) is proportional to the charge per unit length defined in (15), while the right-hand side is proportional to the current defined in (14):

$$\frac{1}{\varepsilon}\frac{\partial q}{\partial z}=-\mu\frac{\partial i}{\partial t}\Rightarrow C\frac{\partial v}{\partial z}=-\varepsilon\mu\frac{\partial i}{\partial t} \tag{22}$$

Since (19) and (22) must be identical, we obtain the general result previously obtained in Section 6-5-6 that the inductance and capacitance per unit length of any arbitrarily shaped transmission line are related as

$$LC=\varepsilon\mu \tag{23}$$

We obtain the second transmission line equation by dotting the lower equation in (12) with $\mathbf{dl}$ and integrating between electrodes:

$$\frac{\partial}{\partial t}\left(\varepsilon\int_1^2 \mathbf{E}_T\cdot\mathbf{dl}\right)=\frac{\partial}{\partial z}\left(\int_1^2(\mathbf{i}_z\times\mathbf{H}_T)\cdot\mathbf{dl}\right)=-\frac{\partial}{\partial z}\left(\int_1^2 \mathbf{H}_T\cdot(\mathbf{i}_z\times\mathbf{dl})\right) \tag{24}$$

to yield from (14)–(16) and (23)

$$\varepsilon\frac{\partial v}{\partial t}=-\frac{1}{\mu}\frac{\partial\lambda}{\partial z}=-\frac{L}{\mu}\frac{\partial i}{\partial z}\Rightarrow\frac{\partial i}{\partial z}=-C\frac{\partial v}{\partial t} \tag{25}$$

## EXAMPLE 8-1 THE COAXIAL TRANSMISSION LINE

Consider the coaxial transmission line shown in Figure 8-3 composed of two perfectly conducting concentric cylinders of radii $a$ and $b$ enclosing a linear medium with permittivity $\varepsilon$ and permeability $\mu$. We solve for the transverse dependence of the fields as if the problem were static, independent of time. If the voltage difference between cylinders is $v$ with the inner cylinder carrying a total current $i$ the static fields are

$$E_r=\frac{v}{r\ln(b/a)},\qquad H_\phi=\frac{i}{2\pi r}$$

The surface charge per unit length $q$ and magnetic flux per unit length $\lambda$ are

$$q = \varepsilon E_r(r=a)2\pi a = \frac{2\pi\varepsilon v}{\ln(b/a)}$$

$$\lambda = \int_a^b \mu H_\phi\, dr = \frac{\mu i}{2\pi}\ln\frac{b}{a}$$

so that the capacitance and inductance per unit length of this structure are

$$C = \frac{q}{v} = \frac{2\pi\varepsilon}{\ln(b/a)}, \qquad L = \frac{\lambda}{i} = \frac{\mu}{2\pi}\ln\frac{b}{a}$$

where we note that as required

$$LC = \varepsilon\mu$$

Substituting $E_r$ and $H_\phi$ into (12) yields the following transmission line equations:

$$\frac{\partial E_r}{\partial z} = -\mu\frac{\partial H_\phi}{\partial t} \Rightarrow \frac{\partial v}{\partial z} = -L\frac{\partial i}{\partial t}$$

$$\frac{\partial H_\phi}{\partial z} = -\varepsilon\frac{\partial E_r}{\partial t} \Rightarrow \frac{\partial i}{\partial z} = -C\frac{\partial v}{\partial t}$$

### 8-1-3 Distributed Circuit Representation

Thus far we have emphasized the field theory point of view from which we have derived relations for the voltage and current. However, we can also easily derive the transmission line equations using a distributed equivalent circuit derived from the following criteria:

(i) The flow of current through a lossless medium between two conductors is entirely by displacement current, in exactly the same way as a capacitor.
(ii) The flow of current along lossless electrodes generates a magnetic field as in an inductor.

Thus, we may discretize the transmission line into many small incremental sections of length $\Delta z$ with series inductance $L\,\Delta z$ and shunt capacitance $C\,\Delta z$, where $L$ and $C$ are the inductance and capacitance per unit lengths. We can also take into account the small series resistance of the electrodes $R\,\Delta z$, where $R$ is the resistance per unit length (ohms per meter) and the shunt conductance loss in the dielectric $G\,\Delta z$, where $G$ is the conductance per unit length (siemens per meter). If the transmission line and dielectric are lossless, $R=0$, $G=0$.

The resulting equivalent circuit for a lossy transmission line shown in Figure 8-5 shows that the current at $z+\Delta z$ and $z$ differ by the amount flowing through the shunt capacitance and conductance:

$$i(z,t)-i(z+\Delta z,t)=C\,\Delta z\frac{\partial v(z,t)}{\partial t}+G\,\Delta z\,v(z,t) \tag{26}$$

Similarly, the voltage difference at $z+\Delta z$ from $z$ is due to the drop across the series inductor and resistor:

$$v(z,t)-v(z+\Delta z,t)=L\Delta z\frac{\partial i(z+\Delta z,t)}{\partial t}+i(z+\Delta z,t)R\,\Delta z \tag{27}$$

By dividing (26) and (27) through by $\Delta z$ and taking the limit as $\Delta z\to 0$, we obtain the lossy transmission line equations:

$$\begin{aligned}\lim_{\Delta z\to 0}\frac{i(z+\Delta z,t)-i(z,t)}{\Delta z}&=\frac{\partial i}{\partial z}=-C\frac{\partial v}{\partial t}-Gv\\ \lim_{\Delta z\to 0}\frac{v(z+\Delta z,t)-v(z,t)}{\Delta z}&=\frac{\partial v}{\partial z}=-L\frac{\partial i}{\partial t}-iR\end{aligned} \tag{28}$$

which reduce to (19) and (25) when $R$ and $G$ are zero.

## 8-1-4 Power Flow

Multiplying the upper equation in (28) by $v$ and the lower by $i$ and then adding yields the circuit equivalent form of Poynting's theorem:

$$\frac{\partial}{\partial z}(vi)=-\frac{\partial}{\partial t}(\tfrac{1}{2}Cv^2+\tfrac{1}{2}Li^2)-Gv^2-i^2R \tag{29}$$

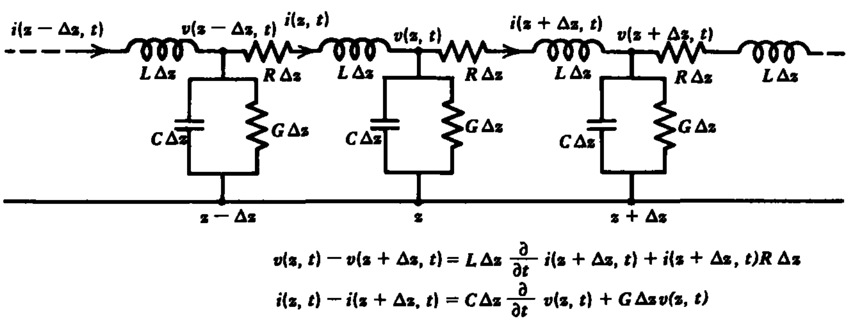



Figure 8-5 Distributed circuit model of a transmission line including small series and shunt resistive losses.

The power flow $vi$ is converted into energy storage ($\frac{1}{2}Cv^2 + \frac{1}{2}Li^2$) or is dissipated in the resistance and conductance per unit lengths.

From the fields point of view the total electromagnetic power flowing down the transmission line at any position $z$ is

$$P(z,t)=\int_S (\mathbf{E}_T\times\mathbf{H}_T)\cdot\mathbf{i}_z\,dS=\int_S \mathbf{E}_T\cdot(\mathbf{H}_T\times\mathbf{i}_z)\,dS \tag{30}$$

where S is the region between electrodes in Figure 8-4. Because the transverse electric field is curl free, we can define the scalar potential

$$\nabla\times\mathbf{E}_T=0\Rightarrow\mathbf{E}_T=-\nabla_T V \tag{31}$$

so that (30) can be rewritten as

$$P(z,t)=\int_S (\mathbf{i}_z\times\mathbf{H}_T)\cdot\nabla_T V\,dS \tag{32}$$

It is useful to examine the vector expansion

$$\nabla_T\cdot[V(\mathbf{i}_z\times\mathbf{H}_T)]=(\mathbf{i}_z\times\mathbf{H}_T)\cdot\nabla_T V+V\overbrace{\nabla_T\cdot(\mathbf{i}_z\times\mathbf{H}_T)}^{0} \tag{33}$$

where the last term is zero because $\mathbf{i}_z$ is a constant vector and $\mathbf{H}_T$ is also curl free:

$$\nabla_T\cdot(\mathbf{i}_z\times\mathbf{H}_T)=\mathbf{H}_T\cdot(\nabla_T\times\mathbf{i}_z)-\mathbf{i}_z\cdot(\nabla_T\times\mathbf{H}_T)=0 \tag{34}$$

Then (32) can be converted to a line integral using the two-dimensional form of the divergence theorem:

$$\begin{aligned}P(z,t)&=\int_S \nabla_T\cdot[V(\mathbf{i}_z\times\mathbf{H}_T)]\,dS\\&=-\int_{\substack{\text{contours on}\\\text{the surfaces of}\\\text{both electrodes}}} V(\mathbf{i}_z\times\mathbf{H}_T)\cdot\mathbf{n}\,ds\end{aligned} \tag{35}$$

where the line integral is evaluated at constant $z$ along the surface of both electrodes. The minus sign arises in (35) because $\mathbf{n}$ is defined inwards in Figure 8-4 rather than outwards as is usual in the divergence theorem. Since we are free to pick our zero potential reference anywhere, we take the outer conductor to be at zero voltage. Then the line integral in (35) is only nonzero over the inner conductor, where $V = v$:

$$
\begin{aligned}
P(z, t) &= -v \oint_{\substack{\text{inner}\\ \text{conductor}}} (\mathbf{i}_z \times \mathbf{H}_T) \cdot \mathbf{n}\, ds \\
&= v \oint_{\substack{\text{inner}\\ \text{conductor}}} (\mathbf{H}_T \times \mathbf{i}_z) \cdot \mathbf{n}\, ds \\
&= v \oint_{\substack{\text{inner}\\ \text{conductor}}} \mathbf{H}_T \cdot (\mathbf{i}_z \times \mathbf{n})\, ds \\
&= v \oint_{\substack{\text{inner}\\ \text{conductor}}} \mathbf{H}_T \cdot \mathbf{ds} \\
&= vi
\end{aligned}
\tag{36}
$$

where we realized that $(\mathbf{i}_z \times \mathbf{n})\, ds = \mathbf{ds}$, defined in Figure 8-4 if $L$ lies along the surface of the inner conductor. The electromagnetic power flowing down a transmission line just equals the circuit power.

### 8-1-5 The Wave Equation

Restricting ourselves now to lossless transmission lines so that $R = G = 0$ in (28), the two coupled equations in voltage and current can be reduced to two single wave equations in $v$ and $i$:

$$
\begin{aligned}
\frac{\partial^2 v}{\partial t^2} &= c^2 \frac{\partial^2 v}{\partial z^2} \\
\frac{\partial^2 i}{\partial t^2} &= c^2 \frac{\partial^2 i}{\partial z^2}
\end{aligned}
\tag{37}
$$

where the speed of the waves is

$$
c = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\varepsilon\mu}} \text{ m/sec} \tag{38}
$$

As we found in Section 7-3-2 the solutions to (37) are propagating waves in the $\pm z$ directions at the speed $c$:

$$
\begin{aligned}
v(z, t) &= \mathrm{V}_+(t - z/c) + \mathrm{V}_-(t + z/c) \\
i(z, t) &= I_+(t - z/c) + I_-(t + z/c)
\end{aligned}
\tag{39}
$$

where the functions $\mathrm{V}_+$, $\mathrm{V}_-$, $I_+$, and $I_-$ are determined by boundary conditions imposed by sources and transmission line terminations. By substituting these solutions back into (28) with $R = G = 0$, we find the voltage and current functions related as

$$\begin{aligned} \mathrm{V}_+ &= I_+ Z_0 \\ \mathrm{V}_- &= -I_- Z_0 \end{aligned} \tag{40}$$

where

$$Z_0 = \sqrt{L/C} \text{ ohm} \tag{41}$$

is known as the characteristic impedance of the transmission line, analogous to the wave impedance $\eta$ in Chapter 7. Its inverse $Y_0 = 1/Z_0$ is also used and is termed the characteristic admittance. In practice, it is difficult to measure $L$ and $C$ of a transmission line directly. It is easier to measure the wave speed $c$ and characteristic impedance $Z_0$ and then calculate $L$ and $C$ from (38) and (41).

The most useful form of the transmission line solutions of (39) that we will use is

$$\begin{aligned} v(z, t) &= \mathrm{V}_+(t - z/c) + \mathrm{V}_-(t + z/c) \\ i(z, t) &= Y_0[\mathrm{V}_+(t - z/c) - \mathrm{V}_-(t + z/c)] \end{aligned} \tag{42}$$

Note the complete duality between these voltage–current solutions and the plane wave solutions in Section 7-3-2 for the electric and magnetic fields.

## 8-2 TRANSMISSION LINE TRANSIENT WAVES

The easiest way to solve for transient waves on transmission lines is through use of physical reasoning as opposed to mathematical rigor. Since the waves travel at a speed $c$, once generated they cannot reach any position $z$ until a time $z/c$ later. Waves traveling in the positive $z$ direction are described by the function $\mathrm{V}_+(t - z/c)$ and waves traveling in the $-z$ direction by $\mathrm{V}_-(t + z/c)$. However, at any time $t$ and position $z$, the voltage is equal to the sum of both solutions while the current is proportional to their difference.

### 8-2-1 Transients on Infinitely Long Transmission Lines

The transmission line shown in Figure 8-6$a$ extends to infinity in the positive $z$ direction. A time varying voltage source $V(t)$ that is turned on at $t = 0$ is applied at $z = 0$ to the line which is initially unexcited. A positively traveling wave $\mathrm{V}_+(t - z/c)$ propagates away from the source. There is no negatively traveling wave, $\mathrm{V}_-(t + z/c) = 0$. These physical

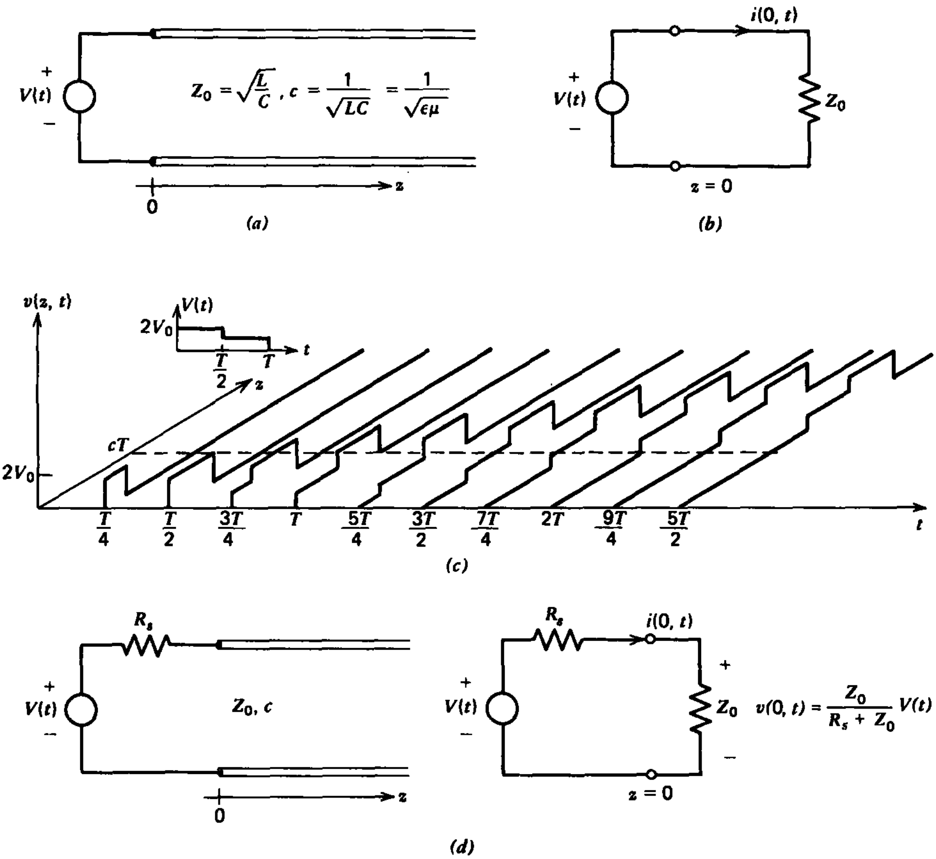



Figure 8-6 ($a$) A semi-infinite transmission line excited by a voltage source at $z=0$. ($b$) To the source, the transmission line looks like a resistor $Z_0$ equal to the characteristic impedance. ($c$) The spatial distribution of the voltage $v(z, t)$ at various times for a staircase pulse of $V(t)$. ($d$) If the voltage source is applied to the transmission line through a series resistance $R_s$, the voltage across the line at $z=0$ is given by the voltage divider relation.

arguments are verified mathematically by realizing that at $t=0$ the voltage and current are zero for $z>0$,

$$\begin{aligned} v(z, t=0) &= \mathrm{V}_+(-z/c)+\mathrm{V}_-(z/c)=0 \\ i(z, t=0) &= Y_0[\mathrm{V}_+(-z/c)-\mathrm{V}_-(z/c)]=0 \end{aligned} \tag{1}$$

which only allows the trivial solutions

$$\mathrm{V}_+(-z/c)=0, \qquad \mathrm{V}_-(z/c)=0 \tag{2}$$

Since $z$ can only be positive, whenever the argument of $\mathrm{V}_+$ is negative and of $\mathrm{V}_-$ positive, the functions are zero. Since $t$ can only be positive, the argument of $\mathrm{V}_-(t+z/c)$ is always positive so that the function is always zero. The argument of $V_+(t-z/c)$ can be positive, allowing a nonzero solution if $t>z/c$ agreeing with our conclusions reached by physical arguments.

With $V_-(t+z/c)=0$, the voltage and current are related as

$$\begin{aligned} v(z, t) &= V_+(t-z/c) \\ i(z, t) &= Y_0 V_+(t-z/c) \end{aligned} \tag{3}$$

The line voltage and current have the same shape as the source, delayed in time for any $z$ by $z/c$ with the current scaled in amplitude by $Y_0$. Thus as far as the source is concerned, the transmission line looks like a resistor of value $Z_0$ yielding the equivalent circuit at $z=0$ shown in Figure 8-6*b*. At $z=0$, the voltage equals that of the source

$$v(0, t) = V(t) = V_+(t) \tag{4}$$

If $V(t)$ is the staircase pulse of total duration $T$ shown in Figure 8-6*c*, the pulse extends in space over the spatial interval:

$$\begin{aligned} 0 \le z \le ct, &\qquad 0 \le t \le T \\ c(t-T) \le z \le ct, &\qquad t > T \end{aligned} \tag{5}$$

The analysis is the same even if the voltage source is in series with a source resistance $R_s$, as in Figure 8-6*d*. At $z=0$ the transmission line still looks like a resistor of value $Z_0$ so that the transmission line voltage divides in the ratio given by the equivalent circuit shown:

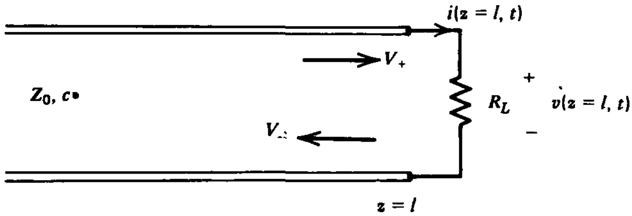
$$\begin{aligned} v(z=0, t) &= \frac{Z_0}{R_s+Z_0} V(t) = V_+(t) \\ i(z=0, t) &= Y_0 V_+(t) = \frac{V(t)}{R_s+Z_0} \end{aligned} \tag{6}$$

The total solution is then identical to that of (3) and (4) with the voltage and current amplitudes reduced by the voltage divider ratio $Z_0/(R_s+Z_0)$.

### 8-2-2 Reflections from Resistive Terminations

#### (a) Reflection Coefficient

All transmission lines must have an end. In Figure 8-7 we see a positively traveling wave incident upon a load resistor $R_L$ at $z=l$. The reflected wave will travel back towards the source at $z=0$ as a $V_-$ wave. At the $z=l$ end the following circuit




$$v(z=l,t) = V_+ + V_- = i(z=l,t)R_L = R_L Y_0 [V_+ - V_-]$$

$$\Rightarrow \Gamma_L = \frac{V_-}{V_+} = \frac{R_L - Z_0}{R_L + Z_0}$$

Figure 8-7 A $V_+$ wave incident upon the end of a transmission line with a load resistor $R_L$ is reflected as a $V_-$ wave.

relations hold:

$$\begin{aligned} v(l,t) &= V_+(t-l/c)+V_-(t+l/c) \\ &= i(l,t)R_L \\ &= Y_0R_L[V_+(t-l/c)-V_-(t+l/c)] \end{aligned} \tag{7}$$

We then find the amplitude of the negatively traveling wave in terms of the incident positively traveling wave as

$$\Gamma_L = \frac{V_-(t+l/c)}{V_+(t-l/c)} = \frac{R_L - Z_0}{R_L + Z_0} \tag{8}$$

where $\Gamma_L$ is known as the reflection coefficient that is of the same form as the reflection coefficient $R$ in Section 7-6-1 for normally incident uniform plane waves on a dielectric.

The reflection coefficient gives us the relative amplitude of the returning $V_-$ wave compared to the incident $V_+$ wave. There are several important limits of (8):

(i) If $R_L = Z_0$, the reflection coefficient is zero ($\Gamma_L = 0$) so that there is no reflected wave and the line is said to be matched.

(ii) If the line is short circuited ($R_L = 0$), then $\Gamma_L = -1$. The reflected wave is equal in amplitude but opposite in sign to the incident wave. In general, if $R_L < Z_0$, the reflected voltage wave has its polarity reversed.

(iii) If the line is open circuited ($R_L = \infty$), then $\Gamma_L = +1$. The reflected wave is identical to the incident wave. In general, if $R_L > Z_0$, the reflected voltage wave is of the same polarity as the incident wave.

**(b) Step Voltage**

A dc battery of voltage $V_0$ with series resistance $R_s$ is switched onto the transmission line at $t = 0$, as shown in Figure 8-8*a*. At $z = 0$, the source has no knowledge of the

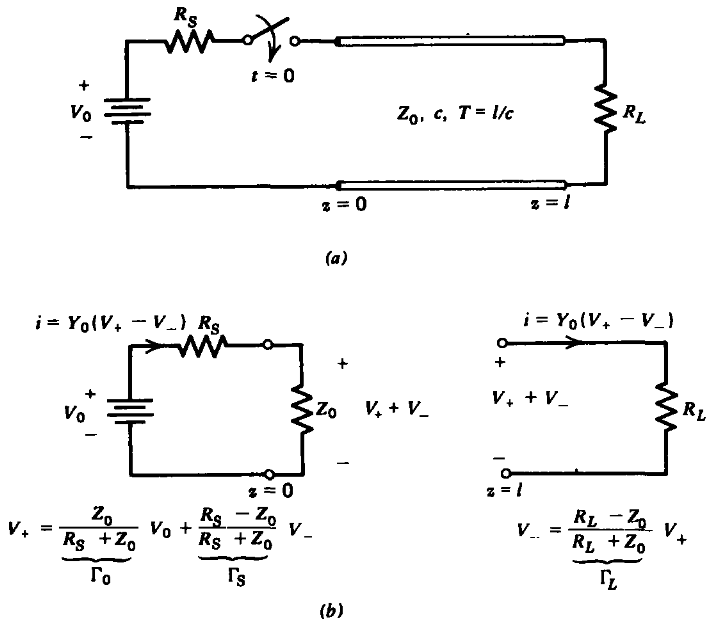



Figure 8-8 ($a$) A dc voltage $V_0$ is switched onto a resistively loaded transmission line through a source resistance $R_s$. ($b$) The equivalent circuits at $z=0$ and $z=l$ allow us to calculate the reflected voltage wave amplitudes in terms of the incident waves.

line's length or load termination, so as for an infinitely long line the transmission line looks like a resistor of value $Z_0$ to the source. There is no $V_-$ wave initially. The $V_+$ wave is determined by the voltage divider ratio of the series source resistance and transmission line characteristic impedance as given by (6).

This $V_+$ wave travels down the line at speed $c$ where it is reflected at $z=l$ for $t>T$, where $T=l/c$ is the transit time for a wave propagating between the two ends. The new $V_-$ wave generated is related to the incident $V_+$ wave by the reflection coefficient $\Gamma_L$. As the $V_+$ wave continues to propagate in the positive $z$ direction, the $V_-$ wave propagates back towards the source. The total voltage at any point on the line is equal to the sum of $V_+$ and $V_-$ while the current is proportional to their difference.

When the $V_-$ wave reaches the end of the transmission line at $z=0$ at time $2T$, in general a new $V_+$ wave is generated, which can be found by solving the equivalent circuit shown in Figure 8-8$b$:

$$v(0,t)+i(0,t)R_s = V_0 \Rightarrow V_+(0,t)+V_-(0,t) + Y_0R_s[V_+(0,t)-V_-(0,t)] = V_0 \tag{9}$$

to yield

$$V_+(0, t) = \Gamma_s V_-(0, t) + \frac{Z_0 V_0}{Z_0 + R_s}, \qquad \Gamma_s = \frac{R_s - Z_0}{R_s + Z_0} \tag{10}$$

where $\Gamma_s$ is just the reflection coefficient at the source end. This new $V_+$ wave propagates towards the load again generating a new $V_-$ wave as the reflections continue.

If the source resistance is matched to the line, $R_s = Z_0$ so that $\Gamma_s = 0$, then $V_+$ is constant for all time and the steady state is reached for $t > 2T$. If the load was matched, the steady state is reached for $t > T$ no matter the value of $R_s$. There are no further reflections from the end of a matched line. In Figure 8-9 we plot representative voltage and current spatial distributions for various times assuming the source is matched to the line for the load being matched, open, or short circuited.

**(i) Matched Line**

When $R_L = Z_0$ the load reflection coefficient is zero so that $V_+ = V_0/2$ for all time. The wavefront propagates down the line with the voltage and current being identical in shape. The system is in the dc steady state for $t \geq T$.

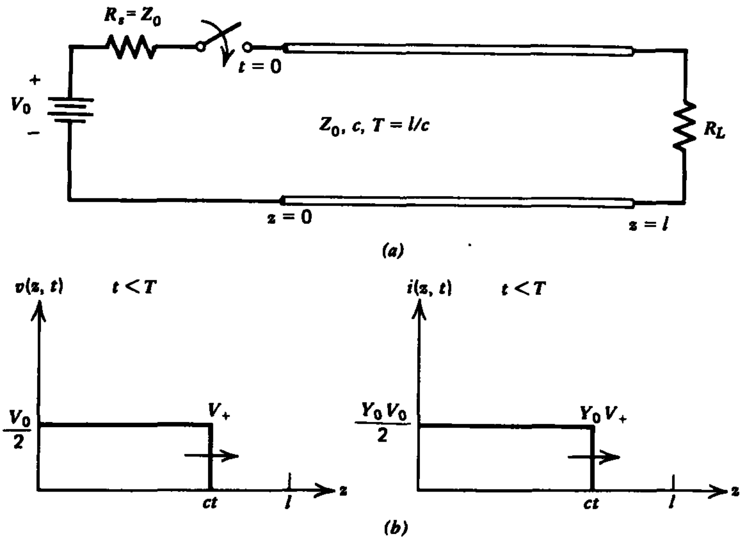
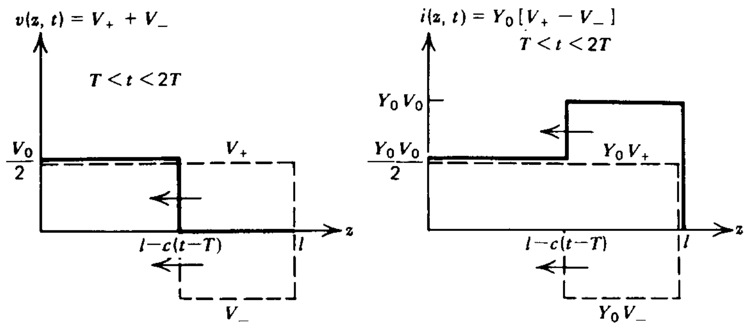
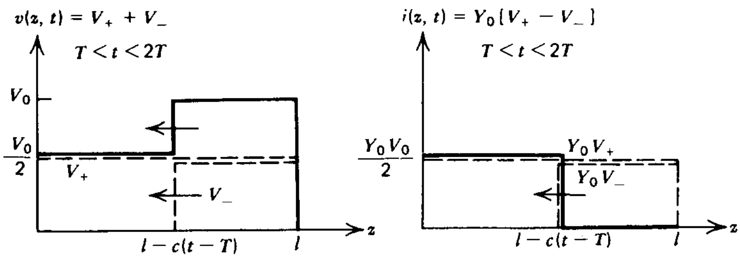



Figure 8-9 (*a*) A dc voltage is switched onto a transmission line with load resistance $R_L$ through a source resistance $R_s$ matched to the line. (*b*) Regardless of the load resistance, half the source voltage propagates down the line towards the load. If the load is also matched to the line ($R_L = Z_0$), there are no reflections and the steady state of $v(z, t \geq T) = V_0/2$, $i(z, t \geq T) = Y_0 V_0/2$ is reached for $t \geq T$. (*c*) If the line is short circuited ($R_L = 0$), then $\Gamma_L = -1$ so that the $V_+$ and $V_-$ waves cancel for the voltage but add for the current wherever they overlap in space. Since the source end is matched, no further reflections arise at $z = 0$ so that the steady state is reached for $t \geq 2T$. (*d*) If the line is open circuited ($R_L = \infty$) so that $\Gamma_L = +1$, the $V_+$ and $V_-$ waves add for the voltage but cancel for the current.




Short circuited line, $R_L = 0$, $(v(z, t \geqslant 2T) = 0,\ i(z, t \geqslant 2T) = Y_0 V_0)$

*(c)*




Open circuited line, $R_L = \infty$, $(v(z, t \geqslant 2T) = V_0,\ i(z, t \geqslant 2T) = 0)$

*(d)*

Figure 8-9

**(ii) Open Circuited Line**

When $R_L = \infty$ the reflection coefficient is unity so that $V_+ = V_-$. When the incident and reflected waves overlap in space the voltages add to a stairstep pulse shape while the current is zero. For $t \geq 2T$, the voltage is $V_0$ everywhere on the line while the current is zero.

**(iii) Short Circuited Line**

When $R_L = 0$ the load reflection coefficient is $-1$ so that $V_+ = -V_-$. When the incident and reflected waves overlap in space, the total voltage is zero while the current is now a stairstep pulse shape. For $t \geq 2T$ the voltage is zero everywhere on the line while the current is $V_0/Z_0$.

## 8-2-3 Approach to the dc Steady State

If the load end is matched, the steady state is reached after one transit time $T = l/c$ for the wave to propagate from the source to the load. If the source end is matched, after one round trip $2T=2l/c$ no further reflections occur. If neither end is matched, reflections continue on forever. However, for nonzero and noninfinite source and load resistances, the reflection coefficient is always less than unity in magnitude so that each successive reflection is reduced in amplitude. After a few round-trips, the changes in $V_+$ and $V_-$ become smaller and eventually negligible. If the source resistance is zero and the load resistance is either zero or infinite, the transient pulses continue to propagate back and forth forever in the lossless line, as the magnitude of the reflection coefficients are unity.

Consider again the dc voltage source in Figure 8-8*a* switched through a source resistance $R_s$ at $t=0$ onto a transmission line loaded at its $z=l$ end with a load resistor $R_L$. We showed in (10) that the $V_+$ wave generated at the $z=0$ end is related to the source and an incoming $V_-$ wave as

$$V_+=\Gamma_0 V_0+\Gamma_s V_-, \qquad \Gamma_0=\frac{Z_0}{R_s+Z_0}, \qquad \Gamma_s=\frac{R_s-Z_0}{R_s+Z_0} \tag{11}$$

Similarly, at $z=l$, an incident $V_+$ wave is converted into a $V_-$ wave through the load reflection coefficient:

$$V_-=\Gamma_L V_+, \qquad \Gamma_L=\frac{R_L-Z_0}{R_L+Z_0} \tag{12}$$

We can now tabulate the voltage at $z=l$ using the following reasoning:

(i) For the time interval $t<T$ the voltage at $z=l$ is zero as no wave has yet reached the end.
(ii) At $z=0$ for $0\leq t\leq 2T$, $V_-=0$ resulting in a $V_+$ wave emanating from $z=0$ with amplitude $V_+=\Gamma_0 V_0$.
(iii) When this $V_+$ wave reaches $z=l$, a $V_-$ wave is generated with amplitude $V_-=\Gamma_L V_+$. The incident $V_+$ wave at $z=l$ remains unchanged until another interval of $2T$, whereupon the just generated $V_-$ wave after being reflected from $z=0$ as a new $V_+$ wave given by (11) again returns to $z=l$.
(iv) Thus, the voltage at $z=l$ only changes at times $(2n-1)T$, $n=1,2,\ldots$, while the voltage at $z=0$ changes at times $2(n-1)T$. The resulting voltage waveforms at the ends are stairstep patterns with steps at these times.

The $n$th traveling $V_+$ wave is then related to the source and the $(n-1)$th $V_-$ wave at $z=0$ as

$$V_{+n}=\Gamma_0 V_0+\Gamma_s V_{-(n-1)} \tag{13}$$

while the $(n-1)$th $V_-$ wave is related to the incident $(n-1)$th $V_+$ wave at $z=l$ as

$$V_{-(n-1)}=\Gamma_L V_{+(n-1)} \tag{14}$$

Using (14) in (13) yields a single linear constant coefficient difference equation in $V_{+n}$:

$$V_{+n}-\Gamma_s\Gamma_L V_{+(n-1)}=\Gamma_0 V_0 \tag{15}$$

For a particular solution we see that $V_{+n}$ being a constant satisfies (15):

$$V_{+n}=C\Rightarrow C(1-\Gamma_s\Gamma_L)=\Gamma_0 V_0\Rightarrow C=\frac{\Gamma_0}{1-\Gamma_s\Gamma_L}V_0 \tag{16}$$

To this solution we can add any homogeneous solution assuming the right-hand side of (15) is zero:

$$V_{+n}-\Gamma_s\Gamma_L V_{+(n-1)}=0 \tag{17}$$

We try a solution of the form

$$V_{+n}=A\lambda^n \tag{18}$$

which when substituted into (17) requires

$$A\lambda^{n-1}(\lambda-\Gamma_s\Gamma_L)=0\Rightarrow\lambda=\Gamma_s\Gamma_L \tag{19}$$

The total solution is then a sum of the particular and homogeneous solutions:

$$V_{+n}=\frac{\Gamma_0}{1-\Gamma_s\Gamma_L}V_0+A(\Gamma_s\Gamma_L)^n \tag{20}$$

The constant $A$ is found by realizing that the first transient wave is

$$V_{+1}=\Gamma_0 V_0=\frac{\Gamma_0}{1-\Gamma_s\Gamma_L}V_0+A(\Gamma_s\Gamma_L) \tag{21}$$

which requires $A$ to be

$$A=-\frac{\Gamma_0 V_0}{1-\Gamma_s\Gamma_L} \tag{22}$$

so that (20) becomes

$$V_{+n}=\frac{\Gamma_0 V_0}{1-\Gamma_s\Gamma_L}[1-(\Gamma_s\Gamma_L)^n] \tag{23}$$

Raising the index of (14) by one then gives the $n$th $V_-$ wave as

$$V_{-n}=\Gamma_L V_{+n} \tag{24}$$

so that the total voltage at $z=l$ after $n$ reflections at times $(2n-1)T$, $n=1,2,\ldots,$ is

$$V_n = V_{+n}+V_{-n} = \frac{V_0\Gamma_0(1+\Gamma_L)}{1-\Gamma_s\Gamma_L}[1-(\Gamma_s\Gamma_L)^n] \tag{25}$$

or in terms of the source and load resistances

$$V_n = \frac{R_L}{R_L+R_s}\, V_0[1-(\Gamma_s\Gamma_L)^n] \tag{26}$$

The steady-state results as $n\to\infty$. If either $R_s$ or $R_L$ are nonzero or noninfinite, the product of $\Gamma_s\Gamma_L$ must be less than unity. Under these conditions

$$\lim_{\substack{n\to\infty\\(|\Gamma_s\Gamma_L|<1)}} (\Gamma_s\Gamma_L)^n = 0 \tag{27}$$

so that in the steady state

$$\lim_{n\to\infty} V_n = \frac{R_L}{R_s+R_L} V_0 \tag{28}$$

which is just the voltage divider ratio as if the transmission line was just a pair of zero-resistance connecting wires. Note also that if either end is matched so that either $\Gamma_s$ or $\Gamma_L$ is zero, the voltage at the load end is immediately in the steady state after the time $T$.

In Figure 8-10 the load is plotted versus time with $R_s=0$ and $R_L=3Z_0$ so that $\Gamma_s\Gamma_L=-\frac{1}{2}$ and with $R_L=\frac{1}{3}Z_0$ so that

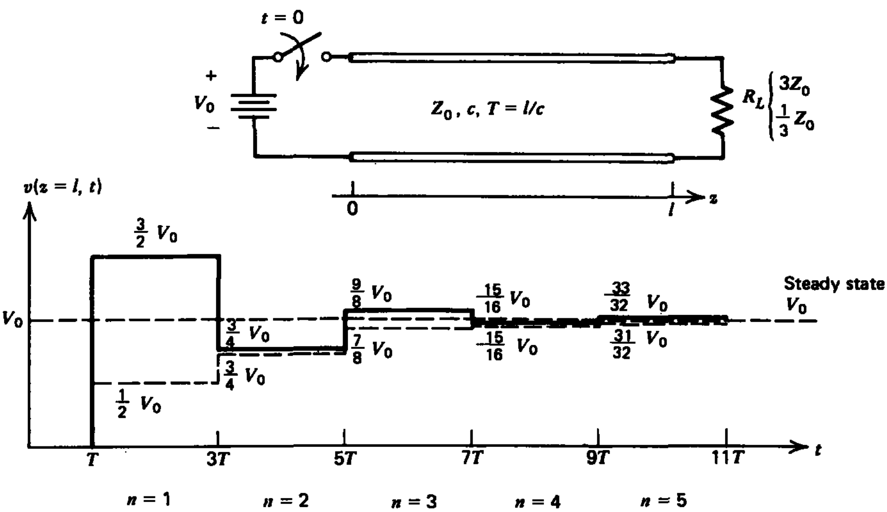



Figure 8-10 The load voltage as a function of time when $R_s=0$ and $R_L=3Z_0$ so that $\Gamma_s\Gamma_L=-\frac{1}{2}$ (solid) and with $R_L=\frac{1}{3}Z_0$ so that $\Gamma_s\Gamma_L=\frac{1}{2}$ (dashed). The dc steady state is the same as if the transmission line were considered a pair of perfectly conducting wires in a circuit.

$\Gamma_s\Gamma_L = +\frac{1}{9}$. Then (26) becomes

$$V_n = \begin{cases} V_0[1-(-\frac{1}{2})^n], & R_L = 3Z_0 \\ V_0[1-(\frac{1}{2})^n], & R_L = \frac{1}{3}Z_0 \end{cases} \tag{29}$$

The step changes in load voltage oscillate about the steady-state value $V_\infty = V_0$. The steps rapidly become smaller having less than one-percent variation for $n > 7$.

If the source resistance is zero and the load resistance is either zero or infinite (short or open circuits), a lossless transmission line never reaches a dc steady state as the limit of (27) does not hold with $\Gamma_s\Gamma_L = \pm 1$. Continuous reflections with no decrease in amplitude results in pulse waveforms for all time. However, in a real transmission line, small losses in the conductors and dielectric allow a steady state to be eventually reached.

Consider the case when $R_s = 0$ and $R_L = \infty$ so that $\Gamma_s\Gamma_L = -1$. Then from (26) we have

$$V_n = \begin{cases} 0, & n \text{ even} \\ 2V_0, & n \text{ odd} \end{cases} \tag{30}$$

which is sketched in Figure 8-11*a*.

For any source and load resistances the current through the load resistor at $z = l$ is

$$\begin{aligned} I_n &= \frac{V_n}{R_L} = \frac{V_0\Gamma_0(1+\Gamma_L)}{R_L(1-\Gamma_s\Gamma_L)}[1-(\Gamma_s\Gamma_L)^n] \\ &= \frac{2V_0\Gamma_0}{R_L+Z_0}\frac{[1-(\Gamma_s\Gamma_L)^n]}{(1-\Gamma_s\Gamma_L)} \end{aligned} \tag{31}$$

If both $R_s$ and $R_L$ are zero so that $\Gamma_s\Gamma_L = 1$, the short circuit current in (31) is in the indeterminate form 0/0, which can be evaluated using l'Hôpital's rule:

$$\begin{aligned} \lim_{\Gamma_s\Gamma_L \to 1} I_n &= \frac{2V_0\Gamma_0}{R_L+Z_0}\frac{[-n(\Gamma_s\Gamma_L)^{n-1}]}{(-1)} \\ &= \frac{2V_0 n}{Z_0} \end{aligned} \tag{32}$$

As shown by the solid line in Figure 8-11*b*, the current continually increases in a stepwise fashion. As $n$ increases to infinity, the current also becomes infinite, which is expected for a battery connected across a short circuit.

### 8-2-4 Inductors and Capacitors as Quasi-static Approximations to Transmission Lines

If the transmission line was one meter long with a free space dielectric medium, the round trip transit time $2T = 2l/c$

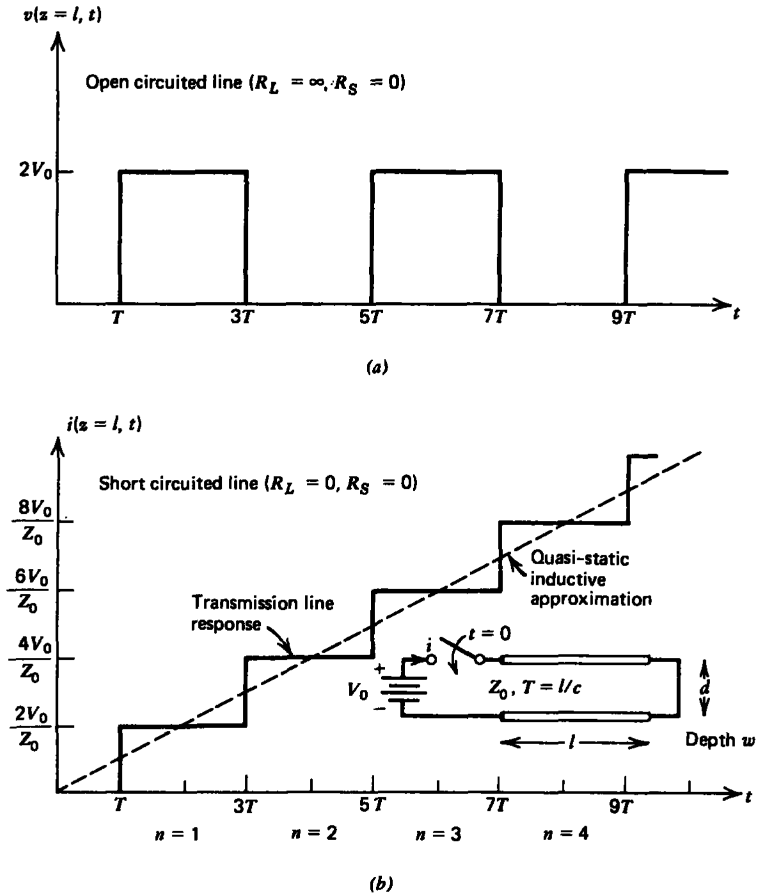



Figure 8-11 The (*a*) open circuit voltage and (*b*) short circuit current at the $z = l$ end of the transmission line for $R_s = 0$. No dc steady state is reached because the system is lossless. If the short circuited transmission line is modeled as an inductor in the quasi-static limit, a step voltage input results in a linearly increasing current (shown dashed). The exact transmission line response is the solid staircase waveform.

is approximately 6 nsec. For many circuit applications this time is so fast that it may be considered instantaneous. In this limit the quasi-static circuit element approximation is valid.

