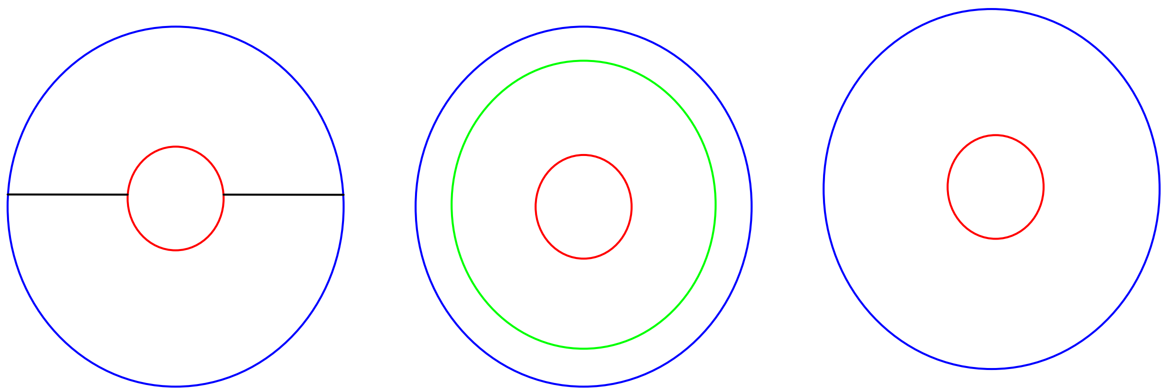
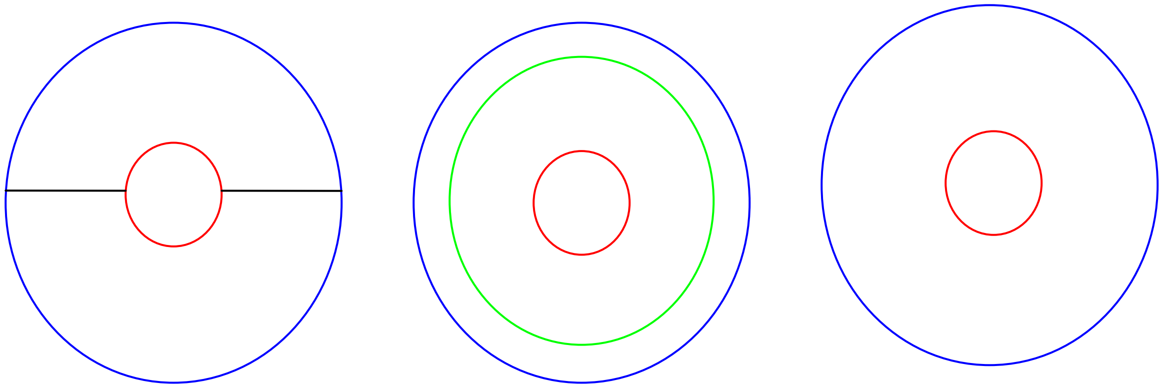
Homework 2 – Due Monday, 14 June

1. There is a small number of simple conductor/dielectric configurations for which we can relatively easily find the capacitance. Students of electromagnetics should be sure that they know how to derive and use all of the expressions for such capacitors. The list of simple capacitors should include the following. You should be able to find the capacitance of any of these configurations.

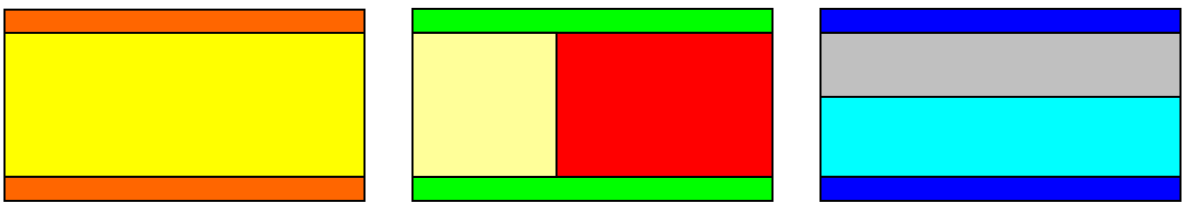


- Spherical Capacitor (radius of inner conductor = *a*, outer conductor = *c*.)
- Spherical Capacitor with more than one dielectric (The radius of the dielectric interface in the second case = *b* and can be anywhere between *a* and *c*. For the first case, the boundary between the two dielectrics exactly splits the space between the plates in half.)

