UNIT XII

Transmission Lines – Transients

A. Common Applications of Transmission Lines with Pulsed or Transient Signals

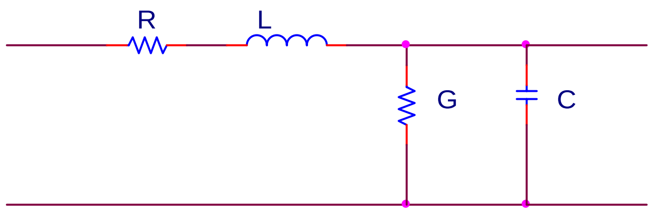
1. Internet LAN
2. Telephone land lines
3. Audio cables
4. Power lines
5. Circuit interconnects
6. Pulse forming networks (pulse power)

B. Two Simple Transmission Line Experiments

1. Lab Experiment
2. PSpice Experiment

C. Circuit Model of Transmission Lines

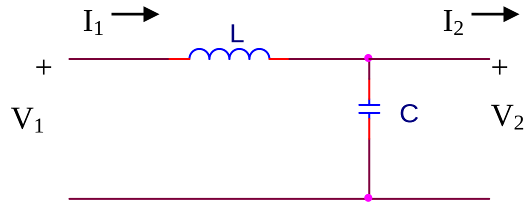
Transmission lines generally consist of two or more parallel or concentric wires. For two wires, the most common configuration is the twisted pair. Concentric wires form coaxial cables. Both configurations are used extensively. Two conductors near one another will have a capacitance, inductance, resistance and conductance. Since we wish to characterize transmission lines of any length, we will use per unit length values for these parameters. A small length of line Δz will have the following circuit configuration.



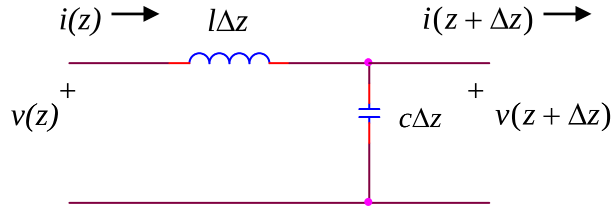
where $R = r\Delta z$, $L = l\Delta z$, $G = g\Delta z$, and $C = c\Delta z$. *r, l, g, and c* are the resistance, inductance, conductance and capacitance per unit length for the line. Note that for a commercially available transmission line (e.g. RG58A/U cable), the values for these per unit length parameters are readily available; most catalogs contain this information.

C.	Transient Analysis

Begin with the voltage and current relationships for the four components. $V = RI$, $V = L\frac{dI}{dt}$, $I = GV$, and $I = C\frac{dV}{dt}$. Each of these expressions holds for the voltage across the component and the current through the component. Also, begin with the simplest case – the lossless transmission line with *R* and *G* both equal to zero. The circuit diagram for a small section of the line of length Δz becomes



where the input and output voltages and currents have been labeled. Applying the equations for the inductor and capacitor, we see that $V_1 - V_2 = L\frac{dI_1}{dt}$ and $I_1 - I_2 = C\frac{dV_2}{dt}$. We should rewrite these expressions and re-label the diagram to more generally describe any section of the line beginning at some location *z*. First, begin with the diagram



where we have also used the per unit length values for the inductance and capacitance. The two equations above then become

$$v(z) - v(z+\Delta z) = l\Delta z\frac{di(z)}{dt}$$
$$i(z) - i(z+\Delta z) = c\Delta z\frac{dv(z+\Delta z)}{dt}$$

The finite difference spatial derivatives of voltage and current are then

$$\frac{v(z+\Delta z) - v(z)}{\Delta z} = -l\frac{di(z)}{dt}$$
$$\frac{i(z+\Delta z) - i(z)}{\Delta z} = -c\frac{dv(z+\Delta z)}{dt}$$

where the minus signs appear because the order of the terms has been reversed. In the limit as $\Delta z \to 0$, the left hand sides of these expressions become the partial derivatives with respect to *z* and, since there are now both *z* and *t* derivatives, the time derivative also becomes a partial derivative

$$\frac{\partial v(z)}{\partial z} = -l\frac{\partial i(z)}{\partial t} \text{ and } \frac{\partial i(z)}{\partial z} = -c\frac{\partial v(z)}{\partial t}$$

Either v or i can be eliminated by combining the two equations

$$\frac{\partial^2 v(z)}{\partial z^2} = -l\frac{\partial^2 i(z)}{\partial z \partial t} \quad \text{and} \quad \frac{\partial^2 i(z)}{\partial t \partial z} = -c\frac{\partial^2 v(z)}{\partial t^2}$$

or

$$\frac{\partial^2 v(z)}{\partial z^2} = -l\frac{\partial^2 i(z)}{\partial z \partial t} = -l\frac{\partial^2 i(z)}{\partial t \partial z} = +lc\frac{\partial^2 v(z)}{\partial t^2}$$

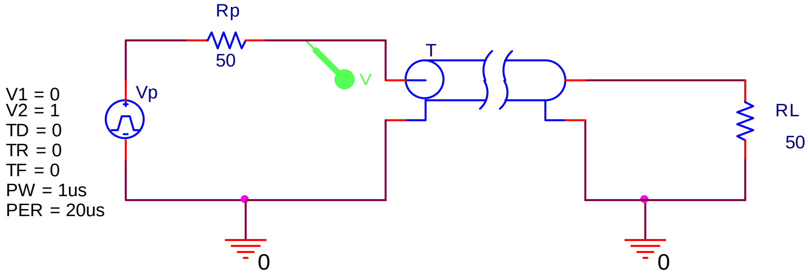
$$\frac{\partial^2 v(z)}{\partial z^2} = lc\frac{\partial^2 v(z)}{\partial t^2}$$