For example, consider again the short circuited transmission line ($R_L = 0$) of length $l$ with zero source resistance. In the magnetic quasi-static limit we would call the structure an inductor with inductance $Ll$ (remember, $L$ is the inductance per unit length) so that the terminal voltage and current are related as

$$v = (Ll)\frac{di}{dt} \tag{33}$$

If a constant voltage $V_0$ is applied at $t=0$, the current is obtained by integration of (33) as

$$i=\frac{V_0}{Ll}t \tag{34}$$

where we use the initial condition of zero current at $t=0$. The linear time dependence of the current, plotted as the dashed line in Figure 8-11*b*, approximates the rising staircase waveform obtained from the exact transmission line analysis of (32).

Similarly, if the transmission line were open circuited with $R_L=\infty$, it would be a capacitor of value $Cl$ in the electric quasi-static limit so that the voltage on the line charges up through the source resistance $R_s$ with time constant $\tau=R_sCl$ as

$$v(t)=V_0(1-e^{-t/\tau}) \tag{35}$$

The exact transmission line voltage at the $z=l$ end is given by (26) with $R_L=\infty$ so that $\Gamma_L=1$:

$$V_n=V_0(1-\Gamma_s^n) \tag{36}$$

where the source reflection coefficient can be written as

$$\begin{aligned}\Gamma_s&=\frac{R_s-Z_0}{R_s+Z_0}\\&=\frac{R_s-\sqrt{L/C}}{R_s+\sqrt{L/C}}\end{aligned} \tag{37}$$

If we multiply the numerator and denominator of (37) through by $Cl$, we have

$$\begin{aligned}\Gamma_s&=\frac{R_sCl-l\sqrt{LC}}{R_sCl+l\sqrt{LC}}\\&=\frac{\tau-T}{\tau+T}=\frac{1-T/\tau}{1+T/\tau}\end{aligned} \tag{38}$$

where

$$T=l\sqrt{LC}=l/c \tag{39}$$

For the quasi-static limit to be valid, the wave transit time $T$ must be much faster than any other time scale of interest so that $T/\tau \ll 1$. In Figure 8-12 we plot (35) and (36) for two values of $T/\tau$ and see that the quasi-static and transmission line results approach each other as $T/\tau$ becomes small.

When the roundtrip wave transit time is so small compared to the time scale of interest so as to appear to be instantaneous, the circuit treatment is an excellent approximation.

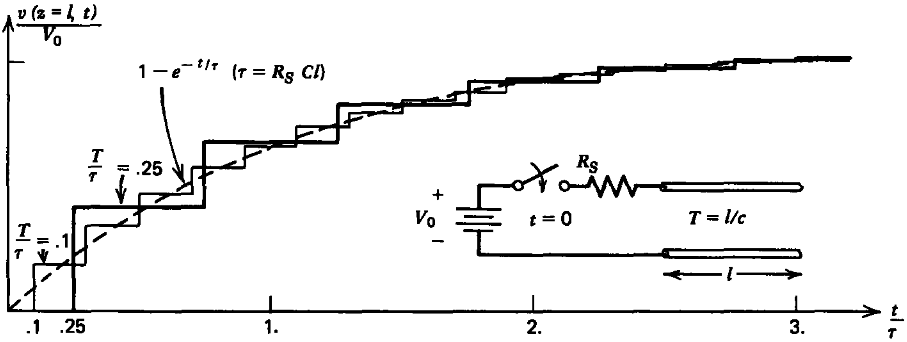



Figure 8-12 The open circuit voltage at $z=l$ for a step voltage applied at $t=0$ through a source resistance $R_s$ for various values of $T/\tau$, which is the ratio of propagation time $T=l/c$ to quasi-static charging time $\tau=R_sCl$. The dashed curve shows the exponential rise obtained by a circuit analysis assuming the open circuited transmission line is a capacitor.

If this propagation time is significant, then the transmission line equations must be used.

### 8-2-5 Reflections from Arbitrary Terminations

For resistive terminations we have been able to relate reflected wave amplitudes in terms of an incident wave amplitude through the use of a reflection coefficient because the voltage and current in the resistor are algebraically related. For an arbitrary termination, which may include any component such as capacitors, inductors, diodes, transistors, or even another transmission line with perhaps a different characteristic impedance, it is necessary to solve a circuit problem at the end of the line. For the arbitrary element with voltage $V_L$ and current $I_L$ at $z=l$, shown in Figure 8-13$a$, the voltage and current at the end of line are related as

$$v(z=l,t)=V_L(t)=V_+(t-l/c)+V_-(t+l/c) \tag{40}$$

$$i(z=l,t)=I_L(t)=Y_0[V_+(t-l/c)-V_-(t+l/c)] \tag{41}$$

We assume that we know the incident $V_+$ wave and wish to find the reflected $V_-$ wave. We then eliminate the unknown $V_-$ in (40) and (41) to obtain

$$2V_+(t-l/c)=V_L(t)+I_L(t)Z_0 \tag{42}$$

which suggests the equivalent circuit in Figure 8-13$b$.

For a particular lumped termination we solve the equivalent circuit for $V_L(t)$ or $I_L(t)$. Since $V_+(t-l/c)$ is already known as it is incident upon the termination, once $V_L(t)$ or

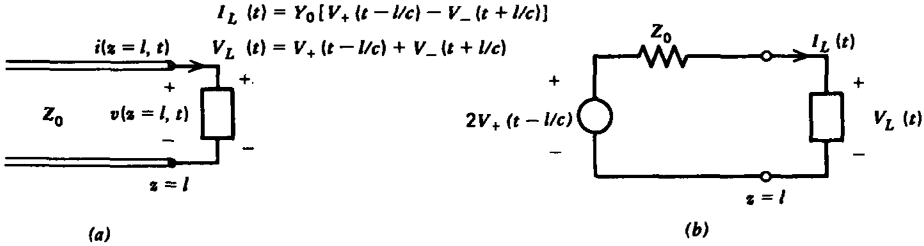



Figure 8-13 A transmission line with an (*a*) arbitrary load at the $z=l$ end can be analyzed from the equivalent circuit in (*b*). Since $V_+$ is known, calculation of the load current or voltage yields the reflected wave $V_-$.

$I_L(t)$ is calculated from the equivalent circuit, $V_-(t+l/c)$ can be calculated as $V_- = V_L - V_+$.

For instance, consider the lossless transmission lines of length $l$ shown in Figure 8-14*a* terminated at the end with either a lumped capacitor $C_L$ or an inductor $L_L$. A step voltage at $t=0$ is applied at $z=0$ through a source resistor matched to the line.

The source at $z=0$ is unaware of the termination at $z=l$ until a time $2T$. Until this time it launches a $V_+$ wave of amplitude $V_0/2$. At $z=l$, the equivalent circuit for the capacitive termination is shown in Figure 8-14*b*. Whereas resistive terminations just altered wave amplitudes upon reflection, inductive and capacitive terminations introduce differential equations.

From (42), the voltage across the capacitor $v_c$ obeys the differential equation

$$Z_0 C_L \frac{dv_c}{dt} + v_c = 2V_+ = V_0, \qquad t > T \tag{43}$$

with solution

$$v_c(t) = V_0[1 - e^{-(t-T)/Z_0 C_L}], \qquad t > T \tag{44}$$

Note that the voltage waveform plotted in Figure 8-14*b* begins at time $T = l/c$.

Thus, the returning $V_-$ wave is given as

$$V_- = v_c - V_+ = V_0/2 + V_0\, e^{-(t-T)/Z_0 C_L} \tag{45}$$

This reflected wave travels back to $z=0$, where no further reflections occur since the source end is matched. The current at $z=l$ is then

$$i_c = C_L \frac{dv_c}{dt} = \frac{V_0}{Z_0} e^{-(t-T)/Z_0 C_L}, \qquad t > T \tag{46}$$

and is also plotted in Figure 8-14*b*.

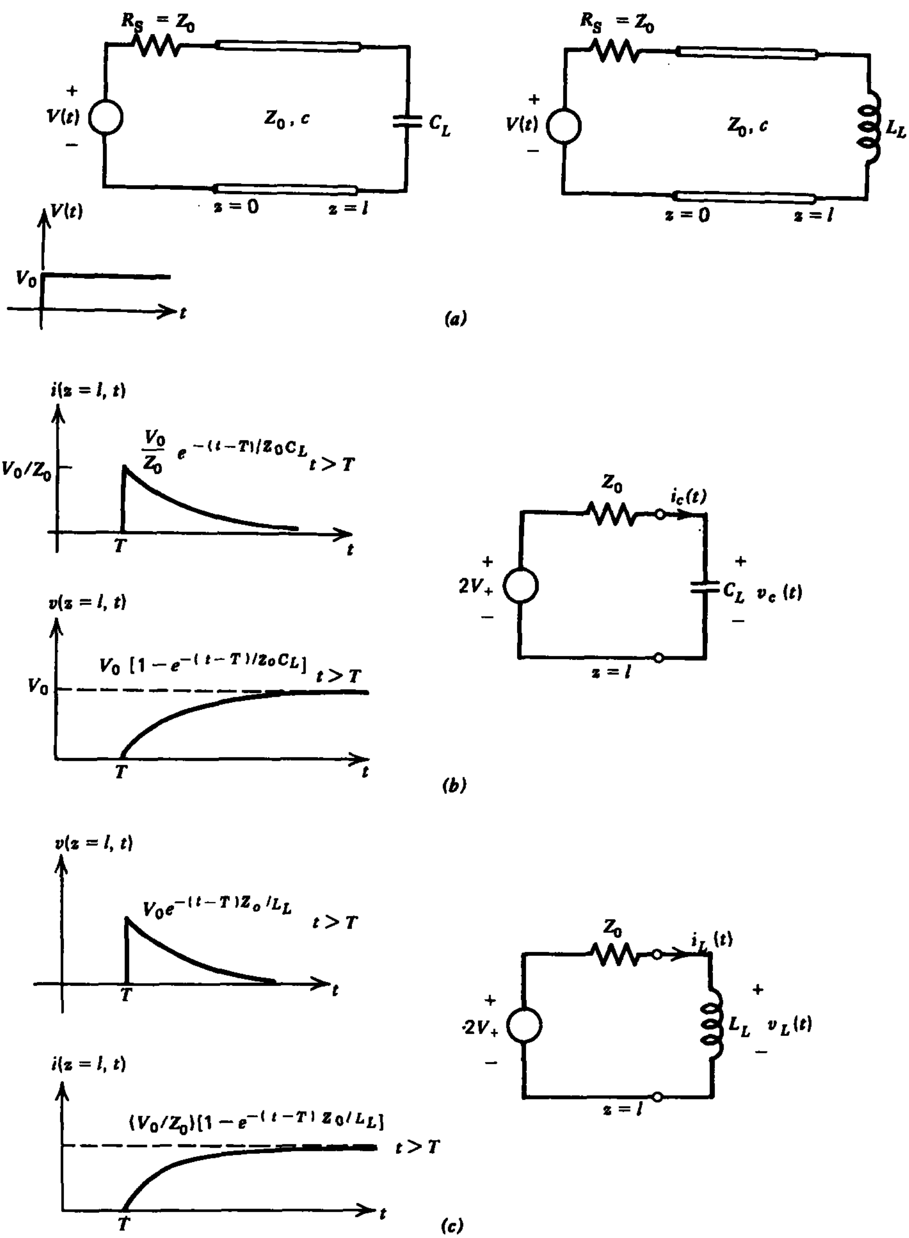



Figure 8-14 (*a*) A step voltage is applied to transmission lines loaded at $z = l$ with a capacitor $C_L$ or inductor $L_L$. The load voltage and current are calculated from the (*b*) resistive-capacitive or (*c*) resistive-inductive equivalent circuits at $z = l$ to yield exponential waveforms with respective time constants $\tau = Z_0 C_L$ and $\tau = L_L/Z_0$ as the solutions approach the dc steady state. The waveforms begin after the initial $V_+$ wave arrives at $z = l$ after a time $T = l/c$. There are no further reflections as the source end is matched.

If the end at $z=0$ were not matched, a new $V_+$ would be generated. When it reached $z=l$, we would again solve the $RC$ circuit with the capacitor now initially charged. The reflections would continue, eventually becoming negligible if $R_s$ is nonzero.

Similarly, the governing differential equation for the inductive load obtained from the equivalent circuit in Figure 8-14$c$ is

$$L_L\frac{di_L}{dt}+i_LZ_0=2V_+=V_0, \qquad t>T \tag{47}$$

with solution

$$i_L=\frac{V_0}{Z_0}(1-e^{-(t-T)Z_0/L_L}), \qquad t>T \tag{48}$$

The voltage across the inductor is

$$v_L=L_L\frac{di_L}{dt}=V_0\,e^{-(t-T)Z_0/L_L}, \qquad t>T \tag{49}$$

Again since the end at $z=0$ is matched, the returning $V_-$ wave from $z=l$ is not reflected at $z=0$. Thus the total voltage and current for all time at $z=l$ is given by (48) and (49) and is sketched in Figure 8-14$c$.

## 8-3 SINUSOIDAL TIME VARIATIONS

### 8-3-1 Solutions to the Transmission Line Equations

Often transmission lines are excited by sinusoidally varying sources so that the line voltage and current also vary sinusoidally with time:

$$\begin{aligned} v(z,t)&=\operatorname{Re}\,[\hat{v}(z)\,e^{j\omega t}]\\ i(z,t)&=\operatorname{Re}\,[\hat{\imath}(z)\,e^{j\omega t}]\end{aligned} \tag{1}$$

Then as we found for TEM waves in Section 7-4, the voltage and current are found from the wave equation solutions of Section 8-1-5 as linear combinations of exponential functions with arguments $t-z/c$ and $t+z/c$:

$$\begin{aligned} v(z,t)&=\operatorname{Re}\,[\hat{V}_+\,e^{j\omega(t-z/c)}+\hat{V}_-\,e^{j\omega(t+z/c)}]\\ i(z,t)&=Y_0\operatorname{Re}\,[\hat{V}_+\,e^{j\omega(t-z/c)}-\hat{V}_-\,e^{j\omega(t+z/c)}]\end{aligned} \tag{2}$$

Now the phasor amplitudes $\hat{V}_+$ and $\hat{V}_-$ are complex numbers and do not depend on $z$ or $t$.

By factoring out the sinusoidal time dependence in (2), the spatial dependences of the voltage and current are

$$\begin{aligned} \hat{v}(z) &= \hat{V}_+ e^{-jkz} + \hat{V}_- e^{+jkz} \\ \hat{i}(z) &= Y_0(\hat{V}_+ e^{-jkz} - \hat{V}_- e^{+jkz}) \end{aligned} \tag{3}$$

where the wavenumber is again defined as

$$k = \omega/c \tag{4}$$

### 8-3-2 Lossless Terminations

#### (a) Short Circuited Line

The transmission line shown in Figure 8-15$a$ is excited by a sinusoidal voltage source at $z=-l$ imposing the boundary condition

$$\begin{aligned} v(z=-l, t) &= V_0 \cos \omega t \\ &= \text{Re}\,(V_0\, e^{j\omega t}) \Rightarrow \hat{v}(z=-l) = V_0 = \hat{V}_+ e^{jkl} + \hat{V}_- e^{-jkl} \end{aligned} \tag{5}$$

Note that to use (3) we must write all sinusoids in complex notation. Then since all time variations are of the form $e^{j\omega t}$, we may suppress writing it each time and work only with the spatial variations of (3).

Because the transmission line is short circuited, we have the additional boundary condition

$$v(z=0, t) = 0 \Rightarrow \hat{v}(z=0) = 0 = \hat{V}_+ + \hat{V}_- \tag{6}$$

which when simultaneously solved with (5) yields

$$\hat{V}_+ = -\hat{V}_- = \frac{V_0}{2j \sin kl} \tag{7}$$

The spatial dependences of the voltage and current are then

$$\begin{aligned} \hat{v}(z) &= \frac{V_0(e^{-jkz} - e^{jkz})}{2j \sin kl} = -\frac{V_0 \sin kz}{\sin kl} \\ \hat{i}(z) &= \frac{V_0 Y_0(e^{-jkz} + e^{+jkz})}{2j \sin kl} = -j\frac{V_0 Y_0 \cos kz}{\sin kl} \end{aligned} \tag{8}$$

The instantaneous voltage and current as functions of space and time are then

$$\begin{aligned} v(z, t) &= \text{Re}\,[\hat{v}(z)\, e^{j\omega t}] = -V_0 \frac{\sin kz}{\sin kl} \cos \omega t \\ i(z, t) &= \text{Re}\,[\hat{i}(z)\, e^{j\omega t}] = \frac{V_0 Y_0 \cos kz \sin \omega t}{\sin kl} \end{aligned} \tag{9}$$

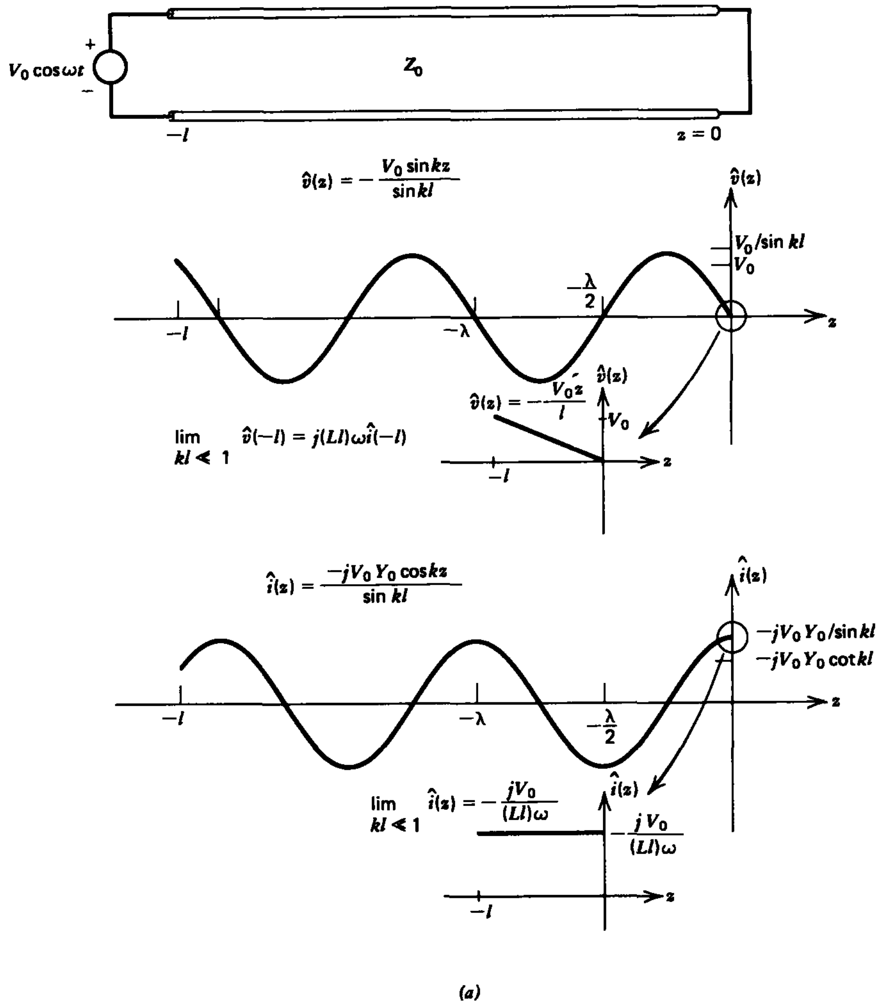
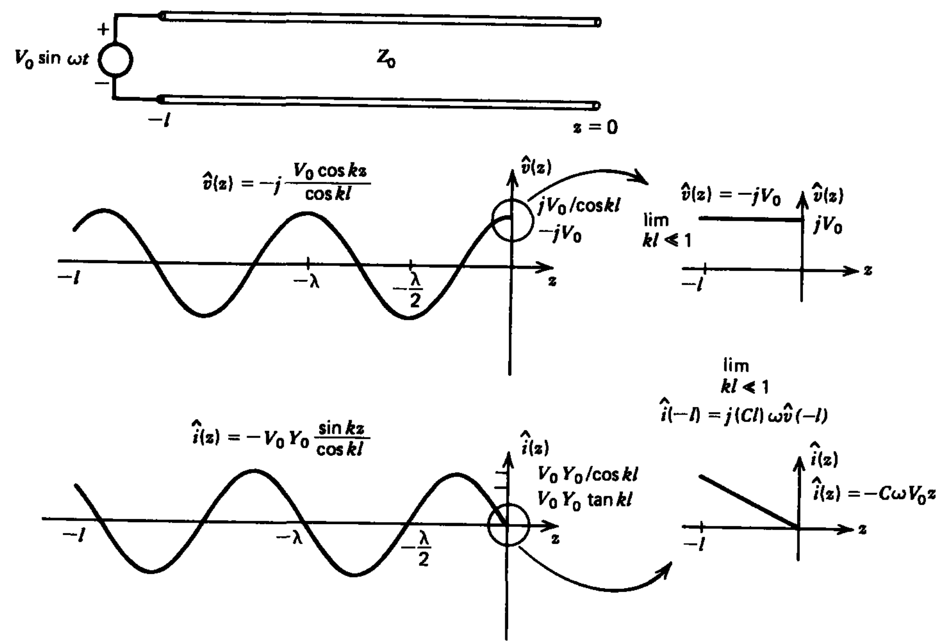


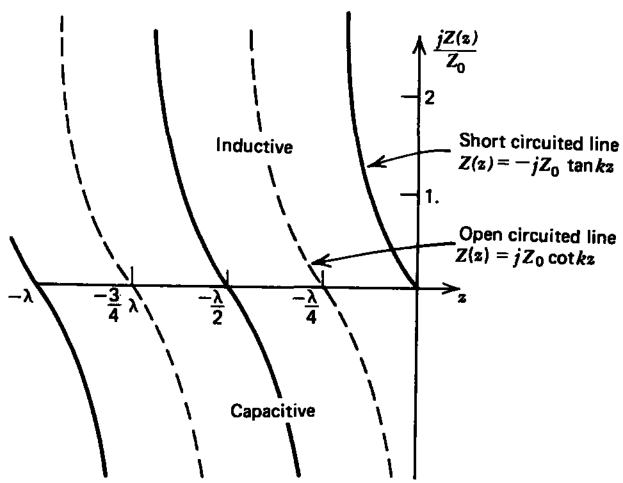

Figure 8-15 The voltage and current distributions on a (*a*) short circuited and (*b*) open circuited transmission line excited by sinusoidal voltage sources at $z=-l$. If the lines are much shorter than a wavelength, they act like reactive circuit elements. (*c*) As the frequency is raised, the impedance reflected back as a function of $z$ can look capacitive or inductive making the transition through open or short circuits.

The spatial distributions of voltage and current as a function of $z$ at a specific instant of time are plotted in Figure 8-15*a* and are seen to be 90° out of phase with one another in space with their distributions periodic with wavelength $\lambda$ given by

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} \tag{10}$$




*(b)*




*(c)*

Figure 8-15

The complex impedance at any position $z$ is defined as

$$Z(z)=\frac{\hat{v}(z)}{\hat{i}(z)} \tag{11}$$

which for this special case of a short circuited line is found from (8) as

$$Z(z)=-jZ_0 \tan kz \tag{12}$$

In particular, at $z=-l$, the transmission line appears to the generator as an impedance of value

$$Z(z=-l)=jZ_0 \tan kl \tag{13}$$

From the solid lines in Figure 8-15$c$ we see that there are various regimes of interest:

(i) When the line is an integer multiple of a half wavelength long so that $kl=n\pi$, $n=1,2,3,\ldots$, the impedance at $z=-l$ is zero and the transmission line looks like a short circuit.

(ii) When the line is an odd integer multiple of a quarter wavelength long so that $kl=(2n-1)\pi/2$, $n=1,2,\ldots$, the impedance at $z=-l$ is infinite and the transmission line looks like an open circuit.

(iii) Between the short and open circuit limits $(n-1)\pi<kl<(2n-1)\pi/2$, $n=1,2,3,\ldots$, $Z(z=-l)$ has a positive reactance and hence looks like an inductor.

(iv) Between the open and short circuit limits $(n-\frac{1}{2})\pi<kl<n\pi$, $n=1,2,\ldots$, $Z(z=-l)$ has a negative reactance and so looks like a capacitor.

Thus, the short circuited transmission line takes on all reactive values, both positive (inductive) and negative (capacitive), including open and short circuits as a function of $kl$. Thus, if either the length of the line $l$ or the frequency is changed, the impedance of the transmission line is changed.

Examining (8) we also notice that if $\sin kl=0$, $(kl=n\pi$, $n=1,2,\ldots)$, the voltage and current become infinite (in practice the voltage and current become large limited only by losses). Under these conditions, the system is said to be resonant with the resonant frequencies given by

$$\omega_n=n\pi c/l, \qquad n=1,2,3,\ldots \tag{14}$$

Any voltage source applied at these frequencies will result in very large voltages and currents on the line.

### (b) Open Circuited Line

If the short circuit is replaced by an open circuit, as in Figure 8-15$b$, and for variety we change the source at $z=-l$ to

$V_0 \sin \omega t$ the boundary conditions are

$$\begin{aligned} i(z=0, t) &= 0 \\ v(z=-l, t) &= V_0 \sin \omega t = \text{Re}\,(-jV_0\, e^{j\omega t}) \end{aligned} \tag{15}$$

Using (3) the complex amplitudes obey the relations

$$\begin{aligned} \hat{i}(z=0) &= 0 = Y_0(\hat{V}_+ - \hat{V}_-) \\ \hat{v}(z=-l) &= -jV_0 = \hat{V}_+\, e^{jkl} + \hat{V}_-\, e^{-jkl} \end{aligned} \tag{16}$$

which has solutions

$$\hat{V}_+ = \hat{V}_- = \frac{-jV_0}{2\cos kl} \tag{17}$$

The spatial dependences of the voltage and current are then

$$\begin{aligned} \hat{v}(z) &= \frac{-jV_0}{2\cos kl}(e^{-jkz} + e^{jkz}) = \frac{-jV_0}{\cos kl}\cos kz \\ \hat{i}(z) &= \frac{-jV_0Y_0}{2\cos kl}(e^{-jkz} - e^{+jkz}) = -\frac{V_0Y_0}{\cos kl}\sin kz \end{aligned} \tag{18}$$

with instantaneous solutions as a function of space and time:

$$\begin{aligned} v(z,t) &= \text{Re}\,[\hat{v}(z)\, e^{j\omega t}] = \frac{V_0 \cos kz}{\cos kl}\sin \omega t \\ i(z,t) &= \text{Re}\,[\hat{i}(z)\, e^{j\omega t}] = -\frac{V_0Y_0}{\cos kl}\sin kz \cos \omega t \end{aligned} \tag{19}$$

The impedance at $z=-l$ is

$$Z(z=-l) = \frac{\hat{v}(-l)}{\hat{i}(-l)} = -jZ_0 \cot kl \tag{20}$$

Again the impedance is purely reactive, as shown by the dashed lines in Figure 8-15$c$, alternating signs every quarter wavelength so that the open circuit load looks to the voltage source as an inductor, capacitor, short or open circuit depending on the frequency and length of the line.

Resonance will occur if

$$\cos kl = 0 \tag{21}$$

or

$$kl = (2n-1)\pi/2, \qquad n = 1, 2, 3, \ldots \tag{22}$$

so that the resonant frequencies are

$$\omega_n = \frac{(2n-1)\pi c}{2l} \tag{23}$$

### 8-3-3 Reactive Circuit Elements as Approximations to Short Transmission Lines

Let us re-examine the results obtained for short and open circuited lines in the limit when $l$ is much shorter than the wavelength $\lambda$ so that in this long wavelength limit the spatial trigonometric functions can be approximated as

$$\lim_{kl\ll 1}\begin{cases}\sin kz \approx kz\\ \cos kz\approx 1\end{cases} \tag{24}$$

Using these approximations, the voltage, current, and impedance for the short circuited line excited by a voltage source $V_0 \cos \omega t$ can be obtained from (9) and (13) as

$$\lim_{kl\ll 1}\begin{cases} v(z,t)=-\dfrac{V_0 z}{l}\cos\omega t, & v(-l,t)=V_0\cos\omega t\\[2mm] i(z,t)=\dfrac{V_0Y_0}{kl}\sin\omega t, & i(-l,t)=\dfrac{V_0\sin\omega t}{(Ll)\omega}\\[2mm] Z(-l)=jZ_0kl=j\dfrac{\omega Z_0 l}{c}=j\omega(Ll) \end{cases} \tag{25}$$

We see that the short circuited transmission line acts as an inductor of value $(Ll)$ (remember that $L$ is the inductance per unit length), where we used the relations

$$Z_0=\frac{1}{Y_0}=\sqrt{\frac{L}{C}},\qquad c=\frac{1}{\sqrt{LC}} \tag{26}$$

Note that at $z=-l$,

$$v(-l,t)=(Ll)\frac{di(-l,t)}{dt} \tag{27}$$

Similarly for the open circuited line we obtain:

$$\lim_{kl\ll 1}\begin{cases} v(z,t)=V_0\sin\omega t\\ i(z,t)=-V_0Y_0kz\cos\omega t, & i(-l,t)=(Cl)\omega V_0\cos\omega t\\[2mm] Z(-l)=\dfrac{-jZ_0}{kl}=\dfrac{-j}{(Cl)\omega} \end{cases} \tag{28}$$

For the open circuited transmission line, the terminal voltage and current are simply related as for a capacitor,

$$i(-l,t)=(Cl)\frac{dv(-l,t)}{dt} \tag{29}$$

with capacitance given by $(Cl)$.

In general, if the frequency of excitation is low enough so that the length of a transmission line is much shorter than the wavelength, the circuit approximations of inductance and capacitance are appropriate. However, it must be remembered that if the frequencies of interest are so high that the length of a circuit element is comparable to the wavelength, it no longer acts like that element. In fact, as found in Section 8-3-2, a capacitor can even look like an inductor, a short circuit, or an open circuit at high enough frequency while vice versa an inductor can also look capacitive, a short or an open circuit.

In general, if the termination is neither a short nor an open circuit, the voltage and current distribution becomes more involved to calculate and is the subject of Section 8-4.

## 8-3-4 Effects of Line Losses

### (a) Distributed Circuit Approach

If the dielectric and transmission line walls have Ohmic losses, the voltage and current waves decay as they propagate. Because the governing equations of Section 8-1-3 are linear with constant coefficients, in the sinusoidal steady state we assume solutions of the form

$$\begin{aligned} v(z,t) &= \operatorname{Re}(\hat{V}\, e^{j(\omega t - kz)}) \\ i(z,t) &= \operatorname{Re}(\hat{I}\, e^{j(\omega t - kz)}) \end{aligned} \tag{30}$$

where now $\omega$ and $k$ are not simply related as the nondispersive relation in (4). Rather we substitute (30) into Eq. (28) in Section 8-1-3:

$$\begin{aligned} \frac{\partial i}{\partial z} &= -C\frac{\partial v}{\partial t} - Gv \Rightarrow -jk\hat{I} = -(Cj\omega + G)\hat{V} \\ \frac{\partial v}{\partial z} &= -L\frac{\partial i}{\partial t} - iR \Rightarrow -jk\hat{V} = -(Lj\omega + R)\hat{I} \end{aligned} \tag{31}$$

which requires that

$$\frac{\hat{V}}{\hat{I}} = \frac{jk}{(Cj\omega + G)} = \frac{Lj\omega + R}{jk} \tag{32}$$

We solve (32) self-consistently for $k$ as

$$k^2 = -(Lj\omega + R)(Cj\omega + G) = LC\omega^2 - j\omega(RC + LG) - RG \tag{33}$$

The wavenumber is thus complex so that we find the real and imaginary parts from (33) as

$$\begin{aligned} k = k_r + jk_i \Rightarrow k_r^2 - k_i^2 &= LC\omega^2 - RG \\ 2k_rk_i &= -\omega(RC + LG) \end{aligned} \tag{34}$$

In the low loss limit where $\omega RC \ll 1$ and $\omega LG \ll 1$, the spatial decay of $k_i$ is small compared to the propagation wavenumber $k_r$. In this limit we have the following approximate solution:

$$\lim_{\substack{\omega RC \ll 1 \\ \omega LG \ll 1}} \begin{cases} k_r \approx \pm \omega\sqrt{LC} = \pm \omega/c \\ k_i = -\dfrac{\omega(RC+LG)}{2k_r} \approx \mp \dfrac{1}{2}\left[R\sqrt{\dfrac{C}{L}} + G\sqrt{\dfrac{L}{C}}\right] \\ \qquad \approx \mp \frac{1}{2}(RY_0 + GZ_0) \end{cases} \tag{35}$$

We use the upper sign for waves propagating in the $+z$ direction and the lower sign for waves traveling in the $-z$ direction.

**(b) Distortionless lines**

Using the value of $k$ of (33),

$$k = \pm[-(Lj\omega + R)(Cj\omega + G)]^{1/2} \tag{36}$$

in (32) gives us the frequency dependent wave impedance for waves traveling in the $\pm z$ direction as

$$\frac{\hat{V}}{\hat{I}} = \pm\left(\frac{Lj\omega + R}{Cj\omega + G}\right)^{1/2} = \pm\sqrt{\frac{L}{C}}\left(\frac{j\omega + R/L}{j\omega + G/C}\right)^{1/2} \tag{37}$$

If the line parameters are adjusted so that

$$\frac{R}{L} = \frac{G}{C} \tag{38}$$

the impedance in (37) becomes frequency independent and equal to the lossless line impedance. Under the conditions of (38) the complex wavenumber reduces to

$$k_r = \pm\omega\sqrt{LC}, \qquad k_i = \mp\sqrt{RG} \tag{39}$$

Although the waves are attenuated, all frequencies propagate at the same phase and group velocities as for a lossless line

$$\begin{aligned} v_p &= \frac{\omega}{k_r} = \pm\frac{1}{\sqrt{LC}} \\ v_g &= \frac{d\omega}{dk_r} = \pm\frac{1}{\sqrt{LC}} \end{aligned} \tag{40}$$

Since all the Fourier components of a pulse excitation will travel at the same speed, the shape of the pulse remains unchanged as it propagates down the line. Such lines are called distortionless.

**(c) Fields Approach**

If $R=0$, we can directly find the TEM wave solutions using the same solutions found for plane waves in Section 7-4-3. There we found that a dielectric with permittivity $\varepsilon$ and small Ohmic conductivity $\sigma$ has a complex wavenumber:

$$\lim_{\sigma/\omega\varepsilon \ll 1} k \approx \pm\left(\frac{\omega}{c}-\frac{j\sigma\eta}{2}\right) \tag{41}$$

Equating (41) to (35) with $R=0$ requires that $GZ_0=\sigma\eta$.

The tangential component of **H** at the perfectly conducting transmission line walls is discontinuous by a surface current. However, if the wall has a large but noninfinite Ohmic conductivity $\sigma_w$, the fields penetrate in with a characteristic distance equal to the skin depth $\delta=\sqrt{2/\omega\mu\sigma_w}$. The resulting $z$-directed current gives rise to a $z$-directed electric field so that the waves are no longer purely TEM.

Because we assume this loss to be small, we can use an approximate perturbation method to find the spatial decay rate of the fields. We assume that the fields between parallel plane electrodes are essentially the same as when the system is lossless except now being exponentially attenuated as $e^{-\alpha z}$, where $\alpha=-k_i$:

$$\begin{aligned} E_x(z,t)&=\mathrm{Re}\,[\hat{E}\,e^{j(\omega t-k_r z)}\,e^{-\alpha z}] \\ H_y(z,t)&=\mathrm{Re}\left[\frac{\hat{E}}{\eta}\,e^{j(\omega t-k_r z)}\,e^{-\alpha z}\right], \qquad k_r=\frac{\omega}{c} \end{aligned} \tag{42}$$

From the real part of the complex Poynting's theorem derived in Section 7-2-4, we relate the divergence of the time-average electromagnetic power density to the time-average dissipated power:

$$\nabla\cdot\langle \mathbf{S}\rangle=-\langle P_d\rangle \tag{43}$$

Using the divergence theorem we integrate (43) over a volume of thickness $\Delta z$ that encompasses the entire width and thickness of the line, as shown in Figure 8-16:

$$\begin{aligned} \int_V \nabla\cdot\langle \mathbf{S}\rangle\,dV &= \oint_S \langle \mathbf{S}\rangle\cdot d\mathbf{S} \\ &= \int_{z+\Delta z} \langle S_z(z+\Delta z)\rangle\,dS \\ &\quad -\int_z \langle S_z(z)\rangle\,dS = -\int_V \langle P_d\rangle\,dV \end{aligned} \tag{44}$$

The power $\langle P_d\rangle$ is dissipated in the dielectric and in the walls. Defining the total electromagnetic power as

$$\langle P(z)\rangle=\int_z \langle S_z(z)\rangle\,dS \tag{45}$$

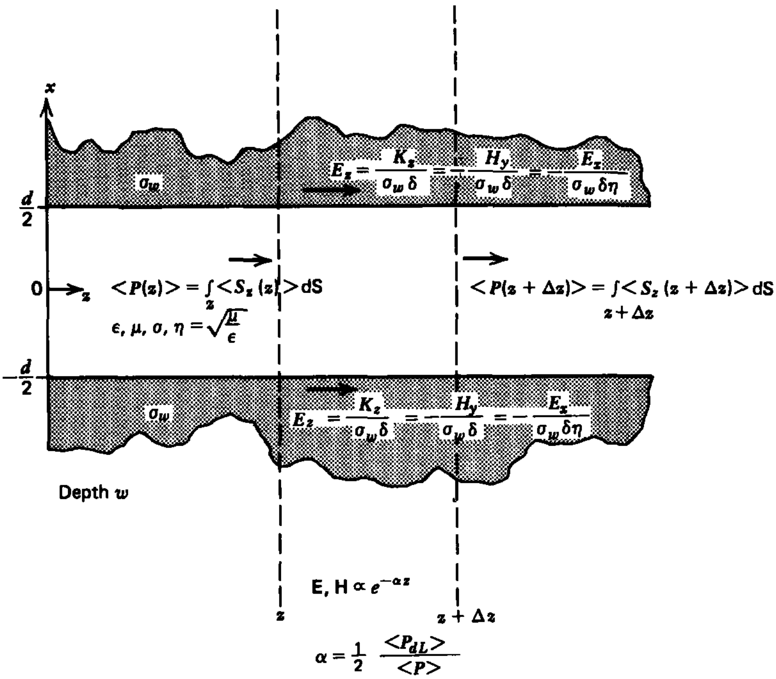



Figure 8-16 A transmission line with lossy walls and dielectric results in waves that decay as they propagate. The spatial decay rate $\alpha$ of the fields is approximately proportional to the ratio of time average dissipated power per unit length $<P_{dL}>$ to the total time average electromagnetic power flow $<P>$ down the line.

(44) can be rewritten as

$$<P(z+\Delta z)> - <P(z)> = -\int <P_d>\, dx\, dy\, dz \tag{46}$$

Dividing through by $dz = \Delta z$, we have in the infinitesimal limit

$$\lim_{\Delta z \to 0} \frac{<P(z+\Delta z)> - <P(z)>}{\Delta z} = \frac{d<P>}{dz} = -\int_S <P_d>\, dx\, dy$$
$$= -<P_{dL}> \tag{47}$$

where $<P_{dL}>$ is the power dissipated per unit length. Since the fields vary as $e^{-\alpha z}$, the power flow that is proportional to the square of the fields must vary as $e^{-2\alpha z}$ so that

$$\frac{d<P>}{dz} = -2\alpha <P> = -<P_{dL}> \tag{48}$$

which when solved for the spatial decay rate is proportional to the ratio of dissipated power per unit length to the total electromagnetic power flowing down the transmission line:

$$\alpha = \frac{1}{2}\frac{\langle P_{dL}\rangle}{\langle P\rangle} \tag{49}$$

For our lossy transmission line, the power is dissipated both in the walls and in the dielectric. Fortunately, it is not necessary to solve the complicated field problem within the walls because we already approximately know the magnetic field at the walls from (42). Since the wall current is effectively confined to the skin depth $\delta$, the cross-sectional area through which the current flows is essentially $w\delta$ so that we can define the surface conductivity as $\sigma_w\delta$, where the electric field at the wall is related to the lossless surface current as

$$\mathbf{K}_w = \sigma_w\delta\mathbf{E}_w \tag{50}$$

The surface current in the wall is approximately found from the magnetic field in (42) as

$$K_z = -H_y = -E_x/\eta \tag{51}$$

The time-average power dissipated in the wall is then

$$\langle P_{dL}\rangle_{\text{wall}} = \frac{w}{2}\operatorname{Re}(\mathbf{E}_w \cdot \mathbf{K}_w^*) = \frac{1}{2}\frac{|\mathbf{K}_w|^2 w}{\sigma_w\delta} = \frac{1}{2}\frac{|\hat{E}|^2 w}{\sigma_w\delta\eta^2} \tag{52}$$

The total time-average dissipated power in the walls and dielectric per unit length for a transmission line system of depth $w$ and plate spacing $d$ is then

$$\begin{aligned}\langle P_{dL}\rangle &= 2\langle P_{dL}\rangle_{\text{wall}} + \tfrac{1}{2}\sigma|\hat{E}|^2 wd\\ &= \tfrac{1}{2}|\hat{E}|^2 w\left(\frac{2}{\eta^2\sigma_w\delta} + \sigma d\right)\end{aligned} \tag{53}$$

where we multiply (52) by two because of the losses in both electrodes. The time-average electromagnetic power is

$$\langle P\rangle = \frac{1}{2}\frac{|\hat{E}|^2}{\eta}wd \tag{54}$$

so that the spatial decay rate is found from (49) as

$$\alpha = -k_i = \frac{1}{2}\left(\frac{2}{\eta^2\sigma_w\delta} + \sigma d\right)\frac{\eta}{d} = \frac{1}{2}\left(\sigma\eta + \frac{2}{\eta\sigma_w\delta d}\right) \tag{55}$$

Comparing (55) to (35) we see that

$$\begin{aligned}&GZ_0 = \sigma\eta, \qquad RY_0 = \frac{2}{\eta\sigma_w\delta d}\\ &\Rightarrow Z_0 = \frac{1}{Y_0} = \frac{d}{w}\eta, \qquad G = \frac{\sigma w}{d}, \qquad R = \frac{2}{\sigma_w w\delta}\end{aligned} \tag{56}$$

## 8-4 ARBITRARY IMPEDANCE TERMINATIONS

### 8-4-1 The Generalized Reflection Coefficient

A lossless transmission line excited at $z=-l$ with a sinusoidal voltage source is now terminated at its other end at $z=0$ with an arbitrary impedance $Z_L$, which in general can be a complex number. Defining the load voltage and current at $z=0$ as

$$\begin{aligned} v(z=0,t) &= v_L(t)=\mathrm{Re}\,(\mathrm{V}_L\,e^{j\omega t})\\ i(z=0,t) &= i_L(t)=\mathrm{Re}\,(I_L\,e^{j\omega t}), \qquad I_L=\mathrm{V}_L/Z_L \end{aligned} \tag{1}$$

where $\mathrm{V}_L$ and $I_L$ are complex amplitudes, the boundary conditions at $z=0$ are

$$\begin{aligned} \mathrm{V}_+ + \mathrm{V}_- &= \mathrm{V}_L\\ Y_0(\mathrm{V}_+ - \mathrm{V}_-) &= I_L=\mathrm{V}_L/Z_L \end{aligned} \tag{2}$$

We define the reflection coefficient as the ratio

$$\Gamma_L=\mathrm{V}_-/\mathrm{V}_+ \tag{3}$$

and solve as

$$\Gamma_L=\frac{Z_L-Z_0}{Z_L+Z_0} \tag{4}$$

Here in the sinusoidal steady state with reactive loads, $\Gamma_L$ can be a complex number as $Z_L$ may be complex. For transient pulse waveforms, $\Gamma_L$ was only defined for resistive loads. For capacitative and inductive terminations, the reflections were given by solutions to differential equations in time. Now that we are only considering sinusoidal time variations so that time derivatives are replaced by $j\omega$, we can generalize $\Gamma_L$ for the sinusoidal steady state.

