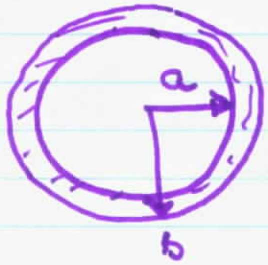
H.W.3 Solution

1. a. Spherical (r, θ, ϕ)

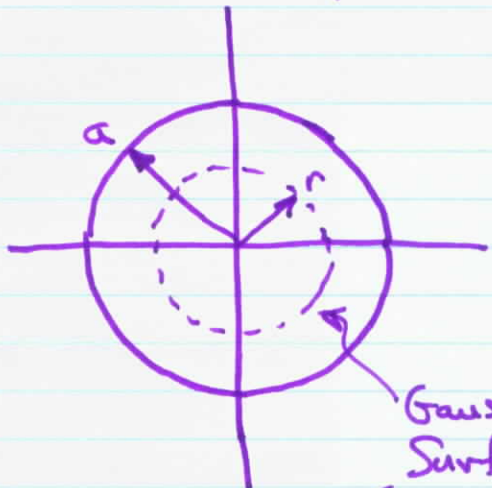
b.



$d\vec{S} = \hat{r}\, r^2 \sin\theta\, d\theta\, d\phi$

This is hard to draw
Anything reasonable is OK.

c. Because $\rho = \rho(r)$ only $\Rightarrow$ $\vec{E} = \hat{r}\, E_r(r)$



Gaussian Surface
(can also be outside the charge

$$\oint \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \int \rho_v \, dv$$

$$E_r(r)\, 4\pi r^2 = E \times \text{area}$$

Inside charge $(0 \le r \le a)$

$$\int \rho \, dv = \rho_0 \frac{4}{3}\pi r^3$$

Outside charge $(a \le r)$

$$\int \rho \, dv = \rho_0 \frac{4}{3}\pi a^3$$

For $0 \le r \le a$

$$E_r(r) = \frac{\rho_0 \frac{4}{3}\pi r^{\cancel{3}}}{\cancel{4\pi r^2}} \frac{1}{\varepsilon_0}$$

$$= \frac{\rho_0 r}{3\varepsilon_0} \quad \Rightarrow \quad \vec{E} = \hat{r}\, \frac{\rho_0 r}{3\varepsilon_0}$$

For $a \le r$

$$E_r(r) = \frac{\rho_0 \frac{4}{3}\pi a^3}{4\pi r^2 \varepsilon_0} = \frac{\rho_0 a^3}{3\varepsilon_0 r^2} \qquad \vec{E} = \frac{\rho_0 a^3}{3\varepsilon_0 r^2}\,\hat{r}$$

d. $\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + 0 + 0$

For $0 \le r \le a$

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\rho_0 r}{3\varepsilon_0}\right) = \frac{1}{r^2}\frac{\rho_0}{3\varepsilon_0}\frac{\partial}{\partial r}(r^3)$$

$$= \frac{\cancel{3r^2}\rho_0}{\cancel{r^2}\cancel{3}\varepsilon_0} = \frac{\rho_0}{\varepsilon_0} \checkmark$$

For $a \le r$

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\frac{\rho_0 a^3}{3\varepsilon_0 \cancel{r^2}}\right) = 0 \checkmark$$

e. $$\underbrace{V(r) - V(c)}_{V(c)=0} = -\int_c^r \vec{E} \cdot d\vec{l}$$

For $a \le r$

$$V(r) = -\int_c^r \frac{\rho_0 a^3}{3\varepsilon_0 r^2}\hat{r}\cdot\hat{r}\,dr = +\frac{\rho_0 a^3}{3\varepsilon_0}\frac{1}{r}\Big|_c^r$$

$$= \frac{\rho_0 a^3}{3\varepsilon_0}\left(\frac{1}{r} - \frac{1}{c}\right)$$

note that this holds for $r<c$ & $r>c$

For $r = a$ $\quad V(a) = \frac{\rho_0 a^3}{3\varepsilon_0}\left(\frac{1}{a} - \frac{1}{c}\right)$

