Fields and Waves I

Quiz 2

Spring 2003

KEY

1. (20) __________________
2. (10) __________________
3. (25) __________________
4. (20) __________________
5. (15) __________________
6. (10) __________________

Total __________________

Name __________________

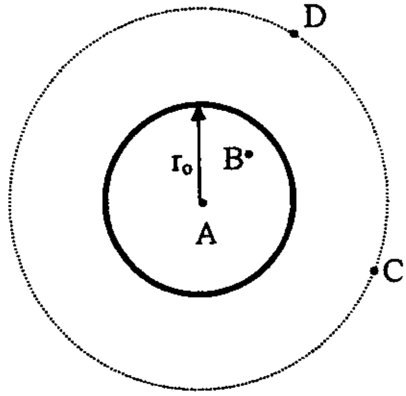
A couple useful equations:

Notes:

1) In the Multiple Choices section, each question may have more than one correct answer. Circle all correct answers.
2) For the Multiple Choices, you may add some comments to justify your answer.
3) If you are stuck on part of a problem, but understand the following parts, assume an expression that you would consider reasonable and use it for the remaining parts. Partial credit will depend on using a reasonable expression.

1) Multiple Choices (20)

i) The following figure represents a cylindrical shell of surface charge, located at $r = r_o$. The dashed line is concentric with the charge distribution. The voltages at the four points indicated are considered in the question.



Which of the following are true:

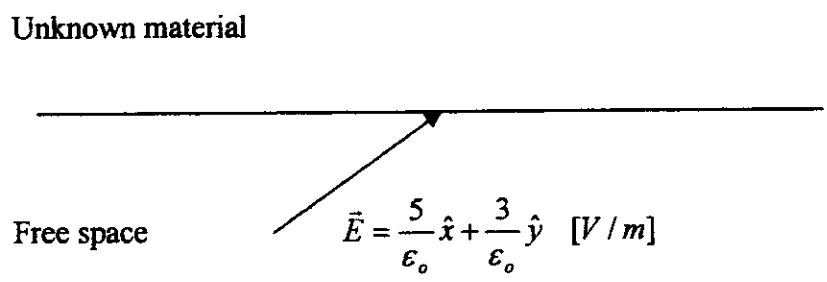
a. $|V_B - V_A| > 0$
b. $|V_D - V_B| > 0$
c. $|V_D - V_C| > 0$
d. $|V_D - V_B| > |V_D - V_A|$

ii) Which of the following is a true statement about Maxwell's equations relating to electrostatics:

a. In a region where there is no charge, the total electric flux entering a volume must be equal to the total electric flux leaving that volume.
b. The continuity of the tangental electric field across a dielectric-dielectric boundary can be derived from the curl free property ($\nabla \times \vec{E} = 0$) of static electric fields.
c. The voltage difference between two points depends on the integration path between the points.
d. If a closed surface encloses zero net charge, the field on that surface must be zero as well.

iii) The capacitance of two fixed conductors with free space between them:

a. Increases if the voltage difference between the conductors decreases.
b. Increases if the magnitude of the charge on a conductor increases.
c. Increases if a dielectric material is placed between the conductors.
d. Increases if the stored electric field energy, W_E, increases.



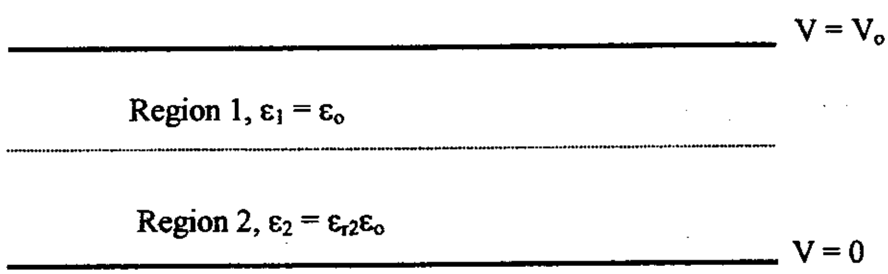
iv) The above figure indicates an electric field line incident upon a boundary between free space and some unknown material. The electric field is given on free space side, but very close to the boundary. The boundary is located at the position $y = 0$.

a. It is possible that the unknown material is a perfect conductor .

b. If the unknown material is a dielectric, the electric field on that side of the boundary will have a <u>smaller</u> total magnitude.

c. If the unknown material is a dielectric, the electric field on that side of the boundary will have a <u>larger</u> total magnitude.

d. If there is a surface charge density, $\rho_s = 3$ [C/m^2] at $y = 0$, the electric field in the unknown material is parallel to the boundary.



v) A parallel plate capacitor is shown above. A dielectric material fills half of the volume, as indicated by the dashed line. Which of the following are true:

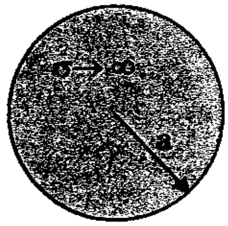
a. $\vec{E}$ in region 1 is larger than $\vec{E}$ in region 2.

b. $\vec{D}$ in region 1 is larger than $\vec{D}$ in region 2.

c. The magnitude of ρ_s on the top plate is larger than the magnitude of ρ_s on the bottom plate.

d. The total charge on the bottom plate is zero since it is grounded.