It is convenient to further define the generalized reflection coefficient as

$$\Gamma(z)=\frac{\mathrm{V}_-\,e^{jkz}}{\mathrm{V}_+\,e^{-jkz}}=\frac{\mathrm{V}_-}{\mathrm{V}_+}\,e^{2jkz}=\Gamma_L\,e^{2jkz} \tag{5}$$

where $\Gamma_L$ is just $\Gamma(z=0)$. Then the voltage and current on the line can be expressed as

$$\begin{aligned} \hat{v}(z) &= \mathrm{V}_+\,e^{-jkz}[1+\Gamma(z)]\\ \hat{\imath}(z) &= Y_0\mathrm{V}_+\,e^{-jkz}[1-\Gamma(z)] \end{aligned} \tag{6}$$

The advantage to this notation is that now the impedance along the line can be expressed as

$$Z_n(z)=\frac{Z(z)}{Z_0}=\frac{\hat{v}(z)}{\hat{\imath}(z)Z_0}=\frac{1+\Gamma(z)}{1-\Gamma(z)} \tag{7}$$

where $Z_n$ is defined as the normalized impedance. We can now solve (7) for $\Gamma(z)$ as

$$\Gamma(z)=\frac{Z_n(z)-1}{Z_n(z)+1} \tag{8}$$

Note the following properties of $Z_n(z)$ and $\Gamma(z)$ for passive loads:

(i) $Z_n(z)$ is generally complex. For passive loads its real part is allowed over the range from zero to infinity while its imaginary part can extend from negative to positive infinity.
(ii) The magnitude of $\Gamma(z)$, $|\Gamma_L|$ must be less than or equal to 1 for passive loads.
(iii) From (5), if $z$ is increased or decreased by a half wavelength, $\Gamma(z)$ and hence $Z_n(z)$ remain unchanged. Thus, if the impedance is known at any position, the impedance of all-points integer multiples of a half wavelength away have the same impedance.
(iv) From (5), if $z$ is increased or decreased by a quarter wavelength, $\Gamma(z)$ changes sign, while from (7) $Z_n(z)$ goes to its reciprocal$\Rightarrow 1/Z_n(z)=Y_n(z)$.
(v) If the line is matched, $Z_L=Z_0$, then $\Gamma_L=0$ and $Z_n(z)=1$. The impedance is the same everywhere along the line.

## 8-4-2 Simple Examples

**(a) Load Impedance Reflected Back to the Source**

Properties (iii)–(v) allow simple computations for transmission line systems that have lengths which are integer multiples of quarter or half wavelengths. Often it is desired to maximize the power delivered to a load at the end of a transmission line by adding a lumped admittance $Y$ across the line. For the system shown in Figure 8-17$a$, the impedance of the load is reflected back to the generator and then added in parallel to the lumped reactive admittance $Y$. The normalized load impedance of $(R_L+jX_L)/Z_0$ inverts when reflected back to the source by a quarter wavelength to $Z_0/(R_L+jX_L)$. Since this is the normalized impedance the actual impedance is found by multiplying by $Z_0$ to yield $Z(z=-\lambda/4)=Z_0^2/(R_L+jX_L)$. The admittance of this reflected load then adds in parallel to $Y$ to yield a total admittance of $Y+(R_L+jX_L)/Z_0^2$. If $Y$ is pure imaginary and of opposite sign to the reflected load susceptance with value $-jX_L/Z_0^2$, maximum power is delivered to the line if the source resistance $R_S$ also equals the resulting line input impedance, $R_S=Z_0^2/R_L$. Since $Y$ is purely

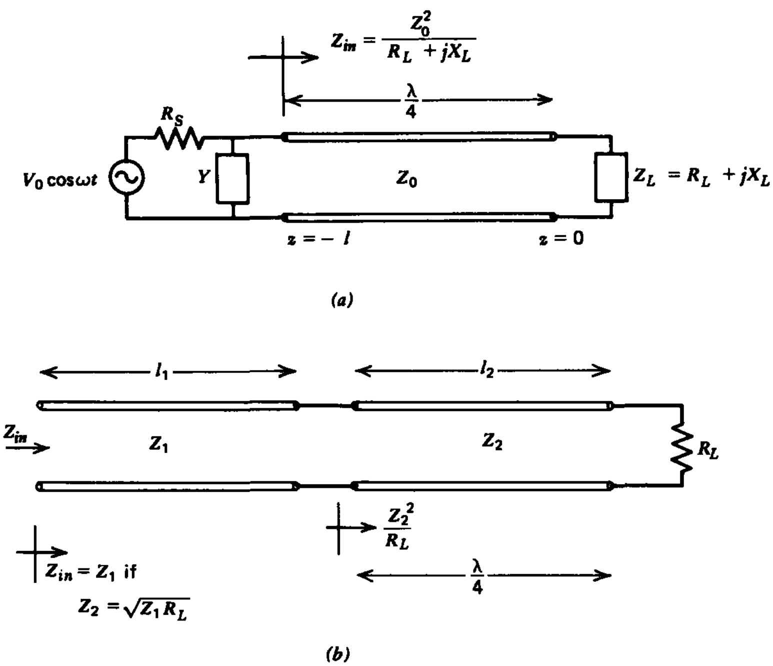



Figure 8-17 The normalized impedance reflected back through a quarter-wave-long line inverts. (*a*) The time-average power delivered to a complex load can be maximized if $Y$ is adjusted to just cancel the reactive admittance of the load reflected back to the source with $R_s$ equaling the resulting input resistance. (*b*) If the length $l_2$ of the second transmission line shown is a quarter wave long or an odd integer multiple of $\lambda/4$ and its characteristic impedance is equal to the geometric average of $Z_1$ and $R_L$, the input impedance $Z_{in}$ is matched to $Z_1$.

reactive and the transmission line is lossless, half the time-average power delivered by the source is dissipated in the load:

$$<P> = \frac{1}{8}\frac{V_0^2}{R_S} = \frac{1}{8}\frac{R_L V_0^2}{Z_0^2} \tag{9}$$

Such a reactive element $Y$ is usually made from a variable length short circuited transmission line called a stub. As shown in Section 8-3-2*a*, a short circuited lossless line always has a pure reactive impedance.

To verify that the power in (9) is actually dissipated in the load, we write the spatial distribution of voltage and current along the line as

$$\begin{aligned} \hat{v}(z) &= V_+ e^{-jkz}(1 + \Gamma_L e^{2jkz}) \\ \hat{i}(z) &= Y_0 V_+ e^{-jkz}(1 - \Gamma_L e^{2jkz}) \end{aligned} \tag{10}$$

where the reflection coefficient for this load is given by (4) as

$$\Gamma_L = \frac{R_L + jX_L - Z_0}{R_L + jX_L + Z_0} \tag{11}$$

At $z = -l = -\lambda/4$ we have the boundary condition

$$\begin{aligned} \hat{v}(z = -l) = V_0/2 &= \mathrm{V}_+ e^{jkl}(1 + \Gamma_L e^{-2jkl}) \\ &= j\mathrm{V}_+(1 - \Gamma_L) \end{aligned} \tag{12}$$

which allows us to solve for $\mathrm{V}_+$ as

$$\mathrm{V}_+ = \frac{-jV_0}{2(1-\Gamma_L)} = \frac{-jV_0}{4Z_0}(R_L + jX_L + Z_0) \tag{13}$$

The time-average power dissipated in the load is then

$$\begin{aligned} \langle P_L \rangle &= \tfrac{1}{2}\,\mathrm{Re}\,[\hat{v}(z=0)\hat{i}^*(z=0)] \\ &= \tfrac{1}{2}|\hat{i}(z=0)|^2 R_L \\ &= \tfrac{1}{2}|\mathrm{V}_+|^2 |1 - \Gamma_L|^2 Y_0^2 R_L \\ &= \tfrac{1}{8} V_0^2 Y_0^2 R_L \end{aligned} \tag{14}$$

which agrees with (9).

**(b) Quarter Wavelength Matching**

It is desired to match the load resistor $R_L$ to the transmission line with characteristic impedance $Z_1$ for any value of its length $l_1$. As shown in Figure 8-17*b*, we connect the load to $Z_1$ via another transmission line with characteristic impedance $Z_2$. We wish to find the values of $Z_2$ and $l_2$ necessary to match $R_L$ to $Z_1$.

This problem is analogous to the dielectric coating problem of Example 7-1, where it was found that reflections could be eliminated if the coating thickness between two different dielectric media was an odd integer multiple of a quarter wavelength and whose wave impedance was equal to the geometric average of the impedance in each adjacent region. The normalized load on $Z_2$ is then $Z_{n2} = R_L/Z_2$. If $l_2$ is an odd integer multiple of a quarter wavelength long, the normalized impedance $Z_{n2}$ reflected back to the first line inverts to $Z_2/R_L$. The actual impedance is obtained by multiplying this normalized impedance by $Z_2$ to give $Z_2^2/R_L$. For $Z_{in}$ to be matched to $Z_1$ for any value of $l_1$, this impedance must be matched to $Z_1$:

$$Z_1 = Z_2^2/R_L \Rightarrow Z_2 = \sqrt{Z_1 R_L} \tag{15}$$

### 8-4-3 The Smith Chart

Because the range of allowed values of $\Gamma_L$ must be contained within a unit circle in the complex plane, all values of $Z_n(z)$ can be mapped by a transformation within this unit circle using (8). This transformation is what makes the substitutions of (3)–(8) so valuable. A graphical aid of this mathematical transformation was developed by P. H. Smith in 1939 and is known as the Smith chart. Using the Smith chart avoids the tedium in problem solving with complex numbers.

Let us define the real and imaginary parts of the normalized impedance at some value of $z$ as

$$Z_n(z) = r + jx \tag{16}$$

The reflection coefficient similarly has real and imaginary parts given as

$$\Gamma(z) = \Gamma_r + j\Gamma_i \tag{17}$$

Using (7) we have

$$r + jx = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i} \tag{18}$$

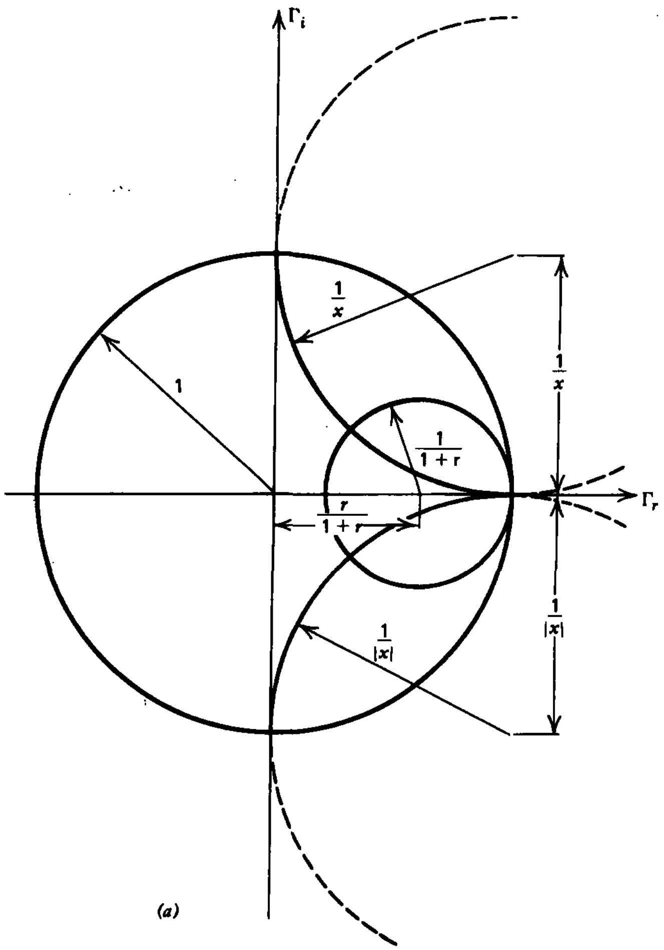
Multiplying numerator and denominator by the complex conjugate of the denominator $(1-\Gamma_r + j\Gamma_i)$ and separating real and imaginary parts yields

$$\begin{aligned} r &= \frac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} \\ x &= \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2} \end{aligned} \tag{19}$$

Since we wish to plot (19) in the $\Gamma_r - \Gamma_i$ plane we rewrite these equations as

$$\begin{aligned} \left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 &= \frac{1}{(1+r)^2} \\ (\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 &= \frac{1}{x^2} \end{aligned} \tag{20}$$

Both equations in (20) describe a family of orthogonal circles. The upper equation is that of a circle of radius $1/(1+r)$ whose center is at the position $\Gamma_i = 0$, $\Gamma_r = r/(1+r)$. The lower equation is a circle of radius $|1/x|$ centered at the position $\Gamma_r = 1$, $\Gamma_i = 1/x$. Figure 8-18*a* illustrates these circles for a particular value of $r$ and $x$, while Figure 8-18*b* shows a few representative values of $r$ and $x$. In Figure 8-19, we have a complete Smith chart. Only those parts of the circles that lie within the unit circle in the $\Gamma$ plane are considered for passive

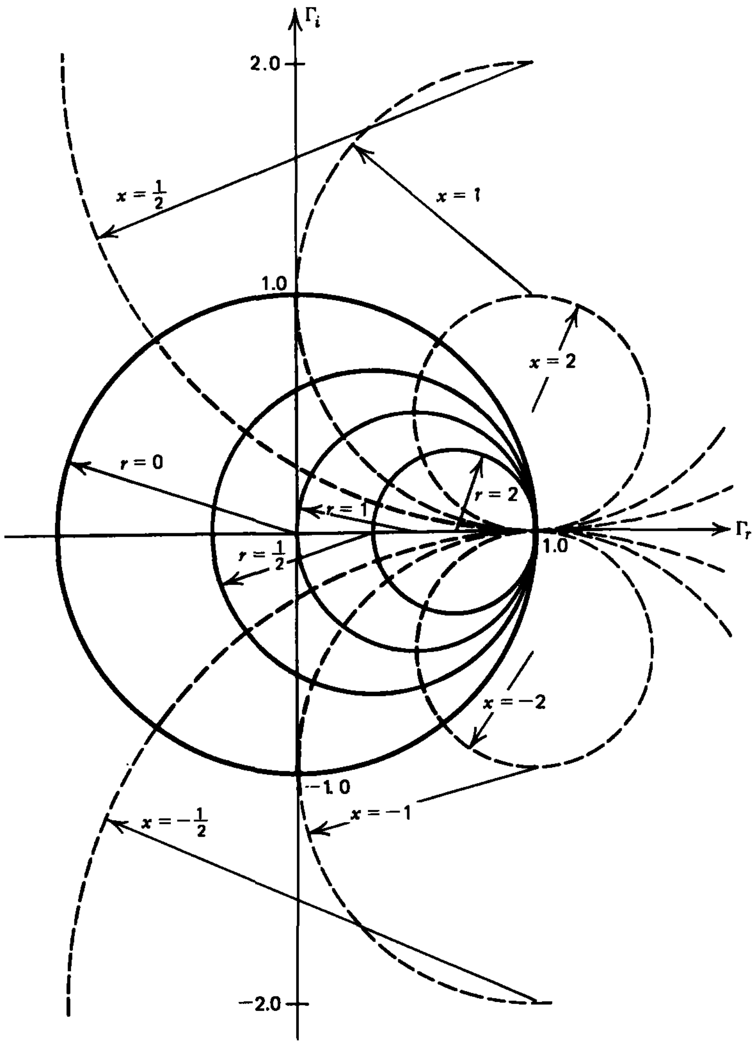



(a)

Figure 8-18 For passive loads the Smith chart is constructed within the unit circle in the complex $\Gamma$ plane. (*a*) Circles of constant normalized resistance $r$ and reactance $x$ are constructed with the centers and radii shown. (*b*) Smith chart construction for various values of $r$ and $x$.

resistive-reactive loads. The values of $\Gamma(z)$ themselves are usually not important and so are not listed, though they can be easily found from (8). Note that all circles pass through the point $\Gamma_r = 1$, $\Gamma_i = 0$.

The outside of the circle is calibrated in wavelengths toward the generator, so if the impedance is known at any point on the transmission line (usually at the load end), the impedance at any other point on the line can be found using just a compass and a ruler. From the definition of $\Gamma(z)$ in (5) with $z$ negative, we move clockwise around the Smith chart when heading towards the source and counterclockwise when moving towards the load.




*(b)*

Figure 8-18

In particular, consider the transmission line system in Figure 8-20*a*. The normalized load impedance is $Z_n = 1 + j$. Using the Smith chart in Figure 8-20*b*, we find the load impedance at position $A$. The effective impedance reflected back to $z = -l$ must lie on the circle of constant radius returning to $A$ whenever $l$ is an integer multiple of a half wavelength. The table in Figure 8-20 lists the impedance at $z = -l$ for various line lengths. Note that at point $C$, where $l = \lambda/4$, that the normalized impedance is the reciprocal of

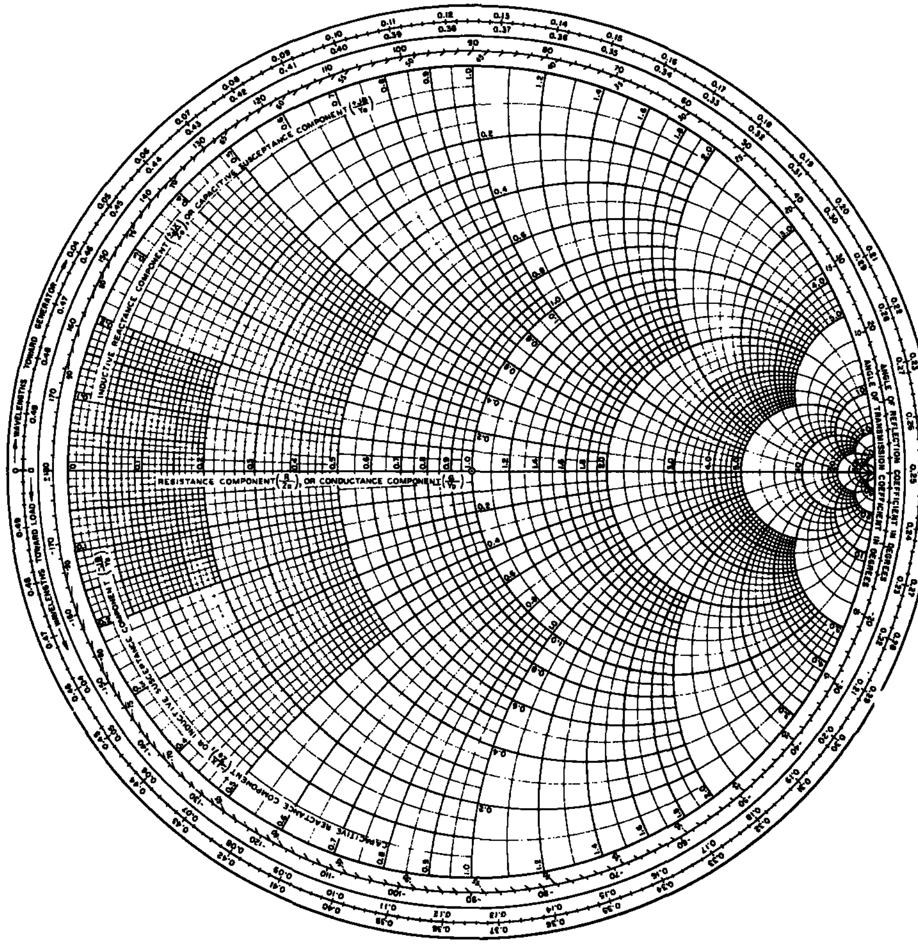


Figure 8-19 A complete Smith chart.

that at $A$. Similarly the normalized impedance at $B$ is the reciprocal of that at $D$.

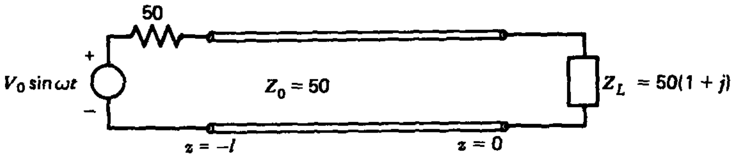
The current from the voltage source is found using the equivalent circuit shown in Figure 8-20$c$ as

$$i = |\hat{I}| \sin(\omega t - \phi) \tag{21}$$

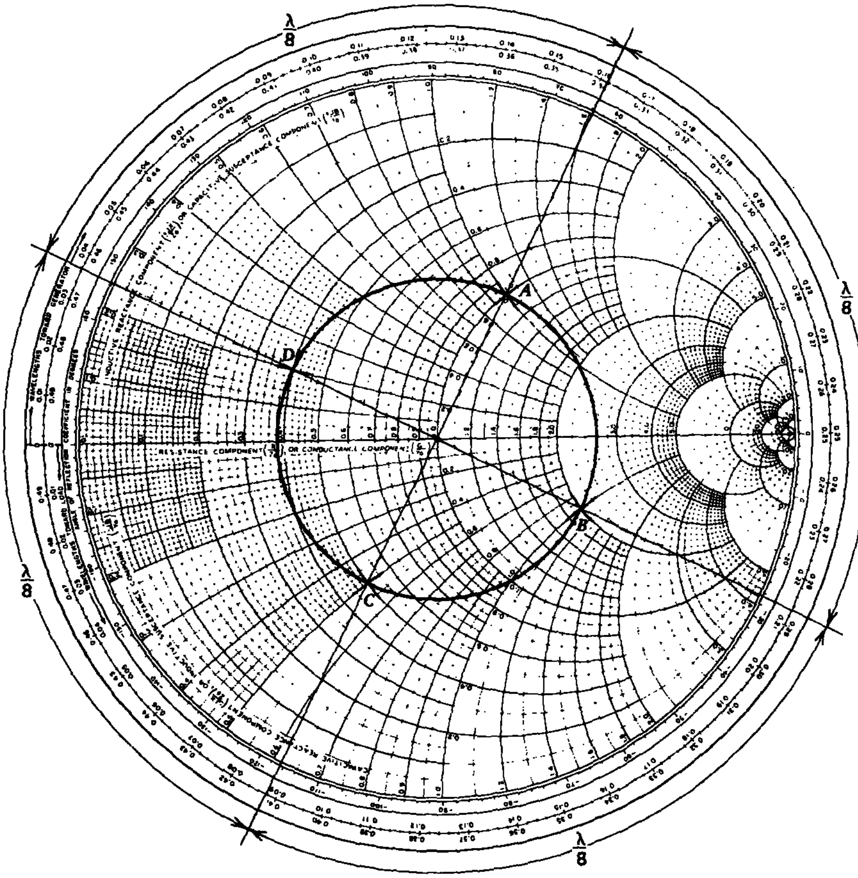
where the current magnitude and phase angle are

$$|\hat{I}| = \frac{V_0}{|50 + Z(z=-l)|}, \qquad \phi = \tan^{-1} \frac{\operatorname{Im}[Z(z=-l)]}{50 + \operatorname{Re}[Z(z=-l)]} \tag{22}$$

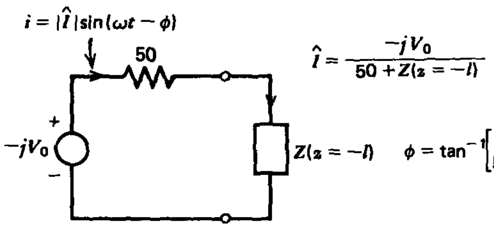
Representative numerical values are listed in Figure 8-20.




(a)




(b)




$$\hat{I} = \frac{-jV_0}{50 + Z(z = -l)}$$

| Point | $l$ | $Z_n(z = -l)$ | $\lvert\hat{I}\rvert Z_0/V_0$ | $\phi$ |
|---|---|---|---|---|
| A | $0, \lambda/2$ | $1 + j$ | 0.447 | 26.6° |
| B | $\lambda/8$ | $2 - j$ | 0.316 | −18.4° |
| C | $\lambda/4$ | $\frac{1}{2}(1 - j)$ | 0.632 | −18.4° |
| D | $\frac{3}{8}\lambda$ | $.4 + .2j$ | 0.707 | 8.1° |

$$\phi = \tan^{-1}\left[\frac{\operatorname{Im}(Z(z = -l))}{50 + \operatorname{Re}(Z(z = -l))}\right]$$

(c)

Figure 8-20 (*a*) The load impedance at $z = 0$ reflected back to the source is found using the (*b*) Smith chart for various line lengths. Once this impedance is known the source current is found by solving the simple series circuit in (*c*).

### 8-4-4 Standing Wave Parameters

The impedance and reflection coefficient are not easily directly measured at microwave frequencies. In practice, one slides an ac voltmeter across a slotted transmission line and measures the magnitude of the peak or rms voltage and not its phase angle.

From (6) the magnitude of the voltage and current at any position $z$ is

$$\begin{aligned} |\hat{v}(z)| &= |V_+|\,|1+\Gamma(z)| \\ |\hat{\imath}(z)| &= Y_0|V_+|\,|1-\Gamma(z)| \end{aligned} \tag{23}$$

From (23), the variations of the voltage and current magnitudes can be drawn by a simple construction in the $\Gamma$ plane, as shown in Figure 8-21. Note again that $|V_+|$ is just a real number independent of $z$ and that $|\Gamma(z)| \leq 1$ for a passive termination. We plot $|1+\Gamma(z)|$ and $|1-\Gamma(z)|$ since these terms are proportional to the voltage and current magnitudes, respectively. The following properties from this con-

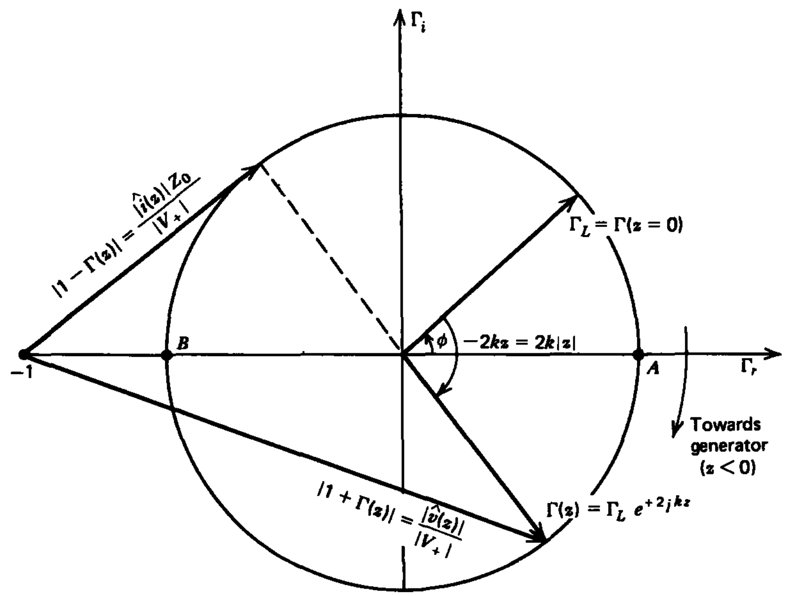



Figure 8-21 The voltage and current magnitudes along a transmission line are respectively proportional to the lengths of the vectors $|1+\Gamma(z)|$ and $|1-\Gamma(z)|$ in the complex $\Gamma$ plane.

struction are apparent:

(i) The magnitude of the current is smallest and the voltage magnitude largest when $\Gamma(z)=1$ at point $A$ and vice versa when $\Gamma(z)=-1$ at point $B$.
(ii) The voltage and current are in phase at the points of maximum or minimum magnitude of either at points $A$ or $B$.
(iii) A rotation of $\Gamma(z)$ by an angle $\pi$ corresponds to a change of $\lambda/4$ in $z$, thus any voltage (or current) maximum is separated by $\lambda/4$ from its nearest minima on either side.

By plotting the lengths of the phasors $|1\pm\Gamma(z)|$, as in Figure 8-22, we obtain a plot of what is called the standing wave pattern on the line. Observe that the curves are not sinusoidal. The minima are sharper than the maxima so the minima are usually located in position more precisely by measurement than the maxima.

From Figures 8-21 and 8-22, the ratio of the maximum voltage magnitude to the minimum voltage magnitude is defined as the voltage standing wave ratio, or VSWR for short:

$$\frac{|\hat{v}(z)|_{\max}}{|\hat{v}(z)|_{\min}}=\frac{1+|\Gamma_L|}{1-|\Gamma_L|}=\mathrm{VSWR} \tag{24}$$

The VSWR is measured by simply recording the largest and smallest readings of a sliding voltmeter. Once the VSWR is measured, the reflection coefficient magnitude can be calculated from (24) as

$$|\Gamma_L|=\frac{\mathrm{VSWR}-1}{\mathrm{VSWR}+1} \tag{25}$$

The angle $\phi$ of the reflection coefficient

$$\Gamma_L=|\Gamma_L|\,e^{j\phi} \tag{26}$$

can also be determined from these standing wave measurements. According to Figure 8-21, $\Gamma(z)$ must swing clockwise through an angle $\phi+\pi$ as we move from the load at $z=0$ toward the generator to the first voltage minimum at $B$. The shortest distance $d_{\min}$ that we must move to reach the first voltage minimum is given by

$$2kd_{\min}=\phi+\pi \tag{27}$$

or

$$\frac{\phi}{\pi}=4\frac{d_{\min}}{\lambda}-1 \tag{28}$$

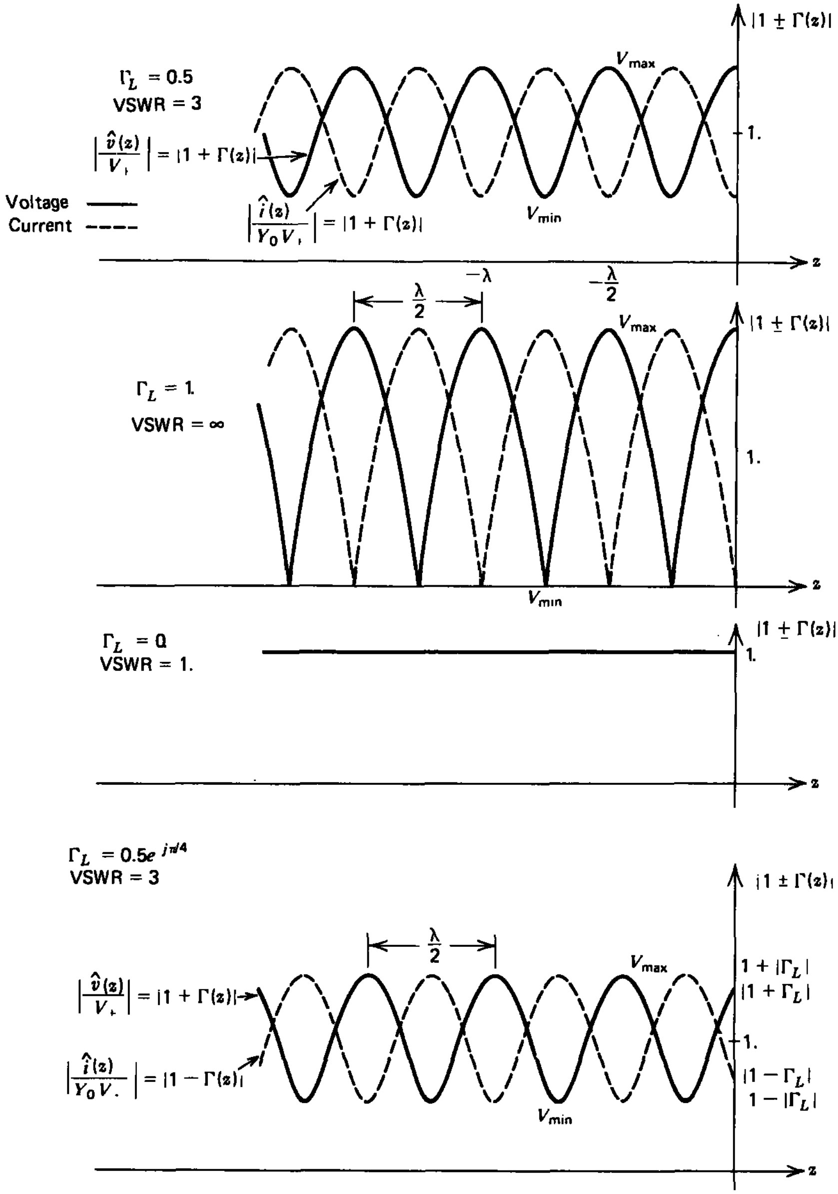



**Figure 8-22** Voltage and current standing wave patterns plotted for various values of the VSWR.

A measurement of $d_{min}$, as well as a determination of the wavelength (the distance between successive minima or maxima is $\lambda/2$) yields the complex reflection coefficient of the load using (25) and (28). Once we know the complex reflection coefficient we can calculate the load impedance from (7). These standing wave measurements are sufficient to determine the terminating load impedance $Z_L$. These measurement properties of the load reflection coefficient and its relation to the load impedance are of great importance at high frequencies where the absolute measurement of voltage or current may be difficult. Some special cases of interest are:

(i) Matched line—If $\Gamma_L = 0$, then VSWR = 1. The voltage magnitude is constant everywhere on the line.
(ii) Short or open circuited line—If $|\Gamma_L| = 1$, then VSWR = $\infty$. The minimum voltage on the line is zero.
(iii) The peak normalized voltage $|\hat{v}(z)/V_+|$ is $1+|\Gamma_L|$ while the minimum normalized voltage is $1-|\Gamma_L|$.
(iv) The normalized voltage at $z=0$ is $|1+\Gamma_L|$ while the normalized current $|\hat{\imath}(z)/Y_0V_+|$ at $z=0$ is $|1-\Gamma_L|$.
(v) If the load impedance is real ($Z_L = R_L$), then (4) shows us that $\Gamma_L$ is real. Then evaluating (7) at $z=0$, where $\Gamma(z=0)=\Gamma_L$, we see that when $Z_L > Z_0$ that VSWR = $Z_L/Z_0$ while if $Z_L < Z_0$, VSWR = $Z_0/Z_L$.

For a general termination, if we know the VSWR and $d_{\min}$, we can calculate the load impedance from (7) as

$$Z_L = Z_0\frac{1+|\Gamma_L|\,e^{j\phi}}{1-|\Gamma_L|\,e^{j\phi}}$$

$$= Z_0\frac{[\text{VSWR}+1+(\text{VSWR}-1)\,e^{j\phi}]}{[\text{VSWR}+1-(\text{VSWR}-1)\,e^{j\phi}]} \tag{29}$$

Multiplying through by $e^{-j\phi/2}$ and then simplifying yields

$$Z_L = \frac{Z_0[\text{VSWR}-j\tan(\phi/2)]}{[1-j\,\text{VSWR}\tan(\phi/2)]}$$

$$= \frac{Z_0[1-j\,\text{VSWR}\tan kd_{\min}]}{[\text{VSWR}-j\tan kd_{\min}]} \tag{30}$$

## EXAMPLE 8-2 VOLTAGE STANDING WAVE RATIO

The VSWR on a 50-Ohm (characteristic impedance) transmission line is 2. The distance between successive voltage minima is 40 cm while the distance from the load to the first minima is 10 cm. What is the reflection coefficient and load impedance?

**SOLUTION**

---

We are given

$$\text{VSWR}=2$$

$$kd_{\min}=\frac{2\pi(10)}{2(40)}=\frac{\pi}{4}$$

The reflection coefficient is given from (25)–(28) as

$$\Gamma_L=\tfrac{1}{3}e^{-j\pi/2}=\frac{-j}{3}$$

while the load impedance is found from (30) as

$$Z_L=\frac{50(1-2j)}{2-j}$$

$$=40-30j\text{ ohm}$$

## 8-5 STUB TUNING

In practice, most sources are connected to a transmission line through a series resistance matched to the line. This eliminates transient reflections when the excitation is turned on or off. To maximize the power flow to a load, it is also necessary for the load impedance reflected back to the source to be equal to the source impedance and thus equal to the characteristic impedance of the line, $Z_0$. This matching of the load to the line for an arbitrary termination can only be performed by adding additional elements along the line.

Usually these elements are short circuited transmission lines, called stubs, whose lengths can be varied. The reactance of the stub can be changed over the range from $-j\infty$ to $j\infty$ simply by varying its length, as found in Section 8-3-2, for the short circuited line. Because stubs are usually connected in parallel to a transmission line, it is more convenient to work with admittances rather than impedances as admittances in parallel simply add.

### 8-5-1 Use of the Smith Chart for Admittance Calculations

Fortunately the Smith chart can also be directly used for admittance calculations where the normalized admittance is defined as

$$Y_n(z)=\frac{Y(z)}{Y_0}=\frac{1}{Z_n(z)} \tag{1}$$

If the normalized load admittance $Y_{nL}$ is known, straightforward impedance calculations first require the computation

$$Z_{nL} = 1/Y_{nL} \tag{2}$$

so that we could enter the Smith chart at $Z_{nL}$. Then we rotate by the required angle corresponding to $2kz$ and read the new $Z_n(z)$. Then we again compute its reciprocal to find

$$Y_n(z) = 1/Z_n(z) \tag{3}$$

The two operations of taking the reciprocal are tedious. We can use the Smith chart itself to invert the impedance by using the fact that the normalized impedance is inverted by a $\lambda/4$ section of line, so that a rotation of $\Gamma(z)$ by 180° changes a normalized impedance into its reciprocal. Hence, if the admittance is given, we enter the Smith chart with a given value of normalized admittance $Y_n$ and rotate by 180° to find $Z_n$. We then rotate by the appropriate number of wavelengths to find $Z_n(z)$. Finally, we again rotate by 180° to find $Y_n(z) = 1/Z_n(z)$. We have actually rotated the reflection coefficient by an angle of $2\pi + 2kz$. Rotation by $2\pi$ on the Smith chart, however, brings us back to wherever we started, so that only the $2kz$ rotation is significant. As long as we do an even number of $\pi$ rotations by entering the Smith chart with an admittance and leaving again with an admittance, we can use the Smith chart with normalized admittances exactly as if they were normalized impedances.

## EXAMPLE 8-3 USE OF THE SMITH CHART FOR ADMITTANCE CALCULATIONS

The load impedance on a 50-Ohm line is

$$Z_L = 50(1 + j)$$

What is the admittance of the load?

### SOLUTION

---

By direct computation we have

$$Y_L = \frac{1}{Z_L} = \frac{1}{50(1+j)} = \frac{(1-j)}{100}$$

To use the Smith chart we find the normalized impedance at $A$ in Figure 8-23:

$$Z_{nL} = 1 + j$$

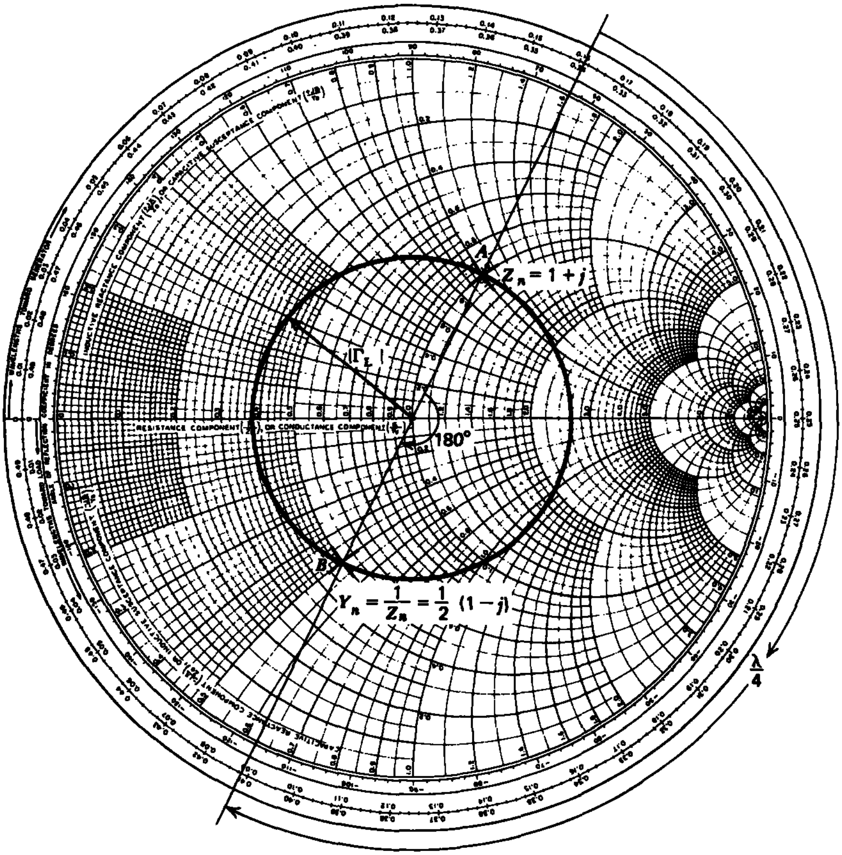



Figure 8-23 The Smith chart offers a convenient way to find the reciprocal of a complex number using the property that the normalized impedance reflected back by a quarter wavelength inverts. Thus, the normalized admittance is found by locating the normalized impedance and rotating this point by 180° about the constant $|\Gamma_L|$ circle.

The normalized admittance that is the reciprocal of the normalized impedance is found by locating the impedance a distance $\lambda/4$ away from the load end at $B$:

$$Y_{nL} = 0.5(1-j) \Rightarrow Y_L = Y_n Y_0 = (1-j)/100$$

Note that the point $B$ is just 180° away from $A$ on the constant $|\Gamma_L|$ circle. For more complicated loads the Smith chart is a convenient way to find the reciprocal of a complex number.

### 8-5-2 Single-Stub Matching

A termination of value $Z_L = 50(1+j)$ on a 50-Ohm transmission line is to be matched by means of a short circuited stub at a distance $l_1$ from the load, as shown in Figure 8-24*a*. We need to find the line length $l_1$ and the length of the stub $l_2$ such that the impedance at the junction is matched to the line ($Z_{in} = 50$ Ohm). Then we know that all further points to the left of the junction have the same impedance of 50 Ohms.

Because of the parallel connection, it is simpler to use the Smith chart as an admittance transformation. The normalized load admittance can be computed using the Smith chart by rotating by 180° from the normalized load impedance at $A$, as was shown in Figure 8-23 and Example 8-3,

$$Z_{nL} = 1 + j \tag{4}$$

to yield

$$Y_{nL} = 0.5(1-j) \tag{5}$$

at the point $B$.