or following the same steps

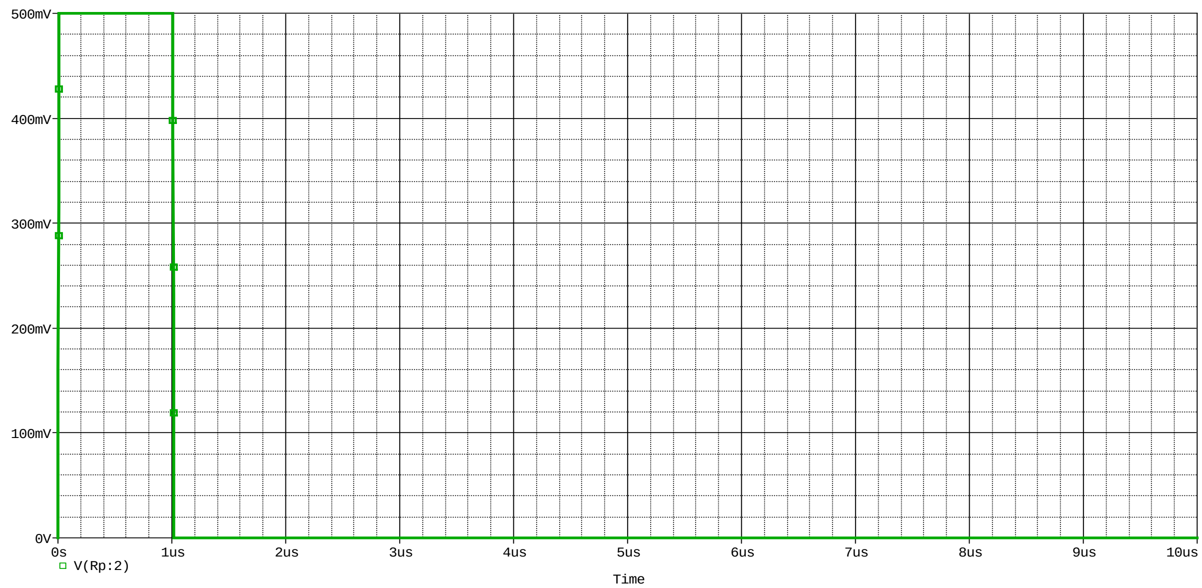
$$\frac{\partial^2 i(z)}{\partial z^2} = lc\frac{\partial^2 i(z)}{\partial t^2}$$

Note: The wave equation is discussed in section 10.7 of Boyce and DiPrima.

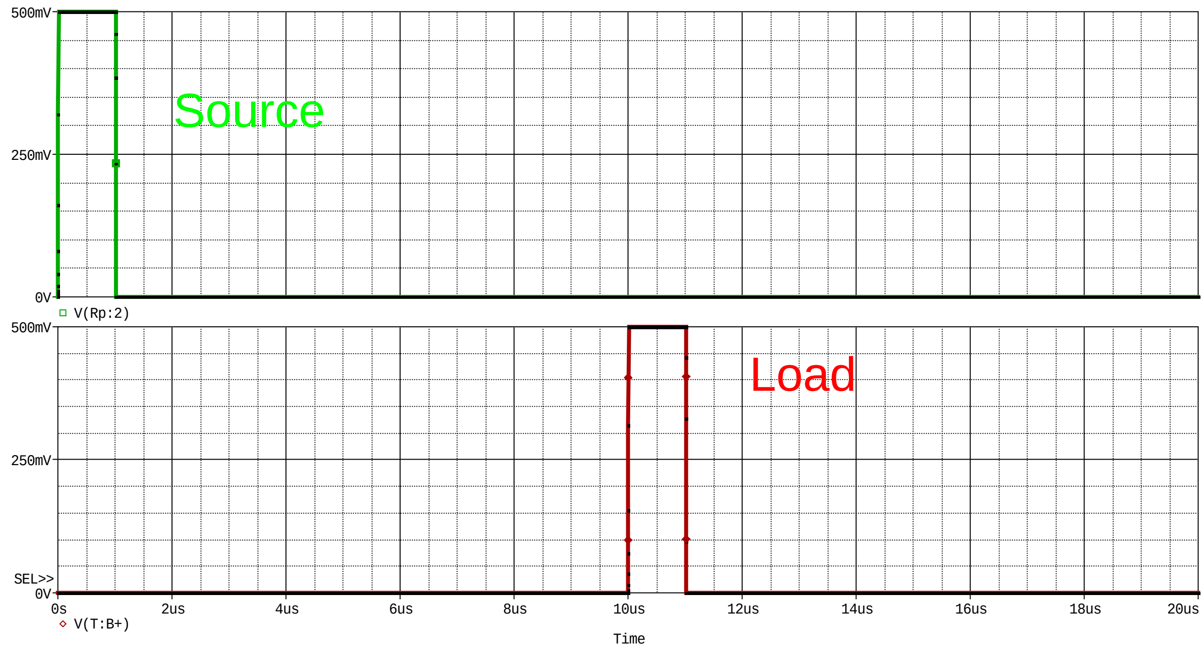
This equation has been studied for a very long time, so we know a great deal about it. However, before we find the general solution, we will do another experiment to see if we can identify some of the characteristics of the solution. Set up the following configuration in PSpice:



Specify a characteristic impedance of the transmission line of 50 Ohms and a delay time of $10\mu s$. Have the pulsed voltage source produce a short ($1\mu s$) pulse with an amplitude of $1V$.



Note that the source voltage appears to the right of the resistor, since this resistance is the internal impedance of the voltage source. Now we will observe the voltage at both the source and the load ends of the transmission line.



Note that the voltage at the load end looks exactly like the voltage at the source end with one significant difference – it is delayed in time by 10 microseconds. We can also do this experiment with the other model of the transmission line (Tlossy) and be more realistic. If you do this, you will observe that the delay is linearly proportional to the length of the line. This means that the pulse propagation speed is a constant or is a fundamental characteristic of the line. To see what the velocity is, repeat the pulse experiment using Tlossy and an inductance per unit length of 0.195 micro Henries and a capacitance per unity length of 78 pico Farads. Try different lengths until the delay is 10 microseconds. You should find that the length must be 2564 meters or 2.564 km and, thus, the velocity of propagation on the line is 2.564×10^8 m/s. Notice how close to the speed of light this is. In fact, this is exactly the speed of light *c* in the material that is used as an insulator for the transmission line (which is always a bit less than *c*). It is also given, in general, by the following expression $u = \frac{1}{\sqrt{lc}}$. If the voltage source at the source is written generally as $v_{source} = f(t)$, the voltage at the load end is $v_{load} = f\left(t - \frac{z}{u}\right)$. In general, $v(z) = f\left(t - \frac{z}{u}\right)$ is the voltage anywhere on the line. It will be easier to keep track of things if we change notation just a bit and use $V = V\left(t - \frac{z}{u}\right)$ as the voltage on the line and $I = I\left(t - \frac{z}{u}\right)$ as the current on the line.

This is the way we indicate mathematically that the voltage and current at any point on the transmission line looks exactly like the voltage or current at the source but delayed in time by

$t = \frac{z}{u}$. If our thinking has been correct, these expressions should satisfy the wave equations we derived above.

We begin by evaluating the first and second derivatives of $V\left(t - \frac{z}{u}\right)$

$$\frac{\partial V}{\partial z} = -\frac{1}{u}V'\left(t - \frac{z}{u}\right) \text{ and } \frac{\partial V}{\partial t} = V'\left(t - \frac{z}{u}\right)$$
$$\frac{\partial^2 V}{\partial z^2} = +\frac{1}{u^2}V''\left(t - \frac{z}{u}\right) \text{ and } \frac{\partial^2 V}{\partial t^2} = V''\left(t - \frac{z}{u}\right)$$

which we can use to see if we have the correct solution to the wave equation.

$$\frac{\partial^2 V}{\partial z^2} = +\frac{1}{u^2}V'' = lc\frac{\partial^2 V}{\partial t^2} = lcV''$$

We see that $V = V\left(t - \frac{z}{u}\right)$ is a solution as long as $u = \frac{1}{\sqrt{lc}}$. The simple case analyzed with PSpice showed that this is the velocity on the line, so we have the correct solution.

