UNIT XIII

Transmission Lines – Sinusoidal Steady-State

A. Common Applications of Transmission Lines with Steady-State Signals

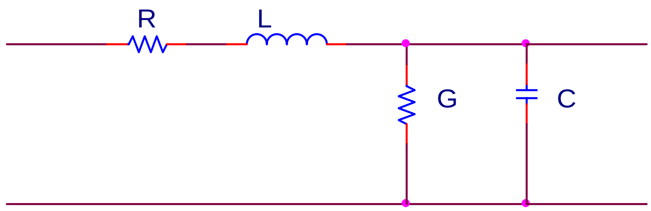
   a. Telephone land lines
   b. Audio cables
   c. Power lines
   d. Cable TV
   e. Circuit Interconnects

B. Two Simple Transmission Line Experiments

   a. Lab Experiment
   b. PSpice Experiment

C. Circuit Model of Transmission Lines

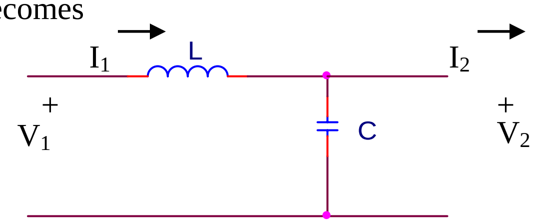
Transmission lines generally consist of two or more parallel or concentric wires. For two wires, the most common configuration is the twisted pair. Concentric wires form coaxial cables. Both configurations are used extensively. Two conductors near one another will have a capacitance, inductance, resistance and conductance. Since we wish to characterize transmission lines of any length, we will use per unit length values for these parameters. A small length of line Δz will have the following circuit configuration.



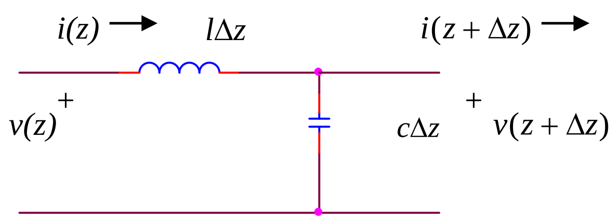
where $R = r\Delta z$, $L = l\Delta z$, $G = g\Delta z$, and $C = c\Delta z$. *r, l, g, and c* are the resistance, inductance, conductance and capacitance per unit length for the line. Note that for a commercially available transmission line (e.g. RG58/U cable), the values for these per unit length parameters are readily available; most catalogs contain this information.

D. Steady-State Analysis

Begin with the voltage and current relationships for the four components using phasor notation. (Assume a time dependence of $e^{j\omega t}$.) $V = RI$, $V = j\omega LI$, $I = GV$, and $I = j\omega CV$. Each of these expressions holds for the voltage across the component and the current through the component. Also, begin with the simplest case – the lossless transmission line with *R* and *G* both equal to zero. The circuit diagram for a small section of the line of length Δz becomes



where the input and output voltages and currents have been labeled. Applying the equations for the inductor and capacitor, we see that $V_1 - V_2 = j\omega LI_1$ and $I_1 - I_2 = j\omega CV_2$. We should rewrite these expressions and re-label the diagram to more generally describe any section of the line beginning at some location *z*. First, begin with the diagram



where we have also used the per unit length values for the inductance and capacitance. The two equations above then become

$$v(z) - v(z + \Delta z) = j\omega l\Delta z i(z)$$

$$i(z) - i(z + \Delta z) = j\omega c\Delta z v(z + \Delta z)$$

The finite difference spatial derivatives of voltage and current are then

$$\frac{v(z + \Delta z) - v(z)}{\Delta z} = -j\omega l i(z)$$

$$\frac{i(z + \Delta z) - i(z)}{\Delta z} = -j\omega c v(z + \Delta z)$$

where the minus signs appear because the order of the terms has been reversed. In the limit as $\Delta z \to 0$, the left hand sides of these expressions become the derivatives with respect to *z* (which we leave in the form of partial derivatives because the time derivative is implied in phasor formalism)

$$\frac{\partial v(z)}{\partial z} = -j\omega l i(z) \text{ and } \frac{\partial i(z)}{\partial z} = -j\omega c v(z)$$

Either *v* or *i* can be eliminated by combining the two equations

$$\frac{\partial^2 v(z)}{\partial z^2} = -j\omega l(-j\omega c)v(z) = -\omega^2 lcv(z)$$

$$\frac{\partial^2 v(z)}{\partial z^2} + \omega^2 lcv(z) = 0$$

or following the same steps

$$\frac{\partial^2 i(z)}{\partial z^2} + \omega^2 lci(z) = 0$$

This equation has been studied for a very long time, so we know a great deal about it. While it is actually a wave equation, in this form it looks like a harmonic oscillator equation, which has the following general solution:

$$v(z) = V_+ e^{-j\beta z} + V_- e^{+j\beta z}$$
$$i(z) = I_+ e^{-j\beta z} + I_- e^{+j\beta z}$$