Now we know from Section 8-3-2 that the short circuited stub can only add an imaginary component to the admittance. Since we want the total normalized admittance to be unity to the left of the stub in Figure 8-24

$$Y_{in} = Y_1 + Y_2 = 1 \tag{6}$$

when $Y_{nL}$ is reflected back to be $Y_1$ it must wind up on the circle whose real part is 1 (as $Y_2$ can only be imaginary), which occurs either at $C$ or back at $A$ allowing $l_1$ to be either $0.25\lambda$ at $A$ or $(0.25+0.177)\lambda = 0.427\lambda$ at $C$ (or these values plus any integer multiple of $\lambda/2$). Then $Y_1$ is either of the following two conjugate values:

$$Y_1 = \begin{cases} 1+j, & l_1 = 0.25\lambda\ (A) \\ 1-j, & l_1 = 0.427\lambda\ (C) \end{cases} \tag{7}$$

For $Y_{in}$ to be unity we must pick $Y_2$ to have an imaginary part to just cancel the imaginary part of $Y_1$:

$$Y_2 = \begin{cases} -j, & l_1 = 0.25\lambda \\ +j, & l_1 = 0.427\lambda \end{cases} \tag{8}$$

which means, since the shorted end has an infinite admittance at $D$ that the stub must be of length such as to rotate the admittance to the points $E$ or $F$ requiring a stub length $l_2$ of $(\lambda/8)(E)$ or $(3\lambda/8)(F)$ (or these values plus any integer multiple

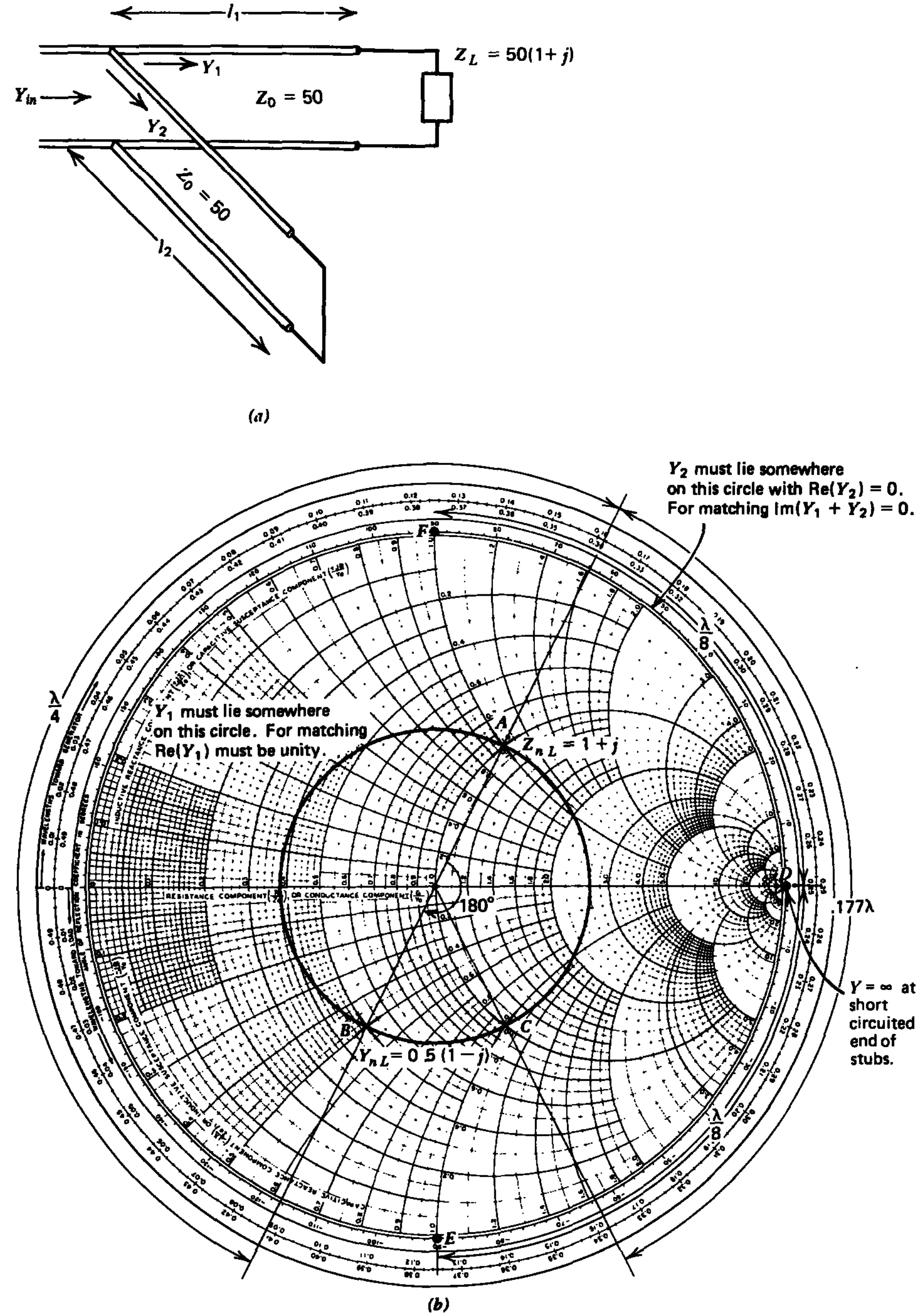
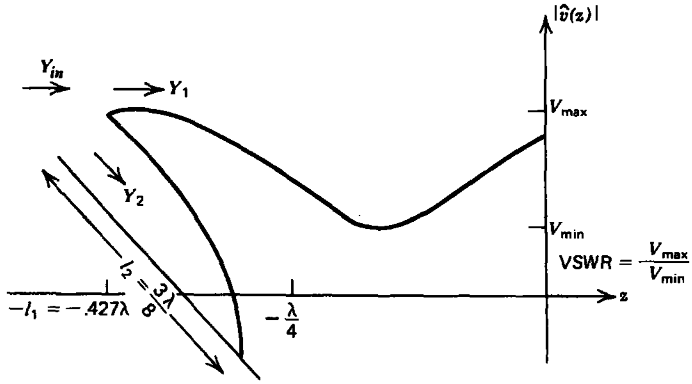



Figure 8-24 (*a*) A single stub tuner consisting of a variable length short circuited line $l_2$ can match any load to the line by putting the stub at the appropriate distance $l_1$ from the load. (*b*) Smith chart construction. (*c*) Voltage standing wave pattern.




(c)

Figure 8-24

of $\lambda/2$). Thus, the solutions can be summarized as

$$l_1 = 0.25\lambda + n\lambda/2, \qquad l_2 = \lambda/8 + m\lambda/2$$

or

$$l_1 = 0.427\lambda + n\lambda/2, \qquad l_2 = 3\lambda/8 + m\lambda/2 \tag{9}$$

where $n$ and $m$ are any nonnegative integers (including zero).

When the load is matched by the stub to the line, the VSWR to the left of the stub is unity, while to the right of the stub over the length $l_1$ the reflection coefficient is

$$\Gamma_L = \frac{Z_{nL} - 1}{Z_{nL} + 1} = \frac{j}{2+j} \tag{10}$$

which has magnitude

$$|\Gamma_L| = 1/\sqrt{5} \approx 0.447 \tag{11}$$

so that the voltage standing wave ratio is

$$\mathrm{VSWR} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} \approx 2.62 \tag{12}$$

The disadvantage to single-stub tuning is that it is not easy to vary the length $l_1$. Generally new elements can only be connected at the ends of the line and not inbetween.

### 8-5-3 Double-Stub Matching

This difficulty of not having a variable length line can be overcome by using two short circuited stubs a fixed length apart, as shown in Figure 8-25$a$. This fixed length is usually $\frac{3}{8}\lambda$. A match is made by adjusting the length of the stubs $l_1$ and

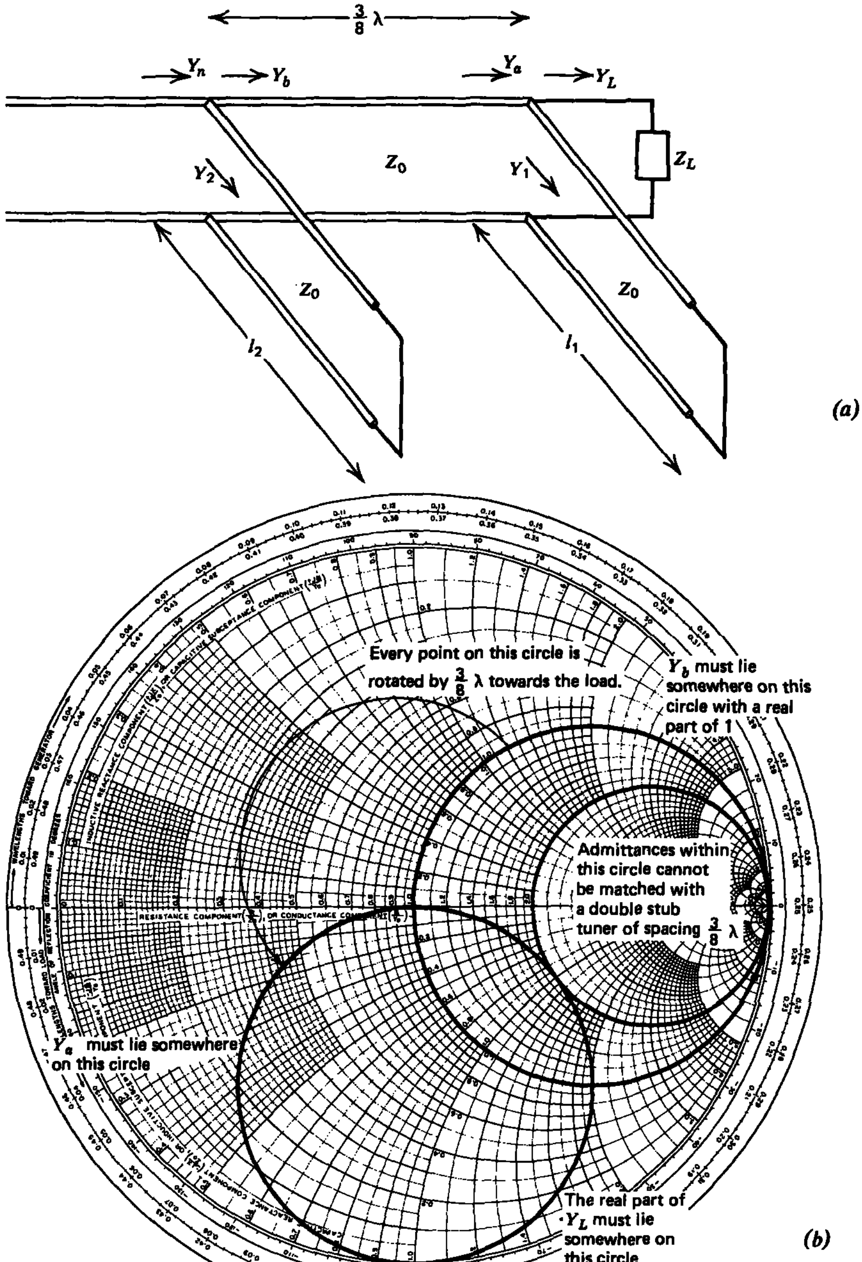



Figure 8-25 (*a*) A double stub tuner of fixed spacing cannot match all loads but is useful because additional elements can only be placed at transmission line terminations and not at any general position along a line as required for a single-stub tuner. (*b*) Smith chart construction. If the stubs are $\frac{3}{8}\lambda$ apart, normalized load admittances whose real part exceeds 2 cannot be matched.

$l_2$. One problem with the double-stub tuner is that not all loads can be matched for a given stub spacing.

The normalized admittances at each junction are related as

$$Y_a = Y_1 + Y_L$$
$$Y_n = Y_2 + Y_b \tag{13}$$

where $Y_1$ and $Y_2$ are the purely reactive admittances of the stubs reflected back to the junctions while $Y_b$ is the admittance of $Y_a$ reflected back towards the load by $\frac{3}{8}\lambda$. For a match we require that $Y_n$ be unity. Since $Y_2$ is purely imaginary, the real part of $Y_b$ must lie on the circle with a real part of unity. Then $Y_a$ must lie somewhere on this circle when each point on the circle is reflected back by $\frac{3}{8}\lambda$. This generates another circle that is $\frac{3}{2}\pi$ back in the counterclockwise direction as we are moving toward the load, as illustrated in Figure 8-25*b*. To find the conditions for a match, we work from left to right towards the load using the following reasoning:

(i) Since $Y_2$ is purely imaginary, the real part of $Y_b$ must lie on the circle with a real part of unity, as in Figure 8-25*b*.

(ii) Every possible point on $Y_b$ must be reflected towards the load by $\frac{3}{8}\lambda$ to find the locus of possible match for $Y_a$. This generates another circle that is $\frac{3}{2}\pi$ back in the counterclockwise direction as we move towards the load, as in Figure 8-25*b*.

Again since $Y_1$ is purely imaginary, the real part of $Y_a$ must also equal the real part of the load admittance. This yields two possible solutions if the load admittance is outside the forbidden circle enclosing all load admittances with a real part greater than 2. Only loads with normalized admittances whose real part is less than 2 can be matched by the double-stub tuner of $\frac{3}{8}\lambda$ spacing. Of course, if a load is within the forbidden circle, it can be matched by a double-stub tuner if the stub spacing is different than $\frac{3}{8}\lambda$.

## EXAMPLE 8-4 DOUBLE-STUB MATCHING

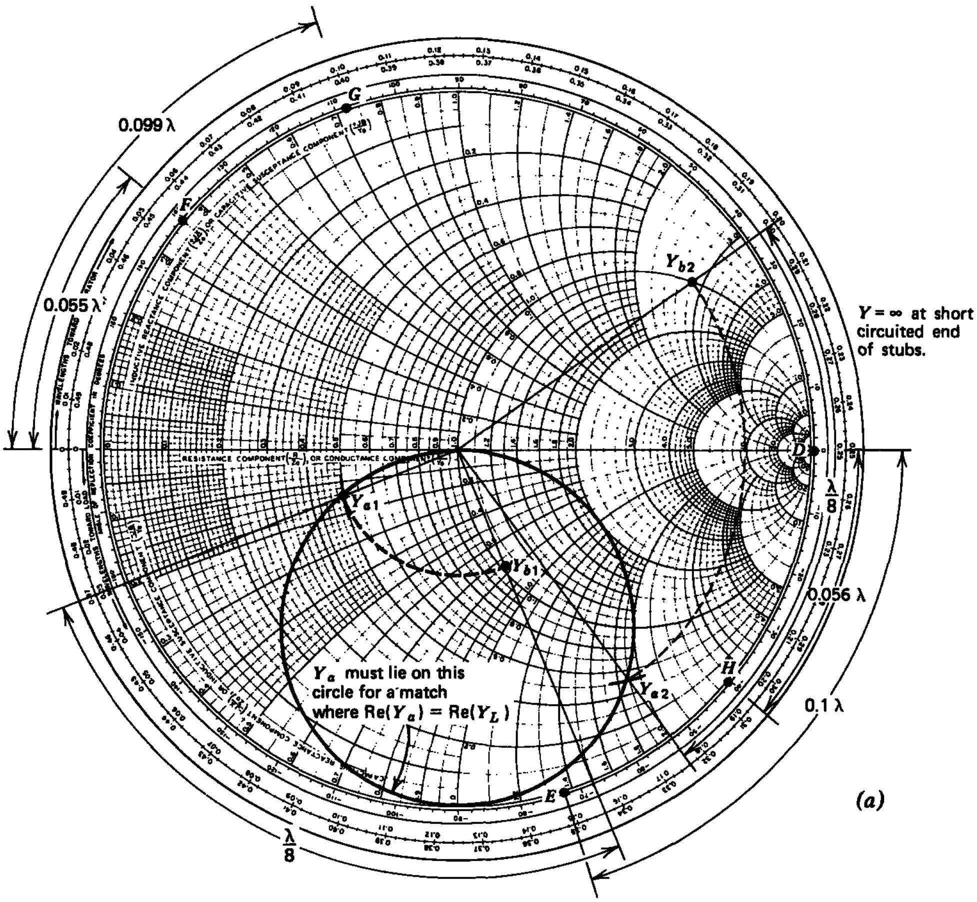
The load impedance $Z_L = 50(1+j)$ on a 50-Ohm line is to be matched by a double-stub tuner of $\frac{3}{8}\lambda$ spacing. What stub lengths $l_1$ and $l_2$ are necessary?

### SOLUTION

---

The normalized load impedance $Z_{nL} = 1+j$ corresponds to a normalized load admittance:

$$Y_{nL} = 0.5(1-j)$$



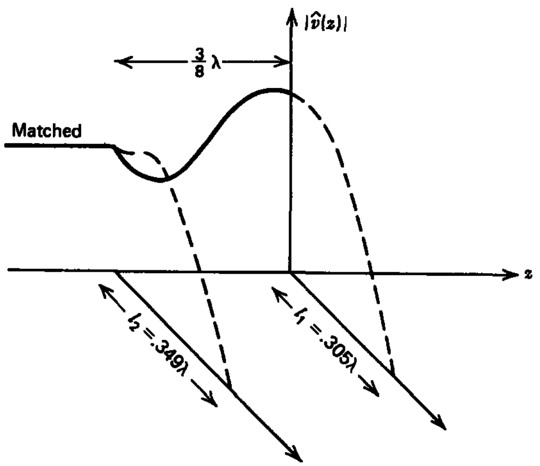




Figure 8-26 (*a*) The Smith chart construction for a double-stub tuner of $\frac{3}{8}\lambda$ spacing with $Z_{nL} = 1 + j$. (*b*) The voltage standing wave pattern.

Then the two solutions for $Y_a$ lie on the intersection of the circle shown in Figure 8-26*a* with the $r=0.5$ circle:

$$Y_{a1}=0.5-0.14j$$

$$Y_{a2}=0.5-1.85j$$

We then find $Y_1$ by solving for the imaginary part of the upper equation in (13):

$$Y_1=j\,\mathrm{Im}\,(Y_a-Y_L)=\begin{cases}0.36j\Rightarrow l_1=0.305\lambda & (F)\\ -1.35j\Rightarrow l_1=0.1\lambda & (E)\end{cases}$$

By rotating the $Y_a$ solutions by $\frac{3}{8}\lambda$ back to the generator (270° clockwise, which is equivalent to 90° counterclockwise), their intersection with the $r=1$ circle gives the solutions for $Y_b$ as

$$Y_{b1}=1.0-0.72j$$

$$Y_{b2}=1.0+2.7j$$

This requires $Y_2$ to be

$$Y_2=-j\,\mathrm{Im}\,(Y_b)=\begin{cases}0.72j\Rightarrow l_2=0.349\lambda & (G)\\ -2.7j\Rightarrow l_2=0.056\lambda & (H)\end{cases}$$

The voltage standing wave pattern along the line and stubs is shown in Figure 8.26*b*. Note the continuity of voltage at the junctions. The actual stub lengths can be those listed plus any integer multiple of $\lambda/2$.

## 8-6 THE RECTANGULAR WAVEGUIDE

We showed in Section 8-1-2 that the electric and magnetic fields for TEM waves have the same form of solutions in the plane transverse to the transmission line axis as for statics. The inner conductor within a closed transmission line structure such as a coaxial cable is necessary for TEM waves since it carries a surface current and a surface charge distribution, which are the source for the magnetic and electric fields. A hollow conducting structure, called a waveguide, cannot propagate TEM waves since the static fields inside a conducting structure enclosing no current or charge is zero.

However, new solutions with electric or magnetic fields along the waveguide axis as well as in the transverse plane are allowed. Such solutions can also propagate along transmission lines. Here the axial displacement current can act as a source of the transverse magnetic field giving rise to transverse magnetic (TM) modes as the magnetic field lies entirely within the transverse plane. Similarly, an axial time varying magnetic field generates transverse electric (TE) modes. The most general allowed solutions on a transmission line are TEM, TM, and TE modes. Removing the inner conductor on a closed transmission line leaves a waveguide that can only propagate TM and TE modes.

### 8-6-1 Governing Equations

To develop these general solutions we return to Maxwell's equations in a linear source-free material:

$$\begin{aligned}
\nabla\times\mathbf{E} &= -\mu\frac{\partial\mathbf{H}}{\partial t}\\
\nabla\times\mathbf{H} &= \varepsilon\frac{\partial\mathbf{E}}{\partial t}\\
\varepsilon\nabla\cdot\mathbf{E} &= 0\\
\mu\nabla\cdot\mathbf{H} &= 0
\end{aligned}\tag{1}$$

Taking the curl of Faraday's law, we expand the double cross product and then substitute Ampere's law to obtain a simple vector equation in $\mathbf{E}$ alone:

$$\begin{aligned}
\nabla\times(\nabla\times\mathbf{E}) &= \nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}\\
&= -\mu\frac{\partial}{\partial t}(\nabla\times\mathbf{H})\\
&= -\varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2}
\end{aligned}\tag{2}$$

Since $\nabla\cdot\mathbf{E}=0$ from Gauss's law when the charge density is zero, (2) reduces to the vector wave equation in $\mathbf{E}$:

$$\nabla^2\mathbf{E}=\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2},\qquad c^2=\frac{1}{\varepsilon\mu}\tag{3}$$

If we take the curl of Ampere's law and perform similar operations, we also obtain the vector wave equation in $\mathbf{H}$:

$$\nabla^2\mathbf{H}=\frac{1}{c^2}\frac{\partial^2\mathbf{H}}{\partial t^2}\tag{4}$$

The solutions for **E** and **H** in (3) and (4) are not independent. If we solve for either **E** or **H**, the other field is obtained from (1). The vector wave equations in (3) and (4) are valid for any shaped waveguide. In particular, we limit ourselves in this text to waveguides whose cross-sectional shape is rectangular, as shown in Figure 8-27.

### 8-6-2 Transverse Magnetic (TM) Modes

We first consider TM modes where the magnetic field has $x$ and $y$ components but no $z$ component. It is simplest to solve (3) for the $z$ component of electric field and then obtain the other electric and magnetic field components in terms of $E_z$ directly from Maxwell's equations in (1).

We thus assume solutions of the form

$$E_z = \mathrm{Re}\,[\hat{E}_z(x, y)\, e^{j(\omega t - k_z z)}] \tag{5}$$

where an exponential $z$ dependence is assumed because the cross-sectional area of the waveguide is assumed to be uniform in $z$ so that none of the coefficients in (1) depends on $z$. Then substituting into (3) yields the Helmholtz equation:

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - \left(k_z^2 - \frac{\omega^2}{c^2}\right)\hat{E}_z = 0 \tag{6}$$

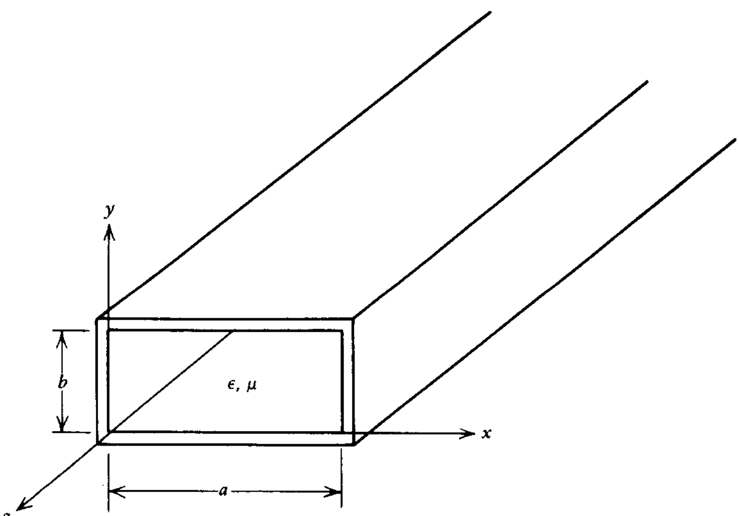



Figure 8-27 A lossless waveguide with rectangular cross section.

This equation can be solved by assuming the same product solution as used for solving Laplace's equation in Section 4-2-1, of the form

$$\hat{E}_z(x, y) = X(x)Y(y) \tag{7}$$

where $X(x)$ is only a function of the $x$ coordinate and $Y(y)$ is only a function of $y$. Substituting this assumed form of solution into (6) and dividing through by $X(x)Y(y)$ yields

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=k_z^2-\frac{\omega^2}{c^2} \tag{8}$$

When solving Laplace's equation in Section 4-2-1 the right-hand side was zero. Here the reasoning is the same. The first term on the left-hand side in (8) is only a function of $x$ while the second term is only a function of $y$. The only way a function of $x$ and a function of $y$ can add up to a constant for all $x$ and $y$ is if each function alone is a constant,

$$\begin{aligned}\frac{1}{X}\frac{d^2X}{dx^2}&=-k_x^2\\ \frac{1}{Y}\frac{d^2Y}{dy^2}&=-k_y^2\end{aligned} \tag{9}$$

where the separation constants must obey the relation

$$k_x^2+k_y^2+k_z^2=k^2=\omega^2/c^2 \tag{10}$$

When we solved Laplace's equation in Section 4-2-6, there was no time dependence so that $\omega=0$. Then we found that at least one of the wavenumbers was imaginary, yielding decaying solutions. For finite frequencies it is possible for all three wavenumbers to be real for pure propagation. The values of these wavenumbers will be determined by the dimensions of the waveguide through the boundary conditions.

The solutions to (9) are sinusoids so that the transverse dependence of the axial electric field $\hat{E}_z(x, y)$ is

$$\hat{E}_z(x, y)=(A_1\sin k_xx+A_2\cos k_xx)(B_1\sin k_yy+B_2\cos k_yy) \tag{11}$$

Because the rectangular waveguide in Figure 8-27 is composed of perfectly conducting walls, the tangential component of electric field at the walls is zero:

$$\begin{aligned}\hat{E}_z(x, y=0)&=0, \qquad \hat{E}_z(x=0, y)=0\\ \hat{E}_z(x, y=b)&=0, \qquad \hat{E}_z(x=a, y)=0\end{aligned} \tag{12}$$

These boundary conditions then require that $A_2$ and $B_2$ are zero so that (11) simplifies to

$$\hat{E}_z(x, y)=E_0\sin k_xx\sin k_yy \tag{13}$$

where $E_0$ is a field amplitude related to a source strength and the transverse wavenumbers must obey the equalities

$$\begin{aligned} k_x &= m\pi/a, \qquad m = 1, 2, 3, \ldots \\ k_y &= n\pi/b, \qquad n = 1, 2, 3, \ldots \end{aligned} \tag{14}$$

Note that if either $m$ or $n$ is zero in (13), the axial electric field is zero. The waveguide solutions are thus described as $\text{TM}_{mn}$ modes where both $m$ and $n$ are integers greater than zero.

The other electric field components are found from the $z$ component of Faraday's law, where $\mathbf{H}_z = 0$ and the charge-free Gauss's law in (1):

$$\begin{aligned} \frac{\partial E_y}{\partial x} &= \frac{\partial E_x}{\partial y} \\ \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} &= 0 \end{aligned} \tag{15}$$

By taking $\partial/\partial x$ of the top equation and $\partial/\partial y$ of the lower equation, we eliminate $E_x$ to obtain

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} = -\frac{\partial^2 E_z}{\partial y\, \partial z} \tag{16}$$

where the right-hand side is known from (13). The general solution for $E_y$ must be of the same form as (11), again requiring the tangential component of electric field to be zero at the waveguide walls,

$$\hat{E}_y(x = 0, y) = 0, \qquad \hat{E}_y(x = a, y) = 0 \tag{17}$$

so that the solution to (16) is

$$\hat{E}_y = -\frac{jk_y k_z E_0}{k_x^2 + k_y^2} \sin k_x x \cos k_y y \tag{18}$$

We then solve for $E_x$ using the upper equation in (15):

$$\hat{E}_x = -\frac{jk_x k_z E_0}{k_x^2 + k_y^2} \cos k_x x \sin k_y y \tag{19}$$

where we see that the boundary conditions

$$\hat{E}_x(x, y = 0) = 0, \qquad \hat{E}_x(x, y = b) = 0 \tag{20}$$

are satisfied.

The magnetic field is most easily found from Faraday's law

$$\hat{\mathbf{H}}(x, y) = -\frac{1}{j\omega\mu}\nabla\times\hat{\mathbf{E}}(x, y) \tag{21}$$

to yield

$$\begin{aligned}
\hat{H}_x &= -\frac{1}{j\omega\mu}\left(\frac{\partial\hat{E}_z}{\partial y} - \frac{\partial\hat{E}_y}{\partial z}\right)\\
&= -\frac{k_y k^2}{j\omega\mu(k_x^2 + k_y^2)}E_0 \sin k_x x \cos k_y y\\
&= \frac{j\omega\varepsilon k_y}{k_x^2 + k_y^2}E_0 \sin k_x x \cos k_y y\\
\hat{H}_y &= -\frac{1}{j\omega\mu}\left(\frac{\partial\hat{E}_x}{\partial z} - \frac{\partial\hat{E}_z}{\partial x}\right)\\
&= \frac{k_x k^2 E_0}{j\omega\mu(k_x^2 + k_y^2)}\cos k_x x \sin k_y y\\
&= -\frac{j\omega\varepsilon k_x}{k_x^2 + k_y^2}E_0 \cos k_x x \sin k_y y\\
\hat{H}_z &= 0
\end{aligned} \tag{22}$$

Note the boundary conditions of the normal component of **H** being zero at the waveguide walls are automatically satisfied:

$$\begin{aligned}
\hat{H}_y(x, y=0) &= 0, \qquad \hat{H}_y(x, y=b) = 0\\
\hat{H}_x(x=0, y) &= 0, \qquad \hat{H}_y(x=a, y) = 0
\end{aligned} \tag{23}$$

The surface charge distribution on the waveguide walls is found from the discontinuity of normal **D** fields:

$$\begin{aligned}
\hat{\sigma}_f(x=0, y) &= \varepsilon\hat{E}_x(x=0, y) = -\frac{jk_z k_x\varepsilon}{k_x^2 + k_y^2}E_0 \sin k_y y\\
\hat{\sigma}_f(x=a, y) &= -\varepsilon\hat{E}_x(x=a, y) = \frac{jk_z k_x\varepsilon}{k_x^2 + k_y^2}E_0 \cos m\pi \sin k_y y\\
\hat{\sigma}_f(x, y=0) &= \varepsilon\hat{E}_y(x, y=0) = -\frac{jk_z k_y\varepsilon}{k_x^2 + k_y^2}E_0 \sin k_x x\\
\hat{\sigma}_f(x, y=b) &= -\varepsilon\hat{E}_y(x, y=b) = \frac{jk_z k_y\varepsilon}{k_x^2 + k_y^2}E_0 \cos n\pi \sin k_x x
\end{aligned} \tag{24}$$

Similarly, the surface currents are found by the discontinuity in the tangential components of **H** to be purely $z$ directed:

$$\begin{aligned}
\hat{K}_z(x, y=0) &= -\hat{H}_x(x, y=0) = \frac{k_y k^2 E_0 \sin k_x x}{j\omega\mu(k_x^2 + k_y^2)} \\
\hat{K}_z(x, y=b) &= \hat{H}_x(x, y=b) = -\frac{k_y k^2 E_0}{j\omega\mu(k_x^2 + k_y^2)} \sin k_x x \cos n\pi \\
\hat{K}_z(x=0, y) &= \hat{H}_y(x=0, y) = \frac{k_x k^2 E_0}{j\omega\mu(k_x^2 + k_y^2)} \sin k_y y \\
\hat{K}_z(x=a, y) &= -\hat{H}_y(x=a, y) = -\frac{k_x k^2 E_0 \cos m\pi \sin k_y y}{j\omega\mu(k_x^2 + k_y^2)}
\end{aligned} \tag{25}$$

We see that if $m$ or $n$ are even, the surface charges and surface currents on opposite walls are of opposite sign, while if $m$ or $n$ are odd, they are of the same sign. This helps us in plotting the field lines for the various $\text{TM}_{mn}$ modes shown in Figure 8-28. The electric field is always normal and the magnetic field tangential to the waveguide walls. Where the surface charge is positive, the electric field points out of the wall, while it points in where the surface charge is negative. For higher order modes the field patterns shown in Figure 8-28 repeat within the waveguide.

Slots are often cut in waveguide walls to allow the insertion of a small sliding probe that measures the electric field. These slots must be placed at positions of zero surface current so that the field distributions of a particular mode are only negligibly disturbed. If a slot is cut along the $z$ direction on the $y=b$ surface at $x=a/2$, the surface current given in (25) is zero for TM modes if $\sin(k_x a/2)=0$, which is true for the $m=$ even modes.

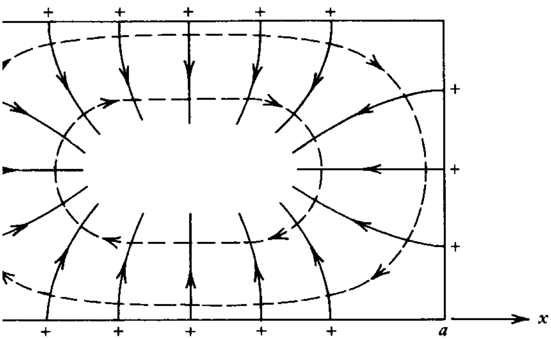
### 8-6-3 Transverse Electric (TE) Modes

When the electric field lies entirely in the $xy$ plane, it is most convenient to first solve (4) for $H_z$. Then as for TM modes we assume a solution of the form

$$H_z = \text{Re}\,[\hat{H}_z(x, y)\, e^{j(\omega t - k_z z)}] \tag{26}$$

which when substituted into (4) yields

$$\frac{\partial^2 \hat{H}_z}{\partial x^2} + \frac{\partial^2 \hat{H}_z}{\partial y^2} - \left(k_z^2 - \frac{\omega^2}{c^2}\right)\hat{H}_z = 0 \tag{27}$$




TM$_{11}$

Electric field (———)

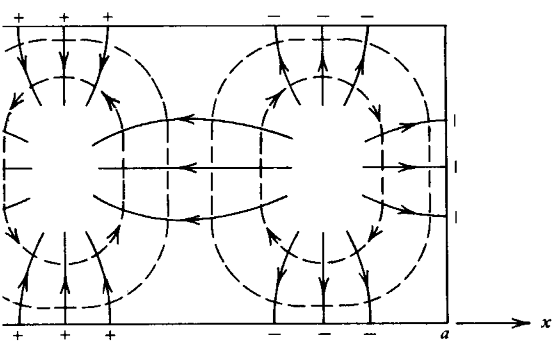
$$\hat{E}_x = \frac{-jk_x k_z E_0}{k_x^2 + k_y^2} \cos k_x x \sin k_y y$$

$$\hat{E}_y = \frac{-jk_y k_z E_0}{k_x^2 + k_y^2} \sin k_x x \cos k_y y$$

$$\hat{E}_z = E_0 \sin k_x x \sin k_y y$$

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{k_y}{k_x} \frac{\tan k_x x}{\tan k_y y}$$

$$\Rightarrow \frac{[\cos k_x x]^{(k_y/k_x)^2}}{\cos k_y y} = \text{const}$$




TM$_{21}$

Magnetic field (- - - -)

$$\hat{H}_x = \frac{j\omega\varepsilon k_y}{k_x^2 + k_y^2} E_0 \sin k_x x \cos k_y y$$

$$\hat{H}_y = \frac{-j\omega\varepsilon k_x}{k_x^2 + k_y^2} E_0 \cos k_x x \sin k_y y$$

$$\frac{dy}{dx} = \frac{H_y}{H_x} = \frac{-k_x \cot k_x x}{k_y \cot k_y y}$$

$$\Rightarrow \sin k_x x \sin k_y y = \text{const}$$

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}, \quad k_z = \left[\frac{\omega^2}{c^2} - \right.$$

8-28 The transverse electric and magnetic field lines for the TM$_{11}$ and TM$_{21}$ modes. The elect
z directed where the field lines converge.

Again this equation is solved by assuming a product solution and separating to yield a solution of the same form as (11):

$$\hat{H}_z(x, y) = (A_1 \sin k_x x + A_2 \cos k_x x)(B_1 \sin k_y y + B_2 \cos k_y y) \quad (28)$$

The boundary conditions of zero normal components of **H** at the waveguide walls require that

$$\begin{aligned} \hat{H}_x(x=0, y) &= 0, \qquad \hat{H}_x(x=a, y) = 0 \\ \hat{H}_y(x, y=0) &= 0, \qquad \hat{H}_y(x, y=b) = 0 \end{aligned} \quad (29)$$

Using identical operations as in (15)–(20) for the TM modes the magnetic field solutions are

$$\begin{aligned} \hat{H}_x &= \frac{jk_z k_x H_0}{k_x^2 + k_y^2} \sin k_x x \cos k_y y, \qquad k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b} \\ \hat{H}_y &= \frac{jk_z k_y H_0}{k_x^2 + k_y^2} \cos k_x x \sin k_y y \\ \hat{H}_z &= H_0 \cos k_x x \cos k_y y \end{aligned} \quad (30)$$

The electric field is then most easily obtained from Ampere's law in (1),

$$\hat{\mathbf{E}} = \frac{1}{j\omega\varepsilon} \nabla \times \hat{\mathbf{H}} \quad (31)$$

to yield

$$\begin{aligned} \hat{E}_x &= \frac{1}{j\omega\varepsilon}\left(\frac{\partial}{\partial y}\hat{H}_z - \frac{\partial}{\partial z}\hat{H}_y\right) \\ &= -\frac{k_y k^2 H_0}{j\omega\varepsilon(k_x^2 + k_y^2)} \cos k_x x \sin k_y y \\ &= \frac{j\omega\mu k_y}{k_x^2 + k_y^2} H_0 \cos k_x x \sin k_y y \\ \hat{E}_y &= \frac{1}{j\omega\varepsilon}\left(\frac{\partial \hat{H}_x}{\partial z} - \frac{\partial \hat{H}_z}{\partial x}\right) \\ &= \frac{k_x k^2 H_0}{j\omega\varepsilon(k_x^2 + k_y^2)} \sin k_x x \cos k_y y \\ &= -\frac{j\omega\mu k_x}{k_x^2 + k_y^2} H_0 \sin k_x x \cos k_y y \\ \hat{E}_z &= 0 \end{aligned} \quad (32)$$

We see in (32) that as required the tangential components of the electric field at the waveguide walls are zero. The surface charge densities on each of the walls are:

$$\begin{aligned}
\hat{\sigma}_f(x=0,y) &= \varepsilon\hat{E}_x(x=0,y) = \frac{-k_y k^2 H_0}{j\omega(k_x^2+k_y^2)}\sin k_y y\\
\hat{\sigma}_f(x=a,y) &= -\varepsilon\hat{E}_x(x=a,y) = \frac{k_y k^2 H_0}{j\omega(k_x^2+k_y^2)}\cos m\pi \sin k_y y\\
\hat{\sigma}_f(x,y=0) &= \varepsilon\hat{E}_y(x,y=0) = \frac{k_x k^2 H_0}{j\omega(k_x^2+k_y^2)}\sin k_x x\\
\hat{\sigma}_f(x,y=b) &= -\varepsilon\hat{E}_y(x,y=b) = -\frac{k_x k^2 H_0}{j\omega(k_x^2+k_y^2)}\cos n\pi \sin k_x x
\end{aligned} \tag{33}$$

For TE modes, the surface currents determined from the discontinuity of tangential **H** now flow in closed paths on the waveguide walls:

$$\begin{aligned}
\hat{\mathbf{K}}(x=0,y) &= \mathbf{i}_x\times\hat{\mathbf{H}}(x=0,y)\\
&= \mathbf{i}_z\hat{H}_y(x=0,y)-\mathbf{i}_y\hat{H}_z(x=0,y)\\
\hat{\mathbf{K}}(x=a,y) &= -\mathbf{i}_x\times\hat{\mathbf{H}}(x=a,y)\\
&= -\mathbf{i}_z\hat{H}_y(x=a,y)+\mathbf{i}_y\hat{H}_z(x=a,y)\\
\hat{\mathbf{K}}(x,y=0) &= \mathbf{i}_y\times\hat{\mathbf{H}}(x,y=0)\\
&= -\mathbf{i}_z\hat{H}_x(x,y=0)+\mathbf{i}_x\hat{H}_z(x,y=0)\\
\hat{\mathbf{K}}(x,y=b) &= -\mathbf{i}_y\times\hat{\mathbf{H}}(x,y=b)\\
&= \mathbf{i}_z\hat{H}_x(x,y=b)-\mathbf{i}_x\hat{H}_z(x,y=b)
\end{aligned} \tag{34}$$

Note that for TE modes either $n$ or $m$ (but not both) can be zero and still yield a nontrivial set of solutions. As shown in Figure 8-29, when $n$ is zero there is no variation in the fields in the $y$ direction and the electric field is purely $y$ directed while the magnetic field has no $y$ component. The $\mathrm{TE}_{11}$ and $\mathrm{TE}_{21}$ field patterns are representative of the higher order modes.

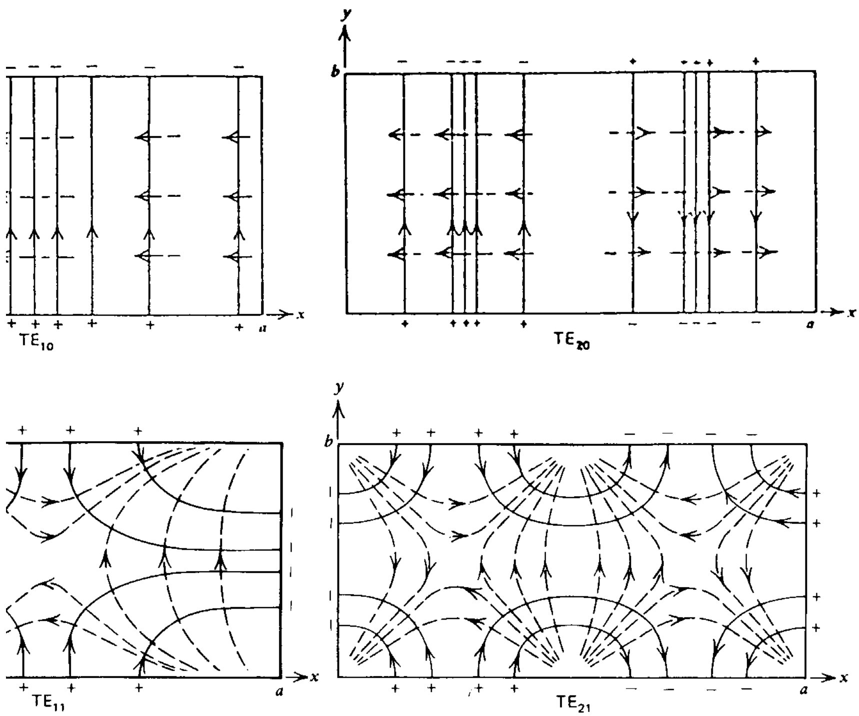
## 8-6-4 Cut-Off

The transverse wavenumbers are

$$k_x = \frac{m\pi}{a}, \qquad k_y = \frac{n\pi}{b} \tag{35}$$

so that the axial variation of the fields is obtained from (10) as

$$k_z = \left[\frac{\omega_2}{c^2} - k_x^2 - k_y^2\right]^{1/2} = \left[\frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{1/2} \tag{36}$$




Electric field (———

$$\hat{E}_x = \frac{j\omega\mu k_y}{k_x^2 + k_y^2} H_0 \text{ c}$$

$$\hat{E}_y = \frac{-j\omega\mu k_x}{k_x^2 + k_y^2} H_0 \text{ si}$$

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n}{b}$$

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{-k_x}{k_y}\,\frac{\tan}{\tan}$$

$$\Rightarrow \cos k_x x \cos k$$

Magnetic field (- - -

$$\hat{H}_x = \frac{jk_z k_x H_0}{k_x^2 + k_y^2} \sin$$

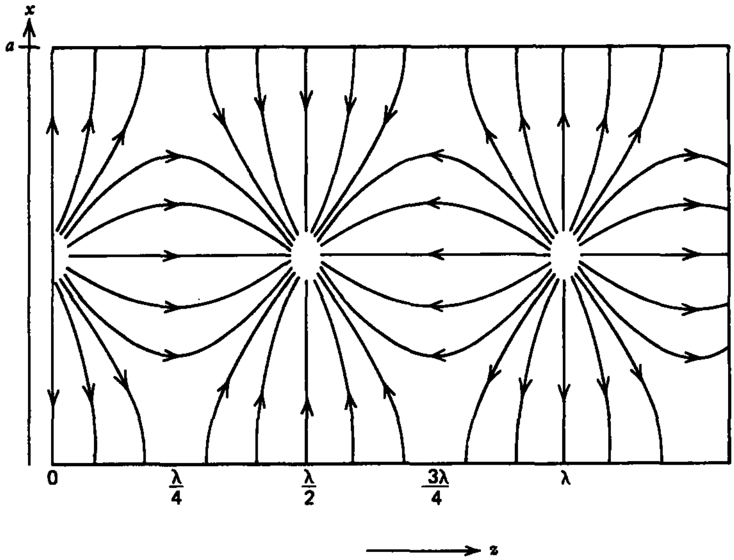
$$\hat{H}_y = \frac{jk_z k_y H_0}{k_x^2 + k_y^2} \cos$$

$$\hat{H}_z = H_0 \cos k_x x \text{ c}$$

$$\frac{dy}{dx} = \frac{H_y}{H_x} = \frac{k_y}{k_x}\,\frac{\cot}{\cot}$$

$$\Rightarrow \frac{[\sin k_x x]^{(k_y/k}}{\sin k_y y}$$

ıe transverse electric and magnetic field lines for various TE modes. The magnetic field is purely z
TE$_{10}$ mode is called the dominant mode since it has the lowest cut-off frequency. (*b*) Surface current




(b)

Figure 8-29

Thus, although $k_x$ and $k_y$ are real, $k_z$ can be either pure real or pure imaginary. A real value of $k_z$ represents power flow down the waveguide in the z direction. An imaginary value of $k_z$ means exponential decay with no time-average power flow. The transition from propagating waves ($k_z$ real) to evanescence ($k_z$ imaginary) occurs for $k_z = 0$. The frequency when $k_z$ is zero is called the cut-off frequency $\omega_c$:

$$\omega_c = c\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{1/2} \tag{37}$$

This frequency varies for each mode with the mode parameters $m$ and $n$. If we assume that $a$ is greater than $b$, the lowest cut-off frequency occurs for the $TE_{10}$ mode, which is called the dominant or fundamental mode. No modes can propagate below this lowest critical frequency $\omega_{c0}$:

$$\omega_{c0} = \frac{\pi c}{a} \Rightarrow f_{c0} = \frac{\omega_{c0}}{2\pi} = \frac{c}{2a}\ \text{Hz} \tag{38}$$

If an air-filled waveguide has $a = 1$ cm, then $f_{c0} = 1.5 \times 10^{10}$ Hz, while if $a = 10$ m, then $f_{c0} = 15$ MHz. This explains why we usually cannot hear the radio when driving through a tunnel. As the frequency is raised above $\omega_{c0}$, further modes can propagate.