2) Simple Problem (10)



A solid spherical perfect conductor with radius, a, is shown in the above figure. The surrounding medium is free space. A total charge, Q [C], is distributed on the conductor.

What is the charge density? (5)

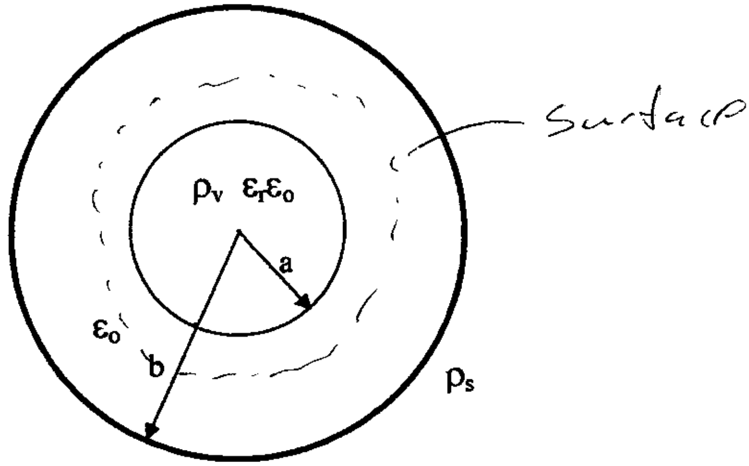
$$\rho_s = \frac{Q}{4\pi a^2}$$

What is the electric field, $\vec{E}$, everywhere? (5)

$$\iiint \bar{E} \cdot d\bar{S} = \frac{Q}{\epsilon_0} \qquad d\bar{S} \rightarrow r^2 \sin\theta \, \cancel{dr} \, d\theta \, d\phi$$

$$E = \frac{Q}{4\pi\epsilon r^2} \hat{r} \quad [V/m]$$

3) Gauss's Law (25 points)



The above figure represents two spherical charge distributions, a volume charge for the region $r < a$ and a surface charge located at $r = b$, where $a < b$. The total charges of each distribution are equal and opposite. The volume charge is distributed throughout a dielectric material with permittivity, $\varepsilon_r\varepsilon_o$, The rest of the domain is free space.

$$\rho = \begin{cases} \rho_o r & r < a \quad [C/m^3] \\ \rho_s & r = b \quad [C/m^2] \end{cases}$$

Indicate the regions where you would solve for the field. (1)

$r<a, \quad a<r<b, \quad b<r$

On the above figure, sketch a surface you would use to apply Gauss's Law. (1)

Determine the surface charge density, ρ_s, in terms of the known volume charge density and geometry. (3)

$$\int \rho_s \, ds = -\int \rho_v \, dv$$

$$\rho_s 4\pi b^2 = \cancel{\tfrac{4}{3}\pi a^3 \rho}\ 4\pi \frac{a^4}{4}\rho_o$$

$$\rho_s = \frac{a^4}{4b^2}\rho_o \ [C/m^2]$$

Determine the displacement field, $\vec{D}$, everywhere. (5)

$r<a$ $\int \vec{D} \cdot d\vec{S} \rightarrow 4\pi r^2 D_r$

$\int \rho \, dv \rightarrow \int_0^{2\pi}\int_0^{\pi}\int_0^{r} \rho_o r^3 \sin\theta \, dr \, d\theta \, d\phi$

$= \pi r^4 \rho_o$

$D_r = \frac{r^2}{4} \rho_o \hat{r} \, [C/m^2]$

$a<r<b$

$\int \rho \, dv = \pi a^4 \rho_o$

$D_r = \frac{a^4}{4r^2} \rho_o \hat{r} \, [C/m^2]$

$b<r$

$D_r = 0$

Determine the electric field, $\vec{E}$, everywhere. (5)

$$E_r = \begin{cases} \frac{r^2}{\epsilon_o 4} \rho_o \hat{r} & r<a \\ \frac{a^4}{4\epsilon_o r^2} \rho_o \hat{r} & a<r<b \\ 0 & b<r \end{cases} \quad [V/m]$$

Name ____________ ECSE-2100 Spring 2003 Section ____________

Verify your field solution is correct using both differential Maxwell's equations that apply to electrostatics. (5)

$\nabla \times E = 0$ by observation

$r < a$

$$\nabla \cdot D \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^4}{4}\rho_0\right) = \rho_0 r \quad \checkmark$$

