Gauss's Law

In differential form $\nabla \bullet \vec{D} = \rho$. Since this equation is applied to an arbitrary point in space, we can use it to determine the charge density once we know the field.

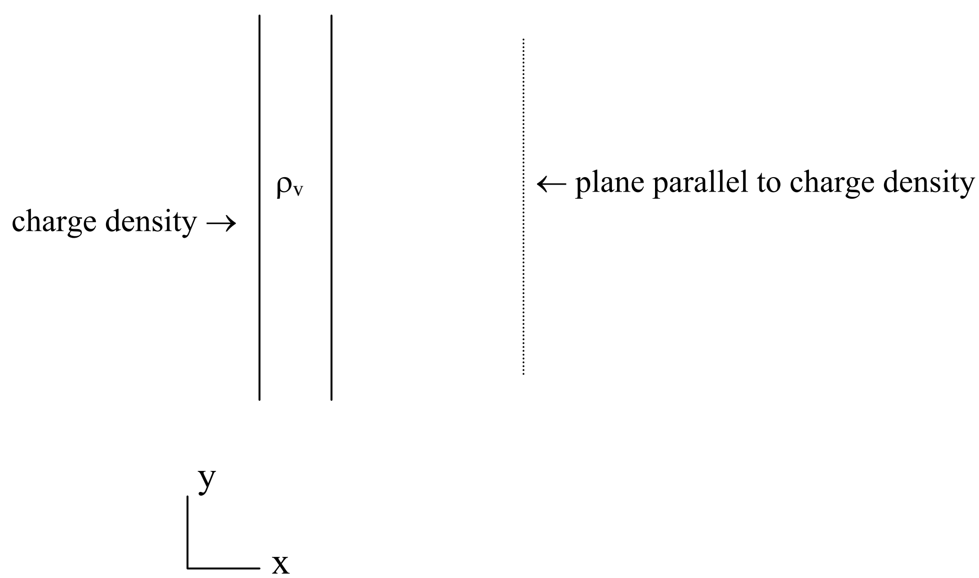
(We can use this equation to solve for the field if we know the charge density. There must be some symmetry so that we can eliminate two of the vector components of $\vec{D}$. However, during this course, we don't utilize this technique.)

In integral form $\int_S \vec{E} \bullet d\vec{S} = \frac{1}{\varepsilon} \int_V \rho_V dV$. This equation is used to determine the total field (flux) passing through a surface due to the charge enclosed. We can apply this integral if we know the charge distribution **and** we can make assumptions about the behavior of $\vec{E}$. Note, the vector $\vec{D}$ and $\vec{E}$ are related by the permittivity, ε.

In this course, we limit our field distributions and geometries to the following symmetric problems:

1. Planar charge density – the charge is an infinite sheet or slab and has a constant density on a plane. By symmetry, the electric field on any plane parallel to the plane of the charge density will be constant and perpendicular to that plane.

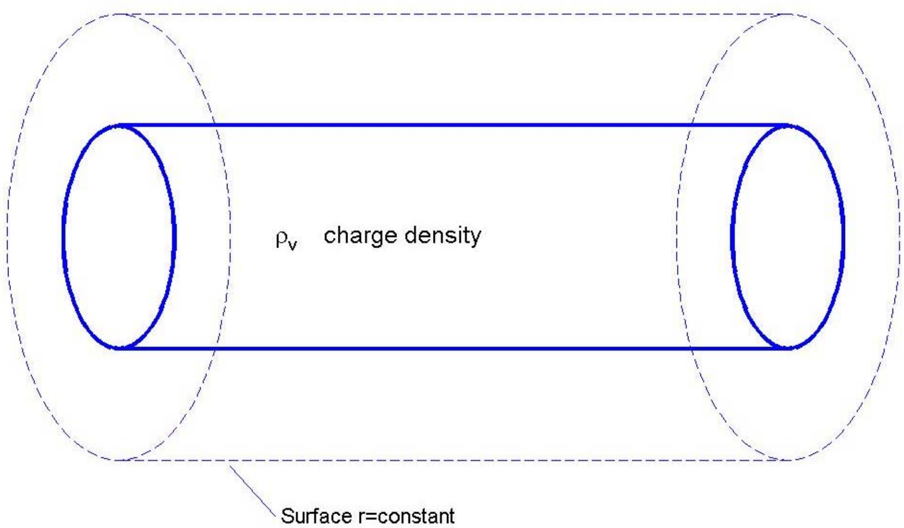
Example: Slab of charge



The field will point in the $\hat{x}$ (or -$\hat{x}$) direction for this geometry. For symmetry to apply, the charge density can only be a function of *x*.