The phase and group velocity of the waves are

$$v_p = \frac{\omega}{k_z} = \frac{\omega}{\left[\frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2\right]^{1/2}}$$
$$v_g = \frac{d\omega}{dk_z} = \frac{k_z c^2}{\omega} = \frac{c^2}{v_p} \Rightarrow v_g v_p = c^2 \tag{39}$$

At cut-off, $v_g = 0$ and $v_p = \infty$ with their product always a constant.

### 8-6-5 Waveguide Power Flow

The time-averaged power flow per unit area through the waveguide is found from the Poynting vector:

$$<S> = \tfrac{1}{2}\,\mathrm{Re}\,(\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*) \tag{40}$$

**(a) Power Flow for the TM Modes**

Substituting the field solutions found in Section 8-6-2 into (40) yields

$$\begin{aligned} <S> &= \tfrac{1}{2}\,\mathrm{Re}\,[(\hat{E}_x \mathbf{i}_x + \hat{E}_y \mathbf{i}_y + \hat{E}_z \mathbf{i}_z)\, e^{-jk_z z} \times (\hat{H}_x^* \mathbf{i}_x + \hat{H}_y^* \mathbf{i}_y)\, e^{+jk_z^* z}] \\ &= \tfrac{1}{2}\,\mathrm{Re}\,[(\hat{E}_x \hat{H}_y^* - \hat{E}_y \hat{H}_x^*)\mathbf{i}_z + \hat{E}_z(\hat{H}_x^* \mathbf{i}_y - \hat{H}_y^* \mathbf{i}_x)]\, e^{-j(k_z - k_z^*)z} \end{aligned} \tag{41}$$

where we remember that $k_z$ may be imaginary for a particular mode if the frequency is below cut-off. For propagating modes where $k_z$ is real so that $k_z = k_z^*$, there is no $z$ dependence in (41). For evanescent modes where $k_z$ is pure imaginary, the $z$ dependence of the Poynting vector is a real decaying exponential of the form $e^{-2|k_z|z}$. For either case we see from (13) and (22) that the product of $\hat{E}_z$ with $\hat{H}_x$ and $\hat{H}_y$ is pure imaginary so that the real parts of the $x$- and $y$-directed time average power flow are zero in (41). Only the $z$-directed power flow can have a time average:

$$\begin{aligned} <S> = \frac{\omega\varepsilon |E_0|^2}{2(k_x^2 + k_y^2)}\,\mathrm{Re}\,[k_z\, e^{-j(k_z - k_z^*)z}(k_x^2 \cos^2 k_x x \sin^2 k_y y \\ + k_y^2 \sin^2 k_x x \cos^2 k_y y)]\mathbf{i}_z \end{aligned} \tag{42}$$

If $k_z$ is imaginary, we have that $<S> = 0$ while a real $k_z$ results in a nonzero time-average power flow. The total $z$-directed power flow is found by integrating (42) over the cross-sectional area of the waveguide:

$$\begin{aligned}<P> &= \int_{x=0}^{a}\int_{y=0}^{b} <S_z>\,dx\,dy\\ &= \frac{\omega\varepsilon k_z ab E_0^2}{8(k_x^2+k_y^2)}\end{aligned} \tag{43}$$

where it is assumed that $k_z$ is real, and we used the following identities:

$$\begin{aligned}\int_0^a \sin^2\frac{m\pi x}{a}\,dx &= \frac{a}{m\pi}\left(\frac{1}{2}\frac{m\pi x}{a}-\frac{1}{4}\sin\frac{2m\pi x}{a}\right)\Big|_0^a\\ &= \begin{cases} a/2, & m\neq 0\\ 0, & m=0\end{cases}\\ \int_0^a \cos^2\frac{m\pi x}{a}\,dx &= \frac{a}{m\pi}\left(\frac{1}{2}\frac{m\pi x}{a}+\frac{1}{4}\sin\frac{2m\pi x}{a}\right)\Big|_0^a\\ &= \begin{cases} a/2, & m\neq 0\\ a, & m=0\end{cases}\end{aligned} \tag{44}$$

For the TM modes, both $m$ and $n$ must be nonzero.

**(b) Power Flow for the TE Modes**

The same reasoning is used for the electromagnetic fields found in Section 8-6-3 substituted into (40):

$$\begin{aligned}<\mathbf{S}> &= \tfrac{1}{2}\operatorname{Re}\left[(\hat{E}_x\mathbf{i}_x+\hat{E}_y\mathbf{i}_y)\,e^{-jk_z z}\times(\hat{H}_x^*\mathbf{i}_x+\hat{H}_y^*\mathbf{i}_y+\hat{H}_z^*\mathbf{i}_z)\,e^{+jk_z^* z}\right]\\ &= \tfrac{1}{2}\operatorname{Re}\left[(\hat{E}_x\hat{H}_y^*-\hat{E}_y\hat{H}_x^*)\mathbf{i}_z-\hat{H}_z^*(\hat{E}_x\mathbf{i}_y-\hat{E}_y\mathbf{i}_x)\right]e^{-j(k_z-k_z^*)z}\end{aligned} \tag{45}$$

Similarly, again we have that the product of $H_z^*$ with $\hat{E}_x$ and $\hat{E}_y$ is pure imaginary so that there are no $x$- and $y$-directed time average power flows. The $z$-directed power flow reduces to

$$\begin{aligned}<S_z> &= \frac{1}{2}\frac{\omega\mu H_0^2}{(k_x^2+k_y^2)}(k_y^2\cos^2 k_x x\sin^2 k_y y\\ &\quad +k_x^2\sin^2 k_x x\cos^2 k_y y)\operatorname{Re}(k_z\,e^{-j(k_z-k_z^*)z})\end{aligned} \tag{46}$$

Again we have nonzero $z$-directed time average power flow only if $k_z$ is real. Then the total $z$-directed power is

$$<P> = \int_{x=0}^{a}\int_{y=0}^{b} <S_z>\,dx\,dy = \begin{cases}\dfrac{\omega\mu k_z ab H_0^2}{8(k_x^2+k_y^2)}, & m, n\neq 0\\[2ex] \dfrac{\omega\mu k_z ab H_0^2}{4(k_x^2+k_y^2)}, & m \text{ or } n=0\end{cases} \tag{47}$$

where we again used the identities of (44). Note the factor of 2 differences in (47) for either the $TE_{10}$ or $TE_{01}$ modes. Both $m$ and $n$ cannot be zero as the $TE_{00}$ mode reduces to the trivial spatially constant uncoupled $z$-directed magnetic field.

### 8-6-6 Wall Losses

If the waveguide walls have a high but noninfinite Ohmic conductivity $\sigma_w$, we can calculate the spatial attenuation rate using the approximate perturbation approach described in Section 8-3-4*b*. The fields decay as $e^{-\alpha z}$, where

$$\alpha = \frac{1}{2}\frac{<P_{dL}>}{<P>} \tag{48}$$

where $<P_{dL}>$ is the time-average dissipated power per unit length and $<P>$ is the electromagnetic power flow in the lossless waveguide derived in Section 8-6-5 for each of the modes.

In particular, we calculate $\alpha$ for the $TE_{10}$ mode ($k_x = \pi/a$, $k_y = 0$). The waveguide fields are then

$$\begin{aligned}\hat{\mathbf{H}} &= H_0\left(\mathbf{i}_x \frac{jk_z a}{\pi}\sin\frac{\pi x}{a} + \cos\frac{\pi x}{a}\mathbf{i}_z\right)\\ \hat{\mathbf{E}} &= -\frac{j\omega\mu a}{\pi}H_0 \sin\frac{\pi x}{a}\mathbf{i}_y\end{aligned} \tag{49}$$

The surface current on each wall is found from (34) as

$$\begin{aligned}\hat{\mathbf{K}}(x=0, y) &= \hat{\mathbf{K}}(x=a, y) = -H_0\mathbf{i}_y\\ \hat{\mathbf{K}}(x, y=0) &= -\hat{\mathbf{K}}(x, y=b) = H_0\left(-\mathbf{i}_z\frac{jk_z a}{\pi}\sin\frac{\pi x}{a} + \mathbf{i}_x\cos\frac{\pi x}{a}\right)\end{aligned} \tag{50}$$

With lossy walls the electric field component $\mathbf{E}_w$ within the walls is in the same direction as the surface current proportional by a surface conductivity $\sigma_w\delta$, where $\delta$ is the skin depth as found in Section 8-3-4*b*. The time-average dissipated power density per unit area in the walls is then:

$$\begin{aligned}<P_d(x=0, y)> &= <P_d(x=a, y)>\\ &= \tfrac{1}{2}\operatorname{Re}(\hat{\mathbf{E}}_w \cdot \hat{\mathbf{K}}^*) = \frac{1}{2}\frac{H_0^2}{\sigma_w\delta}\\ <P_d(x, y=0)> &= <P_d(x, y=b)>\\ &= \frac{1}{2}\frac{H_0^2}{\sigma_w\delta}\left[\left(\frac{k_z a}{\pi}\right)^2\sin^2\frac{\pi x}{a} + \cos^2\frac{\pi x}{a}\right]\end{aligned} \tag{51}$$

The total time average dissipated power per unit length $<P_{dL}>$ required in (48) is obtained by integrating each of the terms in (51) along the waveguide walls:

$$
\begin{aligned}
<P_{dL}> &= \int_0^b [<P_d(x=0, y)> + <P_d(x=a, y)>]\, dy \\
&\quad + \int_0^a [<P_d(x, y=0)> + <P_d(x, y=b)>]\, dx \\
&= \frac{H_0^2 b}{\sigma_w \delta} + \frac{H_0^2}{\sigma_w \delta} \int_0^a \left[\left(\frac{k_z a}{\pi}\right)^2 \sin^2 \frac{\pi x}{a} + \cos^2 \frac{\pi x}{a}\right] dx \\
&= \frac{H_0^2}{\sigma_w \delta}\left\{b + \frac{a}{2}\left[\left(\frac{k_z a}{\pi}\right)^2 + 1\right]\right\} = \frac{H_0^2}{\sigma_w \delta}\left[b + \frac{a}{2}\left(\frac{\omega^2 a^2}{\pi^2 c^2}\right)\right]
\end{aligned} \tag{52}
$$

while the electromagnetic power above cut-off for the $TE_{10}$ mode is given by (47),

$$
<P> = \frac{\omega \mu k_z a b H_0^2}{4(\pi/a)^2} \tag{53}
$$

so that

$$
\alpha = \frac{1}{2} \frac{<P_{dL}>}{<P>} = \frac{2\left(\frac{\pi}{a}\right)^2 \left[b + \frac{a}{2}\left(\frac{\omega^2 a^2}{\pi^2 c^2}\right)\right]}{\omega \mu a b k_z \sigma_w \delta} \tag{54}
$$

where

$$
k_z = \left[\frac{\omega^2}{c^2} - \left(\frac{\pi}{a}\right)^2\right]^{1/2}; \qquad \frac{\omega}{c} > \frac{\pi}{a} \tag{55}
$$

## 8-7 DIELECTRIC WAVEGUIDE

We found in Section 7-10-6 for fiber optics that electromagnetic waves can also be guided by dielectric structures if the wave travels from the dielectric to free space at an angle of incidence greater than the critical angle. Waves propagating along the dielectric of thickness $2d$ in Figure 8-30 are still described by the vector wave equations derived in Section 8-6-1.

### 8-7-1 TM Solutions

We wish to find solutions where the fields are essentially confined within the dielectric. We neglect variations with $y$ so that for TM waves propagating in the $z$ direction the $z$ component of electric field is given in Section 8-6-2 as

$$
E_z(x, t) = \begin{cases} \operatorname{Re}\left[A_2\, e^{-\alpha(x-d)}\, e^{j(\omega t - k_z z)}\right], & x \geq d \\ \operatorname{Re}\left[(A_1 \sin k_x x + B_1 \cos k_x x)\, e^{j(\omega t - k_z z)}\right], & |x| \leq d \\ \operatorname{Re}\left[A_3\, e^{\alpha(x+d)}\, e^{j(\omega t - k_z z)}\right], & x \leq -d \end{cases} \tag{1}
$$

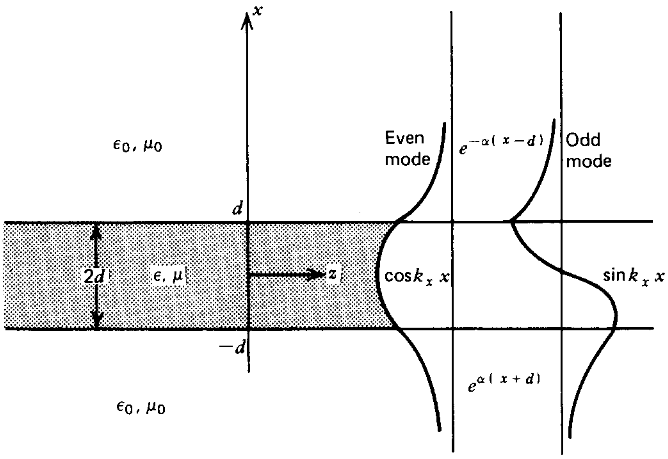



Figure 8-30 TE and TM modes can also propagate along dielectric structures. The fields can be essentially confined to the dielectric over a frequency range if the speed of the wave in the dielectric is less than that outside. It is convenient to separate the solutions into even and odd modes.

where we choose to write the solution outside the dielectric in the decaying wave form so that the fields are predominantly localized around the dielectric.

The wavenumbers and decay rate obey the relations

$$\begin{aligned} k_x^2 + k_z^2 &= \omega^2 \varepsilon\mu \\ -\alpha^2 + k_z^2 &= \omega^2 \varepsilon_0 \mu_0 \end{aligned} \tag{2}$$

The $z$ component of the wavenumber must be the same in all regions so that the boundary conditions can be met at each interface. For propagation in the dielectric and evanescence in free space, we must have that

$$\omega^2 \varepsilon_0 \mu_0 < k_z < \omega^2 \varepsilon\mu \tag{3}$$

All the other electric and magnetic field components can be found from (1) in the same fashion as for metal waveguides in Section 8-6-2. However, it is convenient to separately consider each of the solutions for $E_z$ within the dielectric.

**(a) Odd Solutions**

If $E_z$ in each half-plane above and below the centerline are oppositely directed, the field within the dielectric must vary solely as $\sin k_x x$:

$$\hat{E}_z = \begin{cases} A_2\, e^{-\alpha(x-d)}, & x \geq d \\ A_1 \sin k_x x, & |x| \leq d \\ A_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{4}$$

Then because in the absence of volume charge the electric field has no divergence,

$$\frac{\partial \hat{E}_x}{\partial x} - jk_z\hat{E}_z \Rightarrow \hat{E}_x = \begin{cases} -\dfrac{jk_z}{\alpha} A_2\, e^{-\alpha(x-d)}, & x \geq d \\ -\dfrac{jk_z}{k_x} A_1 \cos k_x x, & |x| \leq d \\ \dfrac{jk_z}{\alpha} A_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{5}$$

while from Faraday's law the magnetic field is

$$\hat{H}_y = -\frac{1}{j\omega\mu}\left(-jk_z\hat{E}_x - \frac{\partial \hat{E}_z}{\partial x}\right)$$

$$\Rightarrow \hat{H}_y = \begin{cases} -\dfrac{j\omega\varepsilon_0 A_2}{\alpha}\, e^{-\alpha(x-d)}, & x \geq d \\ -\dfrac{j\omega\varepsilon A_1}{k_x} \cos k_x x, & |x| \leq d \\ \dfrac{j\omega\varepsilon_0 A_3}{\alpha}\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{6}$$

At the boundaries where $x = \pm d$ the tangential electric and magnetic fields are continuous:

$$E_z(x = \pm d_-) = E_z(x = \pm d_+) \Rightarrow A_1 \sin k_x d = A_2$$

$$-A_1 \sin k_x d = A_3$$

$$H_y(x = \pm d_-) = H_y(x = \pm d_+) \Rightarrow \frac{-j\omega\varepsilon A_1}{k_x} \cos k_x d = \frac{-j\omega\varepsilon_0 A_2}{\alpha} \tag{7}$$

$$\frac{-j\omega\varepsilon A_1}{k_x} \cos k_x d = \frac{j\omega\varepsilon_0 A_3}{\alpha}$$

which when simultaneously solved yields

$$\left.\begin{aligned} \frac{A_2}{A_1} &= \sin k_x d = \frac{\varepsilon\alpha}{\varepsilon_0 k_x} \cos k_x d \\ \frac{A_3}{A_1} &= -\sin k_x d = -\frac{\varepsilon\alpha}{\varepsilon_0 k_x} \cos k_x d \end{aligned}\right\} \Rightarrow \alpha = \frac{\varepsilon_0}{\varepsilon} k_x \tan k_x d \tag{8}$$

The allowed values of $\alpha$ and $k_x$ are obtained by self-consistently solving (8) and (2), which in general requires a numerical method. The critical condition for a guided wave occurs when $\alpha = 0$, which requires that $k_x d = n\pi$ and $k_z^2 = \omega^2 \varepsilon_0 \mu_0$. The critical frequency is then obtained from (2) as

$$\omega^2 = \frac{k_x^2}{\varepsilon\mu - \varepsilon_0\mu_0} = \frac{(n\pi/d)^2}{\varepsilon\mu - \varepsilon_0\mu_0} \tag{9}$$

Note that this occurs for real frequencies only if $\varepsilon\mu > \varepsilon_0\mu_0$.

**(b) Even Solutions**

If $E_z$ is in the same direction above and below the dielectric, solutions are similarly

$$\hat{E}_z = \begin{cases} B_2\, e^{-\alpha(x-d)}, & x \geq d \\ B_1 \cos k_x x, & |x| \leq d \\ B_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{10}$$

$$\hat{E}_x = \begin{cases} -\dfrac{jk_z}{\alpha} B_2\, e^{-\alpha(x-d)}, & x \geq d \\ \dfrac{jk_z}{k_x} B_1 \sin k_x x, & |x| \leq d \\ \dfrac{jk_z}{\alpha} B_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{11}$$

$$\hat{H}_y = \begin{cases} -\dfrac{j\omega\varepsilon_0}{\alpha} B_2\, e^{-\alpha(x-d)}, & x \geq d \\ \dfrac{j\omega\varepsilon}{k_x} B_1 \sin k_x x, & |x| \leq d \\ \dfrac{j\omega\varepsilon_0}{\alpha} B_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{12}$$

Continuity of tangential electric and magnetic fields at $x = \pm d$ requires

$$B_1 \cos k_x d = B_2, \qquad B_1 \cos k_x d = B_3$$

$$\frac{j\omega\varepsilon}{k_x} B_1 \sin k_x d = -\frac{j\omega\varepsilon_0}{\alpha} B_2, \qquad -\frac{j\omega\varepsilon B_1}{k_x} \sin k_x d = \frac{j\omega\varepsilon_0 B_3}{\alpha} \tag{13}$$

or

$$\left.\begin{aligned} \frac{B_2}{B_1} &= \cos k_x d = -\frac{\varepsilon\alpha}{\varepsilon_0 k_x} \sin k_x d \\ \frac{B_3}{B_1} &= \cos k_x d = -\frac{\varepsilon\alpha}{\varepsilon_0 k_x} \sin k_x d \end{aligned}\right\} \Rightarrow \alpha = -\frac{\varepsilon_0 k_x}{\varepsilon} \cot k_x d \tag{14}$$

### 8-7-2 TE Solutions

The same procedure is performed for the TE solutions by first solving for $H_z$.

**(a) Odd Solutions**

$$\hat{H}_z = \begin{cases} A_2\, e^{-\alpha(x-d)}, & x \geq d \\ A_1 \sin k_x x, & |x| \leq d \\ A_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{15}$$

$$\hat{H}_x = \begin{cases} -\dfrac{jk_z}{\alpha} A_2\, e^{-\alpha(x-d)}, & x \geq d \\ -\dfrac{jk_z}{k_x} A_1 \cos k_x x, & |x| \leq d \\ \dfrac{jk_z}{\alpha} A_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{16}$$

$$\hat{E}_y = \begin{cases} \dfrac{j\omega\mu_0}{\alpha} A_2\, e^{-\alpha(x-d)}, & x \geq d \\ \dfrac{j\omega\mu}{k_x} A_1 \cos k_x x, & |x| \leq d \\ -\dfrac{j\omega\mu_0}{\alpha} A_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{17}$$

where continuity of tangential **E** and **H** across the boundaries requires

$$\alpha = \frac{\mu_0}{\mu} k_x \tan k_x d \tag{18}$$

**(b) Even Solutions**

$$\hat{H}_z = \begin{cases} B_2\, e^{-\alpha(x-d)}, & x \geq d \\ B_1 \cos k_x x, & |x| \leq d \\ B_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{19}$$

$$\hat{H}_x = \begin{cases} -\dfrac{jk_z}{\alpha} B_2\, e^{-\alpha(x-d)}, & x \geq d \\ \dfrac{jk_z}{k_x} B_1 \sin k_x x, & |x| \leq d \\ \dfrac{jk_z}{\alpha} B_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{20}$$

$$\hat{E}_y = \begin{cases} \dfrac{j\omega\mu_0}{\alpha} B_2\, e^{-\alpha(x-d)}, & x \geq d \\ -\dfrac{j\omega\mu}{k_x} B_1 \sin k_x x, & |x| \leq d \\ -\dfrac{j\omega\mu_0}{\alpha} B_3\, e^{\alpha(x+d)}, & x \leq -d \end{cases} \tag{21}$$

where $\alpha$ and $k_x$ are related as

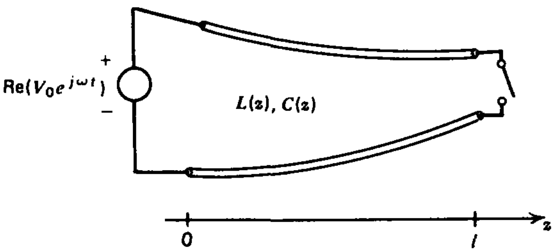
$$\alpha = -\frac{\mu_0}{\mu} k_x \cot k_x d \tag{22}$$

## PROBLEMS

Section 8-1

1. Find the inductance and capacitance per unit length and the characteristic impedance for the wire above plane and two wire line shown in Figure 8-3. (**Hint:** See Section 2-6-4c.)

2. The inductance and capacitance per unit length on a lossless transmission line is a weak function of $z$ as the distance between electrodes changes slowly with $z$.




(a) For this case write the transmission line equations as single equations in voltage and current.

(b) Consider an exponential line, where

$$L(z) = L_0\, e^{\alpha z}, \qquad C(z) = C_0\, e^{-\alpha z}$$

If the voltage and current vary sinusoidally with time as

$$v(z, t) = \mathrm{Re}\,[\hat{v}(z)\, e^{j\omega t}], \qquad i(z, t) = \mathrm{Re}\,[\hat{\imath}(z)\, e^{j\omega t}]$$

find the general form of solution for the spatial distributions of $\hat{v}(z)$ and $\hat{\imath}(z)$.

(c) The transmission line is excited by a voltage source $V_0 \cos \omega t$ at $z = 0$. What are the voltage and current distributions if the line is short or open circuited at $z = l$?

(d) For what range of frequency do the waves strictly decay with distance? What is the cut-off frequency for wave propagation?

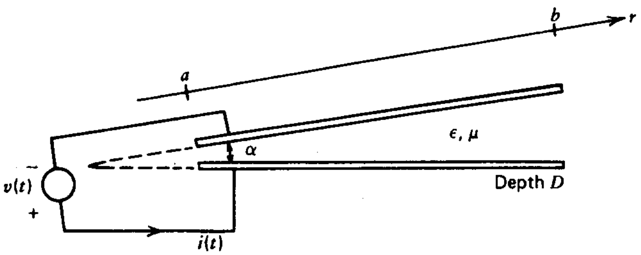
(e) What are the resonant frequencies of the short circuited line?

(f) What condition determines the resonant frequencies of the open circuited line.

3. Two conductors of length $l$ extending over the radial distance $a \leq r \leq b$ are at a constant angle $\alpha$ apart.

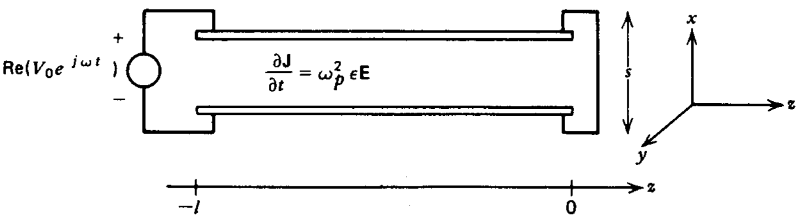
(a) What are the electric and magnetic fields in terms of the voltage and current?

(b) Find the inductance and capacitance per unit length. What is the characteristic impedance?




4. A parallel plate transmission line is filled with a conducting plasma with constitutive law

$$\frac{\partial \mathbf{J}}{\partial t}=\omega_p^2\varepsilon\mathbf{E}$$




(a) How are the electric and magnetic fields related?

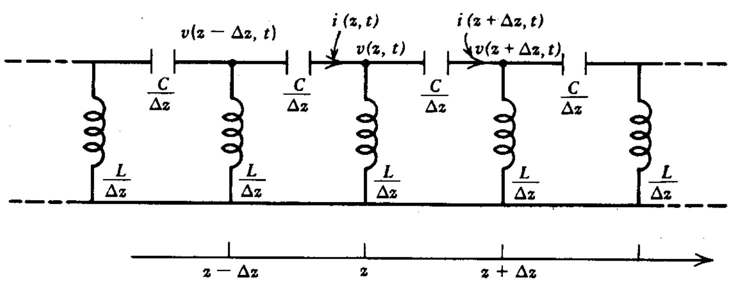
(b) What are the transmission line equations for the voltage and current?

(c) For sinusoidal signals of the form $e^{j(\omega t-kz)}$, how are $\omega$ and $k$ related? Over what frequency range do we have propagation or decay?

(d) The transmission line is short circuited at $z=0$ and excited by a voltage source $V_0\cos\omega t$ at $z=-l$. What are the voltage and current distributions?

(e) What are the resonant frequencies of the system?

5. An unusual type of distributed system is formed by series capacitors and shunt inductors.




(a) What are the governing partial differential equations relating the voltage and current?

(b) What is the dispersion relation between $\omega$ and $k$ for signals of the form $e^{j(\omega t-kz)}$?

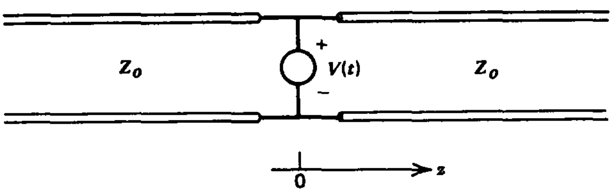
(c) What are the group and phase velocities of the waves? Why are such systems called "backward wave"?

(d) A voltage $V_0 \cos \omega t$ is applied at $z=-l$ with the $z=0$ end short circuited. What are the voltage and current distributions along the line?

(e) What are the resonant frequencies of the system?

Section 8-2

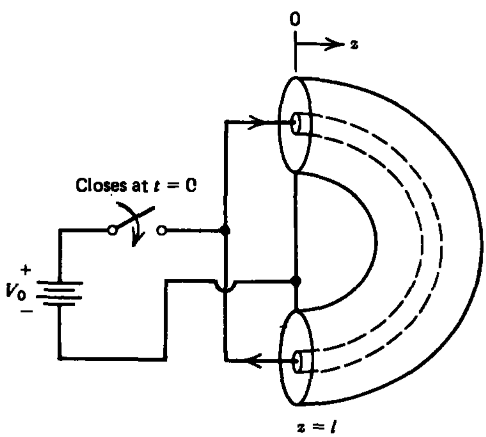
6. An infinitely long transmission line is excited at its center by a step voltage $V_0$ turned on at $t=0$. The line is initially at rest.




(a) Plot the voltage and current distributions at time $T$.

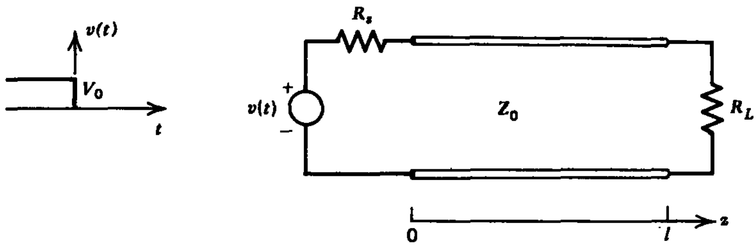
(b) At this time $T$ the voltage is set to zero. Plot the voltage and current everywhere at time $2T$.

7. A transmission line of length $l$ excited by a step voltage source has its ends connected together. Plot the voltage and current at $z=l/4$, $l/2$, and $3l/4$ as a function of time.



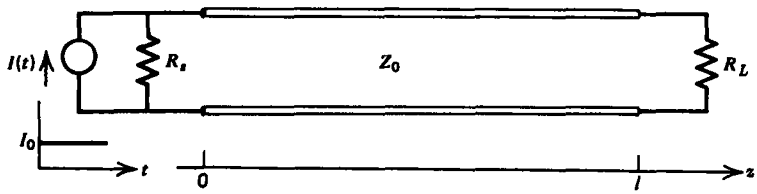

8. The dc steady state is reached for a transmission line loaded at $z=l$ with a resistor $R_L$ and excited at $z=0$ by a dc voltage $V_0$ applied through a source resistor $R_s$. The voltage source is suddenly set to zero at $t=0$.

(a) What is the initial voltage and current along the line?




(b) Find the voltage at the $z = l$ end as a function of time. (**Hint:** Use difference equations.)

9. A step current source turned on at $t = 0$ is connected to the $z = 0$ end of a transmission line in parallel with a source resistance $R_s$. A load resistor $R_L$ is connected at $z = l$.




(a) What is the load voltage and current as a function of time? (**Hint:** Use a Thevenin equivalent network at $z = 0$ with the results of Section 8-2-3.)

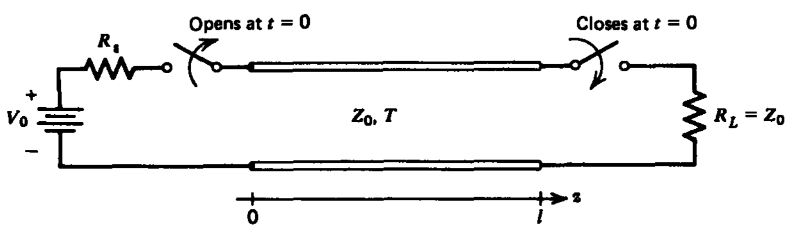
(b) With $R_s = \infty$ plot versus time the load voltage when $R_L = \infty$ and the load current when $R_L = 0$.

(c) If $R_s = \infty$ and $R_L = \infty$, solve for the load voltage in the quasi-static limit assuming the transmission line is a capacitor. Compare with (b).

(d) If $R_s$ is finite but $R_L = 0$, what is the time dependence of the load current?

(e) Repeat (d) in the quasi-static limit where the transmission line behaves as an inductor. When are the results of (d) and (e) approximately equal?

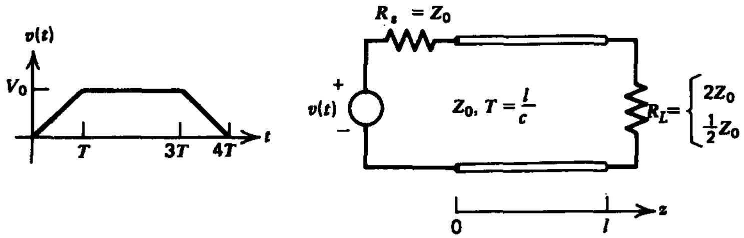
10. Switched transmission line systems with an initial dc voltage can be used to generate high voltage pulses of short time duration. The line shown is charged up to a dc voltage $V_0$ when at $t = 0$ the load switch is closed and the source switch is opened.



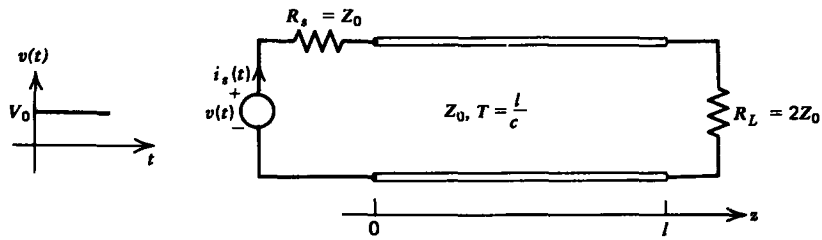
(a) What are the initial line voltage and current? What are $V_+$ and $V_-$?

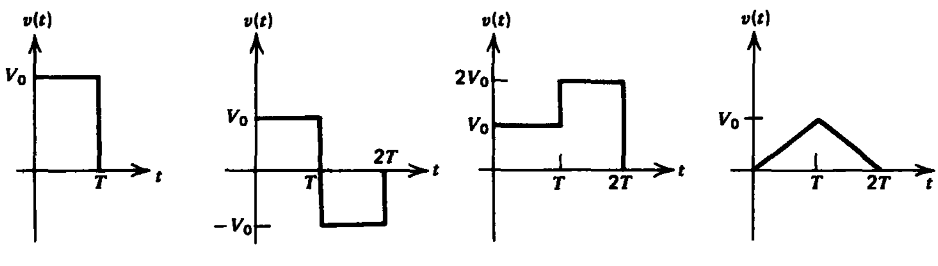
(b) Sketch the time dependence of the load voltage.

11. For the trapezoidal voltage excitation shown, plot versus time the current waveforms at $z=0$ and $z=l$ for $R_L=2Z_0$ and $R_L=\frac{1}{2}Z_0$.




12. A step voltage is applied to a loaded transmission line with $R_L=2Z_0$ through a matching source resistor.




(b) (c)

(a) Sketch the source current $i_s(t)$.

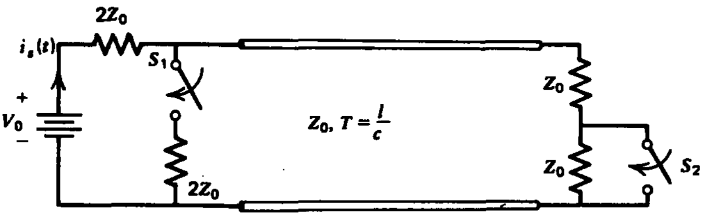
(b) Using superposition of delayed step voltages find the time dependence of $i_s(t)$ for the various pulse voltages shown.

(c) By integrating the appropriate solution of (b), find $i_s(t)$ if the applied voltage is the triangle wave shown.

13. A dc voltage has been applied for a long time to the transmission line circuit shown with switches $S_1$ and $S_2$ open

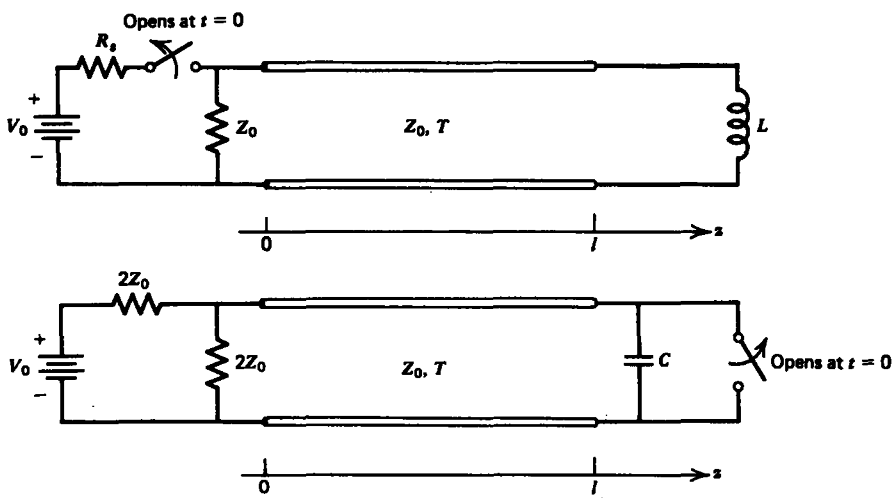
when at $t=0$:

(a) $S_2$ is suddenly closed with $S_1$ kept open;
(b) $S_1$ is suddenly closed with $S_2$ kept open;
(c) Both $S_1$ and $S_2$ are closed.



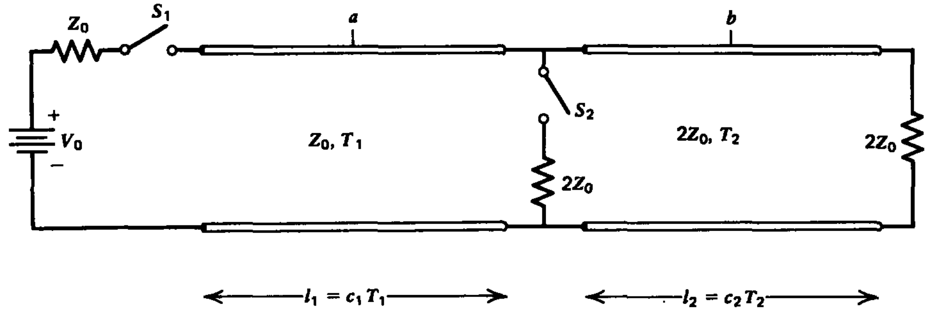

For each of these cases plot the source current $i_s(t)$ versus time.

14. For each of the transmission line circuits shown, the switch opens at $t=0$ after the dc voltage has been applied for a long time.



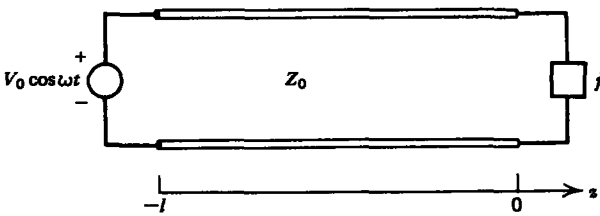

(a) What are the transmission line voltages and currents right before the switches open? What are $V_+$ and $V_-$ at $t=0$?
(b) Plot the voltage and current as a function of time at $z=l/2$.

15. A transmission line is connected to another transmission line with double the characteristic impedance.
(a) With switch $S_2$ open, switch $S_1$ is suddenly closed at $t=0$. Plot the voltage and current as a function of time halfway down each line at points $a$ and $b$.
(b) Repeat (a) if $S_2$ is closed.




Section 8-3

16. A transmission line is excited by a voltage source $V_0 \cos \omega t$ at $z=-l$. The transmission line is loaded with a purely reactive load with impedance $jX$ at $z=0$.

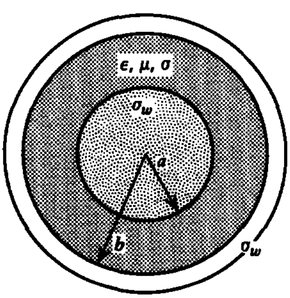



(a) Find the voltage and current distribution along the line.

(b) Find an expression for the resonant frequencies of the system if the load is capacitive or inductive. What is the solution if $|X| = Z_0$?

(c) Repeat (a) and (b) if the transmission line is excited by a current source $I_0 \cos \omega t$ at $z=-l$.

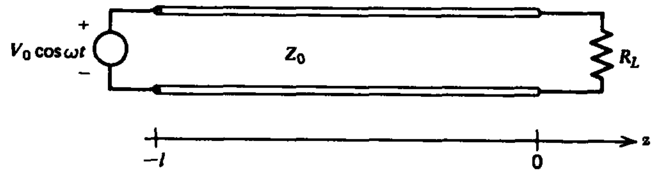
17. (a) Find the resistance and conductance per unit lengths for a coaxial cable whose dielectric has a small Ohmic conductivity $\sigma$ and walls have a large conductivity $\sigma_w$. (**Hint:** The skin depth $\delta$ is much smaller than the radii or thickness of either conductor.)




(b) What is the decay rate of the fields due to the losses?

(c) If the dielectric is lossless ($\sigma = 0$) with a fixed value of outer radius $b$, what value of inner radius $a$ will minimize the decay rate? (**Hint:** $1+1/3.6 \approx \ln 3.6$.)

18. A transmission line of length $l$ is loaded by a resistor $R_L$.



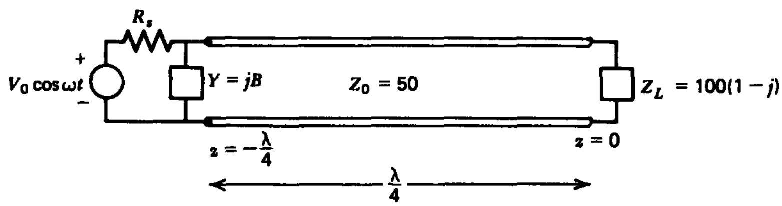

(a) Find the voltage and current distributions along the line.

(b) Reduce the solutions of (a) when the line is much shorter than a wavelength.

(c) Find the approximate equivalent circuits in the long wavelength limit ($kl \ll 1$) when $R_L$ is very small ($R_L \ll Z_0$) and when it is very large ($R_L \gg Z_0$).

Section 8-4

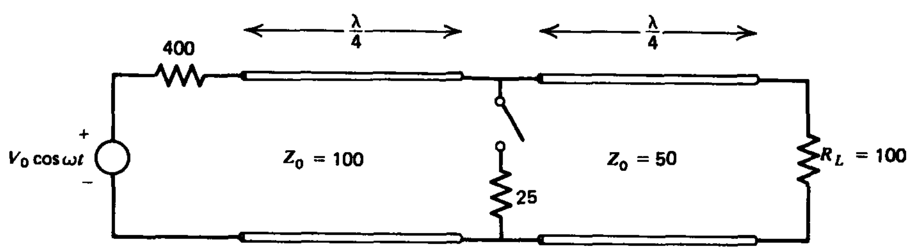
19. For the transmission line shown:




(a) Find the values of lumped reactive admittance $Y = jB$ and non-zero source resistance $R_s$ that maximizes the power delivered by the source. (**Hint:** Do not use the Smith chart.)

(b) What is the time-average power dissipated in the load?

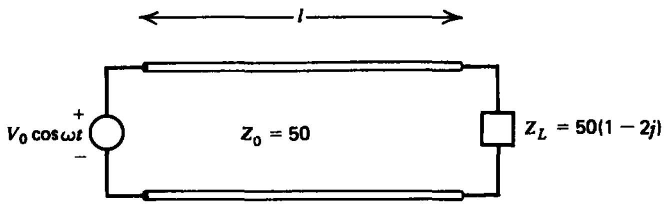
20. (a) Find the time-average power delivered by the source for the transmission line system shown when the switch is open or closed. (**Hint:** Do not use the Smith chart.)




(b) For each switch position, what is the time average power dissipated in the load resistor $R_L$?

(c) For each switch position what is the VSWR on each line?

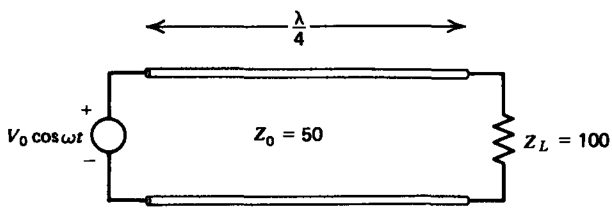
21. (a) Using the Smith chart find the source current delivered (magnitude and phase) for the transmission line system shown, for $l = \lambda/8$, $\lambda/4$, $3\lambda/8$, and $\lambda/2$.




(b) For each value of $l$, what are the time-average powers delivered by the source and dissipated in the load impedance $Z_L$?

(c) What is the VSWR?

22. (a) Without using the Smith chart find the voltage and current distributions for the transmission line system shown.




(b) What is the VSWR?

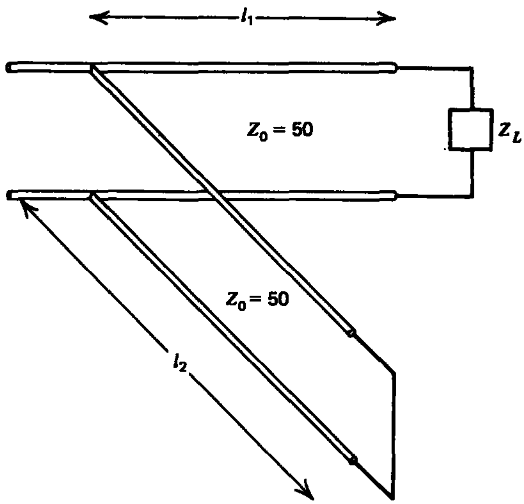
(c) At what positions are the voltages a maximum or a minimum? What is the voltage magnitude at these positions?

23. The VSWR on a 100-Ohm transmission line is 3. The distance between successive voltage minima is 50 cm while the distance from the load to the first minima is 20 cm. What are the reflection coefficient and load impedance?