One of the first topics discussed in any differential equation text (***see section 1.3 of Boyce and DiPrima***) is the order of the equation, which is the order of the highest derivative that appears in the equation. The wave equation is a second order, partial differential equation and, thus, it will have two solutions. This information usually appears to be only of academic interest, with little real-world relevance. However, for the wave equation, this very important characteristic has a beautiful direct connection to reality. To see this, consider the following communications system in which box 1 talks to box 2.

Most of the time, we want both boxes to be able to initiate communication. Thus, pulses can go from box 1 to box 2 or from box 2 to box 1. If we define the positive *z* direction from left to right, the velocity for the former case is positive $u > 0$ and for the latter case it is negative $u < 0$. There is nothing fundamentally different about communicating in each direction, so the solutions must look the same, except for the sign of the velocity. We can write this information as

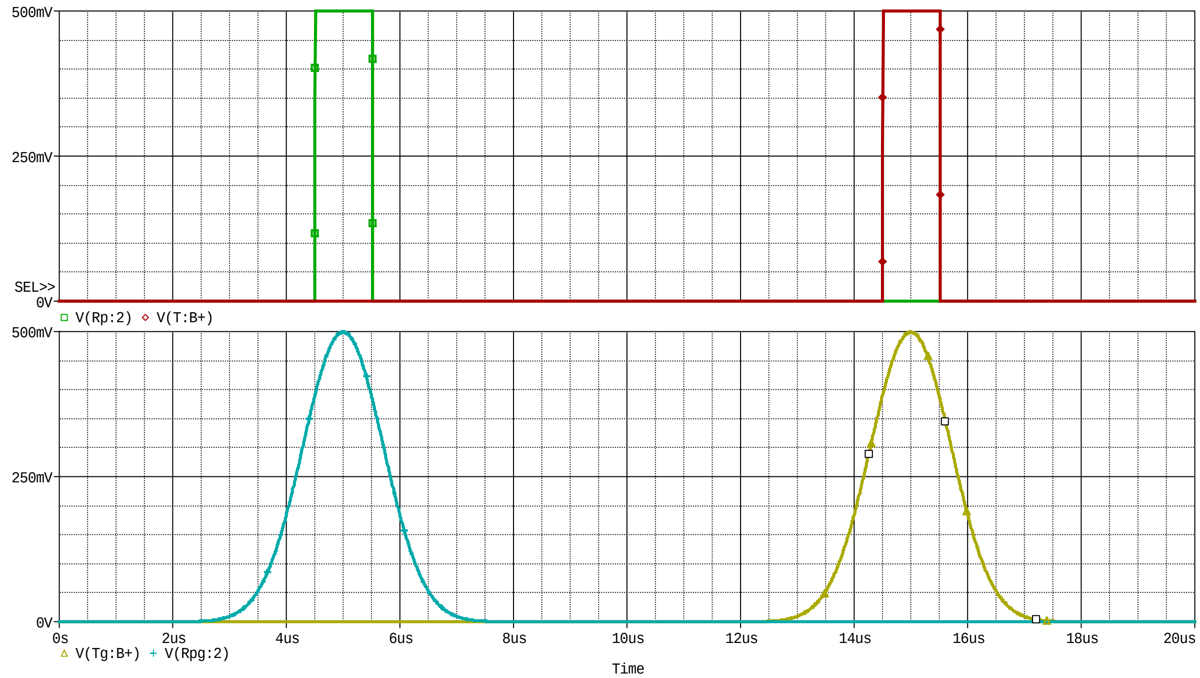
$$V_+ = V\left(t - \frac{z}{u}\right) \qquad V_- = V\left(t + \frac{z}{u}\right)$$
$$I_+ = I\left(t - \frac{z}{u}\right) \qquad I_- = I\left(t + \frac{z}{u}\right)$$

or

$$V_\pm = V\left(t \mp \frac{z}{u}\right) \qquad I_\pm = I\left(t \mp \frac{z}{u}\right)$$

It is left as an exercise for the reader to plug these expressions into the wave equation to show that they indeed do work. The elegance of this description is that we can see that there are two essentially identical solutions, one for each of the two directions of communication.

Notice that the solutions to the wave equation are very general. At no point was the pulse shape specified. Thus, we should see that essentially any pulse shape should propagate the same way as long as the transmission line is lossless. Trying this with PSpice, we see the following:



Both a square pulse and a Gaussian pulse are shown. The pulses at the left are as observed at the source end and at the right at the load end. These very different pulses clearly propagate at exactly the same speed.

Since both pulses can exist on the line at the same time, the most general solution for the voltage and current waves must include both.

$$V(z,t) = V_+\left(t - \frac{z}{u}\right) + V_-\left(t + \frac{z}{u}\right)$$

$$I(z,t) = I_+\left(t - \frac{z}{u}\right) + I_-\left(t + \frac{z}{u}\right)$$

It is not necessary to solve for V and I separately since there is a connection between the two. Start with one of the relations that couple voltage and current:

$$\frac{\partial V}{\partial z} = -l\frac{\partial I}{\partial t}$$

Try the following solution for the current

$$I(z,t) = \frac{1}{lu}V_+\left(t - \frac{z}{u}\right) - \frac{1}{lu}V_-\left(t + \frac{z}{u}\right)$$

then

$$\frac{\partial V}{\partial z} = -\frac{1}{u}V_+{}' + \frac{1}{u}V_-{}' = -l\frac{\partial I}{\partial t} = -l\left\{\frac{1}{lu}V_+{}' - \frac{1}{lu}V_-{}'\right\} = -\frac{1}{u}V_+{}' + \frac{1}{u}V_-{}' = \frac{\partial V}{\partial z} \quad QED$$

There is a more convenient way to write the current in terms of impedance, as is discussed below.

Transmission Line Impedance and the Analysis of Pulse Propagation

For a transmission line with a resistive load R_L the ratio of the voltage to the current at the load is given by $\frac{V}{I} = R_L$. At any location on the line, this ratio is given by $\frac{V(z,t)}{I(z,t)} = Z(z,t)$ which we can call the generalized impedance. Looking at only the positive traveling pulse, we see that $Z_+ = \frac{V_+(z,t)}{I_+(z,t)} = lu = \sqrt{\frac{l}{c}}$. For the negative traveling pulse, $Z_- = \frac{V_-(z,t)}{I_-(z,t)} = -lu = -\sqrt{\frac{l}{c}}$. Since both pulses are have the same ratio (disregarding the sign), we define this as the characteristic impedance of the line $Z_o = \sqrt{\frac{l}{c}}$. For a standard Cable TV coaxial cable, we find that $Z_o = 75\Omega$, while for the RG58A/U cables we use in the classroom or lab, $Z_o = 50\Omega$. The ratio of the total voltage (both positive and negative pulses) to the total current on the line is called the total impedance and is given by $Z_o \frac{V_+(z,t) + V_-(z,t)}{V_+(z,t) - V_-(z,t)}$. We generically represent transmission lines as shown below, giving their characteristic impedance and length.