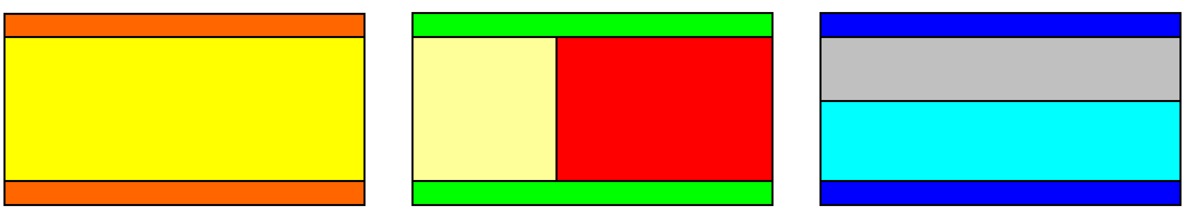
- Two-wire transmission line (end view is shown, the radius of the conductors = *a*, the distance between the centers of the two conductors = *d*)



- Coaxial Cable (radius of inner conductor = *a*, outer conductor = *c*.)
- Coaxial Cable with more than one dielectric (The radius of the dielectric interface in the second case = *b* and can be anywhere between *a* and *c*. For the first case, the boundary between the two dielectrics exactly splits the space between the pates in half.).



- Parallel Plate Capacitor (the separation between the plates = *d*, the area of the plates = *A*
- Parallel Plate Capacitor only partially filled with dielectric or with more than one dielectric (for the second case, the boundary between the two dielectrics can be anywhere between the sides; for the third case the boundary between the two dielectrics can be anywhere between the top and the bottom)



- Parallel plate transmission line (end view is shown)

In this problem, we want to look at some variations of these standard configurations. We will consider three standard capacitors: parallel plate, cylindrical (coaxial) and spherical. Note that questions a and b are stated differently than question c. This is mostly a semantic difference, since the individual parts of the questions can be asked in almost any order. The first two are stated in terms of the charge since that is the fundamental source of the electric field while the third is stated in terms of the voltage since that can actually be measured.

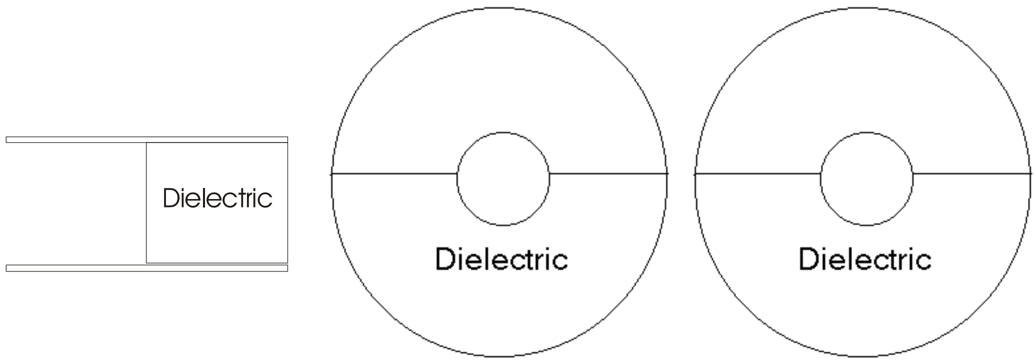
a. Assume that there is a grounded planar conductor at $x = 0$ with a charge $-Q$ and a planar conductor at $x = d$ with a charge $+Q$. The area of each plate = A. Using Gauss' Law, find the electric flux density $\vec{D}$ in the region between the plates ($0 < x < d$) in terms of Q. Write your answer for both the case where there is no dielectric between the conductors (ε_o) and where there is a dielectric between the conductors (ε). Note that the two expressions should be the same. Now write the expressions for the electric field $\vec{E}$ in the region between the conductors for both cases. Use your solution for the electric field to determine the voltage difference between the plates V. Which voltage is larger (for ε or for ε_o) and what is the ratio of the larger voltage to the smaller voltage?