For $0 \le r \le a$

$$V(r) - V(a) = -\int_a^r \frac{\rho_0 r}{3\varepsilon_0}\,dr$$

$$= -\frac{\rho_0}{3\varepsilon_0}\left(\frac{r^2 - a^2}{2}\right)$$

$$= \frac{\rho_0}{6\varepsilon_0}(a^2 - r^2)$$

f. $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right)$

For $r \geq a$ $= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\frac{\rho_0 a^3}{3\varepsilon_0}\left(-\frac{1}{\cancel{r^2}}\right)\right)$

$= \frac{1}{r^2}\frac{\partial}{\partial r}(\text{constant}) = 0$

For $0 \leq r \leq a$ $\nabla^2 V(a) = 0$ $\frac{\partial}{\partial r}(\text{const}) = 0$

$\nabla^2 V = \nabla^2 \frac{\rho_0}{6\varepsilon_0}(a^2 - r^2)$

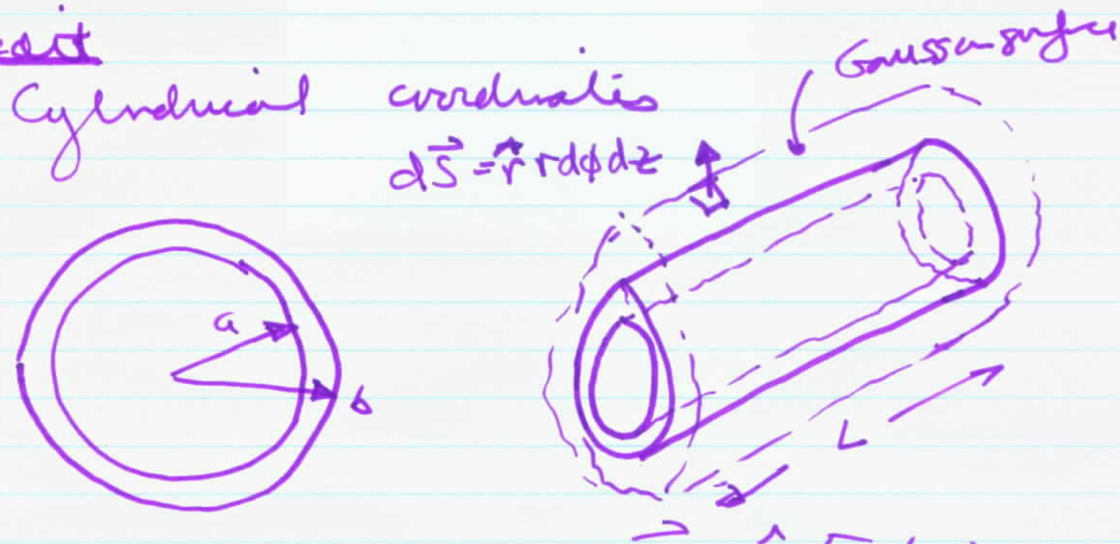
$= -\nabla^2 \frac{\rho_0}{6\varepsilon_0} r^2 = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{\rho_0 r^2}{6\varepsilon_0}\right)\right)$

$= -\frac{\rho_0}{6\varepsilon_0}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\, 2r\right)$

$= -\frac{\rho_0}{6\varepsilon_0}\frac{1}{\cancel{r^2}}\, 6\cancel{r^2} = -\frac{\rho_0}{\varepsilon_0}$ ✓