At the load end, we have a choice of designating the location as *z=0* or *z=d* for a line of length *d*. The choice depends on whether we are more interested in the load or the source. Let us choose the latter in this case. Then at *z=d*,

$Z_o \frac{V_+(d,t) + V_-(d,t)}{V_+(d,t) - V_-(d,t)} = R_L$. This can be rewritten to determine the ratio of the negative traveling voltage to the positive traveling voltage as a result of reflection off of the load $\frac{V_-}{V_+} = \frac{R_L - Z_o}{R_L + Z_o}$.

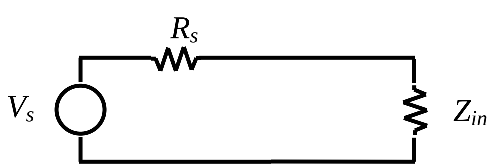
We call this ratio the reflection coefficient at the load and write it as

$$\Gamma_L = \frac{V_-}{V_+} = \frac{R_L - Z_o}{R_L + Z_o}$$

If we are given a positive traveling pulse, we can use this expression to determine the magnitude of the negative traveling pulse, if there is one. For example, if there is a *1V* pulse traveling in the positive z direction down the line and if $R_L = Z_o$, then $\Gamma_L = \frac{Z_o - Z_o}{Z_o + Z_o} = 0$. We call this a matched load, since the load impedance equals the line impedance. When this is the case, there is no reflected pulse and the *1V* pulse appears at the load. The reader is encouraged to try this with PSpice to see that no reflection occurs when the load is matched.

At the source end of the line, we generally find a voltage source with some finite internal impedance.

To determine the voltage that appears at the input of the line, we have to solve the voltage divider expression $V_{in} = \frac{Z_{in}}{Z_{in} + R_s} V_s$ where Z_{in} is the input impedance of the line. If we assume that there are no pulses on the line before time $t=0$, then at this time there will only be a positive traveling pulse on the line since it takes a finite time for the pulse to propagate to the load end, reflect off of the load, and propagate back. Thus, from the expression for the total impedance of the line, $Z_{in} = Z_o \frac{V_+(z,t) + V_-(z,t)}{V_+(z,t) - V_-(z,t)} = Z_o \frac{V_+(z,t)}{V_+(z,t)} = Z_o$. Thus, the initial input impedance of the line is the characteristic impedance. Because of this, Z_o is also called the surge impedance since it is the impedance seen by any large change in voltage or current. Given this input impedance, the initial voltage seen on the line is $V_{in} = V(0,t) = \frac{Z_{in}}{Z_{in} + R_s} V_s = \frac{Z_o}{Z_o + R_s} V_s$. From the point of view of the source, the transmission line looks like a lumped impedance Z_{in}.



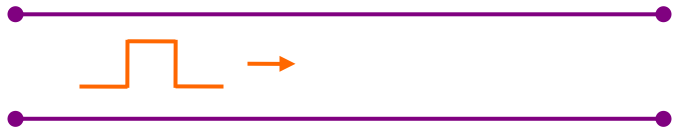
Recall from Circuits that maximum power transfer occurs when the two resistances in this diagram are equal. Thus, the load is matched to the line when $R_s = Z_{in} = Z_o$. We will see below that matching adds another benefit. Note that to launch a *1V* pulse on the line, the initial pulse amplitude has to be *2V* for the matched case.

Thus far, we have determined the initial pulse amplitude launched on the line and what happens when the pulse reaches the load. If the pulse at least partially reflects from the load (due to a mismatch), then a negative traveling pulse will result. When this pulse gets back to the source end, it will see the source impedance in the same way as the positive traveling pulse saw the load impedance. Thus, another reflection will occur unless the source is matched to the line. The reflection coefficient at the source is given by

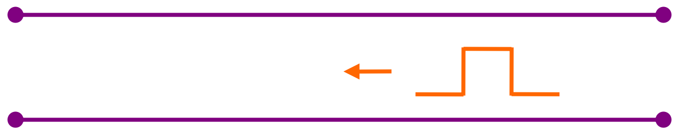
$$\Gamma_s = \frac{R_s - Z_o}{R_s + Z_o}$$

Here we see the additional benefit of matching the source to the line. Should there be a reflection off of the load, resulting in a negative traveling pulse returning to the source, a matched source will eliminate any further reflections. Ideally, we want one pulse to be launched on the line and nothing returned to the source, but we cannot always achieve this.

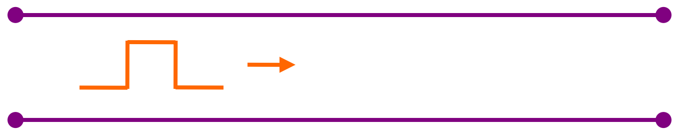
Reviewing, we can determine the magnitude of the initial positive traveling pulse from the voltage divider relationship.



This pulse propagates down to the load and can reflect resulting in a negative traveling pulse whose magnitude is given by the product of the voltage divider expression and the reflection coefficient at the load.

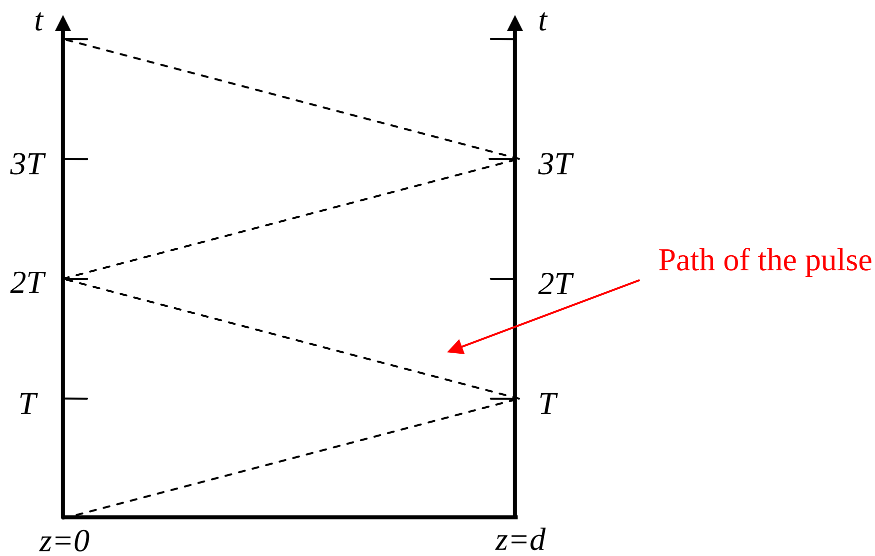


When this pulse reaches the source end, it can reflect again resulting in another positive traveling pulse.