$a < r < b$

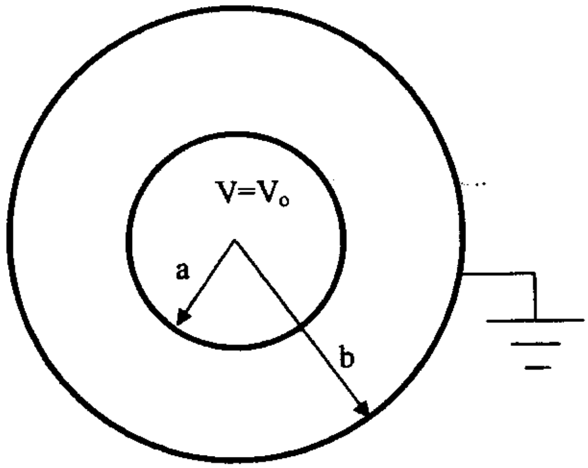
$$\nabla \cdot D \Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{a^4\rho_0}{4}\right) = 0 \quad \checkmark$$

Determine the total energy, W_E, stored in the region $r < a$. (5)

$$W_E = \frac{1}{\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho_0^2 \frac{r^4}{16} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{4\pi}{16\epsilon_0}\frac{a^7}{7}\rho_0^2 \quad [J]$$

4) Laplace, Field, Capacitance (20)



A cylindrical capacitor is shown above. A dielectric material, $\varepsilon_r = 4$, fills the space between the conductor. The voltage on the inner conductor is V_o and the outer conductor is grounded. The dimensions are shown.

Determine the voltage as a function of position. (5)

$$\nabla^2 V = 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

$$\Rightarrow r\frac{\partial V}{\partial r} = C_1$$

$$V(r) = C_1 \ln r + C_2$$

$$\left.\begin{aligned} V(b) = 0 &= C_1 \ln b + C_2 \\ V(a) = V_o &= C_1 \ln a + C_2 \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= \frac{V_o}{\ln\left(\frac{a}{b}\right)} \\ C_2 &= \frac{-V_o \ln b}{\ln\left(\frac{a}{b}\right)} \end{aligned}$$

$$V(r) = V_0 \frac{\ln\left(\frac{r}{b}\right)}{\ln\left(\frac{a}{b}\right)}$$

Determine the electric field, $\vec{E}$, as a function of position. (5)

$$\bar{E} = -\nabla V \Rightarrow -\frac{\partial}{\partial r}\left\{ V_0 \frac{\ln\left(\frac{r}{b}\right)}{\ln\left(\frac{a}{b}\right)} \right\} = -V_0 \frac{1}{r\ln\left(\frac{a}{b}\right)}$$

$[V/m]$

Determine the charge density on the conductors. (5)

note

$$|\rho_s| = |D_n| \qquad \rho_{s_a} = -V_0 \frac{4\epsilon_0}{a\ln\left(\frac{a}{b}\right)} [C/m^2] \quad \left(\ln\left(\frac{a}{b}\right) < 0\right)$$

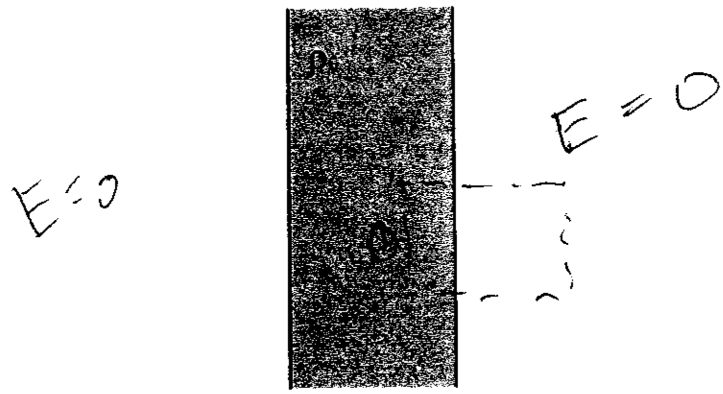
$$\rho_{s_b} = V_0 \frac{4\epsilon_0}{b\ln\left(\frac{a}{b}\right)}$$

Determine the capacitance, C. (5)

$$Q_{tot} = \frac{4\pi V_0 \, 4\epsilon_0}{\ln\left(\frac{a}{b}\right)}$$

$$C = \frac{Q_{tot}}{V_0} = \frac{16\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}$$

5) Field, Voltage (15)



x = -a x = a

A volume charge density, $\rho_v = \rho_o x$ [C/m^3] in the region $-a < x < a$. The charge elsewhere is zero. The domain is free space.