b. Assume that there is a grounded cylindrical conductor at r = b with a charge per unit length $-\rho_{Lb}$ and a conductor at r = a with a charge per unit length ρ_{La} First, with the assumption that the total charge per unit length is zero, determine the relationship between ρ_{La} and ρ_{Lb}. Then, using Gauss' Law, find the electric flux density $\vec{D}$ in the region between the plates ($a < r < b$) in terms of ρ_{La}. Write your answer for both the case where there is no dielectric between the conductors (ε_o) and where there is a dielectric between the conductors (ε). Note that the two expressions should be the same. Now write the expressions for the electric field $\vec{E}$ in the region between the conductors for both cases. Use your solution for the electric field to determine the voltage difference between the conductors V. Which voltage is larger (for ε or for ε_o) and what is the ratio of the larger voltage to the smaller voltage?

c. {*Note that this question is stated differently than the first two. In the first two questions, you were given the charge, while in this question you are given the voltage difference. Either approach can be used to find capacitance. However, we are usually given the voltage instead of the charge, since there is no simple practical device for directly measuring the charge.* } Assume that there is a grounded spherical conductor at $r = b$ and a conductor at $r = a$ with a voltage V_o. Write the electric field $\vec{E}$ in the region between the plates ($a < r < b$) in terms of V_o. Write your answer for both the case where there is no dielectric between the conductors (ε_o) and where there is a dielectric between the conductors (ε) Note that the two expressions should be the same. Now write the expressions for the electric flux density $\vec{D}$ in the region between the conductors. Draw a Gaussian surface that can be used to find the electric field $\vec{E}$ in the region of interest and then evaluate Gauss' Law in integral form to show that your solution is correct.

d. Before proceeding with the last few questions in this problem, assume that the voltage on the plates was given for the parallel plate and coaxial cases and rewrite all of your answers. That is, assume that the voltage difference between the plates is V_o and express Q, $\vec{E}$ and $\vec{D}$ in terms of V_o. You should have two expressions for each case since you have considered both ε and ε_o. Clearly label your answers.

Now, assume that, for each of these configurations, half of the space between the conductors is filled with the dielectric and half is empty (air), as shown below. Note that, for the following questions, you will not be analyzing the coax, but its figure is shown for completeness.



e. Using your expressions for the electric field (written as a function of V_o), show that the boundary condition for the tangential component of the electric field is satisfied at the interface between the dielectric and empty regions for the first and third cases (not the coax).

f. Using the boundary condition for the normal component of $\vec{D}$ at a conductor-dielectric interface, determine the charge density and then the total charge on the grounded conductor for the first and third case. (Not the coax). Your answer for the surface charge density should be different in each material.

g. From the total charge for first and third case and the voltage difference between the two conductors, determine the capacitance. (Again, not the coax) You will have to add up the charge separately in each material.