This process can go on forever if the source and load are not matched to the line. However, since the reflection coefficient is always less than or equal to 1, eventually the pulses become so small that we can ignore them. The only exception is when the reflection coefficients have unit magnitude and the line is perfectly lossless. Then the pulses go on forever. Fortunately, this never happens in the real world.

We need one more piece of information to fully characterize the pulses on the line – the time it takes to propagate from one end of the line to the other $T = \frac{d}{u}$. This tells us when a pulse launched at time $t=0$ will appear at the load. However, we don't yet have a simple method for systematically incorporating and representing all of the pulses, reflection events, etc. Such a method must allow us to show both spatial and temporal information simultaneously since what we observe on the line depends on both where we look and when. The most popular diagram for this purpose goes by several names, but we will call it a *bounce diagram*. It is shown below.



In this diagram, the path in space and time followed by the pulse, or any part of the pulse, is shown by the dashed line. Note that the diagram clearly shows that an observer (the 'scope) at the load will not see a pulse until a time *T* later than it is launched on the line at the source end. Also, since pulses are of finite length, the lead end of the pulse will have already reflected before the trailing end. Thus, an observer at the load will simultaneously see the sum of both the incident (positive traveling) and reflected (negative traveling) pulses and not the individual pulses.

The step-by-step procedure for using the bounce diagram:

1. Draw the basic bounce diagram shown above and label it with the value of $T = \frac{d}{u}$ determined from the line length and velocity.
2. Determine the initial pulse amplitude using the voltage divider relationship and label the first leg of the path with this value.

$$V_{in} = \frac{Z_o}{Z_o + R_s} V_s$$

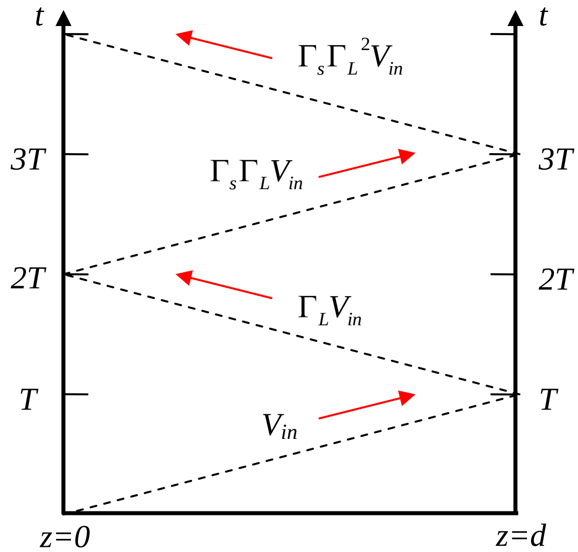
3. Determine the reflection coefficient at the load

$$\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o}$$

   and label the second leg of the path with $\Gamma_L V_{in}$.
4. Determine the reflection coefficient at the source

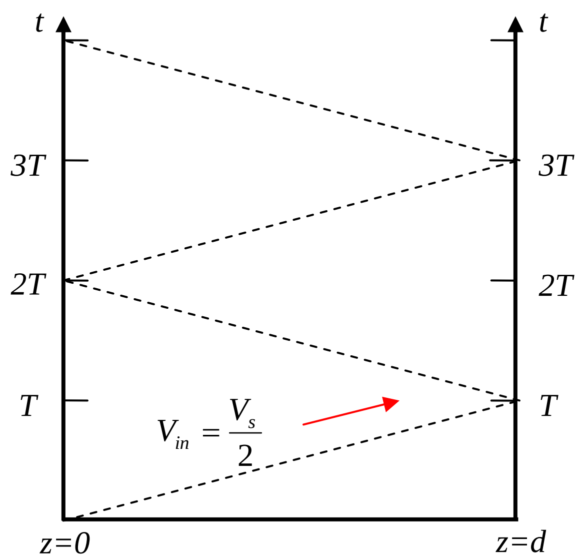
$$\Gamma_s = \frac{R_s - Z_o}{R_s + Z_o}$$

   and label the remaining legs of the path (a fully labeled diagram is shown below).



From this diagram, we can see that the pulse amplitude observed at the load will be equal to $V_{in}\left(1+\Gamma_L\right)$ and, thus, can be larger or smaller than the initial pulse, depending on the sign of Γ_L.

Example 1. Matched Line (R_s=R_L=Z_o) Length = d and velocity = u.

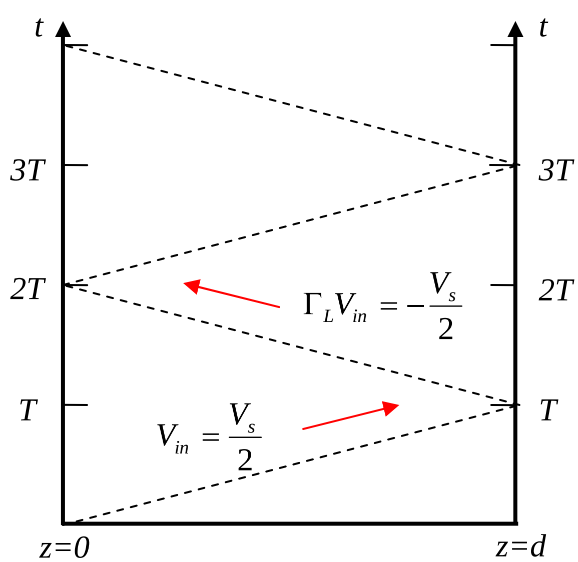


1. $T = \dfrac{d}{u}$
2. $V_{in} = \dfrac{Z_o}{Z_o + R_s} V_s = \dfrac{Z_o}{Z_o + Z_o} V_s = \dfrac{V_s}{2}$

3. $\Gamma_L = \dfrac{R_L - Z_o}{R_L + Z_o} = \dfrac{Z_o - Z_o}{Z_o + Z_o} = 0$ so analysis stops here.

Example 2. Matched Source (R_s=Z_o), Short Circuit Load (R_L=0) Length = d and velocity = u.