As we will see, the first term in each expression characterizes a voltage or current wave propagating in the positive z direction while the second term is a wave in the negative z direction. It may be useful to recall here that almost every useful analytic solution to a practical engineering problem can be written in the form of an exponential, sine or cosine with real or imaginary arguments. Thus, it should be no surprise that the solution looks like this. We will see below that this expression does indeed give us the kind of a solution we expect for signal propagation on a transmission line.

To confirm that the correct solution has been identified, we plug the expressions into the appropriate equations.

$$\frac{\partial v(z)}{\partial z} = V_+ \left(-j\beta\right) e^{-j\beta z} + V_- \left(+j\beta\right) e^{+j\beta z}$$
$$\frac{\partial^2 v(z)}{\partial z^2} = V_+ \left(-j\beta\right)\left(-j\beta\right) e^{-j\beta z} + V_- \left(+j\beta\right)\left(+j\beta\right) e^{+j\beta z}$$
$$\frac{\partial^2 v(z)}{\partial z^2} = V_+ \left(-\beta^2\right) e^{-j\beta z} + V_- \left(-\beta^2\right) e^{+j\beta z} = -\beta^2 v(z)$$

which verifies the solution as long as $\beta^2 = \omega^2 lc$. We call β the propagation constant for the wave, which will relate simply to the wavelength. The same analysis confirms the solution for the current wave. However, it is more convenient to evaluate the current wave from the voltage wave.

$$\frac{\partial v(z)}{\partial z} = V_+ \left(-j\beta\right) e^{-j\beta z} + V_- \left(+j\beta\right) e^{+j\beta z} = -j\omega l i(z)$$
$$\frac{V_+ \left(\beta\right) e^{-j\beta z} + V_- \left(-\beta\right) e^{+j\beta z}}{\omega l} = i(z)$$
$$\frac{V_+ e^{-j\beta z} - V_- e^{+j\beta z}}{\omega l / \beta} = i(z)$$
$$i(z) = \frac{V_+ e^{-j\beta z} - V_- e^{+j\beta z}}{\sqrt{l/c}}$$

The ratio of the voltage to the current is the impedance at any point on the line

$$Z(z) = \frac{v(z)}{i(z)} = \frac{V_+ e^{-j\beta z} + V_- e^{+j\beta z}}{V_+ e^{-j\beta z} - V_- e^{+j\beta z}} \sqrt{\frac{l}{c}}$$

Thus, the term $\sqrt{\frac{l}{c}}$ has the units of impedance (Ohms) so it is called the characteristic impedance of the lossless transmission line: $Z_o = \sqrt{\frac{l}{c}}$. For lossy transmission lines, we will see that this has a more general form. The general form of the voltage and current solutions is then

$$v(z) = V_+ e^{-j\beta z} + V_- e^{+j\beta z}$$
$$i(z) = \frac{V_+ e^{-j\beta z} - V_- e^{+j\beta z}}{Z_o} = \frac{V_+}{Z_o} e^{-j\beta z} - \frac{V_-}{Z_o} e^{+j\beta z}$$

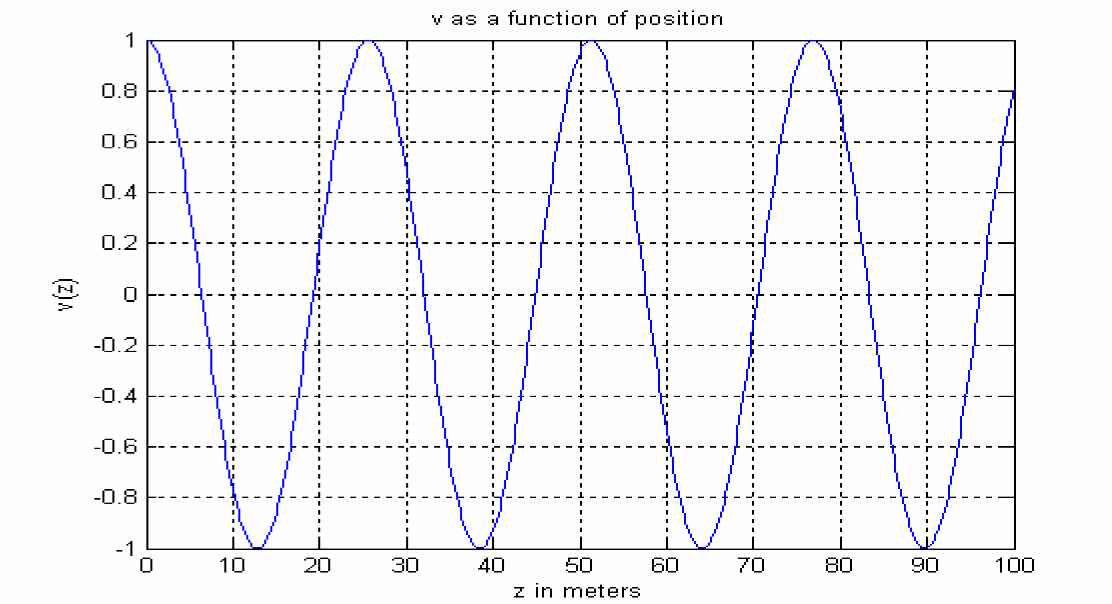
So far, we have only found the voltage and current in terms of phasor notation. To better understand what these expressions represent, we need to convert them to a space-time form in the usual manner for phasors. That is, we multiply the expression by $e^{j\omega t}$ and take the real part.