<u>Extra Credit</u>

g. Cylindrical coordinates

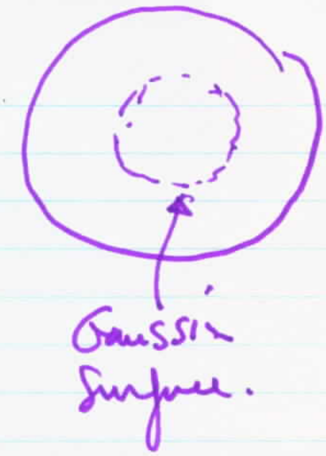


Because $\rho = \rho(r) \Rightarrow \vec{E} = \hat{r} E_r(r)$

$\oint \vec{E}\cdot d\vec{S} = E_r(r)\, 2\pi r L$ $= E \times \text{area}$

Inside charge $\int \rho \, dv = \pi r^2 L \rho_0$

Outside charge $\int \rho \, dv = \pi a^2 L \rho_0$



For $0 \le r \le a$

$$E_r(r)\, 2\pi r L = \frac{\pi r^2 \cdot L \rho_0}{\varepsilon_0}$$

$$E_r(r) = \frac{\cancel{\pi} r^{\cancel{2}} \cancel{L} \rho_0}{2 \cancel{\pi} \cancel{r} \cancel{L} \varepsilon_0} = \frac{r \rho_0}{2 \varepsilon_0} \qquad \vec{E} = \hat{r} \frac{r \rho_0}{2 \varepsilon_0}$$

For $a \le r$

$$E_r(r)\, 2\pi r L = \frac{\pi a^2 L \rho_0}{\varepsilon_0}$$

$$E_r(r) = \frac{\cancel{\pi} a^2 \cancel{L} \rho_0}{2 \cancel{\pi} r \cancel{L} \varepsilon_0} = \frac{a^2 \rho_0}{2 \varepsilon_0 r} \qquad \vec{E} = \hat{r} \frac{a^2 \rho_0}{2 \varepsilon_0 r}$$

$$\nabla \cdot \vec{E} = \frac{1}{r} \frac{\partial}{\partial r}(r E_r) + 0 + 0$$

For $0 \le r \le a$

$$\nabla \cdot \vec{E} = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{r \rho_0}{2 \varepsilon_0} \right) = \frac{1}{r} \frac{\rho_0}{2 \varepsilon_0} \frac{\partial}{\partial r} r^2$$

$$= \frac{1}{\cancel{r}} \frac{\rho_0}{\cancel{2} \varepsilon_0} \cancel{2} \cancel{r} = \frac{\rho_0}{\varepsilon_0} \checkmark$$

For $a \le r$

$$\nabla \cdot \vec{E} = \frac{1}{r} \frac{\partial}{\partial r}\left(\underbrace{\cancel{r} \frac{a^2 \rho_0}{2 \varepsilon_0 \cancel{r}}}_{\text{const}} \right) = 0 \checkmark$$

$$V(r) - \underset{\substack{\parallel \\ 0}}{V(c)} = -\int_c^r \vec{E} \cdot d\vec{l}$$

For $a \leq r$

$$V(r) = -\int_c^r \frac{a^2 \rho_0}{2\varepsilon_0 r}\, dr = \frac{a^2 \rho_0}{2\varepsilon_0} \ln \frac{c}{r}$$

at $r = a$

$$V(a) = \frac{a^2 \rho_0}{2\varepsilon_0} \ln \frac{c}{a}$$

For $r \leq a$

$$V(r) - V(a) = -\int_a^r \frac{\rho_0 r}{2\varepsilon_0}\, dr$$

$$= \frac{\rho_0}{2\varepsilon_0} \left(\frac{a^2 - r^2}{2} \right)$$

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial V}{\partial r}$$

For $r \geq a$ $\quad \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \left(\frac{a^2 \rho_0}{2\varepsilon_0} \ln \frac{c}{r} \right)$

$$= \frac{1}{r} \frac{\partial}{\partial r} \underbrace{r \frac{a^2 \rho_0}{2\varepsilon_0} \left(\frac{1}{r} \right)}_{\text{const}} = 0 \quad \checkmark$$

For $r \leq a$ $\quad \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \left(\underbrace{V(a) + \frac{\rho_0}{2\varepsilon_0} \frac{a^2}{2}}_{\text{const}} - \frac{\rho_0}{2\varepsilon_0} \frac{r^2}{2} \right)$

$$= \frac{1}{r} \frac{\partial}{\partial r} r \left(- \frac{\rho_0}{\cancel{2}\varepsilon_0} \frac{\cancel{2} r}{2} \right) = -\frac{\rho_0}{\varepsilon_0} \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{r^2}{2} \right)$$

$$= -\frac{\rho_0}{\varepsilon_0} \frac{1}{r} \frac{2r}{2} = -\frac{\rho_0}{\varepsilon_0} \quad \checkmark$$