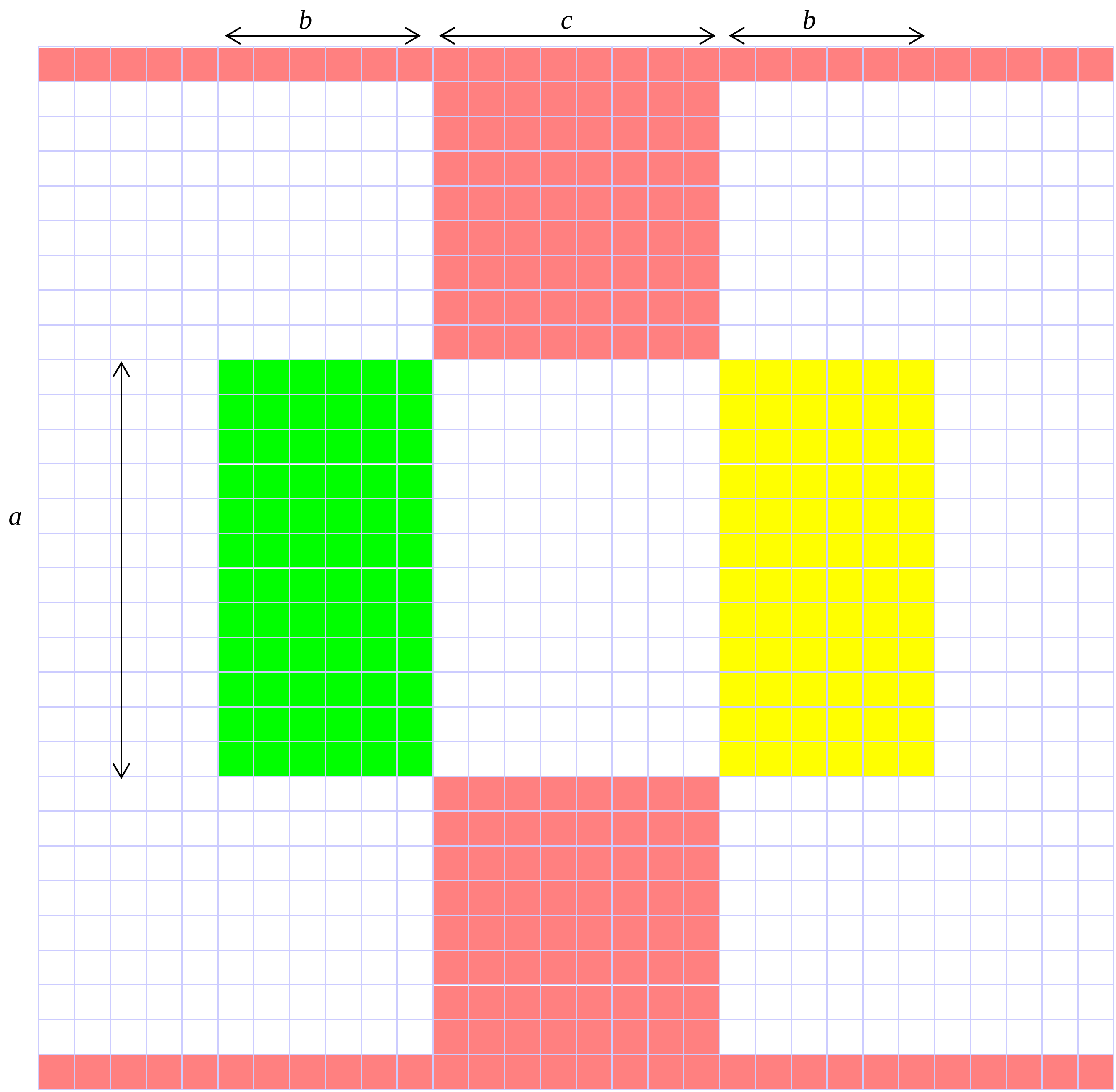
h. For the specific case of the spherical capacitor half filled with dielectric where $a = 20mm$, $b = 30mm$, $\varepsilon = 5\varepsilon_o$ find the numerical value for the capacitance.

2. Nearly all practical electromagnetics problems are analyzed using numerical methods rather than the analytical methods we have been addressing. One of the most common techniques is called the Finite Difference Method, which we have considered in class. In this problem, we will try to find the capacitance of a simple two-dimensional problem with a somewhat odd geometry. To address this geometry, we will use a spreadsheet to solve Laplace's equation.