$$v(z,t) = \mathrm{Re}\left(v(z)e^{j\omega t}\right) = \mathrm{Re}\left(V_+ e^{-j\beta z} e^{j\omega t} + V_- e^{+j\beta z} e^{j\omega t}\right)$$
$$v(z,t) = \mathrm{Re}\left(|V_+| e^{j\theta_+} e^{-j\beta z} e^{j\omega t} + |V_-| e^{j\theta_-} e^{+j\beta z} e^{j\omega t}\right)$$

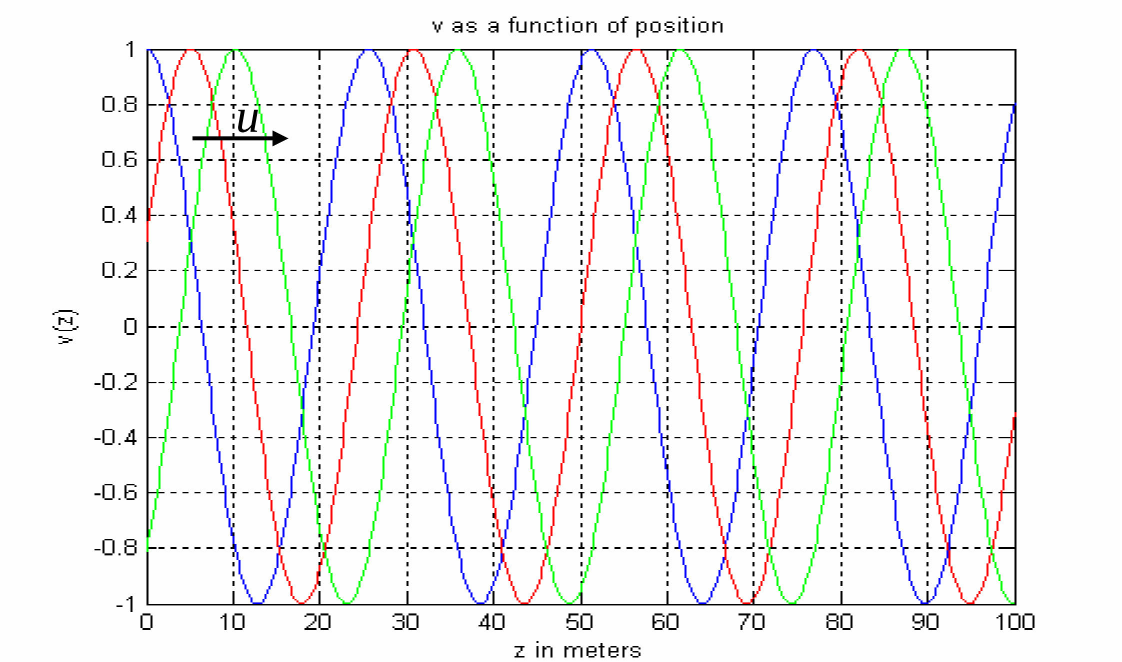
Note that, in general, we have to assume that the amplitudes are also complex, although this is not always the case.

$$v(z,t) = |V_+|\cos(\omega t - \beta z + \theta_+) + |V_-|\cos(\omega t + \beta z + \theta_-)$$

Consider just the first term $|V_+|\cos(\omega t - \beta z + \theta_+)$ and choose some realistic values for the parameters. For example, for an experiment that can be done easily with a standard set of lab equipment (including RG58/U coaxial cables), let $f = 10\ MHz$, $l = 0.196\mu H$, $c = 78pF$, and $V_+ = 1V$. Then, $\omega = 2\pi f = 2x10^7\pi = 6.283x10^7$, and $\beta = \omega\sqrt{lc} = 0.245$. We are free to choose θ_+ to be anything we want since we can pick the time when $t = 0$. Begin by plotting the first term as a function of z at time $t = 0$. (Generated with Matlab.)