Spreucil Case Only

2. Plastic Shell is between $r=a$ & $r=b$

Geometry is the same so

For $0 \le r \le a$ $\qquad \vec{E} = \hat{r}\,\dfrac{\rho_0 r}{2\varepsilon_0}$

For $a \le r < b$ $\qquad \vec{E} = \hat{r}\,\dfrac{\rho_0 a^3}{3\varepsilon r^2}$

← Note

For $b \le r \le c \ldots$ to ∞

$$\vec{E} = \hat{r}\,\frac{\rho_0 a^3}{3\varepsilon_0 r^2}$$

The form of the solution in each region is the same except for the ε $\Rightarrow$

$0 \le r \le a$ $\qquad \nabla\cdot\vec{E} = \dfrac{\rho_0}{\varepsilon_0}$ ✓

$a \le r \le b$ $\qquad \nabla\cdot\vec{E} = 0$ ✓

$b \le r$ $\qquad \nabla\cdot\vec{E} = 0$ ✓

$$V(r) - \underset{0}{\underset{\shortparallel}{V(c)}} = -\int_c^r E_r\,dr$$

$r \ge b$ $\qquad V(r) = \dfrac{\rho_0 a^3}{3\varepsilon_0}\left(\dfrac{1}{r} - \dfrac{1}{c}\right)$

$$V(b) = \frac{\rho_0 a^3}{3\varepsilon_0}\left(\frac{1}{b} - \frac{1}{c}\right)$$

$a \le r \le b$ $\qquad V(r) - V(b) = \dfrac{\rho_0 a^3}{3\varepsilon}\left(\dfrac{1}{r} - \dfrac{1}{b}\right)$

← note

$$V(a) = V(b) + \frac{\rho_0 a^3}{3\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

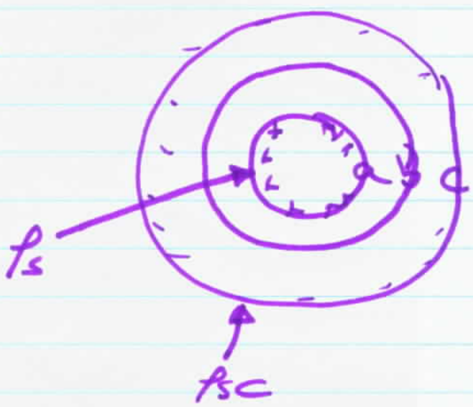
$0 \le r \le a \qquad V(r) - V(a) = -\int_a^r \frac{\rho_0 r}{3\varepsilon_0}\, dr$

$$V(r) = V(a) + \frac{\rho_0}{6\varepsilon_0}(a^2 - r^2)$$

The form of the functions is the same so that the $\nabla^2 V = -\rho/\varepsilon_0$ in the charge region & zero elsewhere

Cylindrical case – not necessary to repeat.

3.



$\rho_s 4\pi a^2 = \rho_0 \frac{4}{3}\pi a^3$

$\rho_s = \rho_0 \frac{a}{3}$

$\rho_s \cancel{4\pi} a^2 = -\rho_{sc} \cancel{4\pi} c^2$

$\rho_{sc} = -\rho_s \frac{a^2}{c^2}$

Total charge at $r = a$ = total charge at $r = c$

4. The $V(r)$ & $\vec{E}(r)$ will be exactly the same in the regions $a \le r \le b$, $b \le r \le c$

The big difference is that $\vec{E} = 0$ outside $r = c$ & $V = 0$ outside $r \ge c$

$\vec{E} = 0$ for $r > c$ because no net charge is enclosed. $V = 0$ because $\vec{E} = 0$ & $V(c) = 0$.