2. Cylindrical charge density – the charge is a cylindrical column or shell and has a constant density on a surface r = *constant*. Note, on that surface the charge is constant but that does not imply it is a surface charge. By symmetry the electrical field will only point in the radial direction, $E \rightarrow E_r$

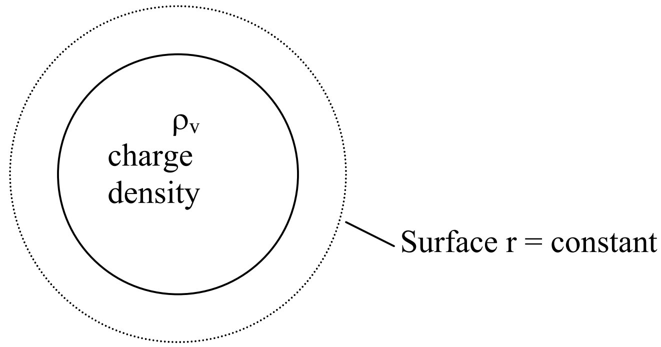
Example: Cylindrical volume charge



The field will point in the $\hat{r}$ (or -$\hat{r}$) direction for this geometry. For symmetry to apply, the charge density can only be a function of r.

3. Spherical charge density – the charge is a spherical volume or shell and has a constant density on a surface $r = constant$. Note, on that surface the charge is constant but that does not imply it is a surface charge. By symmetry the electrical field will only point in the radial direction, $E \rightarrow E_r$

Example: Spherical volume charge



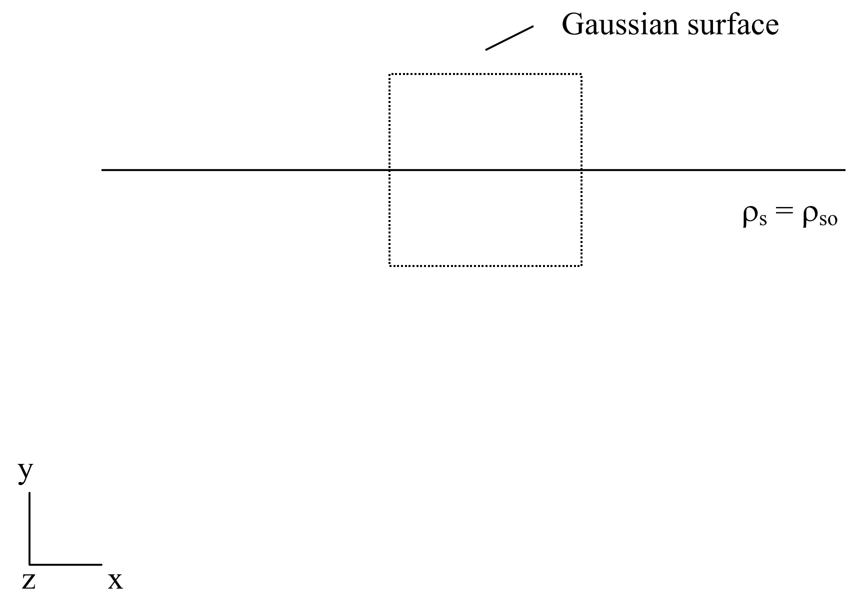
The field will point in the $\hat{r}$ (or -$\hat{r}$) direction for this geometry. For symmetry to apply, the charge density can only be a function of r.

An important point to recognize is that these types of problems are presented since they are simple geometries that we are capable of solving in class. These symmetric problems provide us with a simplification of the left side of the Gauss's Law equation, which is very difficult to achieve in practice. It is rare that an interesting problem will have an exact analytic solution. Occasionally, we may be able to simplify the approach when considering a solution in a given region, but that solution may not be valid outside of that domain.

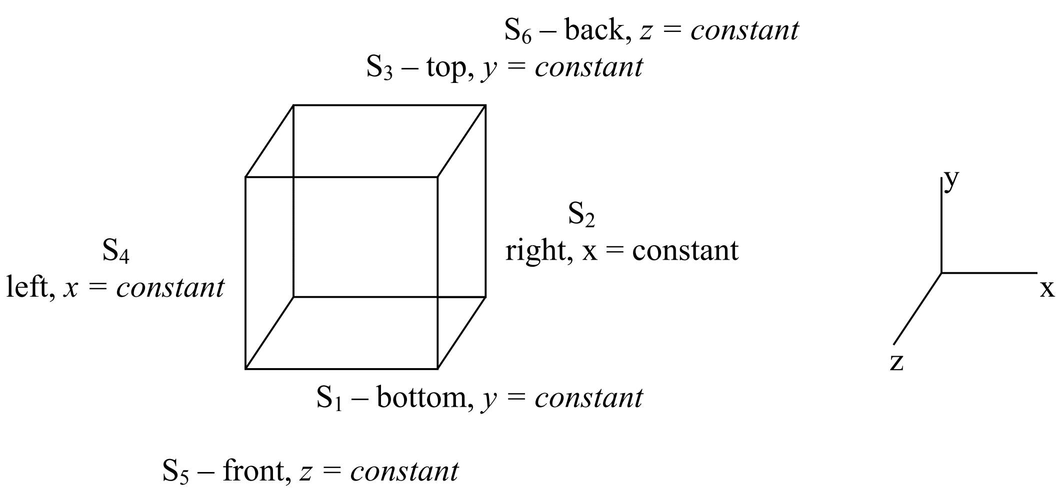
To solve an integral form of Gauss's Law, we need to perform the following steps