We see that this produces a sinusoidal function of position that repeats in a distance a little more than 25 meters. (25.641 meters to be exact). Waves vary in space such that they repeat every wavelength. Thus, it appears that the wavelength of the voltage expression is 25+ meters. Next, plot the expression at three distinct times $t = 0$, $t = 0.02x10^{-6}s$, and $t = 0.04x10^{-6}s$.



Note that as time increases, the sine wave moves to the right (toward positive *z*). This is enough information to figure out the velocity of the wave. At $t = 0$, the first peak occurs at $z = 0$. At $t = 0.04x10^{-6}s$, the first peak occurs at a distance slightly greater than 10 meters (10.256 meters). Thus, the voltage wave propagates at a velocity given by

$$u = \frac{10.256}{0.02x10^{-6}} = 2.564x10^{8}\, {}^{m}\!/\!_{s}$$

which is the speed of light in the insulator material of the transmission line. The velocity of the voltage wave was found by following one of the sinusoidal peaks as it moves in *z*. We could also have found this result from the general expression for the voltage since such a peak in the sinusoid occurs when the derivative of the argument is set equal to zero. That is, for both forms of the argument found in the voltage expression,

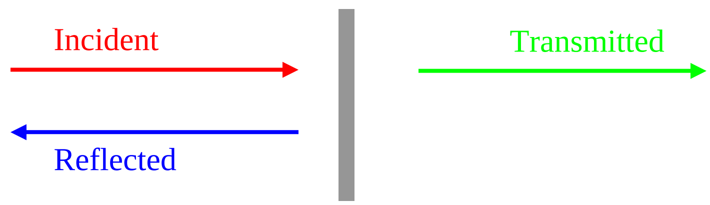
$$\frac{\partial}{\partial t}\left(\omega t \mp \beta z + \theta_{\pm}\right) = \omega \mp \beta \frac{\partial z}{\partial t} + 0 = 0$$

or

$$u = \frac{\partial z}{\partial t} = \pm\frac{\omega}{\beta} = \pm\frac{1}{\sqrt{lc}}$$

Plugging the *l* and *c* for the transmission line into this expression gives exactly the same value for the velocity *u*. As we will see when we address finding the inductance and capacitance per unit length for standard cables, this velocity is indeed the speed of light in the insulator material. Recall from your Physics courses that the wavelength of light is given by $\lambda = {}^{u}\!/\!_{f}$ which we can rewrite as $\lambda = {}^{u}\!/\!_{f} = \frac{\omega 2\pi}{\beta\omega} = \frac{2\pi}{\beta}$. Thus, β has a relationship to the wavelength λ like ω has to T, the period of the sine wave. For the example above, $\lambda = {}^{u}\!/\!_{f} = {}^{2.564x10^8}\!/\!_{10^7} = 25.64 = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{0.245} = 25.64$ *QED*.

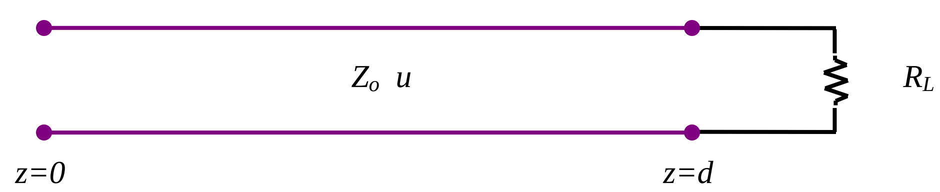
Both signs for *u* are possible because signals can propagate in either direction. In fact, signals can simultaneously propagate in both directions. Since the voltage and current signals are waves like light and sound waves, they can reflect off of any changes in the line. For example, if a Tee connector is placed at the end of the line and the signal split into two lines, some of the signal will reflect off of the Tee and some will transmit into the two cables, just as light or sound reflects off of a change in materials. We will see more about this below. However, it is useful at this point to recognize that the solution developed for current and voltage signals will accommodate mismatch conditions since waves can propagate in both directions.



Note that the characteristic impedance, velocity, etc. for the sinusoidal waves are the same as for transients or pulses on transmission lines, as seen in the previous unit. This is to be expected, since a pulse or transient can be split up into individual frequency components (Fourier Transform), each of which propagates like a single frequency wave. As long as each frequency propagates at the same velocity (as they do in a lossline line), characteristics developed for pulses also work for sine waves and vice versa.