Voltage difference $V(a) - \underset{\overset{\shortparallel}{0}}{V(c)} = V(a)$

is just $V(a)$ because $V(c) = 0$

$$V(a) = V(b) + \frac{\rho_0 a^3}{3\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

note that $\rho_0 = \frac{3\rho_s}{a}$ so that

$$V(b) = \frac{\cancel{3}\rho_s}{\cancel{a}} \frac{a^{\cancel{3}2}}{\cancel{3}\varepsilon_0}\left(\frac{1}{b} - \frac{1}{c}\right)$$

$$= \frac{\rho_s a^2}{\varepsilon_0}\left(\frac{1}{b} - \frac{1}{c}\right)$$

Inner $V(a.) = V(b) + \frac{\cancel{3}\rho_s}{\cancel{a}} \frac{a^{\cancel{3}2}}{\cancel{3}\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$

$$= \frac{\rho_s a^2}{\varepsilon_0}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{\rho_s a^2}{\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

If the dielectric switches to the outer region

Outer $V(a) = \frac{\rho_s a^2}{\varepsilon}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{\rho_s a^2}{\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$

$$\frac{1}{b} - \frac{1}{c} = \frac{1}{2.41a} - \frac{1}{3a} = \frac{1}{a} 0.08$$

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a} - \frac{1}{2.41a} = \frac{1}{a} 0.585$$

Inner case $V(a) = \frac{\rho_s a^{\cancel{2}}}{\varepsilon_0}\left[\frac{0.08}{\cancel{a}} + \frac{1}{4\cancel{a}} 0.585\right]$

$$= \frac{\rho_s a}{\varepsilon_0}\left[.228\right]$$

Outer $V(a) = \frac{\rho_s a^2}{\varepsilon_0}\left[\frac{1}{4}\frac{.08}{a} + \frac{.585}{8}\right]$

$$= \frac{\rho_s a}{\varepsilon_0}\left[0.6055\right]$$

The outer case produces the larger voltage difference for the same charge.

5. Extra Credit

Now we assume that the voltage difference $V(a)$ is the same and the charge changes for the two cases.

For the inner case $V(a) = \frac{\rho_s a}{\varepsilon_0}\left[.228\right]$

or $\rho_{s\,in} = \frac{4.39\ \varepsilon_0 V(a)}{a}$

For the outer case $V(a) = \frac{\rho_s a}{\varepsilon_0}\left[0.6055\right]$

$$\rho_{s\,out} = 1.65\ \frac{\varepsilon_0 V(a)}{a}$$

So $\rho_{s\,in} / \rho_{s\,out} = 2.72$

Plotting the E fields for both cases we see that the max E occurs at $r = a$

Inner case

$$E_r(a) = \frac{\rho_0 a^{\cancel{3}}}{3\varepsilon \cancel{a^2}} = \frac{\rho_0 a}{3\varepsilon} = \frac{3\rho_s}{a}\frac{a}{3\varepsilon} = \frac{\rho_s}{\varepsilon}$$

which actually is always the case

Outer case $\quad E_r(a) = \dfrac{\rho_s}{\varepsilon_0}$

Using the charge density values from the previous page,

Inner $\quad E_r(a) = \dfrac{4.39\,\cancel{\varepsilon_0}\,V(a)}{4\cancel{\varepsilon_0}\,a} = 1.097\,\dfrac{V(a)}{a}$