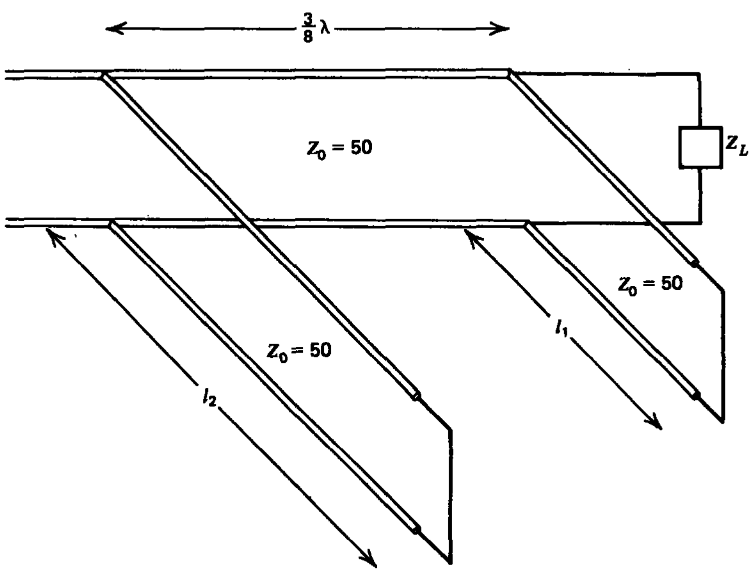
## Section 8-5

24. For each of the following load impedances in the single-stub tuning transmission line system shown, find all values of the length of the line $l_1$ and stub length $l_2$ necessary to match the load to the line.

(a) $Z_L = 100(1 - j)$  (c) $Z_L = 25(2 - j)$
(b) $Z_L = 50(1 + 2j)$  (d) $Z_L = 25(1 + 2j)$



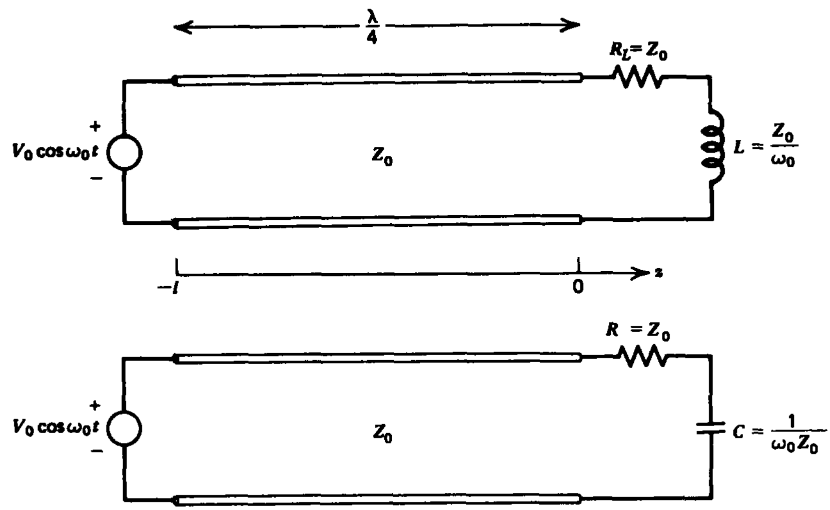

25. For each of the following load impedances in the double-stub tuning transmission line system shown, find stub lengths $l_1$ and $l_2$ to match the load to the line.




(a) $Z_L = 100(1-j)$ (c) $Z_L = 25(2-j)$
(b) $Z_L = 50(1+2j)$ (d) $Z_L = 25(1+2j)$

26. (a) Without using the Smith chart, find the input impedance $Z_{in}$ at $z = -l = -\lambda/4$ for each of the loads shown.

(b) What is the input current $i(z = -l, t)$ for each of the loads?




(c) The frequency of the source is doubled to $2\omega_0$. The line length $l$ and loads $L$ and $C$ remain unchanged. Repeat (a) and (b).

(d) The frequency of the source is halved to $\frac{1}{2}\omega_0$. Repeat (a) and (b).

Section 8-6

27. A rectangular metal waveguide is filled with a plasma with constitutive law

$$\frac{\partial \mathbf{J}_f}{\partial t} = \omega_p^2 \varepsilon \mathbf{E}$$

(a) Find the TE and TM solutions that satisfy the boundary conditions.

(b) What is the wavenumber $k_z$ along the axis? What is the cut-off frequency?

(c) What are the phase and group velocities of the waves?

(d) What is the total electromagnetic power flowing down the waveguide for each of the modes?

(e) If the walls have a large but finite conductivity, what is the spatial decay rate for $TE_{10}$ propagating waves?

28. (a) Find the power dissipated in the walls of a waveguide with large but finite conductivity $\sigma_w$ for the $TM_{mn}$ modes (**Hint:** Use Equation (25).)

(b) What is the spatial decay rate for propagating waves?

29. (a) Find the equations of the electric and magnetic field lines in the $xy$ plane for the TE and TM modes.

(b) Find the surface current field lines on each of the waveguide surfaces for the $TE_{mn}$ modes. **Hint:**

$$\int \tan x\,dx = -\ln \cos x$$

$$\int \cot x\,dx = \ln \sin x$$

(c) For all modes verify the conservation of charge relation on the $x=0$ surface:

$$\nabla_\Sigma \cdot \mathbf{K} + \frac{\partial \sigma_f}{\partial t} = 0$$

30. (a) Find the first ten lowest cut-off frequencies if $a=b=1$ cm in a free space waveguide.

(b) What are the necessary dimensions for a square free space waveguide to have a lowest cut-off frequency of $10^{10}$, $10^8$, $10^6$, $10^4$, or $10^2$ Hz?

31. A rectangular waveguide of height $b$ and width $a$ is short circuited by perfectly conducting planes at $z=0$ and $z=l$.

(a) Find the general form of the TE and TM electric and magnetic fields. (**Hint:** Remember to consider waves traveling in the $\pm z$ directions.)

(b) What are the natural frequencies of this resonator?

(c) If the walls have a large conductivity $\sigma_w$ find the total time-average power $<P_d>$ dissipated in the $TE_{101}$ mode.

(d) What is the total time-average electromagnetic energy $<W>$ stored in the resonator?

(e) Find the $Q$ of the resonator, defined as

$$Q = \frac{\omega_0 <W>}{<P_d>}$$

where $\omega_0$ is the resonant frequency.

Section 8.7

32. (a) Find the critical frequency where the spatial decay rate $\alpha$ is zero for all the dielectric modes considered.

(b) Find approximate values of $\alpha$, $k_x$, and $k_z$ for a very thin dielectric, where $k_x d \ll 1$.

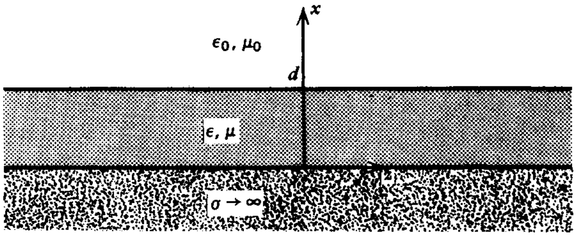
(c) For each of the solutions find the time-average power per unit length in each region.

(d) If the dielectric has a small Ohmic conductivity $\sigma$, what is the approximate attenuation rate of the fields.

33. A dielectric waveguide of thickness $d$ is placed upon a perfect conductor.

(a) Which modes can propagate along the dielectric?

(b) For each of these modes, what are the surface current and charges on the conductor?




(c) Verify the conservation of charge relation:

$$\nabla_\Sigma \cdot \mathbf{K} + \frac{\partial \sigma_f}{\partial t} = 0$$

(d) If the conductor has a large but noninfinite Ohmic conductivity $\sigma_w$, what is the approximate power per unit area dissipated?

(e) What is the approximate attenuation rate of the fields?

# chapter 9

*radiation*

In low-frequency electric circuits and along transmission lines, power is guided from a source to a load along highly conducting wires with the fields predominantly confined to the region around the wires. At very high frequencies these wires become antennas as this power can radiate away into space without the need of any guiding structure.

## 9-1 THE RETARDED POTENTIALS

### 9-1-1 Nonhomogeneous Wave Equations

Maxwell's equations in complete generality are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$

$$\nabla \cdot \mathbf{D} = \rho_f \tag{4}$$

In our development we will use the following vector identities

$$\nabla \times (\nabla V) = 0 \tag{5}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{6}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{7}$$

where $\mathbf{A}$ and $V$ can be any functions but in particular will be the magnetic vector potential and electric scalar potential, respectively.

Because in (3) the magnetic field has no divergence, the identity in (6) allows us to again define the vector potential $\mathbf{A}$ as we had for quasi-statics in Section 5-4:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{8}$$

so that Faraday's law in (1) can be rewritten as

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0 \tag{9}$$

Then (5) tells us that any curl-free vector can be written as the gradient of a scalar so that (9) becomes

$$\mathbf{E}+\frac{\partial \mathbf{A}}{\partial t}=-\nabla V \tag{10}$$

where we introduce the negative sign on the right-hand side so that $V$ becomes the electric potential in a static situation when $\mathbf{A}$ is independent of time. We solve (10) for the electric field and with (8) rewrite (2) for linear dielectric media ($\mathbf{D}=\varepsilon\mathbf{E}$, $\mathbf{B}=\mu\mathbf{H}$):

$$\nabla\times(\nabla\times\mathbf{A})=\mu\mathbf{J}_f+\frac{1}{c^2}\left[-\nabla\left(\frac{\partial V}{\partial t}\right)-\frac{\partial^2\mathbf{A}}{\partial t^2}\right],\qquad c^2=\frac{1}{\varepsilon\mu} \tag{11}$$

The vector identity of (7) allows us to reduce (11) to

$$\nabla^2\mathbf{A}-\nabla\left[\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial V}{\partial t}\right]-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu\mathbf{J}_f \tag{12}$$

Thus far, we have only specified the curl of $\mathbf{A}$ in (8). The Helmholtz theorem discussed in Section 5-4-1 told us that to uniquely specify the vector potential we must also specify the divergence of $\mathbf{A}$. This is called setting the gauge. Examining (12) we see that if we set

$$\nabla\cdot\mathbf{A}=-\frac{1}{c^2}\frac{\partial V}{\partial t} \tag{13}$$

the middle term on the left-hand side of (12) becomes zero so that the resulting relation between $\mathbf{A}$ and $\mathbf{J}_f$ is the nonhomogeneous vector wave equation:

$$\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu\mathbf{J}_f \tag{14}$$

The condition of (13) is called the Lorentz gauge. Note that for static conditions, $\nabla\cdot\mathbf{A}=0$, which is the value also picked in Section 5-4-2 for the magneto-quasi-static field. With (14) we can solve for $\mathbf{A}$ when the current distribution $\mathbf{J}_f$ is given and then use (13) to solve for $V$. The scalar potential can also be found directly by using (10) in Gauss's law of (4) as

$$\nabla^2 V+\frac{\partial}{\partial t}(\nabla\cdot\mathbf{A})=\frac{-\rho_f}{\varepsilon} \tag{15}$$

The second term can be put in terms of $V$ by using the Lorentz gauge condition of (13) to yield the scalar wave equation:

$$\nabla^2 V-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}=\frac{-\rho_f}{\varepsilon} \tag{16}$$

Note again that for static situations this relation reduces to Poisson's equation, the governing equation for the quasi-static electric potential.

### 9-1-2 Solutions to the Wave Equation

We see that the three scalar equations of (14) (one equation for each vector component) and that of (16) are in the same form. If we can thus find the general solution to any one of these equations, we know the general solution to all of them.

As we had earlier proceeded for quasi-static fields, we will find the solution to (16) for a point charge source. Then the solution for any charge distribution is obtained using superposition by integrating the solution for a point charge over all incremental charge elements.

In particular, consider a stationary point charge at $r=0$ that is an arbitrary function of time $Q(t)$. By symmetry, the resulting potential can only be a function of $r$ so that (16) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}=0, \qquad r>0 \tag{17}$$

where the right-hand side is zero because the charge density is zero everywhere except at $r=0$. By multiplying (17) through by $r$ and realizing that

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=\frac{\partial^2}{\partial r^2}(rV) \tag{18}$$

we rewrite (17) as a homogeneous wave equation in the variable $(rV)$:

$$\frac{\partial^2}{\partial r^2}(rV)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(rV)=0 \tag{19}$$

which we know from Section 7-3-2 has solutions

$$rV=f_+\left(t-\frac{r}{c}\right)+f_-\left(t+\frac{r}{c}\right) \tag{20}$$

We throw out the negatively traveling wave solution as there are no sources for $r>0$ so that all waves emanate radially outward from the point charge at $r=0$. The arbitrary function $f_+$ is evaluated by realizing that as $r\to 0$ there can be no propagation delay effects so that the potential should approach the quasi-static Coulomb potential of a point charge:

$$\lim_{r\to 0} V=\frac{Q(t)}{4\pi\varepsilon r}\Rightarrow f_+(t)=\frac{Q(t)}{4\pi\varepsilon} \tag{21}$$

The potential due to a point charge is then obtained from (20) and (21) replacing time $t$ with the retarded time $t-r/c$:

$$V(r,t)=\frac{Q(t-r/c)}{4\pi\varepsilon r} \tag{22}$$

The potential at time $t$ depends not on the present value of charge but on the charge value a propagation time $r/c$ earlier when the wave now received was launched.

The potential due to an arbitrary volume distribution of charge $\rho_f(t)$ is obtained by replacing $Q(t)$ with the differential charge element $\rho_f(t)\,dV$ and integrating over the volume of charge:

$$V(r,t)=\int_{\text{all charge}}\frac{\rho_f(t-r_{QP}/c)}{4\pi\varepsilon r_{QP}}\,dV \tag{23}$$

where $r_{QP}$ is the distance between the charge as a source at point $Q$ and the field point at $P$.

The vector potential in (14) is in the same direction as the current density $\mathbf{J}_f$. The solution for $\mathbf{A}$ can be directly obtained from (23) realizing that each component of $\mathbf{A}$ obeys the same equation as (16) if we replace $\rho_f/\varepsilon$ by $\mu\mathbf{J}_f$:

$$\mathbf{A}(r,t)=\int_{\text{all current}}\frac{\mu\mathbf{J}_f(t-r_{QP}/c)}{4\pi r_{QP}}\,dV \tag{24}$$

## 9-2 RADIATION FROM POINT DIPOLES

### 9-2-1 The Electric Dipole

The simplest building block for a transmitting antenna is that of a uniform current flowing along a conductor of incremental length $dl$ as shown in Figure 9-1. We assume that this current varies sinusoidally with time as

$$i(t)=\operatorname{Re}(\hat{I}e^{j\omega t}) \tag{1}$$

Because the current is discontinuous at the ends, charge must be deposited there being of opposite sign at each end $[q(t)=\operatorname{Re}(\hat{Q}e^{j\omega t})]$:

$$i(t)=\pm\frac{dq}{dt}\Rightarrow\hat{I}=\pm j\omega\hat{Q},\qquad z=\pm\frac{dl}{2} \tag{2}$$

This forms an electric dipole with moment

$$\mathbf{p}=q\,dl\,\mathbf{i}_z \tag{3}$$

If we can find the potentials and fields from this simple element, the solution for any current distribution is easily found by superposition.

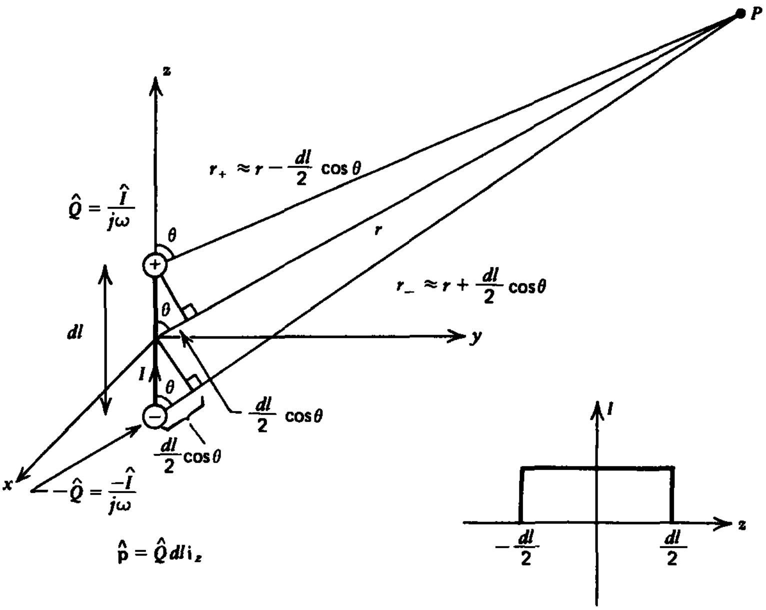



Figure 9-1 A point dipole antenna is composed of a very short uniformly distributed current-carrying wire. Because the current is discontinuous at the ends, equal magnitude but opposite polarity charges accumulate there forming an electric dipole.

By symmetry, the vector potential cannot depend on the angle $\phi$,

$$A_z = \mathrm{Re}\,[\hat{A}_z(r,\theta)\,e^{j\omega t}] \tag{4}$$

and must be in the same direction as the current:

$$A_z(r,t) = \mathrm{Re}\left[\int_{-dl/2}^{+dl/2} \frac{\mu \hat{I}\, e^{j[\omega(t - r_{QP}/c)]}}{4\pi r_{QP}}\,dz\right] \tag{5}$$

Because the dipole is of infinitesimal length, the distance from the dipole to any field point is just the spherical radial distance $r$ and is constant for all points on the short wire. Then the integral in (5) reduces to a pure multiplication to yield

$$\hat{A}_z = \frac{\mu \hat{I}\, dl}{4\pi r} e^{-jkr}, \qquad A_z(r,t) = \mathrm{Re}\,[\hat{A}_z(r)\,e^{j\omega t}] \tag{6}$$

where we again introduce the wavenumber $k = \omega/c$ and neglect writing the sinusoidal time dependence present in all field and source quantities. The spherical components of $\hat{A}_z$

are ($\mathbf{i}_z = \mathbf{i}_r \cos\theta - \mathbf{i}_\theta \sin\theta$):

$$\hat{A}_r = \hat{A}_z \cos\theta, \qquad \hat{A}_\theta = -\hat{A}_z \sin\theta, \qquad \hat{A}_\phi = 0 \tag{7}$$

Once the vector potential is known, the electric and magnetic fields are most easily found from

$$\begin{aligned} \hat{\mathbf{H}} &= \frac{1}{\mu}\nabla\times\hat{\mathbf{A}}, \qquad \mathbf{H}(r,t) = \operatorname{Re}[\hat{\mathbf{H}}(r,\theta)\,e^{j\omega t}] \\ \hat{\mathbf{E}} &= \frac{1}{j\omega\varepsilon}\nabla\times\hat{\mathbf{H}}, \qquad \mathbf{E}(r,t) = \operatorname{Re}[\hat{\mathbf{E}}(r,\theta)\,e^{j\omega t}] \end{aligned} \tag{8}$$

Before we find these fields, let's examine an alternate approach.

### 9-2-2 Alternate Derivation Using the Scalar Potential

It was easiest to find the vector potential for the point electric dipole because the integration in (5) reduced to a simple multiplication. The scalar potential is due solely to the opposite point charges at each end of the dipole,

$$\hat{V} = \frac{\hat{Q}}{4\pi\varepsilon}\left(\frac{e^{-jkr_+}}{r_+} - \frac{e^{-jkr_-}}{r_-}\right) \tag{9}$$

where $r_+$ and $r_-$ are the distances from the respective dipole charges to any field point, as shown in Figure 9-1. Just as we found for the quasi-static electric dipole in Section 3-1-1, we cannot let $r_+$ and $r_-$ equal $r$ as a zero potential would result. As we showed in Section 3-1-1, a first-order correction must be made, where

$$\begin{aligned} r_+ &\approx r - \frac{dl}{2}\cos\theta \\ r_- &\approx r + \frac{dl}{2}\cos\theta \end{aligned} \tag{10}$$

so that (9) becomes

$$\hat{V} \approx \frac{\hat{Q}}{4\pi\varepsilon r}\,e^{-jkr}\left(\frac{e^{jk(dl/2)\cos\theta}}{\left(1 - \frac{dl}{2r}\cos\theta\right)} - \frac{e^{-jk(dl/2)\cos\theta}}{\left(1 + \frac{dl}{2r}\cos\theta\right)}\right) \tag{11}$$

Because the dipole length $dl$ is assumed much smaller than the field distance $r$ and the wavelength, the phase factors in the exponentials are small so they and the $1/r$ dependence in the denominators can be expanded in a first-order Taylor series to result in:

$$\lim_{\substack{k\,dl \ll 1 \\ dl/r \ll 1}} \hat{V} \approx \frac{\hat{Q}}{4\pi\varepsilon r} e^{-jkr}\left[\left(1+j\frac{k\,dl}{2}\cos\theta\right)\left(1+\frac{dl}{2r}\cos\theta\right)\right.$$

$$\left. -\left(1-jk\frac{dl}{2}\cos\theta\right)\left(1-\frac{dl}{2r}\cos\theta\right)\right]$$

$$= \frac{\hat{Q}\,dl}{4\pi\varepsilon r^2} e^{-jkr}\cos\theta(1+jkr) \tag{12}$$

When the frequency becomes very low so that the wavenumber also becomes small, (12) reduces to the quasi-static electric dipole potential found in Section 3-1-1 with dipole moment $\hat{p} = \hat{Q}\,dl$. However, we see that the radiation correction terms in (12) dominate at higher frequencies (large $k$) far from the dipole ($kr \gg 1$) so that the potential only dies off as $1/r$ rather than the quasi-static $1/r^2$. Using the relationships $\hat{Q} = \hat{I}/j\omega$ and $c = 1/\sqrt{\varepsilon\mu}$, (12) could have been obtained immediately from (6) and (7) with the Lorentz gauge condition of Eq. (13) in Section 9-1-1:

$$\hat{V} = \frac{-c^2}{j\omega}\nabla\cdot\hat{A} = \frac{-c^2}{j\omega}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\hat{A}_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\hat{A}_\theta\sin\theta)\right)$$

$$= \frac{\mu\hat{I}\,dl\,c^2}{4\pi j\omega}\frac{(1+jkr)}{r^2}e^{-jkr}\cos\theta$$

$$= \frac{\hat{Q}\,dl}{4\pi\varepsilon r^2}(1+jkr)\,e^{-jkr}\cos\theta \tag{13}$$

### 9-2-3 The Electric and Magnetic Fields

Using (6), the fields are directly found from (8) as

$$\hat{\mathbf{H}} = \frac{1}{\mu}\nabla\times\hat{\mathbf{A}}$$

$$= \mathbf{i}_\phi\frac{1}{\mu r}\left(\frac{\partial}{\partial r}(r\hat{A}_\theta) - \frac{\partial\hat{A}_r}{\partial\theta}\right)$$

$$= -\mathbf{i}_\phi\frac{\hat{I}\,dl}{4\pi}k^2\sin\theta\left(\frac{1}{jkr}+\frac{1}{(jkr)^2}\right)e^{-jkr} \tag{14}$$

$$\begin{aligned}\hat{\mathbf{E}} &= \frac{1}{j\omega\varepsilon}\nabla\times\hat{\mathbf{H}}\\ &= \frac{1}{j\omega\varepsilon}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\hat{H}_\phi\sin\theta)\mathbf{i}_r - \frac{1}{r}\frac{\partial}{\partial r}(r\hat{H}_\phi)\mathbf{i}_\theta\right)\\ &= -\frac{\hat{I}\,dl\,k^2}{4\pi}\sqrt{\frac{\mu}{\varepsilon}}\left\{\mathbf{i}_r\left[2\cos\theta\left(\frac{1}{(jkr)^2}+\frac{1}{(jkr)^3}\right)\right]\right.\\ &\quad\left.+\mathbf{i}_\theta\left[\sin\theta\left(\frac{1}{jkr}+\frac{1}{(jkr)^2}+\frac{1}{(jkr)^3}\right)\right]\right\}e^{-jkr} \qquad (15)\end{aligned}$$

Note that even this simple source generates a fairly complicated electromagnetic field. The magnetic field in (14) points purely in the $\phi$ direction as expected by the right-hand rule for a $z$-directed current. The term that varies as $1/r^2$ is called the induction field or near field for it predominates at distances close to the dipole and exists even at zero frequency. The new term, which varies as $1/r$, is called the radiation field since it dominates at distances far from the dipole and will be shown to be responsible for time-average power flow away from the source. The near field term does not contribute to power flow but is due to the stored energy in the magnetic field and thus results in reactive power.

The $1/r^3$ terms in (15) are just the electric dipole field terms present even at zero frequency and so are often called the electrostatic solution. They predominate at distances close to the dipole and thus are the near fields. The electric field also has an intermediate field that varies as $1/r^2$, but more important is the radiation field term in the $\mathbf{i}_\theta$ component, which varies as $1/r$. At large distances ($kr \gg 1$) this term dominates.

In the far field limit ($kr \gg 1$), the electric and magnetic fields are related to each other in the same way as for plane waves:

$$\lim_{kr\gg 1}\hat{E}_\theta = \sqrt{\frac{\mu}{\varepsilon}}\hat{H}_\phi = \frac{\hat{E}_0}{jkr}\sin\theta\, e^{-jkr}, \qquad \hat{E}_0 = -\frac{\hat{I}\,dl\,k^2}{4\pi}\sqrt{\frac{\mu}{\varepsilon}} \qquad (16)$$

The electric and magnetic fields are perpendicular and their ratio is equal to the wave impedance $\eta = \sqrt{\mu/\varepsilon}$. This is because in the far field limit the spherical wavefronts approximate a plane.

### 9-2-4 Electric Field Lines

Outside the dipole the volume charge density is zero, which allows us to define an electric vector potential $\mathbf{C}$:

$$\nabla\cdot\mathbf{E} = 0 \Rightarrow \mathbf{E} = \nabla\times\mathbf{C} \qquad (17)$$

Because the electric field in (15) only has $r$ and $\theta$ components, $\mathbf{C}$ must only have a $\phi$ component, $C_\phi(r, \theta)$:

$$\mathbf{E}=\nabla\times\mathbf{C}=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta C_\phi)\mathbf{i}_r-\frac{1}{r}\frac{\partial}{\partial r}(rC_\phi)\mathbf{i}_\theta \tag{18}$$

We follow the same procedure developed in Section 4-4-3$b$, where the electric field lines are given by

$$\frac{dr}{r\,d\theta}=\frac{E_r}{E_\theta}=-\frac{\dfrac{\partial}{\partial\theta}(\sin\theta C_\phi)}{\sin\theta\dfrac{\partial}{\partial r}(rC_\phi)} \tag{19}$$

which can be rewritten as an exact differential,

$$\frac{\partial}{\partial r}(r\sin\theta C_\phi)\,dr+\frac{\partial}{\partial\theta}(r\sin\theta C_\phi)\,d\theta=0\Rightarrow d(r\sin\theta C_\phi)=0 \tag{20}$$

so that the field lines are just lines of constant stream-function $r\sin\theta C_\phi$. $C_\phi$ is found by equating each vector component in (18) to the solution in (15):

$$\begin{aligned}&\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\hat{C}_\phi)\\&\quad=\hat{E}_r=-\frac{\hat{I}\,dl\,k^2}{4\pi}\sqrt{\frac{\mu}{\varepsilon}}\left[2\cos\theta\left(\frac{1}{(jkr)^2}+\frac{1}{(jkr)^3}\right)\right]e^{-jkr}\\&-\frac{1}{r}\frac{\partial}{\partial r}(r\hat{C}_\phi)\\&\quad=\hat{E}_\theta=-\frac{\hat{I}\,dl\,k^2}{4\pi}\sqrt{\frac{\mu}{\varepsilon}}\left[\sin\theta\left(\frac{1}{(jkr)}+\frac{1}{(jkr)^2}+\frac{1}{(jkr)^3}\right)\right]e^{-jkr}\end{aligned} \tag{21}$$

which integrates to

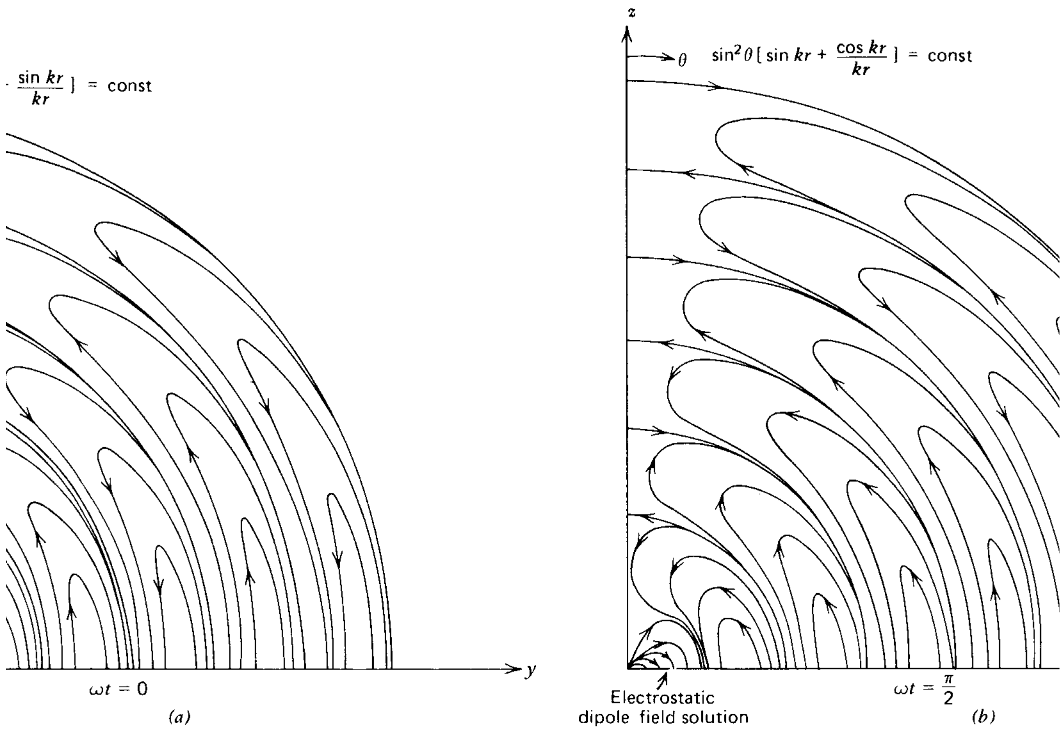
$$\hat{C}_\phi=\frac{\hat{I}\,dl}{4\pi}\sqrt{\frac{\mu}{\varepsilon}}\frac{\sin\theta}{r}\left(1-\frac{j}{(kr)}\right)e^{-jkr} \tag{22}$$

Then assuming $\hat{I}$ is real, the instantaneous value of $C_\phi$ is

$$\begin{aligned}C_\phi&=\operatorname{Re}(\hat{C}_\phi e^{j\omega t})\\&=\frac{\hat{I}\,dl}{4\pi}\sqrt{\frac{\mu}{\varepsilon}}\frac{\sin\theta}{r}\left(\cos(\omega t-kr)+\frac{\sin(\omega t-kr)}{kr}\right)\end{aligned} \tag{23}$$

so that, omitting the constant amplitude factor in (23), the field lines are

$$rC_\phi\sin\theta=\text{const}\Rightarrow\sin^2\theta\left(\cos(\omega t-kr)+\frac{\sin(\omega t-kr)}{kr}\right)=\text{const} \tag{24}$$




The electric field lines for a point electric dipole at $\omega t = 0$ and $\omega t = \pi/2$.

These field lines are plotted in Figure 9-2 at two values of time. We can check our result with the static field lines for a dipole given in Section 3-1-1. Remembering that $k=\omega/c$, at low frequencies,

$$\lim_{\omega\to 0}\begin{cases}\cos(\omega t-kr)\approx 1\\ \dfrac{\sin(\omega t-kr)}{kr}\approx\dfrac{(t-r/c)}{r/c}\approx\dfrac{t}{r/c}-1\end{cases} \tag{25}$$

so that, in the low-frequency limit at a fixed time, (24) approaches the result of Eq. (6) of Section 3–1–1:

$$\lim_{\omega\to 0}\sin^2\theta\left(\frac{ct}{r}\right)=\text{const} \tag{26}$$

Note that the field lines near the dipole are those of a static dipole field, as drawn in Figure 3-2. In the far field limit

$$\lim_{kr\gg 1}\sin^2\theta\cos(\omega t-kr)=\text{const} \tag{27}$$

the field lines repeat with period $\lambda=2\pi/k$.

### 9-2-5 Radiation Resistance

Using the electric and magnetic fields of Section 9-2-3, the time-average power density is

$$\begin{aligned}<S>&=\tfrac{1}{2}\operatorname{Re}(\hat{\mathbf{E}}\times\hat{\mathbf{H}}^*)\\ &=\operatorname{Re}\left\{\frac{|\hat{I}\,dl|^2\eta k^4}{2(4\pi)^2}\left[-\mathbf{i}_\theta\sin 2\theta\left(-\frac{1}{(jkr)^3}+\frac{1}{(jkr)^5}\right)\right]\right.\\ &\qquad\left.+\mathbf{i}_r\sin^2\theta\left(-\frac{1}{(jkr)^2}+\frac{1}{(jkr)^5}\right)\right\}\\ &=\tfrac{1}{2}|\hat{I}\,dl|^2\left(\frac{k}{4\pi}\right)^2\frac{\eta}{r^2}\sin^2\theta\,\mathbf{i}_r\\ &=\frac{1}{2}\frac{|\hat{E}_0|^2}{\eta}\frac{\sin^2\theta}{(kr)^2}\mathbf{i}_r\end{aligned} \tag{28}$$

where $\hat{E}_0$ is defined in (16).

Only the far fields contributed to the time-average power flow. The near and intermediate fields contributed only imaginary terms in (28) representing reactive power.

The power density varies with the angle $\theta$, being zero along the electric dipole's axis ($\theta=0,\pi$) and maximum at right angles to it ($\theta=\pi/2$), illustrated by the radiation power pattern in Fig. 9-3. The strength of the power density is proportional to the length of the vector from the origin to the

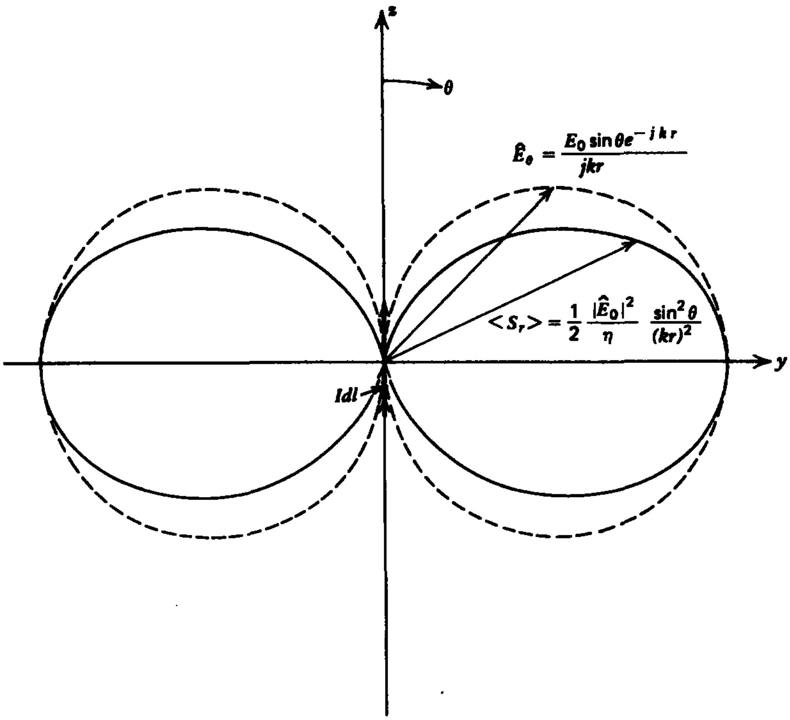



Figure 9-3 The strength of the electric field and power density due to a z-directed point dipole as a function of angle $\theta$ is proportional to the length of the vector from the origin to the radiation pattern.

radiation pattern. These directional properties are useful in beam steering, where the directions of power flow can be controlled.

The total time-average power radiated by the electric dipole is found by integrating the Poynting vector over a spherical surface at any radius $r$:

$$\begin{aligned}
\langle P\rangle &= \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \langle S_r\rangle r^2 \sin\theta\, d\theta\, d\phi \\
&= \tfrac{1}{2}|\hat{I}\,dl|^2\left(\frac{k}{4\pi}\right)^2 \eta 2\pi \int_{\theta=0}^{\pi} \sin^3\theta\, d\theta \\
&= \frac{|\hat{I}\,dl|^2}{16\pi}\eta k^2[-\tfrac{1}{3}\cos\theta(\sin^2\theta+2)]\Big|_0^{\pi} \\
&= \frac{|\hat{I}.dl|^2}{12\pi}\eta k^2
\end{aligned} \tag{29}$$

As far as the dipole is concerned, this radiated power is lost in the same way as if it were dissipated in a resistance $R$,

$$\langle P\rangle=\tfrac{1}{2}|\hat{I}|^2R \tag{30}$$

where this equivalent resistance is called the radiation resistance:

$$R=\frac{\eta}{6\pi}(k\,dl)^2=\frac{2\pi\eta}{3}\left(\frac{dl}{\lambda}\right)^2,\qquad k=\frac{2\pi}{\lambda} \tag{31}$$

In free space $\eta_0=\sqrt{\mu_0/\varepsilon_0}\approx 120\pi$, the radiation resistance is

$$R_0=80\pi^2\left(\frac{dl}{\lambda}\right)^2\quad\text{(free space)} \tag{32}$$

These results are only true for point dipoles, where $dl$ is much less than a wavelength ($dl/\lambda\ll 1$). This verifies the validity of the quasi-static approximation for geometries much smaller than a radiated wavelength, as the radiated power is then negligible.

If the current on a dipole is not constant but rather varies with $z$ over the length, the only term that varies with $z$ for the vector potential in (5) is $\hat{I}(z)$:

$$\hat{A}_z(r)=\mathrm{Re}\left[\int_{-dl/2}^{+dl/2}\frac{\mu\hat{I}(z)\,e^{-jkr_{QP}}}{4\pi r_{QP}}\,dz\right]\approx\mathrm{Re}\left[\frac{\mu\,e^{-jkr_{QP}}}{4\pi r_{QP}}\int_{-dl/2}^{+dl/2}\hat{I}(z)\,dz\right] \tag{33}$$

where, because the dipole is of infinitesimal length, the distance $r_{QP}$ from any point on the dipole to any field point far from the dipole is essentially $r$, independent of $z$. Then, all further results for the electric and magnetic fields are the same as in Section 9-2-3 if we replace the actual dipole length $dl$ by its effective length,

$$dl_{\text{eff}}=\frac{1}{\hat{I}_0}\int_{-dl/2}^{+dl/2}\hat{I}(z)\,dz \tag{34}$$

where $\hat{I}_0$ is the terminal current feeding the center of the dipole.

Generally the current is zero at the open circuited ends, as for the linear distribution shown in Figure 9-4,

$$\hat{I}(z)=\begin{cases}I_0(1-2z/dl), & 0\le z\le dl/2\\ I_0(1+2z/dl), & -dl/2\le z\le 0\end{cases} \tag{35}$$

so that the effective length is half the actual length:

$$dl_{\text{eff}}=\frac{1}{I_0}\int_{-dl/2}^{+dl/2}\hat{I}(z)\,dz=\frac{dl}{2} \tag{36}$$

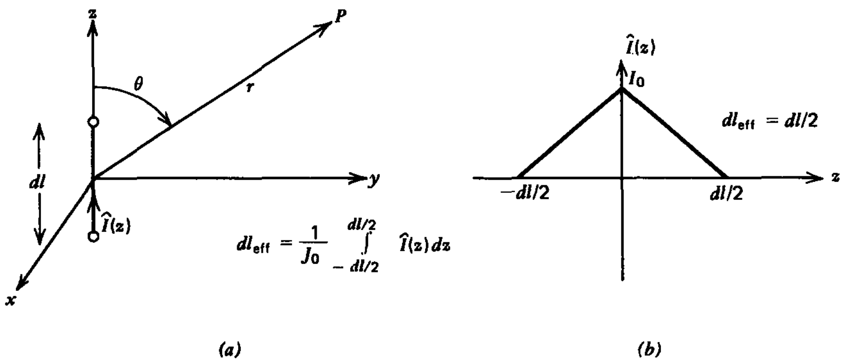



Figure 9-4 (*a*) If a point electric dipole has a nonuniform current distribution, the solutions are of the same form if we replace the actual dipole length $dl$ by an effective length $dl_{eff}$. (*b*) For a triangular current distribution the effective length is half the true length.

Because the fields are reduced by half, the radiation resistance is then reduced by $\frac{1}{4}$:

$$R = \frac{2\pi\eta}{3}\left(\frac{dl_{eff}}{\lambda}\right)^2 = 20\pi^2\sqrt{\frac{\mu_r}{\varepsilon_r}}\left(\frac{dl}{\lambda}\right)^2 \tag{37}$$

In free space the relative permeability $\mu_r$ and relative permittivity $\varepsilon_r$ are unity.

Note also that with a spatially dependent current distribution, a line charge distribution is found over the whole length of the dipole and not just on the ends:

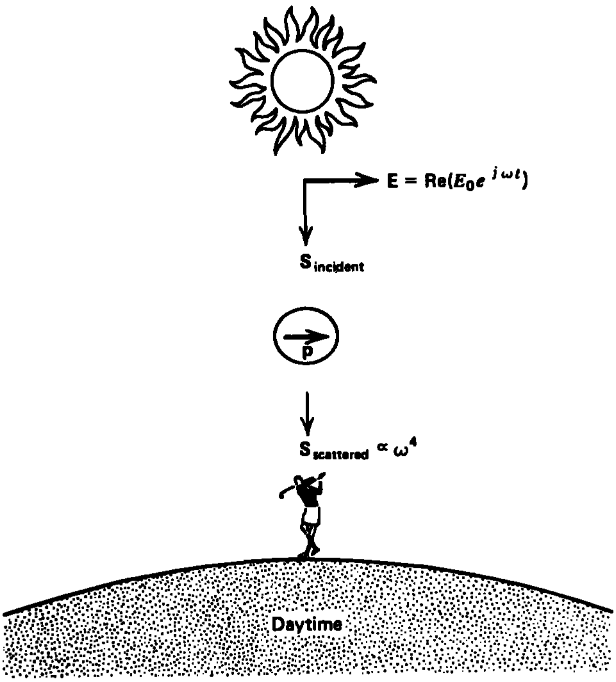
$$\hat{\lambda} = -\frac{1}{j\omega}\frac{d\hat{I}}{dz} \tag{38}$$

For the linear current distribution described by (35), we see that:

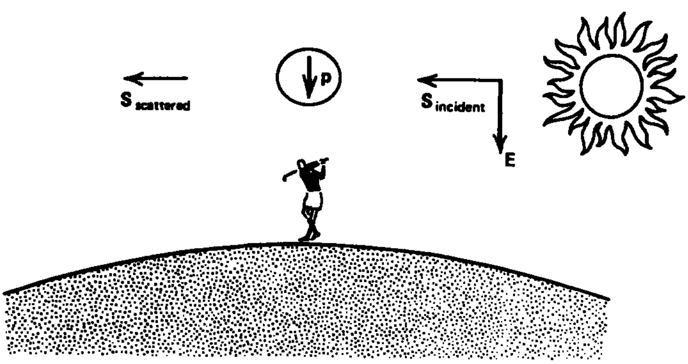
$$\hat{\lambda} = \pm\frac{2I_0}{j\omega\, dl}\begin{cases} 0 \le z \le dl/2 \\ -dl/2 \le z \le 0 \end{cases} \tag{39}$$

### 9-2-6 Rayleigh Scattering (or why is the sky blue?)

If a plane wave electric field Re $[E_0\, e^{j\omega t}\mathbf{i}_x]$ is incident upon an atom that is much smaller than the wavelength, the induced dipole moment also contributes to the resultant field, as illustrated in Figure 9-5. The scattered power is perpendicular to the induced dipole moment. Using the dipole model developed in Section 3-1-4, where a negative spherical electron cloud of radius $R_0$ with total charge $-Q$ surrounds a fixed




*(a)*




*(b)*

Figure 9-5 An incident electric field polarizes dipoles that then re-radiate their energy primarily perpendicular to the polarizing electric field. The time-average scattered power increases with the fourth power of frequency so shorter wavelengths of light are scattered more than longer wavelengths. (*a*) During the daytime an earth observer sees more of the blue scattered light so the sky looks blue (short wavelengths). (*b*) Near sunset the light reaching the observer lacks blue so the sky appears reddish (long wavelength).

positive point nucleus, Newton's law for the charged cloud with mass $m$ is:

$$\frac{d^2x}{dt^2}+\omega_0^2 x=\mathrm{Re}\left(\frac{QE_0}{m}e^{j\omega t}\right), \qquad \omega_0^2=\frac{Q^2}{4\pi\varepsilon m R_0^3} \tag{40}$$

The resulting dipole moment is then

$$\hat{p}=Q\hat{x}=\frac{Q^2E_0/m}{\omega_0^2-\omega^2} \tag{41}$$

where we neglect damping effects. This dipole then re-radiates with solutions given in Sections 9-2-1–9-2-5 using the dipole moment of (41) ($\hat{I}\,dl \to j\omega\hat{p}$). The total time-average power radiated is then found from (29) as

$$<P>=\frac{\omega^4|\hat{p}|^2\eta}{12\pi c^2}=\frac{\omega^4\eta(Q^2E_0/m)^2}{12\pi c^2(\omega_0^2-\omega^2)^2} \tag{42}$$

To approximately compute $\omega_0$, we use the approximate radius of the electron found in Section 3-8-2 by equating the energy stored in Einstein's relativistic formula relating mass to energy:

$$mc^2=\frac{3Q^2}{20\pi\varepsilon R_0}\Rightarrow R_0=\frac{3Q^2}{20\pi\varepsilon mc^2}\approx 1.69\times 10^{-15}\ \mathrm{m} \tag{43}$$

Then from (40)

$$\omega_0=\frac{\sqrt{5/3}\ 20\pi\varepsilon mc^3}{3Q^2}\approx 2.3\times 10^{23}\ \mathrm{radian/sec} \tag{44}$$

is much greater than light frequencies ($\omega \approx 10^{15}$) so that (42) becomes approximately

$$\lim_{\omega_0 \gg \omega} <P> \approx \frac{\eta}{12\pi}\left(\frac{Q^2E_0\omega^2}{mc\omega_0^2}\right)^2 \tag{45}$$

This result was originally derived by Rayleigh to explain the blueness of the sky. Since the scattered power is proportional to $\omega^4$, shorter wavelength light dominates. However, near sunset the light is scattered parallel to the earth rather than towards it. The blue light received by an observer at the earth is diminished so that the longer wavelengths dominate and the sky appears reddish.