Given that we should expect two waves propagating on the line (one in each direction), we will see the same kind of constructive and destructive interference that is observed with any kind of a wave. The locations where the waves add or subtract from one another will not propagate, but will be fixed in space. Thus, the net affect of the two waves will be a standing wave. We call the propagating waves traveling waves to distinguish them from the standing waves. It is much easier to observe standing waves since the sit still than it is traveling waves since they propagate at the speed of light.

Since the primary cause of standing waves is some kind of a mismatch between transmission lines or lines and loads, we consider what happens when we add a simple resistive load at the end of a line.



Transmission lines are usually shown as two parallel wires, even if they really are coaxial cables, with the length, characteristic impedance and velocity indicated. One can also describe the line with length, capacitance and inductance per unit length. Recall that, at this point, only lossless lines are being considered. We will generalize our results to lossy lines after we have fully described this simpler, ideal case. In the diagram above, the input end is labeled as $z = 0$ while the output end is $z = d$. We are free to choose either end as $z = 0$ depending on the problem being addressed. A great deal of the analysis of waves is done by first finding the most convenient representations. Thus, we will be re-writing voltage and current expressions in different forms, as we did when we represented the current wave in terms of the voltage magnitudes and impedance. The next step is to recognize that we will usually know what the positive traveling wave magnitude V_+ is since it is produced by a known source. Thus, we should write the negative traveling wave magnitude V_- in terms of what we know.

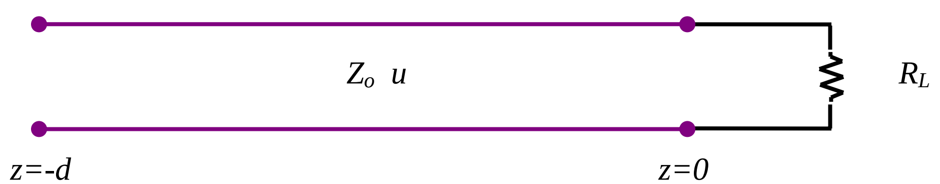
$$V_- = \Gamma_L V_+$$

where Γ_L is called the reflection coefficient at the load since we will only have a negative traveling wave if some or all of the positive traveling wave reflects off of the load. This is only useful if Γ_L is relatively straight forward to determine, which it is. To find it, we recognize that the voltage and current at the load end of the line must be the same as the voltage and current for the load itself. To assure that this is the case, we can set the impedance on the line, evaluated at the load, equal to the load impedance.

$$Z(d) = \frac{v(d)}{i(d)} = Z_o \frac{V_+ e^{-j\beta d} + V_- e^{+j\beta d}}{V_+ e^{-j\beta d} - V_- e^{+j\beta d}} = R_L$$

$$Z_o \frac{V_+\left(e^{-j\beta d} + \Gamma_L e^{+j\beta d}\right)}{V_+\left(e^{-j\beta d} - \Gamma_L e^{+j\beta d}\right)} = R_L$$

This looks like a complicated expression to address. However, at this point we are only concerned with what is happening at the load, so we are free to select the load end as $z = 0$ and the source end as $z = -d$.



Then, we have

$$Z(0) = \frac{v(0)}{i(0)} = Z_o \frac{V_+ e^{-j\beta 0} + V_- e^{+j\beta 0}}{V_+ e^{-j\beta 0} - V_- e^{+j\beta 0}} = R_L$$

$$Z_o \frac{V_+\left(1+\Gamma_L\right)}{V_+\left(1-\Gamma_L\right)} = R_L$$

which can be re-written to solve for Γ_L

$$\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o}$$

Also, re-writing the voltage and current expressions

$$v(z) = V_+\left(e^{-j\beta z} + \Gamma_L e^{+j\beta z}\right)$$

$$i(z) = \frac{V_+}{Z_o}\left(e^{-j\beta z} - \Gamma_L e^{+j\beta z}\right)$$

To analyze a transmission line, specify the frequency f, length d, the characteristic impedance Z_o, the propagation velocity u, the input magnitude V_+, and the load impedance R_L and then determine the propagation constant β and the reflection coefficient Γ_L. The voltage and current waves can then be evaluated. Consider a few simple cases.