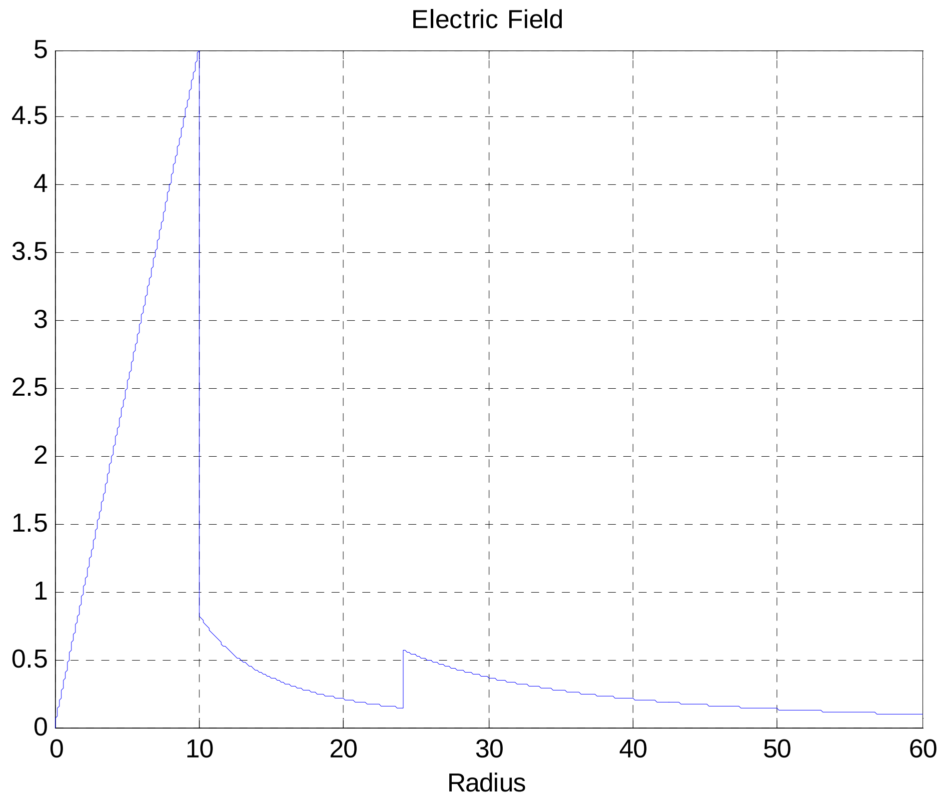
Outer $\quad E_r(a) = \dfrac{1.65\,\cancel{\varepsilon_0}\,V(a)}{\cancel{\varepsilon_0}\,a} = 1.65\,\dfrac{V(a)}{a}$

Thus the outer case produces the larger electric field and, thus, is more likely to have breakdown problems.

Electric Field as Function of Radius for the Various Cases

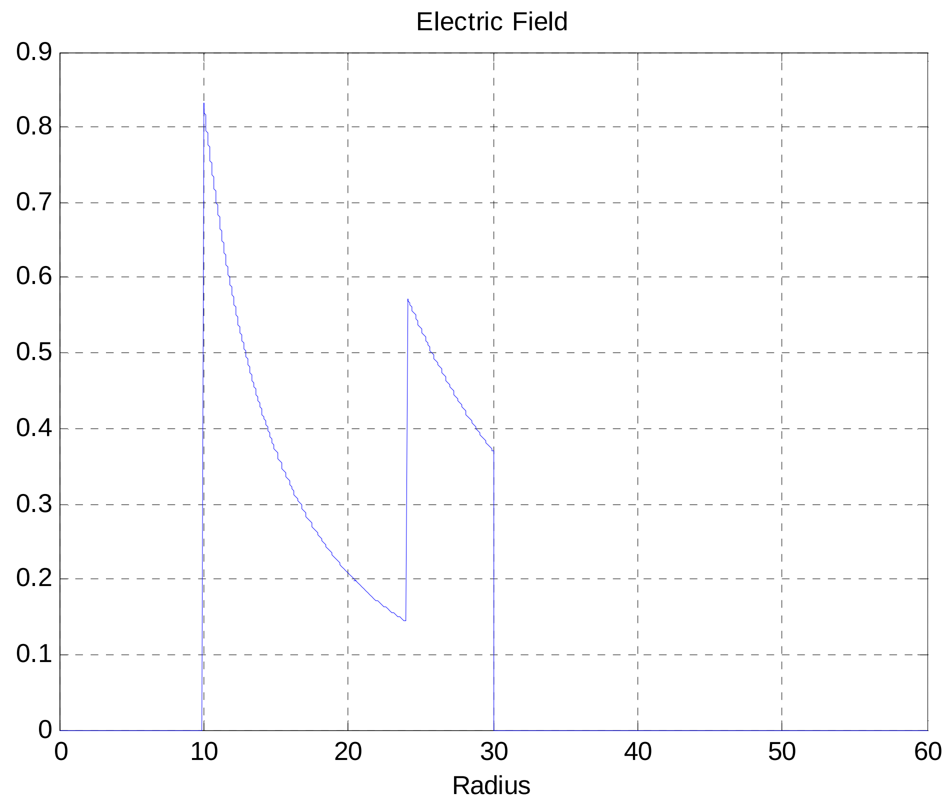
Since we need some real numbers to use Matlab, we choose $a = 10$. Also, we suppress the charge density ρ_o and the permittivity of free space ε_o in all expressions. Thus, the actual electric field should be multiplied by $\frac{\rho_o}{\varepsilon_o}$. We also assume $\varepsilon = 4\varepsilon_o$ in the plastic shell.