The method is reasonably simple. First, identify the cells you will use to represent the conductors. Set the value in these cells equal to the potential on the conductor. Second, all exterior cells must have either a specific potential or be set equal to their nearest interior neighbor. This is equivalent to setting the normal derivative of the potential equal to zero. This boundary condition is quite accurate if the boundary in question is a line of symmetry for the problem. As we will see, this will be the case in this problem. Third, for all interior points, set the voltage in each cell equal to the average of its four nearest neighbors. This is the finite difference equivalent of Laplace's equation. After you enable iteration, the spreadsheet values will eventually converge to something near the correct voltages at each location. Use this information to determine the capacitance per unit length of this configuration, following the method we discussed in class. For your solution, use at least a 31 x 31 array (bigger is generally better, but bigger takes much longer). Whatever information you use from your spreadsheet has to be included with your analysis.

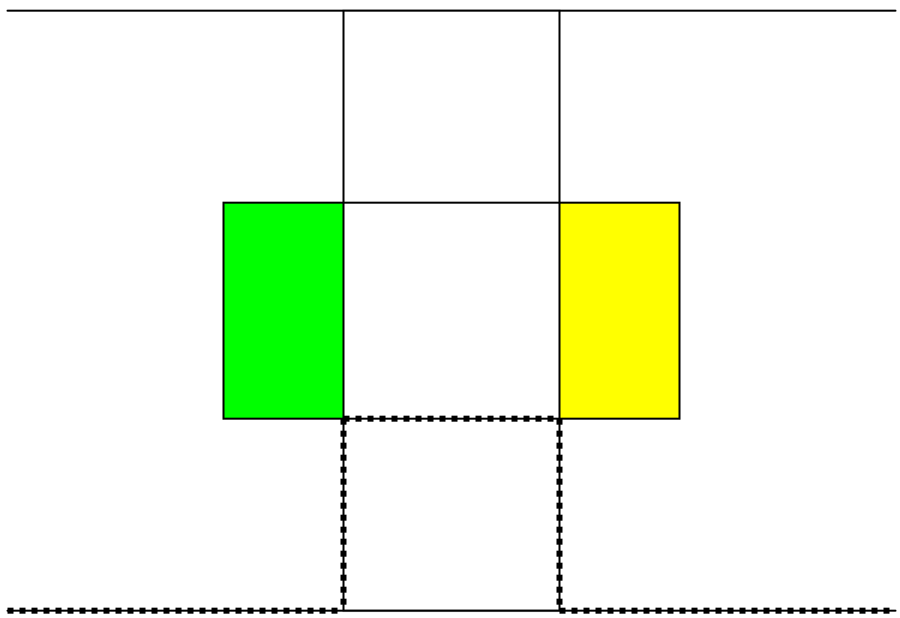
The geometry you are to consider is shown on the next page. The top and bottom electrodes are not totally flat, but have a much thicker part near the middle. The large separation between the plates is h, the width of the plates is w, the other dimensions are given. Assume that it continues for a very long distance into the page, just like a coaxial cable. Thus, you will be finding the capacitance per unit length, just as is the case with the cable. Also, assume that the structure is periodic in the horizontal direction. That is, it repeats over and over. That means that the left and right boundaries are lines of symmetry, so that for the open parts of the boundary, the normal derivative boundary condition is the appropriate choice. Assume that the top electrode is at 100 volts and the bottom is grounded. Also assume that the dielectric constant for the green region between the electrodes is $2.1\varepsilon_o$. and for the yellow region it is $6.4\varepsilon_o$

To summarize – solve for the potential at all points represented by cells and use this information to determine the capacitance per unit length of this structure. Discuss why you think your answer is at least approximately correct. Since you will need actual numbers if you are to use the spreadsheet approach, find the specific numerical solution for the following: $h = 30cm$, $w = 30cm$, $a = 12cm$, $b = 6cm$, $c = 8cm$.



Also, repeat the capacitance calculation for the case with no dielectrics between the plates. That is the green and yellow regions are also air. Compare both your answers to the simple parallel plate case with just flat electrodes.

Determine and plot the surface charge density as a function of position on the dashed line shown. Indicate where the charge density is the largest and explain why this is the case.



Make a 3-D sketch of both configurations showing all surfaces, inside and out. Indicate on these surfaces where positive and negative charges are found. Indicate also the size of each surface and the average charge density on each surface. Note – remember that on a real physical plate, there are six surfaces, so be sure that your sketch shows all of them.