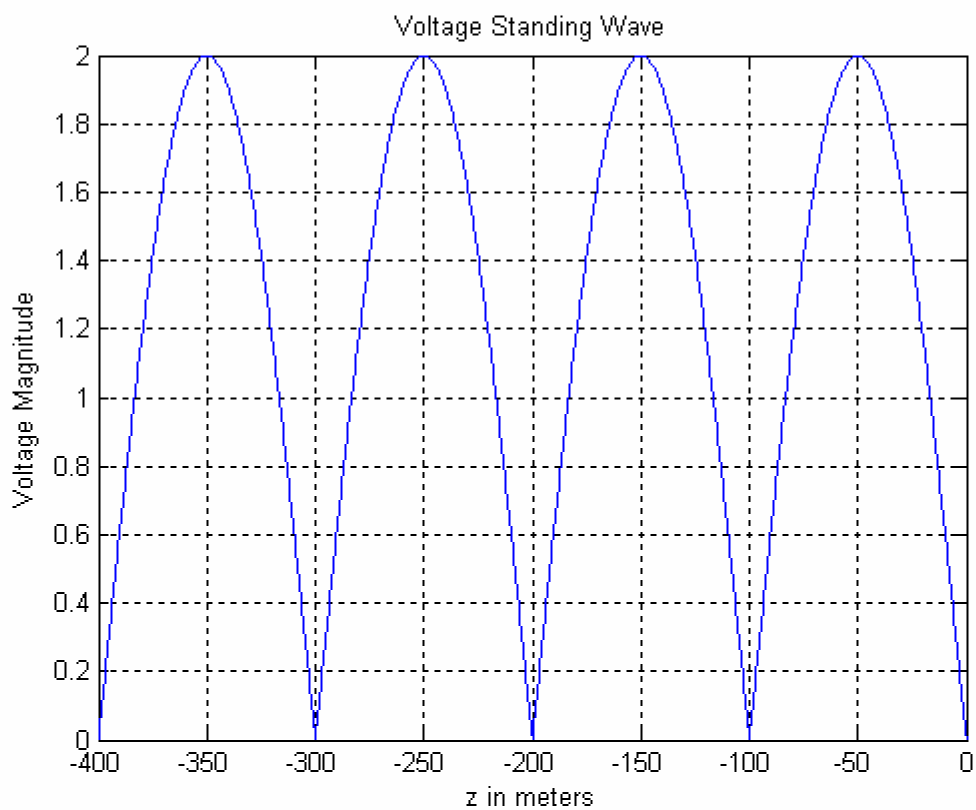
First, assume $f = 1MHz$, $d = 400$ *meters*, $Z_o = 50$ *Ohms*, $u = 2x10^8$ *m/s*, $V_+ = 1$ *V*, and $R_L = 0$. This is the case for a short circuit load. For such a load, we expect that all of the incident wave will reflect, which it does.

$$\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o} = \frac{0-50}{0+50} = -1$$

We will also need

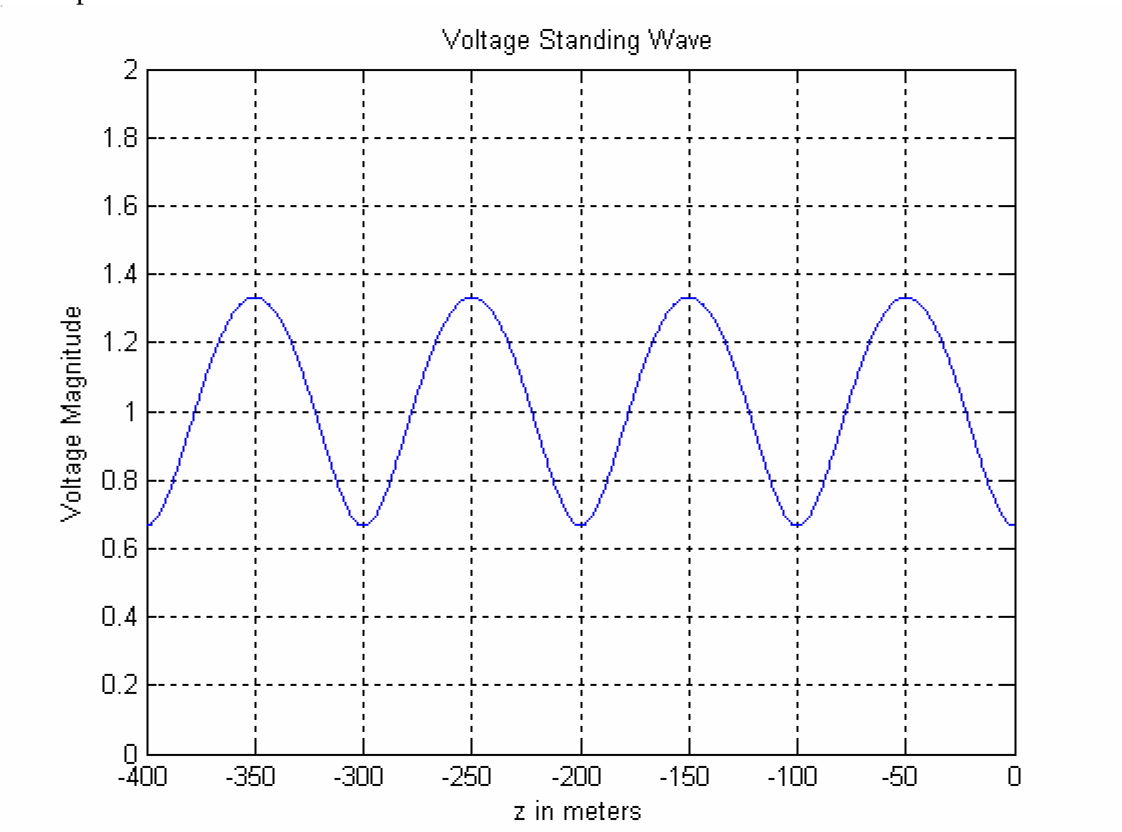
$$\beta = \omega/u = 2\pi f/u = 0.0314$$