First, for the case where there are no conductors and there is a volume charge in the central spherical region.

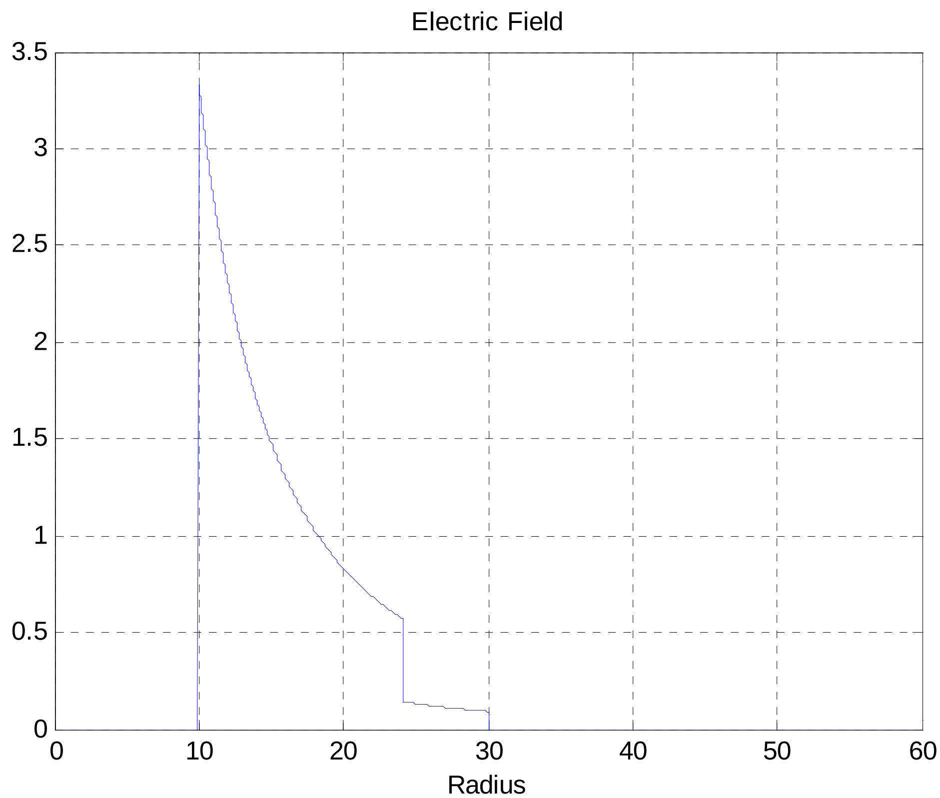


Note that the field increases linearly with radius out to *a*, then drops by a factor of 4 at the boundary with the plastic shell. At *b*, it increases by a factor of 4 and then decays toward zero.

Next, we add the conductor in the central spherical region and outside of $r = c$. The E field must go to zero in the conductors, which we observe to be the case. Also the maximum electric field is lower now, so the plot appears to expand some. Note that the maximum E field is found at $r = a$, even though its value is suppressed by a factor of 4 due to the dielectric.



Finally, we reverse the location of the plastic shell, moving it to the outer region. We clearly see that the maximum E field is much larger in this case.



Before we can compare the actual values of the E fields, however, we must remember that the condition on this problem is that the voltage difference is the same, not the charge. In the plots above the charge is assumed to be the same.