1.) Recognize the coordinate system
2.) Using symmetry, determine which components of the field exist
3.) Create a Gaussian surface such that the sides of the surface are either parallel or perpendicular to the direction of the field in step 2 – remember, the Gaussian surface is arbitrary in size
4.) Determine the total charge inside that surface. The charge distribution can be a volume, surface, line and/or point charge.
5.) Evaluate the flux passing through that surface. If the field is parallel to the surface, then $\vec{E} \bullet d\vec{S} = 0$. If the field is perpendicular to the surface, then $\int_S \vec{E} \bullet d\vec{S} \rightarrow E_i \int_S dS$, where E_i is the field in the component direction. Again, it is important to recognize that this simplification only exists for the symmetric problems we use in class. Real problems are much more complex.
6.) You can now equate the results from step 4 and step 5 to determine the field in the region you define the Gaussian surface.

As an example, consider a sheet of charge that is located on the plane y = 0 and has a charge density ρ_{so}. A sheet of charge is a surface charge density. In order for symmetry to apply, the sheet must have a uniform charge density. The coordinates are given in the lower left. The sheet is infinite in the z-direction (perpendicular to the paper) and in the x-direction.

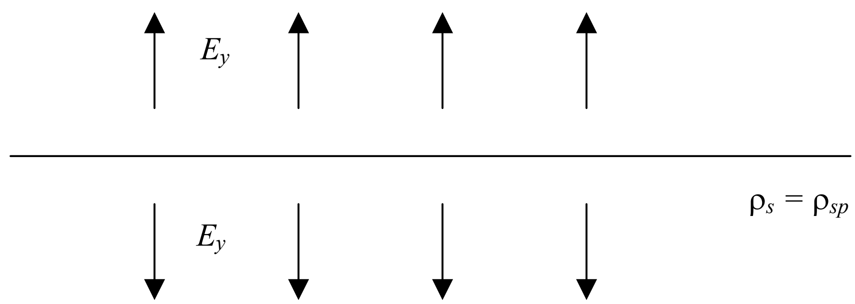


The Gaussian surface is drawn as an overhead view. In actuality, it has six sides:



Step 1) The geometry is Cartesian

Step 2) By symmetry, the field is in the y direction. It is important to note that the field is positive for $y > 0$ and negative for $y < 0$ since the charge distribution is positive. The direction of the field lines indicates positive or negative field.



Step 3) The Gaussian surface is in the previous figures. The box has sides of length, l. Also, the box is symmetric about the surface charge distribution. We know that the field on a plane parallel to the surface charge is constant, so the magnitude of the field at some location $y=C$ must be the same as that at the location $y=-C$. However, we do not know how the field behaves as a function of y.

Step 4) To determine the total charge inside the box, we must integrate across the section of surface charge enclosed by our Gaussian surface.

$$Q_{enc} = \int_0^l \int_0^l \rho_{so} \, dx dz$$

The limits of integration indicate that the charge enclosed is independent of the origin. Again, a uniform charge distribution is necessary for this assumption. Now,

$$Q_{enc} = \rho_{so} l^2$$

The total charge enclosed is dependent on the dimensions of the Gaussian surface.

Step 5) To determine the total flux passing through the Gaussian surface, we must integrate across all six sides that make up that surface,

$$\oint_S \vec{E} \bullet d\vec{S} = \sum_{n=1}^{6} \int_{S_n} \vec{E} \bullet d\vec{S}$$

This equation can be simplified by noticing that the flux passing through sides 2, 4, 5, and 6 is zero. The field is parallel to those surfaces and therefore the dot product between the field and the unit vector normal to the surface is zero, $\vec{E} \bullet d\vec{S} = 0$. In other words, the flux passing through those surfaces is zero. Now,

$$\oint_S \vec{E} \bullet d\vec{S} = \int_{S_1} \vec{E} \bullet d\vec{S} + \int_{S_3} \vec{E} \bullet d\vec{S}$$

On side 1, $d\vec{S} \rightarrow -dxdz\ \hat{y}$ and the field 'points' in the negative y direction.

On side 3, $d\vec{S} \rightarrow dxdz\ \hat{y}$ and the field 'points' in the positive y direction.

For both sides $\vec{E} \bullet d\vec{S} = E_y dxdz$

$$\oint_S \vec{E} \bullet d\vec{S} = 2E_y \int_0^l \int_0^l dxdz$$

The *2* indicates that both integrals yield the same results. Again, the limits of integration indicate an arbitrary sized box.

$$\oint_S \vec{E} \bullet d\vec{S} = 2E_y l^2$$

Step 6) Equate sides to find the field.

$$2E_y l^2 = \rho_{so} l^2$$

$$E_y = \frac{1}{2}\rho_{so}$$

We have the magnitude of the field. Note, the magnitude of the field is independent of y and that the dimensions of the Gaussian surface vanish from the solution. We still need to include the direction, so a complete solution would look like.

$$\vec{E} = \begin{cases} \frac{1}{2}\rho_{so}\hat{y} & y > 0 \\ \frac{-1}{2}\rho_{so}\hat{y} & y < 0 \end{cases}$$