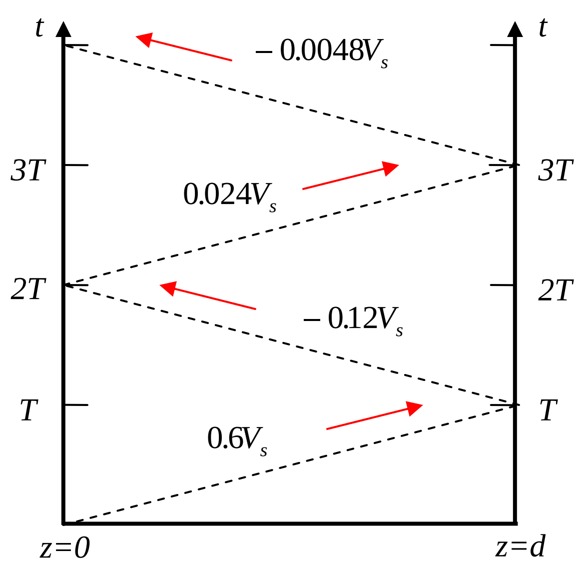
1. $T = \dfrac{d}{u}$
2. $V_{in} = \dfrac{Z_o}{Z_o + R_s} V_s = \dfrac{Z_o}{Z_o + Z_o} V_s = \dfrac{V_s}{2}$ which is the amplitude on the first leg of the path.
3. $\Gamma_L = \dfrac{R_L - Z_o}{R_L + Z_o} = \dfrac{0 - Z_o}{0 + Z_o} = -1$ so the amplitude on the second leg of the path will be $\Gamma_L V_{in} = -V_{in}$
4. $\Gamma_s = \dfrac{R_s - Z_o}{R_s + Z_o} = \dfrac{Z_o - Z_o}{Z_o + Z_o} = 0$ so analysis stops here.



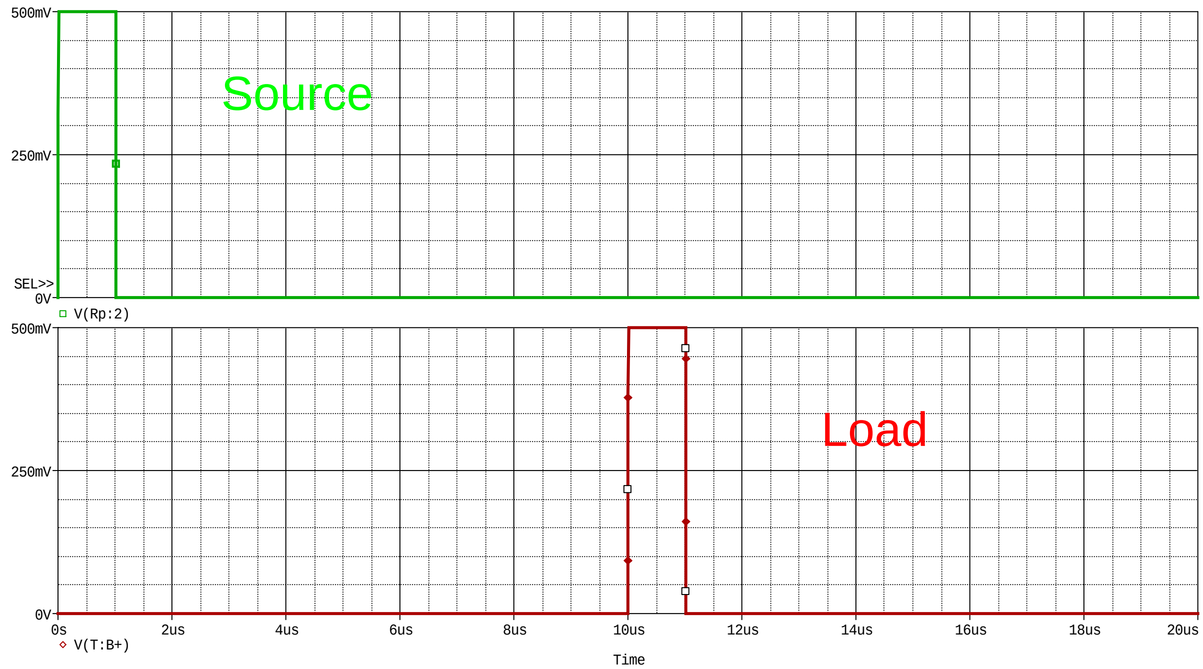
Example 3. Matched Source (R_s=Z_o), Open Circuit Load ($R_L = \infty$) Length = d and velocity = u. This is left as an exercise for the reader.

Example 4. 50Ω Source and Load (R_s=R_L= 50Ω), 75Ω Line ($Z_o = 75\Omega$) Length = d and velocity = u.

1. $T = \dfrac{d}{u}$
2. $V_{in} = \dfrac{Z_o}{Z_o + R_s} V_s = \dfrac{75}{75+50} V_s = \dfrac{3V_s}{5} = 0.6V_s$ which is the amplitude on the first leg of the path.
3. $\Gamma_L = \dfrac{R_L - Z_o}{R_L + Z_o} = \dfrac{50-75}{50+75} = -0.2$ so the amplitude on the second leg of the path will be $\Gamma_L V_{in} = (-0.2)(0.6)V_s = -0.12V_s$
4. $\Gamma_s = \dfrac{R_s - Z_o}{R_s + Z_o} = \dfrac{50-75}{50+75} = -0.2$ so the amplitude on the third leg of the path will be $\Gamma_s \Gamma_L V_{in} = (-0.2)(-0.2)(0.6)V_s = +0.024V_s$
5. On the fourth leg of the path, the amplitude will be sufficiently small to end our analysis for now $\Gamma_s \Gamma_L{}^2 V_{in} = (-0.2)(-0.2)(-0.2)(0.6)V_s = -0.0048V_s$

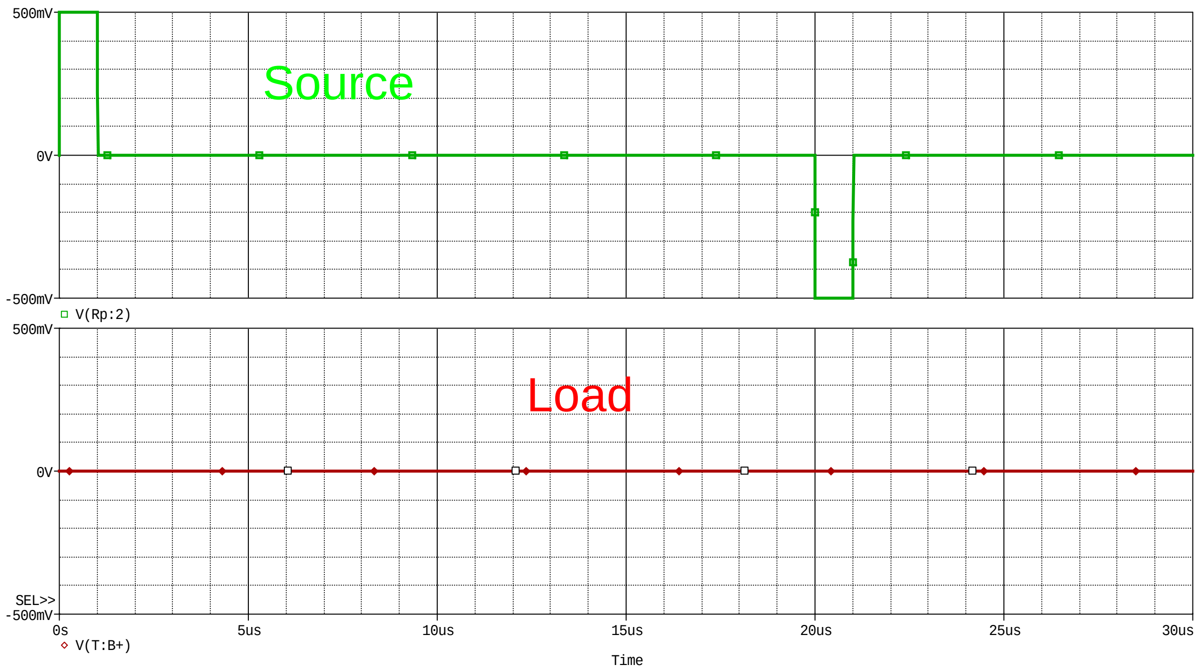


Our real goal here is to predict the voltages that will be observed at the input (source) and output (load) ends of the lines, since these are the only accessible locations. Using PSpice to generate the observed voltages, we have for example 1