The standing wave pattern, determined by taking the absolute value of the voltage can be easily evaluated using Matlab.



We see that the voltage is zero at the load end ($z = 0$), as it must be at a short circuit. At this location, the incident and reflected waves are exactly out of phase and they cancel or $V_- = -V_+$. *50 meters* away from the load, the two waves are exactly in phase and add constructively resulting in a total magnitude that is twice the incident magnitude. Constructive and destructive interference repeats every *100 meters*. For wave phenomena, absolute distances are not really of interest. Rather, what matters is distance as measured in wavelengths. For this case, $\lambda = u/f = 2x10^8 / 1x10^6 = 200$. Thus, we see that the locations where the standing wave is a maximum are separated by $\lambda/2$ as are the locations where the standing wave is a minimum.

Second, assume everything is the same except that $R_L = 25$ *Ohms*. Then the standing wave pattern is.



The reflection coefficient in this case is

$$\Gamma_L = \frac{R_L - Z_o}{R_L + Z_o} = \frac{25 - 50}{25 + 50} = -\frac{1}{3}$$

Thus, the reflected wave cannot fully cancel out the incident wave since it is too small. The locations of the maximas and minimas are the same.

To characterize such patterns in general, we define the Voltage Standing Wave Ratio as

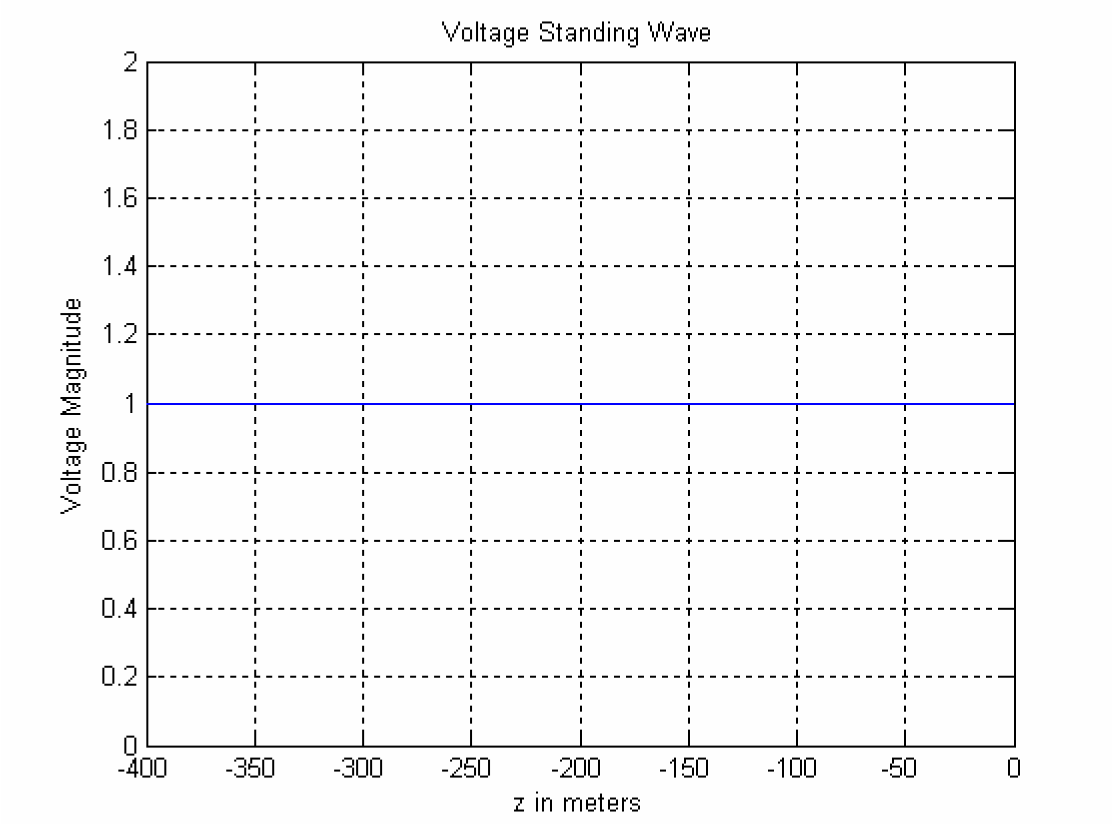
$$VSWR = S = \frac{1 + \Gamma_L}{1 - \Gamma_L}$$