### 9-2-7 Radiation from a Point Magnetic Dipole

A closed sinusoidally varying current loop of very small size flowing in the $z=0$ plane also generates radiating waves. Because the loop is closed, the current has no divergence so that there is no charge and the scalar potential is zero. The vector potential phasor amplitude is then

$$\hat{\mathbf{A}}(r)=\int \frac{\mu \hat{\mathbf{I}}\, e^{-jkr_{QP}}}{4\pi r_{QP}}\, dl \tag{46}$$

We assume the dipole to be much smaller than a wavelength, $k(r_{QP}-r)\ll 1$, so that the exponential factor in (46) can be linearized to

$$\lim_{k(r_{QP}-r)\ll 1} e^{-jkr_{QP}}=e^{-jkr}\, e^{-jk(r_{QP}-r)}\approx e^{-jkr}[1-jk(r_{QP}-r)] \tag{47}$$

Then (46) reduces to

$$\begin{aligned}\hat{\mathbf{A}}(r)&=\int \frac{\mu \hat{\mathbf{I}}}{4\pi}\, e^{-jkr}\left(\frac{1+jkr}{r_{QP}}-jk\right) dl\\ &=e^{-jkr}\int \frac{\mu \hat{\mathbf{I}}}{4\pi}\left(\frac{1+jkr}{r_{QP}}-jk\right) dl\\ &=\frac{\mu}{4\pi}\, e^{-jkr}\left((1+jkr)\int \frac{\hat{\mathbf{I}}\, dl}{r_{QP}}-jk\int \hat{\mathbf{I}}\, dl\right)\end{aligned} \tag{48}$$

where all terms that depend on $r$ can be taken outside the integrals because $r$ is independent of $dl$. The second integral is zero because the vector current has constant magnitude and flows in a closed loop so that its average direction integrated over the loop is zero. This is most easily seen with a rectangular loop where opposite sides of the loop contribute equal magnitude but opposite signs to the integral, which thus sums to zero. If the loop is circular with radius $a$,

$$\hat{I}\, dl=\hat{I}\mathbf{i}_\phi a\, d\phi \Rightarrow \int_0^{2\pi} \mathbf{i}_\phi\, d\phi=\int_0^{2\pi}(-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y)\, d\phi=0 \tag{49}$$

the integral is again zero as the average value of the unit vector $\mathbf{i}_\phi$ around the loop is zero.

The remaining integral is the same as for quasi-statics except that it is multiplied by the factor $(1+jkr)\, e^{-jkr}$. Using the results of Section 5-5-1, the quasi-static vector potential is also multiplied by this quantity:

$$\hat{\mathbf{A}}=\frac{\mu \hat{m}}{4\pi r^2}\sin\theta(1+jkr)\, e^{-jkr}\mathbf{i}_\phi, \qquad \hat{m}=\hat{I}\, dS \tag{50}$$

The electric and magnetic fields are then

$$\hat{\mathbf{H}}=\frac{1}{\mu}\nabla\times\hat{\mathbf{A}}=-\frac{\hat{m}}{4\pi}jk^3\,e^{-jkr}\left\{\mathbf{i}_r\left[2\cos\theta\left(\frac{1}{(jkr)^2}+\frac{1}{(jkr)^3}\right)\right]\right.$$
$$\left.+\mathbf{i}_\theta\left[\sin\theta\left(\frac{1}{jkr}+\frac{1}{(jkr)^2}+\frac{1}{(jkr)^3}\right)\right]\right\} \tag{51}$$
$$\hat{\mathbf{E}}=\frac{1}{j\omega\varepsilon}\nabla\times\hat{\mathbf{H}}=\frac{\hat{m}jk^3}{4\pi}\eta\,e^{-jkr}\sin\theta\left(\frac{1}{(jkr)}+\frac{1}{(jkr)^2}\right)\mathbf{i}_\phi$$

The magnetic dipole field solutions are the dual to those of the electric dipole where the electric and magnetic fields reverse roles if we replace the electric dipole moment with the magnetic dipole moment:

$$\frac{\mathbf{p}}{\varepsilon}=\frac{q\,\mathbf{dl}}{\varepsilon}=\frac{\mathbf{I}\,dl}{j\omega\varepsilon}\to\mathbf{m} \tag{52}$$

## 9-3 POINT DIPOLE ARRAYS

The power density for a point electric dipole varies with the broad angular distribution $\sin^2\theta$. Often it is desired that the power pattern be highly directive with certain angles carrying most of the power with negligible power density at other angles. It is also necessary that the directions for maximum power flow be controllable with no mechanical motion of the antenna. These requirements can be met by using more dipoles in a periodic array.

### 9-3-1 A Simple Two Element Array

To illustrate the basic principles of antenna arrays we consider the two element electric dipole array shown in Figure 9-6. We assume each element carries uniform currents $\hat{I}_1$ and $\hat{I}_2$ and has lengths $dl_1$ and $dl_2$, respectively. The elements are a distance $2a$ apart. The fields at any point $P$ are given by the superposition of fields due to each dipole alone. Since we are only interested in the far field radiation pattern where $\theta_1\approx\theta_2\approx\theta$, we use the solutions of Eq. (16) in Section 9-2-3 to write:

$$\hat{E}_\theta=\eta\hat{H}_\phi=\frac{\hat{E}_1\sin\theta\,e^{-jkr_1}}{jkr_1}+\frac{\hat{E}_2\sin\theta\,e^{-jkr_2}}{jkr_2} \tag{1}$$

where

$$\hat{E}_1=-\frac{\hat{I}_1\,dl_1\,k^2}{4\pi}\eta,\qquad \hat{E}_2=-\frac{\hat{I}_2\,dl_2\,k^2\eta}{4\pi}$$

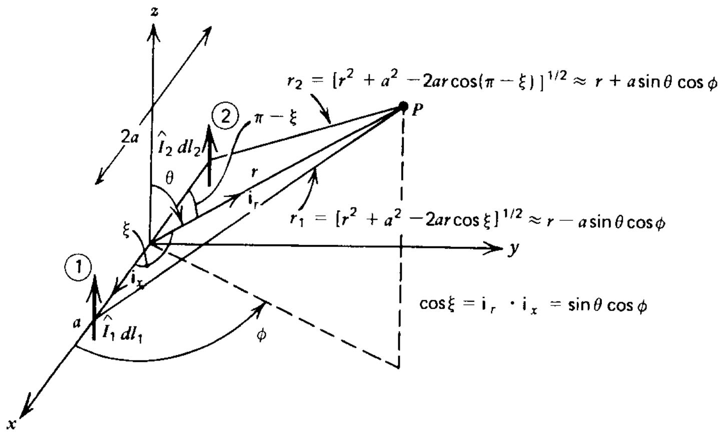



Figure 9-6 The field at any point $P$ due to two-point dipoles is just the sum of the fields due to each dipole alone taking into account the difference in distances to each dipole.

Remember, we can superpose the fields but we cannot superpose the power flows.

From the law of cosines the distances $r_1$ and $r_2$ are related as

$$\begin{aligned} r_2 &= [r^2 + a^2 - 2ar\cos(\pi - \xi)]^{1/2} = [r^2 + a^2 + 2ar\cos\xi]^{1/2} \\ r_1 &= [r^2 + a^2 - 2ar\cos\xi]^{1/2} \end{aligned} \qquad (2)$$

where $\xi$ is the angle between the unit radial vector $\mathbf{i}_r$ and the $x$ axis:

$$\cos\xi = \mathbf{i}_r \cdot \mathbf{i}_x = \sin\theta\cos\phi$$

Since we are interested in the far field pattern, we linearize (2) to

$$\lim_{r \gg a} \begin{cases} r_2 \approx r\left[1 + \dfrac{1}{2}\left(\dfrac{a^2}{r^2} + \dfrac{2a}{r}\sin\theta\cos\phi\right)\right] \approx r + a\sin\theta\cos\phi \\ \\ r_1 \approx r\left[1 + \dfrac{1}{2}\left(\dfrac{a^2}{r^2} - \dfrac{2a}{r}\sin\theta\cos\phi\right)\right] \approx r - a\sin\theta\cos\phi \end{cases} \qquad (3)$$

In this far field limit, the correction terms have little effect in the denominators of (1) but can have significant effect in the exponential phase factors if $a$ is comparable to a wavelength so that $ka$ is near or greater than unity. In this spirit we include the first-order correction terms of (3) in the phase factors of (1), but not anywhere else, so that (1) is rewritten as

$$\hat{E}_\theta = \eta \hat{H}_\phi$$
$$= \underbrace{\frac{jk\eta}{4\pi r} \sin\theta\, e^{-jkr}}_{\text{element factor}} \underbrace{(\hat{I}_1\, dl_1\, e^{jka \sin\theta\cos\phi} + \hat{I}_2\, dl_2\, e^{-jka\sin\theta\cos\phi})}_{\text{array factor}} \quad (4)$$

The first factor is called the element factor because it is the radiation field per unit current element ($\hat{I}\, dl$) due to a single dipole at the origin. The second factor is called the array factor because it only depends on the geometry and excitations (magnitude and phase) of each dipole element in the array.

To examine (4) in greater detail, we assume the two dipoles are identical in length and that the currents have the same magnitude but can differ in phase $\chi$:

$$\begin{gathered} dl_1 = dl_2 \equiv dl \\ \hat{I}_1 = \hat{I}, \qquad \hat{I}_2 = \hat{I}\, e^{j\chi} \Rightarrow \hat{E}_1 = \hat{E}_0, \qquad \hat{E}_2 = \hat{E}_0\, e^{j\chi} \end{gathered} \quad (5)$$

so that (4) can be written as

$$\hat{E}_\theta = \eta \hat{H}_\phi = \frac{2\hat{E}_0}{jkr} e^{-jkr} \sin\theta\, e^{j\chi/2} \cos\left(ka \sin\theta\cos\phi - \frac{\chi}{2}\right) \quad (6)$$

Now the far fields also depend on $\phi$. In particular, we focus attention on the $\theta = \pi/2$ plane. Then the power flow,

$$\lim_{\theta=\pi/2} \langle S_r \rangle = \frac{1}{2\eta} |\hat{E}_\theta|^2 = \frac{2|\hat{E}_0|^2}{\eta (kr)^2} \cos^2\left(ka\cos\phi - \frac{\chi}{2}\right) \quad (7)$$

depends strongly on the dipole spacing $2a$ and current phase difference $\chi$.

### (a) Broadside Array

Consider the case where the currents are in phase ($\chi = 0$) but the dipole spacing is a half wavelength ($2a = \lambda/2$). Then, as illustrated by the radiation pattern in Figure 9-7*a*, the field strengths cancel along the $x$ axis while they add along the $y$ axis. This is because along the $y$ axis $r_1 = r_2$, so the fields due to each dipole add, while along the $x$ axis the distances differ by a half wavelength so that the dipole fields cancel. Wherever the array factor phase ($ka\cos\phi - \chi/2$) is an integer multiple of $\pi$, the power density is maximum, while wherever it is an odd integer multiple of $\pi/2$, the power density is zero. Because this radiation pattern is maximum in the direction perpendicular to the array, it is called a broadside pattern.

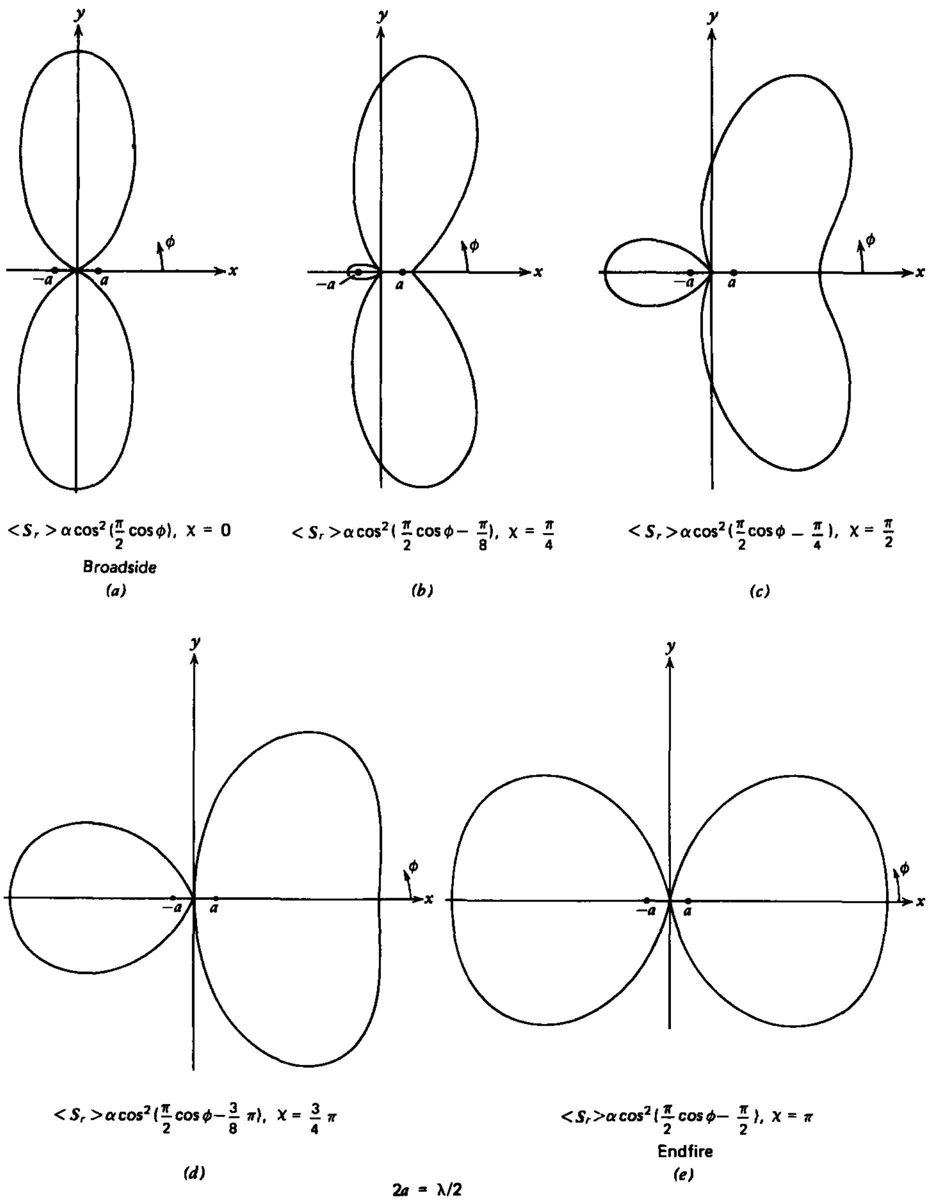



Figure 9-7 The power radiation pattern due to two-point dipoles depends strongly on the dipole spacing and current phases. With a half wavelength dipole spacing ($2a = \lambda/2$), the radiation pattern is drawn for various values of current phase difference in the $\theta = \pi/2$ plane. The broadside array in (*a*) with the currents in phase ($\chi = 0$) has the power lobe in the direction perpendicular to the array while the end-fire array in (*e*) has out-of-phase currents ($\chi = \pi$) with the power lobe in the direction along the array.

**(b) End-fire Array**

If, however, for the same half wavelength spacing the currents are out of phase ($\chi = \pi$), the fields add along the $x$ axis but cancel along the $y$ axis. Here, even though the path lengths along the $y$ axis are the same for each dipole, because the currents are out of phase the fields cancel. Along the $x$ axis the extra $\pi$ phase because of the half wavelength path difference is just canceled by the current phase difference of $\pi$ so that the fields due to each dipole add. The radiation pattern is called end-fire because the power is maximum in the direction along the array, as shown in Figure 9-7$e$.

**(c) Arbitrary Current Phase**

For arbitrary current phase angles and dipole spacings, a great variety of radiation patterns can be obtained, as illustrated by the sequences in Figures 9-7 and 9-8. More power lobes appear as the dipole spacing is increased.

## 9-3-2 An $N$ Dipole Array

If we have $(2N+1)$ equally spaced dipoles, as shown in Figure 9-9, the $n$th dipole's distance to the far field point is approximately,

$$\lim_{r \gg |na|} r_n \approx r - na \sin\theta \cos\phi \tag{8}$$

so that the array factor of (4) generalizes to

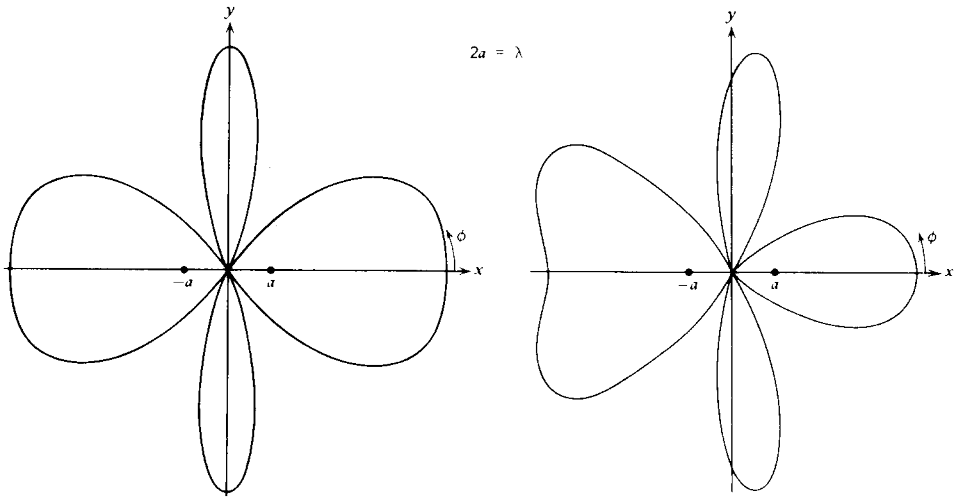
$$AF = \sum_{-N}^{+N} \hat{I}_n \, dl_n \, e^{jkna \sin\theta \cos\phi} \tag{9}$$

where for symmetry we assume that there are as many dipoles to the left (negative $n$) as to the right (positive $n$) of the $z$ axis, including one at the origin ($n = 0$). In the event that a dipole is not present at a given location, we simply let its current be zero. The array factor can be varied by changing the current magnitude or phase in the dipoles. For simplicity here, we assume that all dipoles have the same length $dl$, the same current magnitude $I_0$, and differ in phase from its neighbors by a constant angle $\chi_0$ so that

$$\hat{I}_n = I_0 \, e^{-jn\chi_0}, \qquad -N \leq n \leq N \tag{10}$$

and (9) becomes

$$AF = I_0 \, dl \sum_{-N}^{+N} e^{jn(ka \sin\theta \cos\phi - \chi_0)} \tag{11}$$

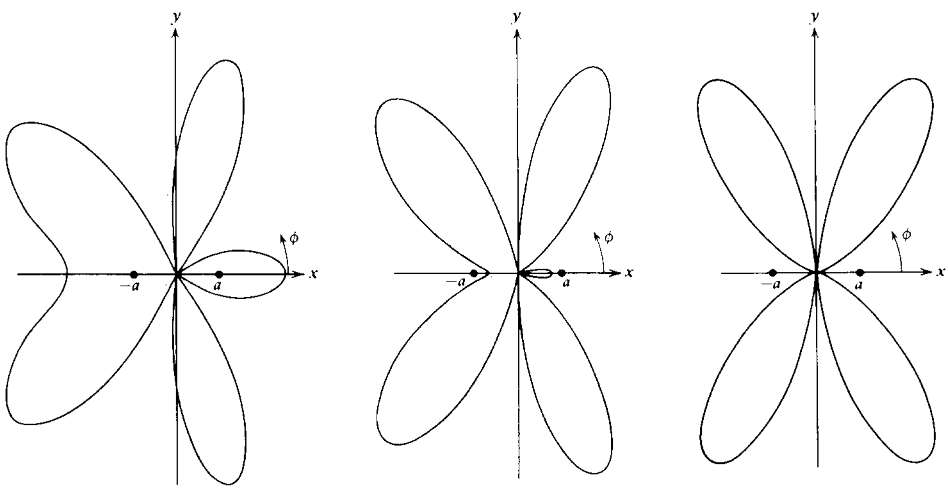



$\langle S_r \rangle \propto \cos^2(\pi \cos\phi)$, $\chi = 0$

(a)

$\langle S_r \rangle \propto \cos^2(\pi \cos\phi - \frac{\pi}{8})$, $\chi = \pi/4$

(b)




$\langle S_r \rangle \propto \cos^2(\pi \cos\phi - \frac{\pi}{4})$, $\chi = \frac{\pi}{2}$

(c)

$\langle S_r \rangle \propto \cos^2(\pi \cos\phi - \frac{3\pi}{8})$, $\chi = \frac{3}{4}\pi$

(d)

$\langle S_r \rangle \propto \cos^2(\pi \cos\phi - \frac{\pi}{2})$, $\chi = \pi$

(e)

Figure 9-8 With a full wavelength dipole spacing ($2a = \lambda$) there are four main power lobes.

Defining the parameter

$$\beta = e^{j(ka \sin\theta \cos\phi - \chi_0)} \tag{12}$$

the geometric series in (11) can be written as

$$S = \sum_{-N}^{+N} \beta^n = \beta^{-N} + \beta^{-N+1} + \cdots + \beta^{-2} + \beta^{-1} + 1 + \beta + \beta^2 + \cdots + \beta^{N-1} + \beta^N \tag{13}$$

If we multiply this series by $\beta$ and subtract from (13), we have

$$S(1-\beta) = \beta^{-N} - \beta^{N+1} \tag{14}$$

which allows us to write the series sum in closed form as

$$S = \frac{\beta^{-N} - \beta^{N+1}}{1-\beta} = \frac{\beta^{-(N+1/2)} - \beta^{(N+1/2)}}{\beta^{-1/2} - \beta^{1/2}} = \frac{\sin[(N+\frac{1}{2})(ka \sin\theta \cos\phi - \chi_0)]}{\sin[\frac{1}{2}(ka \sin\theta \cos\phi - \chi_0)]} \tag{15}$$

In particular, we again focus on the solution in the $\theta = \pi/2$ plane so that the array factor is

$$AF = I_0\, dl \frac{\sin[(N+\frac{1}{2})(ka \cos\phi - \chi_0)]}{\sin[\frac{1}{2}(ka \cos\phi - \chi_0)]} \tag{16}$$

The radiation pattern is proportional to the square of the array factor. Maxima occur where

$$ka \cos\phi - \chi_0 = 2n\pi \qquad n = 0, 1, 2, \ldots \tag{17}$$

The principal maximum is for $n = 0$ as illustrated in Figure 9-10 for various values of $ka$ and $\chi_0$. The larger the number of dipoles $N$, the narrower the principal maximum with smaller amplitude side lobes. This allows for a highly directive beam at angle $\phi$ controlled by the incremental current phase angle $\chi_0$, so that $\cos\phi = \chi_0/ka$, which allows for electronic beam steering by simply changing $\chi_0$.

## 9-4 LONG DIPOLE ANTENNAS

The radiated power, proportional to $(dl/\lambda)^2$, is small for point dipole antennas where the dipole's length $dl$ is much less than the wavelength $\lambda$. More power can be radiated if the length of the antenna is increased. Then however, the fields due to each section of the antenna may not add constructively.

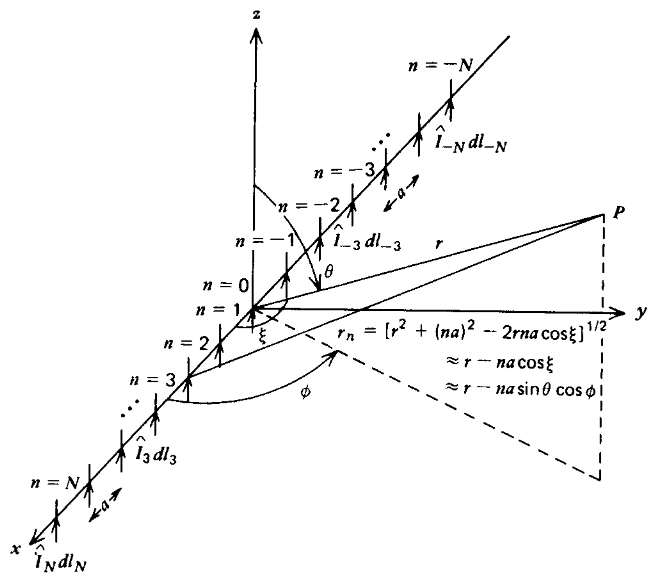



Figure 9-9 A linear point dipole array with $2N+1$ equally spaced dipoles.

## 9-4-1 Far Field Solution

Consider the long dipole antenna in Figure 9-11 carrying a current $\hat{I}(z)$. For simplicity we restrict ourselves to the far field pattern where $r \gg L$. Then, as we found for dipole arrays, the differences in radial distance for each incremental current element of length $dz$ are only important in the exponential phase factors and not in the $1/r$ dependences.

From Section 9-2-3, the incremental current element at position $z$ generates a far electric field:

$$d\hat{E}_\theta = \eta\, d\hat{H}_\phi = \frac{jk\eta}{4\pi}\frac{\hat{I}(z)\,dz}{r}\sin\theta\, e^{-jk(r-z\cos\theta)} \tag{1}$$

where we again assume that in the far field the angle $\theta$ is the same for all incremental current elements.

The total far electric field due to the entire current distribution is obtained by integration over all current elements:

$$\hat{E}_\theta = \eta\hat{H}_\phi = \frac{jk\eta}{4\pi r}\sin\theta\, e^{-jkr}\int_{-L/2}^{+L/2}\hat{I}(z)\, e^{jkz\cos\theta}\,dz \tag{2}$$

If the current distribution is known, the integral in (2) can be directly evaluated. The practical problem is difficult because the current distribution along the antenna is determined by the near fields through the boundary conditions.

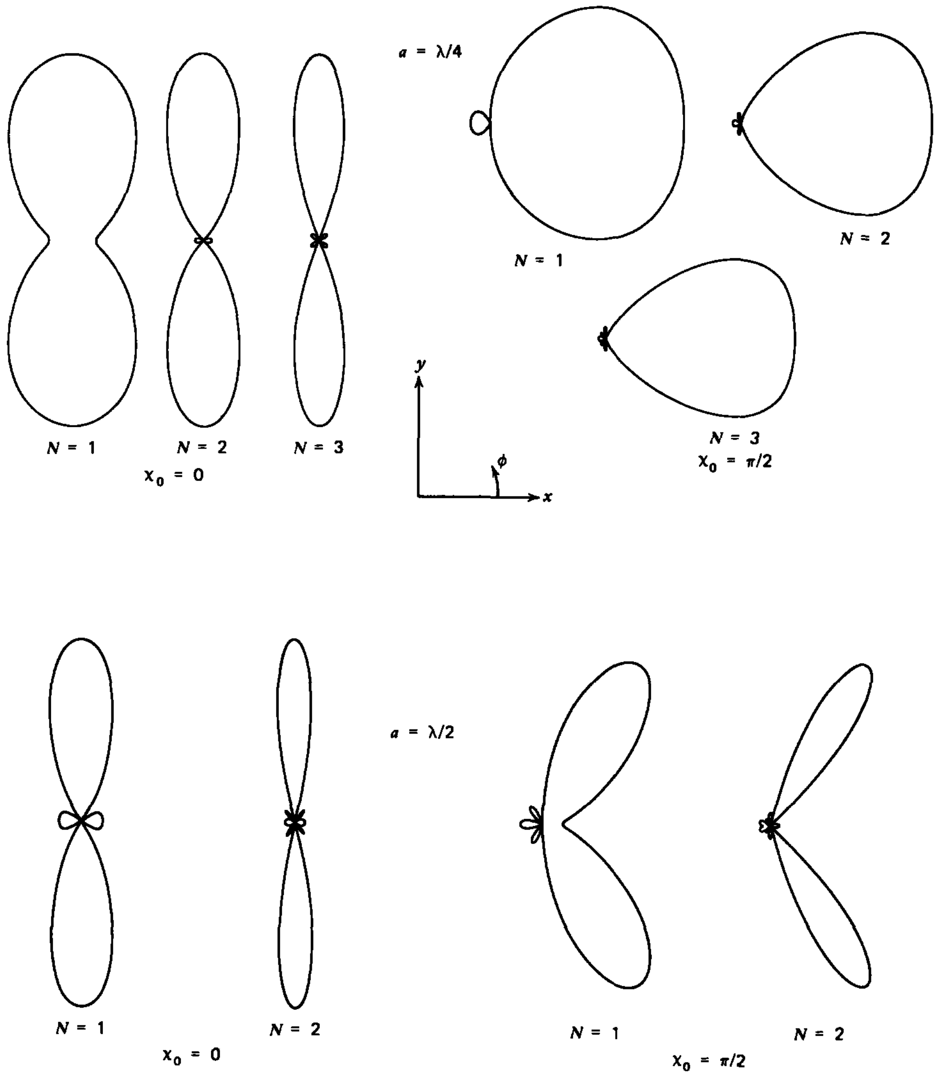



Figure 9-10 The radiation pattern for an $N$ dipole linear array for various values of $N$, dipole spacing $2a$, and relative current phase $\chi_0$ in the $\theta = \pi/2$ plane.

Since the fields and currents are coupled, an exact solution is impossible no matter how simple the antenna geometry. In practice, one guesses a current distribution and calculates the resultant (near and far) fields. If all boundary conditions along the antenna are satisfied, then the solution has been found. Unfortunately, this never happens with the first guess. Thus based on the field solution obtained from the originally guessed current, a corrected current distribution is used and

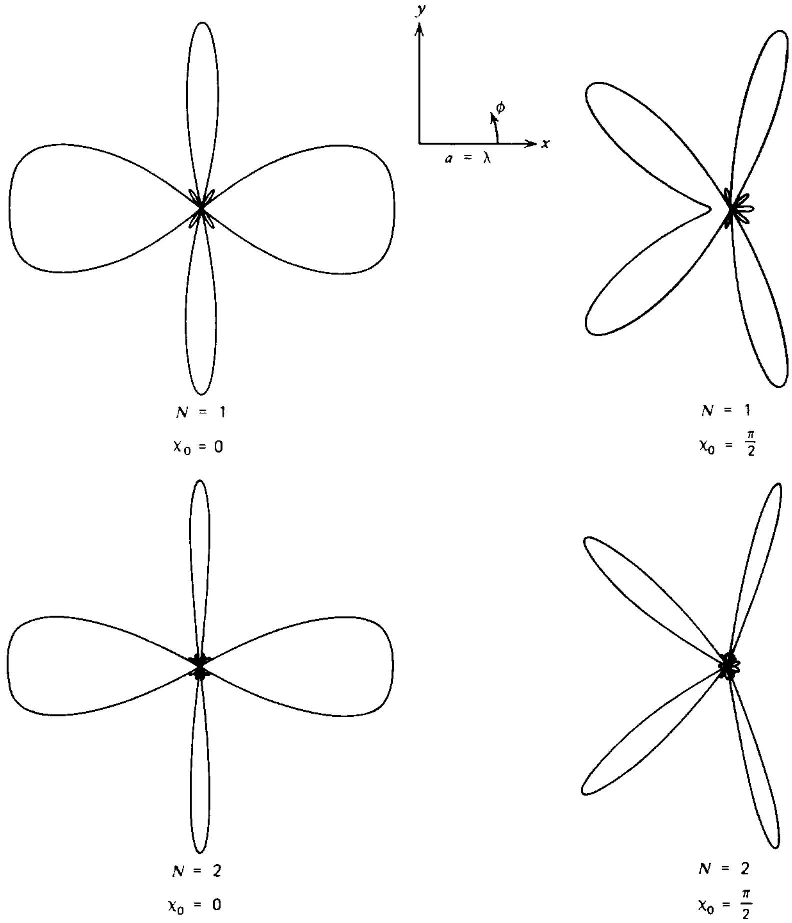



Figure 9-10

the resulting fields are again calculated. This procedure is numerically iterated until convergence is obtained with self-consistent fields and currents.

### 9-4-2 Uniform Current

A particularly simple case is when $\hat{I}(z)=\hat{I}_0$ is a constant. Then (2) becomes:

$$\begin{aligned}\hat{E}_\theta=\eta\hat{H}_\phi&=\frac{jk\eta}{4\pi r}\sin\theta\, e^{-jkr}\hat{I}_0\int_{-L/2}^{+L/2}e^{jkz\cos\theta}\,dz\\&=\frac{jk\eta}{4\pi r}\sin\theta\, e^{-jkr}\hat{I}_0\left.\frac{e^{jkz\cos\theta}}{jk\cos\theta}\right|_{-L/2}^{+L/2}\\&=\frac{\hat{I}_0\eta}{4\pi r}\tan\theta\, e^{-jkr}\left[2j\sin\left(\frac{kL}{2}\cos\theta\right)\right]\end{aligned}\tag{3}$$

The time-average power density is then

$$<S_r> = \frac{1}{2\eta}|\hat{E}_\theta|^2 = \frac{|\hat{E}_0|^2 \tan^2\theta \sin^2[(kL/2)\cos\theta]}{2\eta(kr)^2(kL/2)^2} \tag{4}$$

where

$$\hat{E}_0 = \frac{\hat{I}_0 L \eta k^2}{4\pi} \tag{5}$$

This power density is plotted versus angle $\theta$ in Figure 9-12 for various lengths $L$. The principal maximum always appears at $\theta = \pi/2$, becoming sharper as $L$ increases. For $L > \lambda$, zero power density occurs at angles

$$\cos\theta = \frac{2n\pi}{kL} = \frac{n\lambda}{L}, \qquad n = 1, 2, \ldots \tag{6}$$

Secondary maxima then occur at nearby angles but at much smaller amplitudes compared to the main lobe at $\theta = \pi/2$.

### 9-4-3 Radiation Resistance

The total time-average radiated power is obtained by integrating (4) over all angles:

$$\begin{aligned} <P> &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} <S_r> r^2 \sin\theta\, d\theta\, d\phi \\ &= \frac{|\hat{E}_0|^2\pi}{k^2\eta(kL/2)^2}\int_{\theta=0}^{\pi} \frac{\sin^3\theta}{\cos^2\theta}\sin^2\left(\frac{kL}{2}\cos\theta\right) d\theta \end{aligned} \tag{7}$$

If we introduce the change of variable,

$$v = \frac{kL}{2}\cos\theta, \qquad dv = -\frac{kL}{2}\sin\theta\, d\theta \tag{8}$$

the integral of (7) becomes

$$<P> = \frac{|\hat{E}_0|^2\pi}{k^2\eta(kL/2)^2}\int_{+kL/2}^{-kL/2} \left(\frac{2}{kL}\sin^2 v\, dv - \frac{kL}{2}\frac{\sin^2 v}{v^2}\right) dv \tag{9}$$

The first term is easily integrable as

$$\int \sin^2 v\, dv = \tfrac{1}{2}v - \tfrac{1}{4}\sin 2v \tag{10}$$

The second integral results in a new tabulated function $Si(x)$ called the sine integral, defined as:

$$Si(x) = \int_0^x \frac{\sin t}{t}\, dt \tag{11}$$

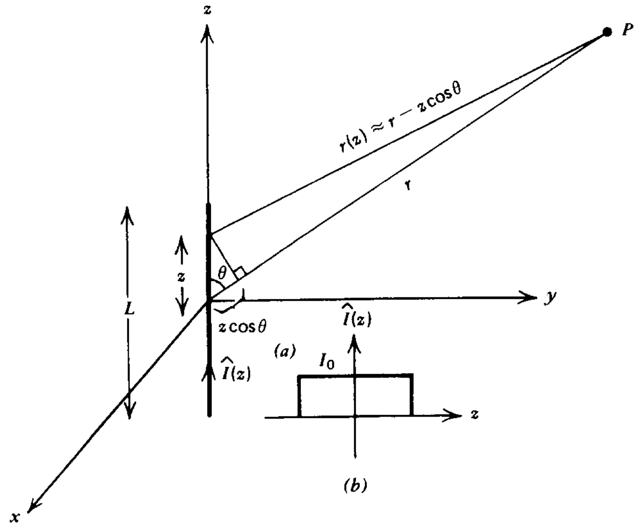



Figure 9-11 (*a*) For a long dipole antenna, each incremental current element at coordinate $z$ is at a slightly different distance to any field point $P$. (*b*) The simplest case study has the current uniformly distributed over the length of the dipole.

which is plotted in Figure 9-13. Then the second integral in (9) can be expanded and integrated by parts:

$$\begin{aligned}\int \frac{\sin^2 v}{v^2}\,dv &= \int \frac{(1-\cos 2v)}{2v^2}\,dv\\ &= -\frac{1}{2v} - \int \cos 2v \frac{dv}{2v^2}\\ &= -\frac{1}{2v} + \frac{\cos 2v}{2v} + \int \frac{\sin 2v\,d(2v)}{2v}\\ &= -\frac{1}{2v} + \frac{\cos 2v}{2v} + Si(2v)\end{aligned} \tag{12}$$

Then evaluating the integrals of (10) and (12) in (9) at the upper and lower limits yields the time-average power as:

$$<P> = \frac{|\hat{E}_0|^2 \pi}{k^2 \eta (kL/2)^2}\left(\frac{\sin kL}{kL} + \cos kL - 2 + kLSi(kL)\right) \tag{13}$$

where we used the fact that the sine integral is an odd function $Si(x) = -Si(x)$.

Using (5), the radiation resistance is then

$$R = \frac{2<P>}{|\hat{I}_0|^2} = \frac{\eta}{2\pi}\left(\frac{\sin kL}{kL} + \cos kL - 2 + kLSi(kL)\right) \tag{14}$$

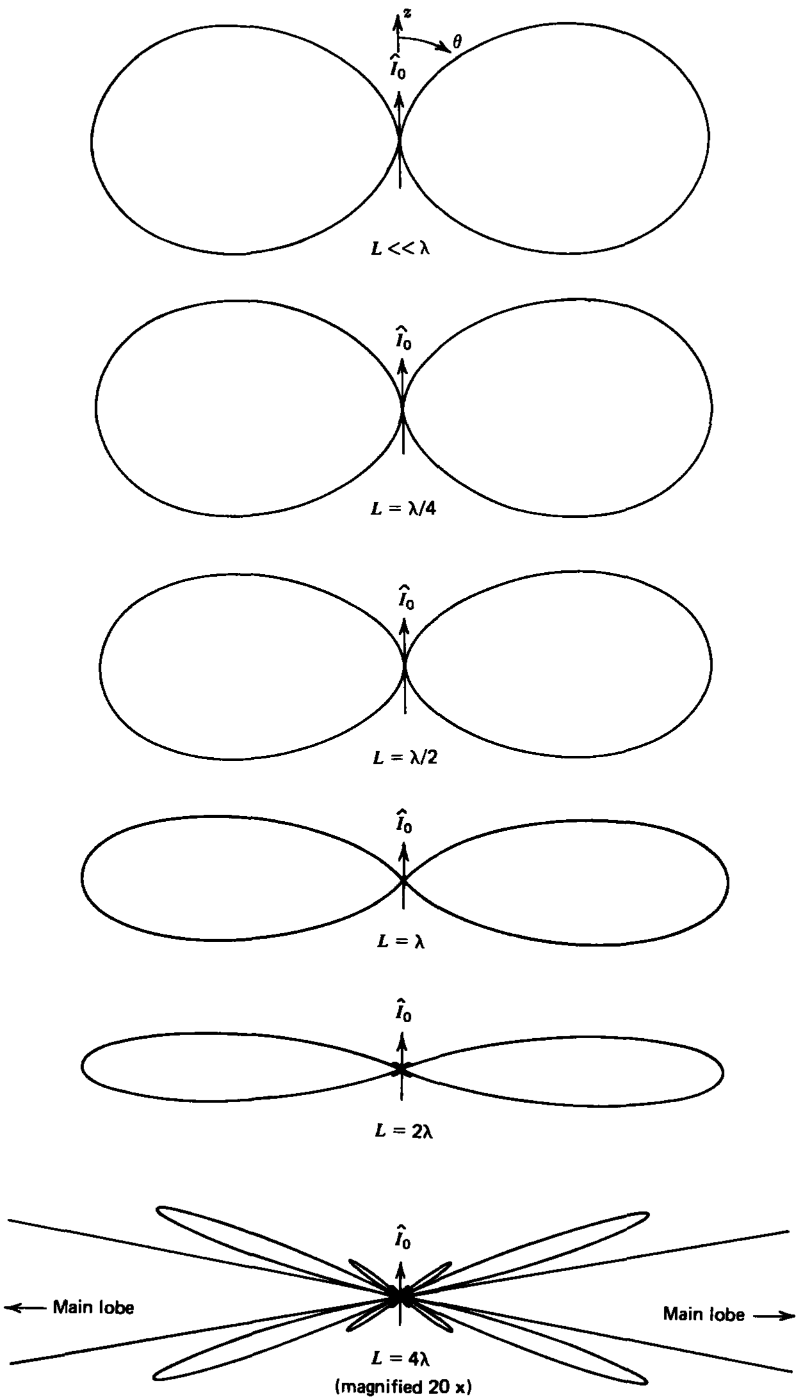



Figure 9-12 The radiation pattern for a long dipole for various values of its length.

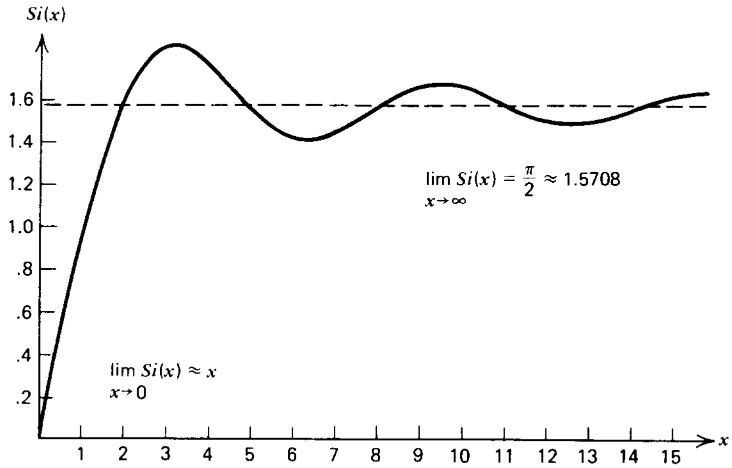



Figure 9-13 The sine integral $Si(x)$ increases linearly for small arguments and approaches $\pi/2$ for large arguments oscillating about this value for intermediate arguments.