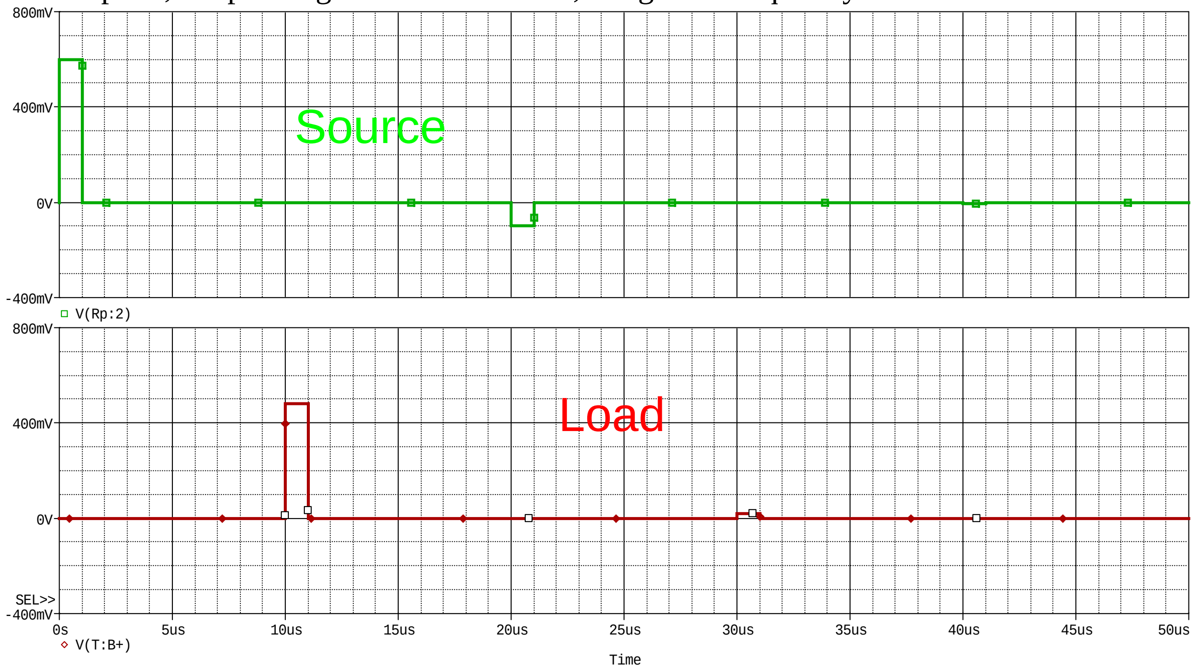


The same pulse appears at both ends, separated by the time *T* which, in this case, is 10 microseconds.

For example 2, we expect to see no voltage at the load, since it is a short circuit. The initial pulse reappears at the source after a time *2T* but inverted since the reflection coefficient is *-1*.



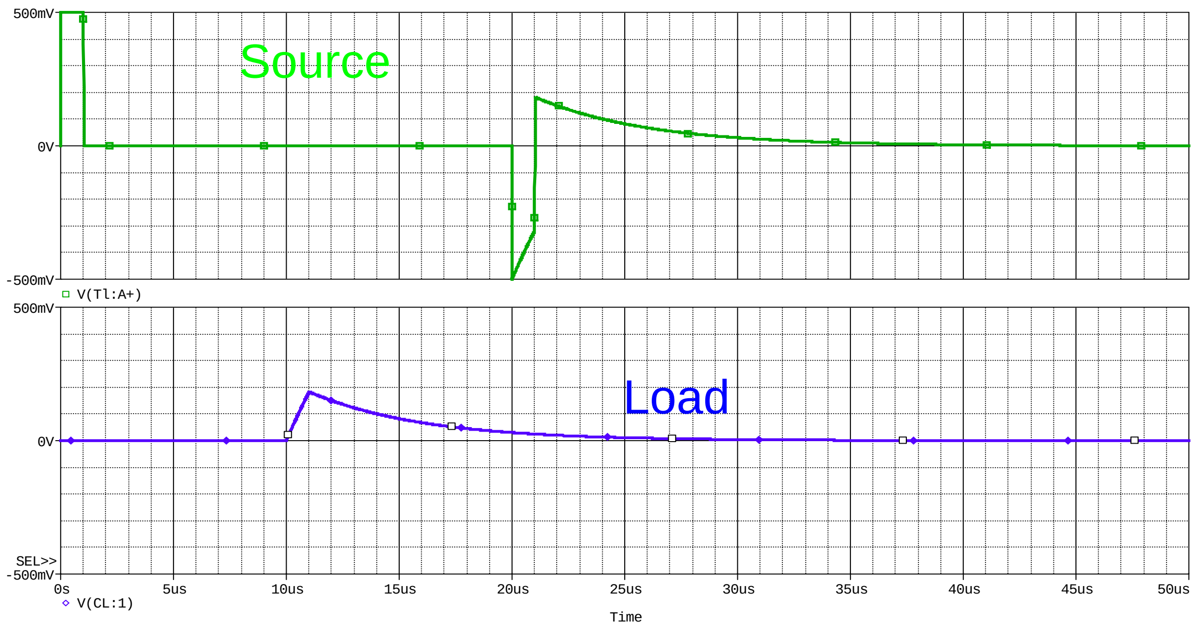
For example 3, the pulses go on for some time, but get small quickly.



Note that the voltages observed include the contributions from two pulses at each point since the incident and reflected pulses are observed simultaneously.

Up to this point, we have not considered non-resistive loads (e.g. capacitors or inductors), nor have we addressed lossy lines (e.g. with r or g not equal to zero). We will address these issues more thoroughly when we consider time-harmonic (single frequency) analysis. For now, we will include only a few more examples using PSpice to show what can happen.