Sketch the figure you would use to apply Gauss's Law. Position your shape exactly where you intend to apply the integral equation. Describe why you chose that shape and location (ie. if no field passes through some of the surfaces, if the field is zero on some of the surfaces, etc.). (4)

Field only exists on side 1
($-\hat{x}$ direction

Determine the electric field, $\vec{E}$, everywhere. (8)

$$\int \bar{E} \cdot d\bar{S} \Rightarrow E_x \,(\text{area})$$

$$Q_{enc} = \int_0^{l}\int_0^{l}\int_x^{a} \rho_o\, x \,dx\,dy\,dz = \frac{\rho_o}{2}(a^2 - x^2)\,(\text{area})$$

$$E_x = -\frac{\rho_o}{2\epsilon_o}(a^2 - x^2)\,\hat{x} \quad [V/m]$$

Determine the voltage difference between $x = -a$ and $x = a$. (3)

$$-\int \bar{E}\cdot d\bar{l} = \int_{-a}^{a} \frac{\rho_o}{2\epsilon_o}(a^2 - x^2) = \cancel{\frac{\rho_o}{2\epsilon}}\left[\ \frac{2}{3}\frac{\rho_o}{\epsilon}\,a^3 \quad [V]\right.$$

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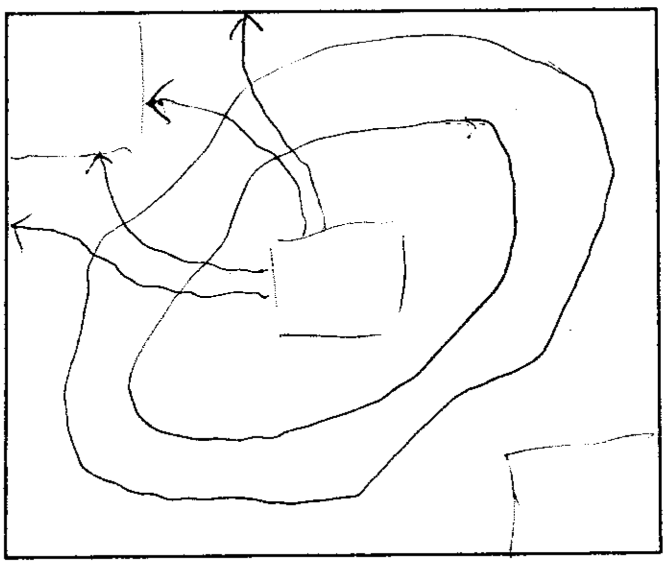
6) Finite Difference (10)

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	3.7	6.68	8.68	9.62	9.51	8.48	6.71	4.14	0	0	0
0	0	0	0	8.13	14.3	18.4	20.3	19.9	17.7	14.2	9.87	4.93	0	0
0	0	0	0	14.5	24.1	30.4	33.1	32.2	28.2	22.5	16.2	9.87	4.14	0
0	3.7	8.13	14.5	25.8	37	46.1	49.5	47.6	40.4	31.5	22.5	14.2	6.71	0
0	6.68	14.3	24.1	37	52.2	67.4	71.3	68.3	54.4	40.4	28.2	17.7	8.48	0
0	8.68	18.4	30.4	46.1	67.4	100	100	100	68.3	47.6	32.2	19.9	9.51	0
0	9.62	20.3	33.1	49.5	71.3	100	100	100	71.3	49.5	33.1	20.3	9.62	0
0	9.51	19.9	32.2	47.6	68.3	100	100	100	67.4	46.1	30.4	18.4	8.68	0
0	8.48	17.7	28.2	40.4	54.4	68.3	71.3	67.4	52.2	37	24.1	14.3	6.68	0
0	6.71	14.2	22.5	31.5	40.4	47.6	49.5	46.1	37	25.8	14.5	8.13	3.7	0
0	4.14	9.87	16.2	22.5	28.2	32.2	33.1	30.4	24.1	14.5	0	0	0	0
0	0	4.93	9.87	14.2	17.7	19.9	20.3	18.4	14.3	8.13	0	0	0	0
0	0	0	4.14	6.71	8.48	9.51	9.62	8.68	6.68	3.7	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

The above Excel spreadsheet data corresponds to a solution of Laplace's equation with cell spacing $h = 2$ [mm]. A dielectric material, $\varepsilon_r = 4$, is in the region between the conductors.

In the box below, sketch the field lines and equipotentials. Indicate where the electric field is largest. Sketch the conductor boundaries as well.(5)



Determine the capacitance per unit length. (The geometry is a cross section, continuing perpendicular to the paper.) (5)

$$C = \frac{\sum h \rho_s}{V_0} = \sum \frac{h D_n}{V_0} = \sum \frac{h \varepsilon E_n}{V_0} \approx \sum \frac{\varepsilon}{V_0} h \left(\frac{E_{n_2} - E_{n_1}}{h} \right) \approx \frac{4\varepsilon}{V_0} [32.6 + 28.7 + 31$$