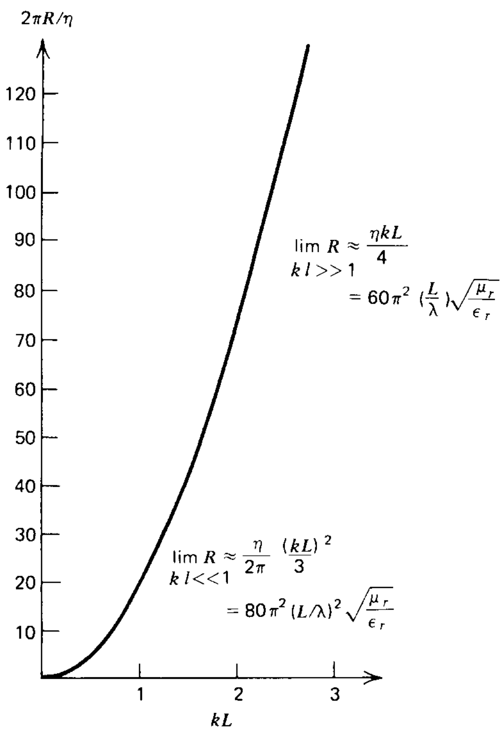



Figure 9-14 The radiation resistance for a dipole antenna carrying a uniformly distributed current increases with the square of its length when it is short ($L/\lambda \ll 1$) and only linearly with its length when it is long ($L/\lambda \gg 1$). For short lengths, the radiation resistance approximates that of a point dipole.

which is plotted versus $kL$ in Fig. 9-14. This result can be checked in the limit as $L$ becomes very small ($kL \ll 1$) since the radiation resistance should approach that of a point dipole given in Section 9-2-5. In this short dipole limit the bracketed terms in (14) are

$$\lim_{kL\ll 1}\begin{cases}\dfrac{\sin kL}{kL}\approx 1-\dfrac{(kL)^2}{6}\\[2ex] \cos kL\approx 1-\dfrac{(kL)^2}{2}\\[2ex] kLSi(kL)\approx (kL)^2\end{cases} \tag{15}$$

so that (14) reduces to

$$\lim_{kL\ll 1} R\approx\frac{\eta}{2\pi}\frac{(kL)^2}{3}=\frac{2\pi\eta}{3}\left(\frac{L}{\lambda}\right)^2=80\pi^2\left(\frac{L}{\lambda}\right)^2\sqrt{\frac{\mu_r}{\varepsilon_r}} \tag{16}$$

which agrees with the results in Section 9-2-5. Note that for large dipoles ($kL \gg 1$), the sine integral term dominates with $Si(kL)$ approaching a constant value of $\pi/2$ so that

$$\lim_{kL\gg 1} R\approx\frac{\eta kL}{4}=60\sqrt{\frac{\mu_r}{\varepsilon_r}}\,\pi^2\frac{L}{\lambda} \tag{17}$$

## PROBLEMS

Section 9-1

1. We wish to find the properties of waves propagating within a linear dielectric medium that also has an Ohmic conductivity $\sigma$.

(a) What are Maxwell's equations in this medium?

(b) Defining vector and scalar potentials, what gauge condition decouples these potentials?

(c) A point charge at $r=0$ varies sinusoidally with time as $Q(t)=\text{Re}\,(\hat{Q}\,e^{j\omega t})$. What is the scalar potential?

(d) Repeat (a)–(c) for waves in a plasma medium with constitutive law

$$\frac{\partial \mathbf{J}_f}{\partial t}=\omega_p^2\varepsilon\mathbf{E}$$

2. An infinite current sheet at $z=0$ varies as $\text{Re}\,[K_0\,e^{j(\omega t-k_x x)}\mathbf{i}_x]$.

(a) Find the vector and scalar potentials.

(b) What are the electric and magnetic fields?

(c) Repeat (a) and (b) if the current is uniformly distributed over a planar slab of thickness $2a$:

$$\mathbf{J}_f = \begin{cases} J_0\, e^{j(\omega t - k_x x)}\mathbf{i}_x, & -a < z < a \\ 0, & |z| > a \end{cases}$$

3. A sphere of radius $R$ has a uniform surface charge distribution $\sigma_f = \text{Re}\,(\hat{\sigma}_0\, e^{j\omega t})$ where the time varying surface charge is due to a purely radial conduction current.

(a) Find the scalar and vector potentials, inside and outside the sphere. (**Hint:** $r_{QP}^2 = r^2 + R^2 - 2rR\cos\theta$; $r_{QP}\, dr_{QP} = rR\sin\theta\, d\theta$.)

(b) What are the electric and magnetic fields everywhere?

Section 9.2

4. Find the effective lengths, radiation resistances and line charge distributions for each of the following current distributions valid for $|z| < dl/2$ on a point electric dipole with short length $dl$:

(a) $\hat{I}(z) = I_0 \cos \alpha z$

(b) $\hat{I}(z) = I_0\, e^{-\alpha|z|}$

(c) $\hat{I}(z) = I_0 \cosh \alpha z$

5. What is the time-average power density, total time-average power, and radiation resistance of a point magnetic dipole?

6. A plane wave electric field $\text{Re}\,(\mathbf{E}_0\, e^{j\omega t})$ is incident upon a perfectly conducting spherical particle of radius $R$ that is much smaller than the wavelength.

(a) What is the induced dipole moment? (**Hint:** See Section 4-4-3.)

(b) If the small particle is, instead, a pure lossless dielectric with permittivity $\varepsilon$, what is the induced dipole moment?

(c) For both of these cases, what is the time-average scattered power?

7. A plane wave magnetic field $\text{Re}\,(\mathbf{H}_0\, e^{j\omega t})$ is incident upon a perfectly conducting particle that is much smaller than the wavelength.

(a) What is the induced magnetic dipole moment? (**Hint:** See Section 5-7-2*ii* and 5-5-1.)

(b) What are the re-radiated electric and magnetic fields?

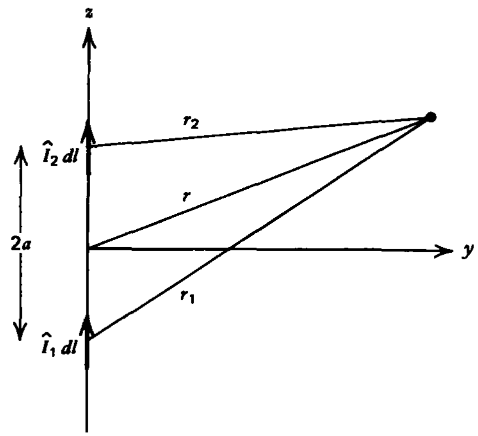
(c) What is the time-average scattered power? How does it vary with frequency?

8. (a) For the magnetic dipole, how are the magnetic field lines related to the vector potential **A**?

(b) What is the equation of these field lines?

Section 9.3

9. Two aligned dipoles $\hat{I}_1\, dl$ and $\hat{I}_2\, dl$ are placed along the $z$ axis a distance $2a$ apart. The dipoles have the same length




while the currents have equal magnitudes but phase difference $\chi$.

(a) What are the far electric and magnetic fields?

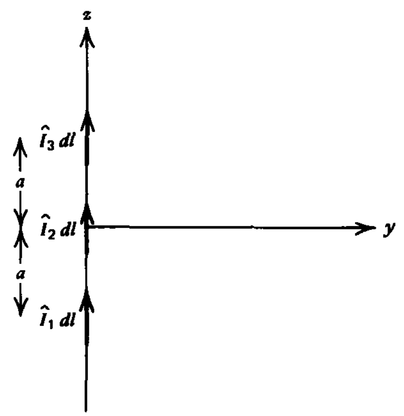
(b) What is the time-average power density?

(c) At what angles is the power density zero or maximum?

(d) For $2a=\lambda/2$, what values of $\chi$ give a broadside or end-fire array?

(e) Repeat (a)–(c) for $2N+1$ equally spaced aligned dipoles along the $z$ axis with incremental phase difference $\chi_0$.

10. Three dipoles of equal length $dl$ are placed along the $z$ axis.




(a) Find the far electric and magnetic fields.

(b) What is the time average power density?

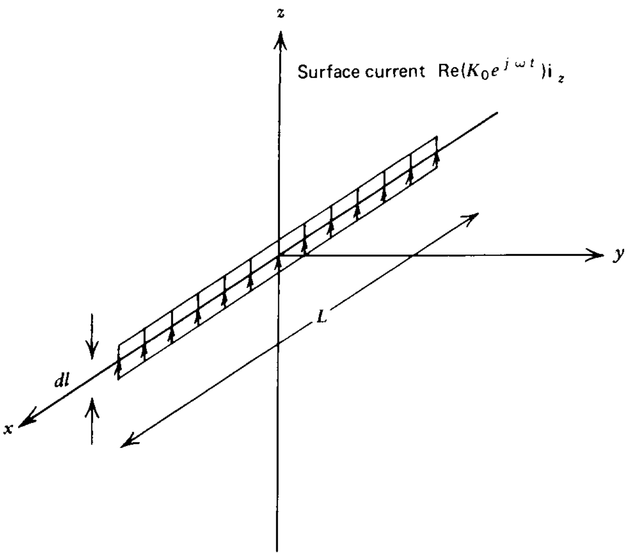
(c) For each of the following cases find the angles where the power density is zero or maximum.

(i) $\hat{I}_1=\hat{I}_3=I_0,\ \hat{I}_2=2I_0$

(ii) $\hat{I}_1=\hat{I}_3=I_0,\ \hat{I}_2=-2I_0$

(iii) $\hat{I}_1=-\hat{I}_3=I_0,\ \hat{I}_2=2jI_0$

11. Many closely spaced point dipoles of length $dl$ placed along the $x$ axis driven in phase approximate a $z$-directed current sheet $\text{Re}(K_0 e^{j\omega t}\mathbf{i}_z)$ of length $L$.




(a) Find the far fields from this current sheet.
(b) At what angles is the power density minimum or maximum?

Section 9.4
12. Find the far fields and time-average power density for each of the following current distributions on a long dipole:

$$\text{(a)}\quad \hat{I}(z)=\begin{cases} I_0(1-2z/L), & 0<z<L/2\\ I_0(1+2z/L), & -L/2<z<0\end{cases}$$

***Hint:***

$$\int z\,e^{az}\,dz=\frac{e^{az}}{a^2}(az-1)$$

$$\text{(b)}\quad \hat{I}(z)=I_0\cos \pi z/L,\quad -L/2<z<L/2$$

**Hint:**

$$\int e^{az}\cos pz\,dz=e^{az}\frac{(a\cos pz+p\sin pz)}{(a^2+p^2)}$$

(c) For these cases find the radiation resistance when $kL\ll 1$.

# SOLUTIONS TO SELECTED PROBLEMS

## Chapter 1

1. Area $= \pi a^2$

3. (a) $\mathbf{A}+\mathbf{B}=6\mathbf{i}_x-2\mathbf{i}_y-6\mathbf{i}_z$

   (b) $\mathbf{A}\cdot\mathbf{B}=6$

   (c) $\mathbf{A}\times\mathbf{B}=-14\mathbf{i}_x+12\mathbf{i}_y-18\mathbf{i}_z$

5. (b) $\mathbf{B}_{\parallel}=2(-\mathbf{i}_x+2\mathbf{i}_y-\mathbf{i}_z)$, $\mathbf{B}_{\perp}=5\mathbf{i}_x+\mathbf{i}_y-3\mathbf{i}_z$

7. (a) $\mathbf{A}\cdot\mathbf{B}=-75$

   (b) $\mathbf{A}\times\mathbf{B}=-100\mathbf{i}_z$

   (c) $\theta=126.87°$

12. (a) $\nabla f=(az+3bx^2y)\mathbf{i}_x+bx^3\mathbf{i}_y+ax\mathbf{i}_z$

14. (a) $\nabla\cdot\mathbf{A}=3$

17. (b) $\Phi=\frac{1}{2}abc$

18. (a) $\nabla\times\mathbf{A}=(x-y^2)\mathbf{i}_x-y\mathbf{i}_y-x^2\mathbf{i}_z$

23. (b) $\nabla f=\dfrac{1}{h_u}\dfrac{\partial f}{\partial u}\mathbf{i}_u+\dfrac{1}{h_v}\dfrac{\partial f}{\partial v}\mathbf{i}_v+\dfrac{1}{h_w}\dfrac{\partial f}{\partial w}\mathbf{i}_w$

   (c) $dV=h_uh_vh_w\,du\,dv\,dw$

   (d) $\nabla\cdot\mathbf{A}=\dfrac{1}{h_uh_vh_w}\left[\dfrac{\partial}{\partial u}(h_vh_wA_u)+\dfrac{\partial}{\partial v}(h_uh_wA_v)+\dfrac{\partial}{\partial w}(h_uh_vA_w)\right]$

   $$(\nabla\times\mathbf{A})_u=\frac{1}{h_vh_w}\left[\frac{\partial(h_wA_w)}{\partial v}-\frac{\partial(h_vA_v)}{\partial w}\right]$$

25. (a) $r_{QP}=\sqrt{30}$, (b) $\mathbf{i}_{QP}=\dfrac{\mathbf{r}_{QP}}{r_{QP}}=\dfrac{\mathbf{i}_x-5\mathbf{i}_y+2\mathbf{i}_z}{\sqrt{30}}$,

   (c) $\mathbf{n}=\dfrac{5\mathbf{i}_x+\mathbf{i}_y}{\sqrt{26}}$

## Chapter 2

3. $E_0=\dfrac{4}{3}\dfrac{\pi R^3\rho_m g}{q}$

4. $Q_2=\dfrac{2\pi\varepsilon_0 d^3Mg}{Q_1\sqrt{l^2-\left(\frac{d}{2}\right)^2}}$

5. (a) $\omega=\left[\dfrac{Q_1Q_2}{4\pi\varepsilon_0R^3m}\right]^{1/2}$

7. (a) $m=\dfrac{m_1m_2}{m_1+m_2}$,

(b) $v=\pm\sqrt{\dfrac{-q_1q_2}{2\pi\varepsilon_0 m}\left(\dfrac{1}{\mathrm{r}}-\dfrac{1}{\mathrm{r}_0}\right)}$,

(d) $t=\dfrac{\pi}{2}\mathrm{r}_0^{3/2}\left[\dfrac{2\pi m\varepsilon_0}{-q_1q_2}\right]^{1/2}$

8. $h=\dfrac{qE_0L^2}{2mv_0^2}$

10. (b) $q=\dfrac{6\sqrt{3}}{7^{3/2}}Q$

12. (a) $q=2\lambda_0 a$, (b) $q=\frac{4}{3}\pi\rho_0 a^3$, (c) $q=2\sigma_0 ab\pi$

15. $\theta=\tan^{-1}\left[\dfrac{Q\sigma_0}{2\varepsilon_0 Mg}\right]$

16. (a) $E_r=\dfrac{\lambda L}{2\pi\varepsilon_0\mathrm{r}\sqrt{L^2+\mathrm{r}^2}}$,

(b) $E_x=\dfrac{\sigma_0}{2\pi\varepsilon_0}\left[\sin^{-1}\left(\dfrac{L^2-x^2}{L^2+x^2}\right)+\dfrac{\pi}{2}\right];\quad x>0$

18. (a) $E_y=\dfrac{-\lambda_0 a^2}{\pi\ \epsilon_0[z^2+a^2]^{3/2}}$

(b) $E_y=\dfrac{-\sigma_0}{\pi\epsilon_0}\left\{\dfrac{-a}{\sqrt{z^2+a^2}}+\ln\left[\dfrac{a+\sqrt{z^2+a^2}}{|z|}\right]\right\}$

20. (a) $E_y=-\dfrac{\lambda_0 a^2}{\pi\varepsilon_0(a^2+z^2)^{3/2}}$

21. $E_y=\dfrac{\lambda_0 z^2}{2\pi\varepsilon_0(a^2+z^2)^{3/2}}$

22. (a) $Q_T=4\pi\varepsilon_0 AR^4$

23. (c)
$$E_x=\begin{cases}\dfrac{\rho_0}{2\varepsilon_0 d}(x^2-d^2) & |x|<d\\ 0 & |x|>d\end{cases}$$

25. (*c*)
$$E_r=\begin{cases}\dfrac{\rho_0\mathrm{r}^2}{3\varepsilon_0 a} & \mathrm{r}<a\\ \dfrac{\rho_0 a^2}{3\varepsilon_0\mathrm{r}} & \mathrm{r}>a\end{cases}$$

26. $\mathbf{E}=\dfrac{\rho_0 d}{2\varepsilon_0}\mathbf{i}_x$

27. $W=-\dfrac{\lambda\sigma_0 l^2}{4\varepsilon_0}$

28. (a) $v_0 \geq \sqrt{\dfrac{qQ}{2\pi\varepsilon_0 Rm}}$, (b) $r = 4R$

29. (a) $\mathbf{E} = -2Ax\mathbf{i}_x, \rho_f = -2A\varepsilon_0$

31. (a) $\Delta v = \dfrac{\sigma_0 a}{\varepsilon_0}$

32. (a) $dq = -\dfrac{Q}{R}dz'$

33. (c) $V \approx \dfrac{q_0 a}{4\pi\varepsilon_0 r^2}\cos\theta$, (d) $r = r_0 \sin^2\theta$

34. (d) $q_c = -\dfrac{qV_p}{V_c}$

36. (a) $E_y = -\dfrac{\sigma_0}{2\pi\varepsilon_0}\ln\left(1 - \dfrac{d}{y}\right)$

38. (a) $x_0 = \sqrt{\dfrac{q}{16\pi\varepsilon_0 E_0}}$, (b) $v_0 > \dfrac{1}{2}\sqrt{\dfrac{q}{m}}\left[\dfrac{qE_0}{\pi\varepsilon_0}\right]^{1/4}$,

(c) $W = \dfrac{q^2}{16\pi\varepsilon_0 d}$

43. (e) $\lambda = \pm\sqrt{\dfrac{R_1}{R_2}}, \alpha = \pm\dfrac{R_1}{R_2}$

44. (g) $q_T = -4\pi\varepsilon_0 R^2\dfrac{\pi^2}{6}$

**Chapter 3**

2. (a) $p_z = \lambda_0 L^2$, (e) $p_z = QR$

4. (a) $\rho_0 = \dfrac{3Q}{\pi R_0^3}$

7. (a) $\mathbf{d} = \dfrac{4\pi\varepsilon_0 R_0^3 \mathbf{E}_0}{Q}$

8. (b) $\dfrac{Q}{L^2} = 2\pi\varepsilon_0 E_0$

10. (a) $\mathbf{p}_{\text{ind}} = \mathbf{p}\dfrac{R^3}{D^3}$

12. (a) $V(x) = \dfrac{V_0}{2}\,\dfrac{\sinh x/l_d}{\sinh l/l_d}$

15. (b) $Q = \dfrac{m\omega RA\sigma}{q}$

17. (a) $D_r = \dfrac{\lambda}{2\pi r}$

19. (a) $\lambda' = -\lambda'' = \dfrac{\lambda(\varepsilon_2 - \varepsilon_1)}{\varepsilon_1 + \varepsilon_2}, \lambda''' = \dfrac{2\varepsilon_2\lambda}{\varepsilon_1 + \varepsilon_2}$

23. (a)
$$E_r = \begin{cases} -\dfrac{P_0 r}{\varepsilon_0 R} & r < R \\ 0 & r > R \end{cases}$$

26. (a) $R = \dfrac{s \ln \dfrac{\sigma_2}{\sigma_1}}{lD(\sigma_2 - \sigma_1)}$

31. $C = \dfrac{2\pi l(\varepsilon_2 a - \varepsilon_1 b)}{(b-a)\ln \dfrac{\varepsilon_2 a}{\varepsilon_1 b}}$

33. $\sigma_f(r = a_1) = \dfrac{\rho_0 a_0^2}{3a_1}(1 - e^{-t/\tau});\ \tau = \varepsilon/\sigma$

35. $\rho_f = \rho_0\, e^{-\sigma r^3/(3\varepsilon A)}$

38. (a) $v(z) = -\dfrac{V_0 \sinh \sqrt{2RG}(z - l)}{\sinh \sqrt{2RG}l}$

41. (b) $\dfrac{\varepsilon\mu}{2}[E^2(l) - E^2(0)] + \varepsilon\dfrac{dv}{dt} = J(t)l$

(c) $E(l) = \dfrac{V_0/l}{1 - \dfrac{\mu t V_0}{2l^2}}$, (f) $\tau = \dfrac{2l^2}{\mu V_0}(1 - e^{-1/2})$

42. (c) $E_i^2 = \left(\dfrac{V_0}{R_0 - R_i}\right)^2 = \dfrac{I}{2\pi\varepsilon\mu l}$

43. (a) $W = -\frac{1}{2}\mathbf{p}\cdot\mathbf{E}$

44. $W = \dfrac{p^2}{12\pi\varepsilon_0 R^3}$

47. (a) $W = \dfrac{-Q^2}{8\pi\varepsilon_0 R}$

48. (a) $W_{\text{init}} = \frac{1}{2}CV_0^2$, (b) $W_{\text{final}} = \frac{1}{4}CV_0^2$

49. (b) $W = -pE(\cos\theta - 1)$

50. $h = \frac{1}{2}(\varepsilon - \varepsilon_0)\dfrac{V_0^2}{\rho_m g s^2}$

52. (b) $f_y=\frac{1}{2}\frac{\varepsilon_0 A}{(s+d)^2}\left[V_0+\frac{P_0 d}{\varepsilon_0}\right]^2$

54. (b) $f_z=\frac{\pi V_0^2}{\ln\frac{b}{a}}(\varepsilon-\varepsilon_0)$

55. $f_x=-\frac{1}{2}\frac{\varepsilon_0 d}{s}V_0^2$

56. (c) $T=\frac{1}{2}v^2\frac{dC}{d\theta}=\frac{-NV_0^2R^2\varepsilon_0}{s}$

57. (a) $v(t)=\frac{\sigma_f Uwt}{4\pi\varepsilon_0 R}$

58. (a) $\rho_f=\rho_0\,e^{-\sigma z/\varepsilon U}$

59. (a) $nC_i>\frac{1}{R}+\frac{2}{R_L}$, (c) $\frac{nC_i}{2}>\frac{1}{R}$, $\omega_0=\frac{\sqrt{3}}{2}\frac{nC_i}{C}$

## Chapter 4

2. (a)
$$V=\begin{cases}\frac{\sigma_0}{2\varepsilon a}\cos ay\,e^{-ax} & x>0\\ \frac{\sigma_0}{2\varepsilon a}\cos ay\,e^{ax} & x<0\end{cases}$$

4. (a) $V=\frac{4V_1}{\pi}\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{\sin\frac{n\pi y}{d}\sinh\frac{n\pi(l-x)}{d}}{n\sinh\frac{n\pi l}{d}}$

7. (a) $V_p=\frac{\rho_0}{\varepsilon_0 a^2}\sin ax$

12.
$$V(\mathrm{r},\phi)=\begin{cases}\left[\frac{P_2-P_1}{2\varepsilon_0}-E_0\right]\mathrm{r}\cos\phi & 0\le \mathrm{r}\le a\\ \left[-E_0\mathrm{r}+\frac{(P_2-P_1)a^2}{2\varepsilon_0\mathrm{r}}\right]\cos\phi & \mathrm{r}>a\end{cases}$$

13. (a) $\mathbf{E}=\left[E_0\left(1+\frac{a^2}{\mathrm{r}^2}\right)\cos\phi+\frac{\lambda(t)}{2\pi\varepsilon r}\right]\mathbf{i}_r-E_0\left(1-\frac{a^2}{\mathrm{r}^2}\right)\sin\phi\,\mathbf{i}_\phi$

(b) $\cos\phi<-\frac{\lambda(t)}{4\pi\varepsilon aE_0}$, (c) $\lambda_{max}=4\pi\varepsilon aE_0$

15. (*a*) $$V(r,z)=\begin{cases}\dfrac{V_0 z}{l\ln\dfrac{b}{a}}\ln\dfrac{r}{a} & a\le r\le b\\[2ex] \dfrac{V_0 z}{l} & b\le r\le c\end{cases}$$

17. (b) $E_0 \geq \sqrt{\dfrac{8\rho_m g R}{27\epsilon}}$

22. $V(2,2)=V(3,2)=V(2,3)=V(3,3)=-4.$

23. (a) $V(2,2)=-1.0000,\ V(3,2)=-.5000,\ V(2,3)$
$=-.5000,\ V(3,3)=.0000$

(b) $V(2,2)=1.2500,\ V(3,2)=-.2500,\ V(2,3)$
$=.2500,\ V(3,3)=-1.2500$

**Chapter 5**

2. (b) $B_0^2>\dfrac{2mV_0}{es^2}$, (e) $B_0^2>\dfrac{8b^2mV_0}{e(b^2-a^2)^2}$

3. $B_0=\dfrac{-mg}{qv_0}$

4. (d) $\dfrac{e}{m}=\dfrac{E_y}{RB_0^2}$

8. (c) $\mathbf{J}=\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B})$

10. (a) $B_z=\dfrac{2\mu_0 I(a^2+b^2)^{1/2}}{\pi ab}$, (c) $B_z=\dfrac{n\mu_0 I}{2\pi a}\tan\dfrac{\pi}{n}$

12. (a) $B_z=\dfrac{\mu_0 K_0\pi}{4}$

13. (a) $B_\phi=\dfrac{\mu_0 IL}{2\pi r\sqrt{L^2+r^2}}$

15. (b) $B_y=\dfrac{\mu_0 J_0 d}{2}$

17. (b) $$B_x=\begin{cases}-\dfrac{\mu_0 J_0}{2a}(y^2-a^2) & |y|<a\\ 0 & |y|>a\end{cases}$$

18. (d) $y=\dfrac{y_0}{2}$ at $x=-\infty$

21. (a) $m_z=\frac{1}{2}q\omega a^2$

23. $\omega_0 = \sqrt{\gamma B_0}$

27. (a) $I' = \dfrac{(\mu_2 - \mu_1)}{(\mu_1 + \mu_2)} I, \; I'' = \dfrac{2\mu_1 I}{\mu_1 + \mu_2}$

34. (a) $\mathbf{H} = \begin{cases} \dfrac{-M_0}{2}\mathbf{i}_x \\ \dfrac{M_0 a^2}{2r^2}[\cos\phi\,\mathbf{i}_r + \sin\phi\, i_\phi] \end{cases}$

35. (a) $H_z(x) = -\dfrac{I}{Dd}(x - d)$, (b) $f_x = \dfrac{1}{2}\mu_0 \dfrac{I^2 s}{D}$

36. (a) $f_x = \frac{1}{2}(\mu - \mu_0) H_0^2 Ds$, (b) $f_x = \mu_0 M_0 Ds[H_0 + M_0]$

## Chapter 6

1. (a) $M = \mu_0[D - \sqrt{D^2 - a^2}]$, (e) $f_r = \mu_0 Ii\left[\dfrac{D}{\sqrt{D^2 - a^2}} - 1\right]$

3. (d) $v(t) = v_0\left[\dfrac{\alpha}{2\beta}\sin\beta t + \cos\beta t\right]e^{-\alpha t/2}; \; \beta = \sqrt{\omega_0^2 - \left(\dfrac{\alpha}{2}\right)^2}$

$$i(t) = \frac{m v_0 \omega_0^2}{B_0 b \beta}\sin\beta t\, e^{-\alpha t/2}$$

(e) $v_0 > \dfrac{B_0 b s}{\sqrt{mL}}$

4. (a) $M = \dfrac{\mu_0 N s}{2\pi}\ln\dfrac{a}{b},\; M = \mu_0 N[R - \sqrt{R^2 - a^2}]$

7. (c) $f_z = \dfrac{3\mu_0 (IdS)^2}{32\pi d^4}$

8. (a) $H_z = K(t)$, (b) $K(t) = K_0\left(\dfrac{x_0}{x_0 - Vt}\right)^{(1 - 1/R_m)}$

9. (a) $i = \dfrac{r\sigma d}{2}\dfrac{dB}{dt}dr$, (c) $P = \dfrac{\pi\sigma d a^4}{8}\left(\dfrac{dB}{dt}\right)^2$

10. $L = \mu_0 N^2[b - \sqrt{b^2 - a^2}]$

14. (a) $\dfrac{v_2}{v_1} = \dfrac{N_2}{2N_1}, \dfrac{i_2}{i_1} = \dfrac{2N_1}{N_2}$

16. (a) $V_{oc} = J_x B_z d\dfrac{(\mu_+^2 n_+ - \mu_-^2 n_-)}{q(\mu_+ n_+ + \mu_- n_-)^2}$

17. (b), (c) $EMF = -\frac{\mu_0 V_0 I}{2\pi}\ln\frac{R_2}{R_1}$,

(d) $EMF = -\frac{(\mu-\mu_0) I V_0}{2\pi}\ln\frac{R_2}{R_1}$

18. (a) $\mathbf{H}=0, \mathbf{B}=\mu_0 M_0 \mathbf{i}_z$, (b) $v_{oc}=\frac{\omega B_z}{2}(b^2-a^2)$

20. (b) $V > \frac{1}{\mu_0 \sigma N D}$

21. (a) $\omega > \frac{(R_r+R_f)}{G}$

(b) $C_{\text{crit}} = \frac{4L_f}{[R_r+R_f-G\omega]^2}$; $C > C_{\text{crit}}(dc)$, $C < C_{\text{crit}}(ac)$

(c) $\omega_0\left[\frac{1}{L_f C}-\left[\frac{R_r+R_f-G\omega}{2L_f}\right]^2\right]^{1/2}$

22. (b) $H_z(x,t)=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{2I_0}{n\pi D}\sin\frac{n\pi x}{d}e^{-\alpha_n t}$

23. (c) $H_y(x,t)=H_0-\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{4H_0}{n\pi}\sin\frac{n\pi x}{d}e^{-\alpha_n t}$

25. (b) $H_z(y,t)=-K_0+\sum_{n=0}^{\infty}\frac{(-1)^n 4K_0}{\pi(2n+1)}\cos\left[\frac{(2n+1)\pi y}{2D}\right]—e^{-\alpha_n t}$

(d) $\hat{H}_z(y)=K_0\left[\frac{[e^{(1+j)y/\delta}+e^{-(1+j)y/\delta}]}{[e^{(1+j)D/\delta}+e^{-(1+j)D/\delta}]}\right]$

26. (a) $H_z(x)=\frac{K_0}{1-e^{R_m}}[2e^{R_m x/l}-(1+e^{R_m})]$

27. (a) $\hat{H}_z(x)=K_0 e^{-\beta x}e^{R_m x/2l}$; $\beta=\frac{R_m}{2l}\sqrt{1+\frac{2jl^2}{R_m^2\delta^2}}$

28. (a)

$$\hat{\mathbf{H}}(x)=\begin{cases}\dfrac{K_0\left\{\left(1-\frac{\mu}{\mu_0}\frac{k}{\gamma}\right)e^{k(x-s)}(\mathbf{i}_z+j\mathbf{i}_x)+\left(1+\frac{\mu}{\mu_0}\frac{k}{\gamma}\right)e^{-k(x-s)}(\mathbf{i}_z-j\mathbf{i}_x)\right\}}{\left[1-\frac{\mu}{\mu_0}\frac{k}{\gamma}\right]e^{-ks}+\left[1+\frac{\mu}{\mu_0}\frac{k}{\gamma}\right]e^{ks}} & 0<x<s\\[2ex] \dfrac{2K_0 e^{-\gamma(x-s)}\left[\mathbf{i}_z-\frac{jk}{\gamma}\mathbf{i}_x\right]}{\left[1-\frac{\mu}{\mu_0}\frac{k}{\gamma}\right]e^{-ks}+\left[1+\frac{\mu}{\mu_0}\frac{k}{\gamma}\right]e^{ks}} & x>s\end{cases}$$

29. (a) $H_y = H_0 \dfrac{\cosh k(x - d/2)}{\cosh kd/2}$

32. (b) $\hat{H}_\phi(r) = \dfrac{I_0}{2\pi a} \dfrac{J_1[(r/\delta)(1-j)]}{J_1[(a/\delta)(1-j)]}$

33. (a) $T = -L_1 I_0^2 \cos^2 \omega_0 t \sin 2\theta$

34. (c) $T = M_0 I_1 I_2 \cos\theta,$

(f) $\theta(t) = \theta_0 \left[\cos\beta t + \dfrac{\alpha}{2\beta}\sin\beta t\right] e^{-\alpha t/2}$

35. (a) $L(x) = \dfrac{\mu_0 x}{2\pi} \ln \dfrac{b}{a},$ (b) $f_x = \dfrac{\mu_0 i^2}{4\pi} \ln \dfrac{b}{a}$

37. $h = \dfrac{I^2(\mu - \mu_0) \ln \dfrac{b}{a}}{4\pi^2 \rho_m g(b^2 - a^2)}$

## Chapter 7

4. (b) $W = 4[P_s E_c + \mu_0 M_s H_c]$

9. (b) $\hat{E}_x(z) = \begin{cases} \dfrac{j\eta_0 J_0 \sin kd\, e^{\mp jk_0(z \mp d)}}{\omega\varepsilon[\eta_0 \sin kd - j\eta \cos kd]} & z > d \\ & z < -d \\ -\dfrac{J_0 \eta \cos kz}{\omega\varepsilon[\eta_0 \sin kd - j\eta \cos kd]} & |z| < d \end{cases}$

10. (b) $E_x = E_0 e^{j(\omega t \mp \beta z)} e^{\mp(\alpha z/2)} \quad \begin{matrix} z > 0 \\ z < 0 \end{matrix}; \ \beta = \sqrt{\omega^2 \varepsilon\mu - \dfrac{\alpha^2}{4}}$

11. (a) $t_1' - t_2' = \dfrac{\gamma v}{c_0^2}(z_2 - z_1),$

(b) $t_1' - t_2' = \gamma(t_1 - t_2),$ (c) $z_2' - z_1' = \gamma L$

12. (a) $u_z' = \dfrac{u_z - v}{1 - vu_z/c_0^2}, \ u_{x,y}' = \dfrac{u_{x,y}\sqrt{1 - (v/c_0)^2}}{1 - vu_z/c_0^2}$

15. (b) $\varepsilon(\omega) = \varepsilon_0\left[1 + \dfrac{\omega_p^2}{\omega_0^2 - \omega^2}\right]$

16. (c) $k^2 = \dfrac{\omega^2}{c^2}\left[1 - \dfrac{\omega_p^2}{\omega(\omega \mp \omega_0)}\right]$

20. (a) $\mathbf{E} = E_0\left[\cos\omega\left(t - \dfrac{z}{c}\right) - \cos\omega\left[\left(1 - \dfrac{2v}{c}\right)\left(t - \dfrac{z}{c}\right)\right]\right]\mathbf{i}_y$

22. $\alpha^2 - k^2 = -\omega^2\varepsilon\mu, \ \boldsymbol{\alpha} \cdot \mathbf{k} = \dfrac{1}{\delta^2}$

26. (a) $L_1+L_2=s_i \sin\theta_i+s_r \sin\theta_r=h_1\tan\theta_i+h_2\tan\theta_r$

31. $\theta_t\approx 41.7°$

33. (a) $\left(\frac{x}{R}\right)^2>1-\frac{(n^2-2)^2}{4}$ ; $\sqrt{2}\le n\le 2$

(b) $R'=\frac{\alpha R}{[\sqrt{n^2(1-\alpha^2)}\sqrt{n^2-\alpha^2}+\alpha^2]}$

**Chapter 8**

2. (c) $\hat{v}(z)=-\frac{V_0\sin\beta(z-l)\,e^{\alpha z/2}}{\sin\beta l}$ (Short circuited end)

4. (c) $\omega^2=\omega_p^2+k^2c^2$, (d) $v(z,t)=-\frac{V_0\sin kz\cos\omega t}{\sin kl}$

5. (b) $k=\frac{1}{\omega\sqrt{LC}}$

14. (a) $V_+=-V_-=\frac{V_0Z_0}{2R_s}$

16. (b) $\tan kl=-XY_0$

21. (c) $VSWR=\frac{1+\sqrt{2}}{\sqrt{2}-1}\approx 5.83$

22. (b) $VSWR=2$

23. $Z_L=170.08-133.29j$

24. (a) $l_1=.137\lambda+\frac{n\lambda}{2}$, $l_2=.089\lambda+\frac{m\lambda}{2}$

$l_1=.279\lambda+\frac{n\lambda}{2}$, $l_2=.411\lambda+\frac{m\lambda}{2}$

25. (a) $l_1=.166\lambda+\frac{n\lambda}{2}$, $l_2=.411\lambda+\frac{m\lambda}{2}$

$l_1=.077\lambda+\frac{n\lambda}{2}$, $l_2=.043\lambda+\frac{m\lambda}{2}$

27. (e) $\alpha=\frac{2(\pi/a)^2[b+(a/2)(\omega^2a^2/\pi^2c^2)]}{\omega\mu abk_z\sigma_w\delta}$

28. (b) $=\frac{2\omega\varepsilon(bk_x^2+ak_y^2)}{\sigma_w\delta k_z ab(k_x^2+k_y^2)}$

29. (a) TE mode:

electric field: $\cos k_x x \cos k_y y = \text{const}$

magnetic field: $\dfrac{\sin (k_x x)^{(k_y/k_x)^2}}{\sin k_y y} = \text{const}$

31. (b) $\dfrac{\omega^2}{c^2} = \left(\dfrac{m\pi}{a}\right)^2 + \left(\dfrac{n\pi}{b}\right)^2 + \left(\dfrac{p\pi}{l}\right)^2,$

32. (a) $\omega^2 = \dfrac{k_x^2}{\varepsilon\mu - \varepsilon_0\mu_0}$

**Chapter 9**

1. (c) $\hat{V} = \dfrac{\hat{Q}}{4\pi\varepsilon r} e^{-jr\sqrt{\omega^2/c^2 - j\omega\mu\sigma}}$

4. (a) $dl_{\text{eff}} = \dfrac{2 \sin (\alpha dl/2)}{\alpha}, \; \hat{\lambda}(z) = \dfrac{I_0\alpha}{j\omega} \sin \alpha z$

6. (a) $\hat{p}_z = 4\pi\varepsilon_0 R^3 \hat{E}_0$

(c) $\langle P \rangle = \dfrac{\omega^4 |\hat{p}_z|^2 \eta}{12\pi c^2}$

7. (a) $m_{\text{ind}} = 2\pi H_0 R^3$

8. (b) $\sin^2\theta \left[\dfrac{\cos (\omega t - kr)}{kr} - \sin (\omega t - kr)\right] = \text{const}$

9. (a) $\hat{E}_\theta \approx \dfrac{2\hat{E}_0}{jkr} \sin \theta \, e^{-j(kr - \chi/2)} \left[\cos \left(ka \cos \theta - \dfrac{\chi}{2}\right)\right]$

11. $\hat{E}_\theta = \dfrac{2jK_0 dl \eta \, e^{-jkr}}{4\pi r \cos \phi} \sin \left(\dfrac{kL}{2} \sin \theta \cos \phi\right)$

12. (a) $\hat{E}_\theta = \dfrac{\eta I_0 \sin \theta \, e^{-jkr}}{j\pi krL \cos^2 \theta} \cos \left[\left(\dfrac{kL}{2} \cos \theta\right) - 1\right]$

# INDEX

## VECTOR IDENTITIES

$$(\mathbf{A}\times\mathbf{B})\cdot\mathbf{C}=\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=(\mathbf{C}\times\mathbf{A})\cdot\mathbf{B}$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

$$\nabla\cdot(\nabla\times\mathbf{A})=0$$

$$\nabla\times(\nabla f)=0$$

$$\nabla(fg)=f\nabla g+g\nabla f$$

$$\nabla(\mathbf{A}\cdot\mathbf{B})=(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})$$

$$\nabla\cdot(f\mathbf{A})=f\nabla\cdot\mathbf{A}+(\mathbf{A}\cdot\nabla)f$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\mathbf{A}-(\mathbf{A}\cdot\nabla)\mathbf{B}$$

$$\nabla\times(f\mathbf{A})=\nabla f\times\mathbf{A}+f\nabla\times\mathbf{A}$$

$$(\nabla\times\mathbf{A})\times\mathbf{A}=(\mathbf{A}\cdot\nabla)\mathbf{A}-\tfrac{1}{2}\nabla(\mathbf{A}\cdot\mathbf{A})$$

$$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$$

## INTEGRAL THEOREMS

*Line Integral of a Gradient*

$$\int_a^b \nabla f\cdot d\mathbf{l}=f(b)-f(a)$$

*Divergence Theorem*:

$$\int_V \nabla\cdot\mathbf{A}\,dV=\oint_S \mathbf{A}\cdot d\mathbf{S}$$

Corollaries

$$\int_V \nabla f\,dV=\oint_S f\,d\mathbf{S}$$

$$\int_V \nabla\times\mathbf{A}\,dV=-\oint_S \mathbf{A}\times d\mathbf{S}$$

*Stokes' Theorem*:

$$\oint_L \mathbf{A}\cdot d\mathbf{l}=\int_S(\nabla\times\mathbf{A})\cdot d\mathbf{S}$$

Corollary

$$\oint_L f\,d\mathbf{l}=-\int_S \nabla f\times d\mathbf{S}$$

## MAXWELL'S EQUATIONS

| *Integral* | *Differential* | *Boundary Conditions* |
|---|---|---|
| **Faraday's Law** | | |
| $\oint_L \mathbf{E}' \cdot \mathbf{dl} = -\frac{d}{dt}\int_S \mathbf{B} \cdot \mathbf{dS}$ | $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ | $\mathbf{n} \times (\mathbf{E}_2' - \mathbf{E}_1') = 0.$ |
| **Ampere's Law with Maxwell's Displacement Current Correction** | | |
| $\oint_L \mathbf{H} \cdot \mathbf{dl} = \int_S \mathbf{J}_f \cdot \mathbf{dS} + \frac{d}{dt}\int_S \mathbf{D} \cdot \mathbf{dS}$ | $\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$ | $\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_f$ |
| **Gauss's Law** | | |
| $\oint_S \mathbf{D} \cdot \mathbf{dS} = \int_V \rho_f \, dV$ | $\nabla \cdot \mathbf{D} = \rho_f$ | $\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_f$ |
| $\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$ | $\nabla \cdot \mathbf{B} = 0$ | $\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0$ |
| **Conservation of Charge** | | |
| $\oint_S \mathbf{J}_f \cdot \mathbf{dS} + \frac{d}{dt}\int_V \rho_f \, dV = 0$ | $\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0$ | $\mathbf{n} \cdot (\mathbf{J}_2 - \mathbf{J}_1) + \frac{\partial \sigma_f}{\partial t} = 0$ |

Usual Linear Constitutive Laws

$\mathbf{D} = \varepsilon \mathbf{E}$

$\mathbf{B} = \mu \mathbf{H}$

$\mathbf{J}_f = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \sigma \mathbf{E}'$ [Ohm's law for moving media with velocity $\mathbf{v}$]

## PHYSICAL CONSTANTS

| Constant | Symbol | Value | units |
|---|---|---|---|
| Speed of light in vacuum | $c$ | $2.9979 \times 10^8 \approx 3 \times 10^8$ | m/sec |
| Elementary electron charge | $e$ | $1.602 \times 10^{-19}$ | coul |
| Electron rest mass | $m_e$ | $9.11 \times 10^{-31}$ | kg |
| Electron charge to mass ratio | $\frac{e}{m_e}$ | $1.76 \times 10^{11}$ | coul/kg |
| Proton rest mass | $m_p$ | $1.67 \times 10^{-27}$ | kg |
| Boltzmann constant | $k$ | $1.38 \times 10^{-23}$ | joule/°K |
| Gravitation constant | $G$ | $6.67 \times 10^{-11}$ | nt-m$^2$/(kg)$^2$ |
| Acceleration of gravity | $g$ | $9.807$ | m/(sec)$^2$ |
| Permittivity of free space | $\varepsilon_0$ | $8.854 \times 10^{-12} \approx \frac{10^{-9}}{36\pi}$ | farad/m |
| Permeability of free space | $\mu_0$ | $4\pi \times 10^{-7}$ | henry/m |
| Planck's constant | $h$ | $6.6256 \times 10^{-34}$ | joule-sec |
| Impedance of free space | $\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ | $376.73 \approx 120\pi$ | ohms |
| Avogadro's number | $N$ | $6.023 \times 10^{23}$ | atoms/mole |