Example 5 50Ω Source and Line Matched ($R_s=Z_o=50\Omega$), Capacitive Load (*C=0.1 microfarad)*. Length = *2564meters* and velocity = $2.564x10^8$ *meters per second.*

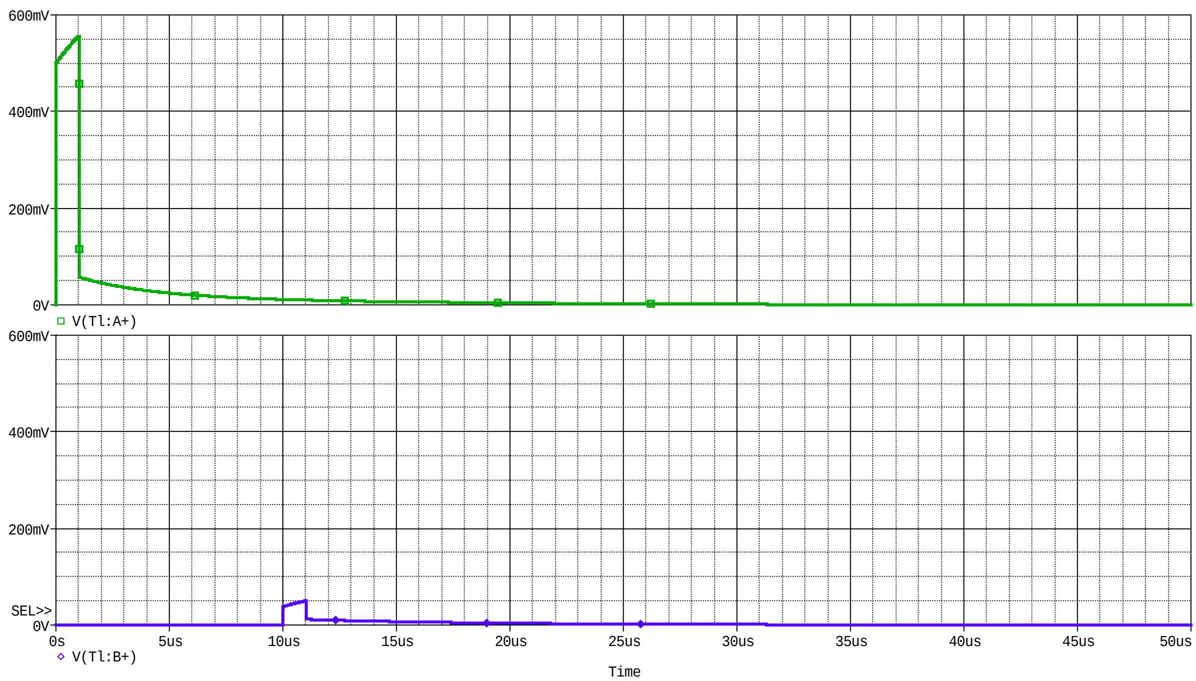


Note that the pulse mostly reflects from the capacitor. This is because it is unable to charge up quickly enough to store all of the energy in the pulse. Since the pulse is fully absorbed in the matched source, once it returns to the source, only one pulse is observed at the load. Since the load takes some time to discharge through the line, an identical decaying voltage is also observed at the source with the usual delay of the transmission line propagation time.

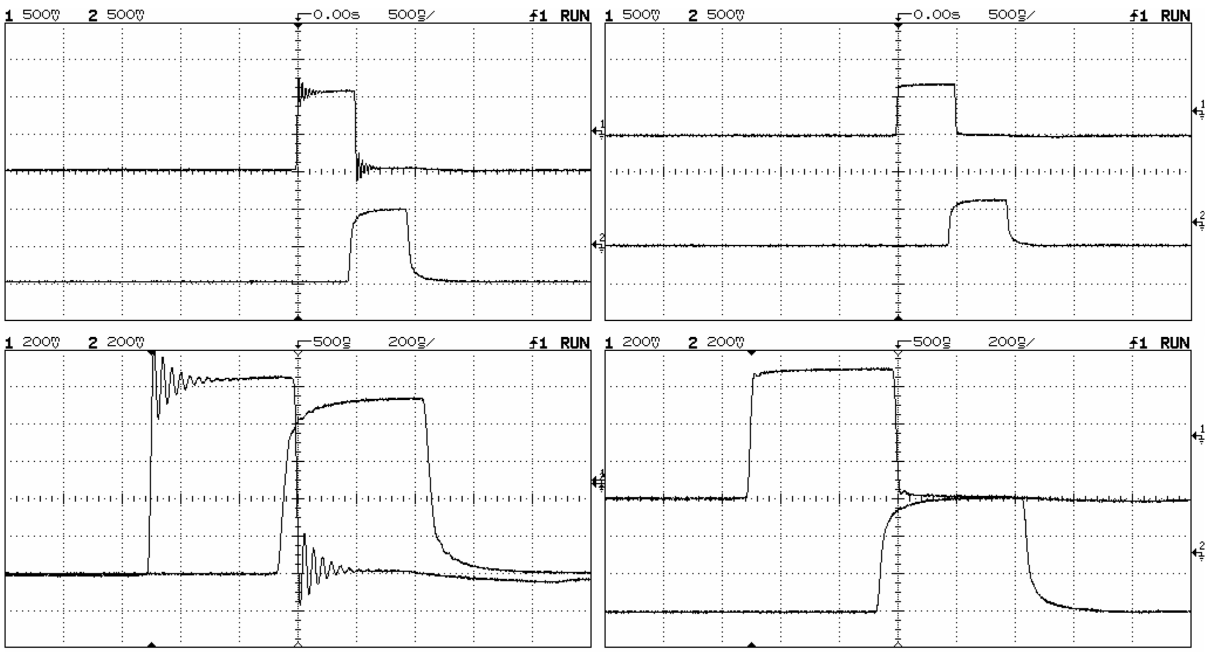
Example 6 Matched Lossy Line ($R_s=R_L=Z_o$) Length = *2564m* and velocity = $2.564x10^8 m/s$. The line is lossy with a distributed resistance of 0.1 ohms per meter.

Even with this small loss, the pulse that appears at the load is much smaller than for the lossless line. Also, the addition of the resistive term slightly mismatches the line, so there will now be a small amount of reflection at each end. Finally, notice that the resistance of the line means that it cannot charge and discharge instantaneously and, thus, finite rise and fall times are observed in the signals. We will return to this effect after we have considered time harmonic analysis of transmission lines. Recall that a finite length pulse contains a broad spectrum of frequencies and, thus, the analysis of pulses can also be done using the results from signal frequency sine wave excitation of the line. One only needs to take the Fourier transform of the initial pulse, determine what happens to each of the frequencies in the pulse and then take the inverse transform of the result.

The line analyzed here is about 2.5 km long (like in a cable TV system). Most practical lines have much more loss than is assumed here. Thus, one can appreciate the necessity for re-amplifying the signal from time-to-time if a useful voltage level is to be obtained at the load. In general, the signal is also distorted, as is the case here.



The reader is encouraged to try other configurations using PSpice. This is certainly much quicker than setting up actual lab experiments. The results from some real *100 meter* cables are shown below. We will discuss these also after steady state analysis.



Extra Bounce Diagram