since the maximum constructive interference has to be when the two waves line up exactly and the magnitude is $V_+\left(1+\Gamma_L\right)$ while the maximum destructive interference has to be when one of the two waves is positive and one is negative and the magnitude is $V_+\left(1-\Gamma_L\right)$ so the ratio will equal *S*. For the short circuit load $S \to \infty$ while for the *25 Ohm* load, $S = \dfrac{1+\Gamma_L}{1-\Gamma_L} = \dfrac{1-\frac{1}{3}}{1+\frac{1}{3}} = 0.5$ which is consistent with the figure.

Third, consider a matched case, where R_L = *50 Ohms*. Then there is no reflected wave

$$\Gamma_L = \frac{50-50}{50+50} = 0$$

and the standing wave pattern is very, very simple

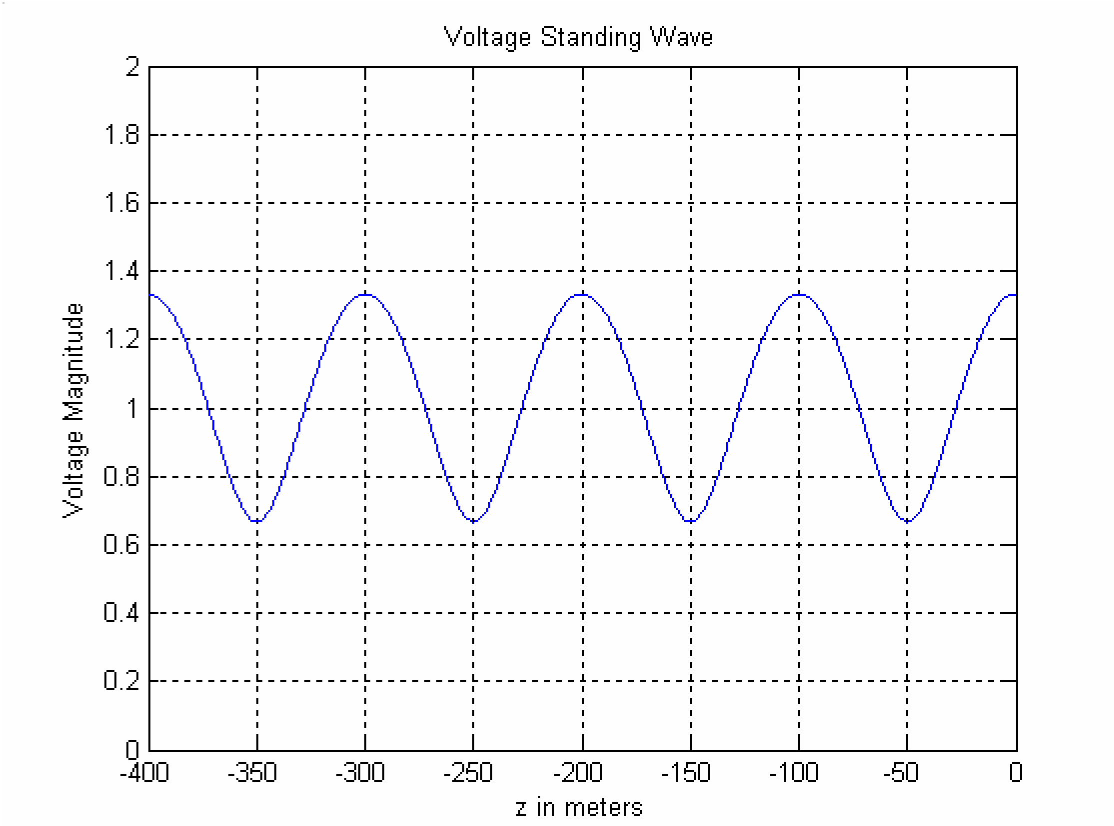


or the magnitude is the same everywhere.

Fourth, the pattern will change when the load impedance exceeds the characteristic impedance. Consider R_L = *100 Ohms*. Then, the reflection coefficient will be

$$\Gamma_L = \frac{100-50}{100+50} = \frac{1}{3}$$

This means that the incident and reflected waves have the same sign at the load end, so the two waves are in phase and the maximum will occur at the load. The first minimum will be a quarter wavelength from the load.

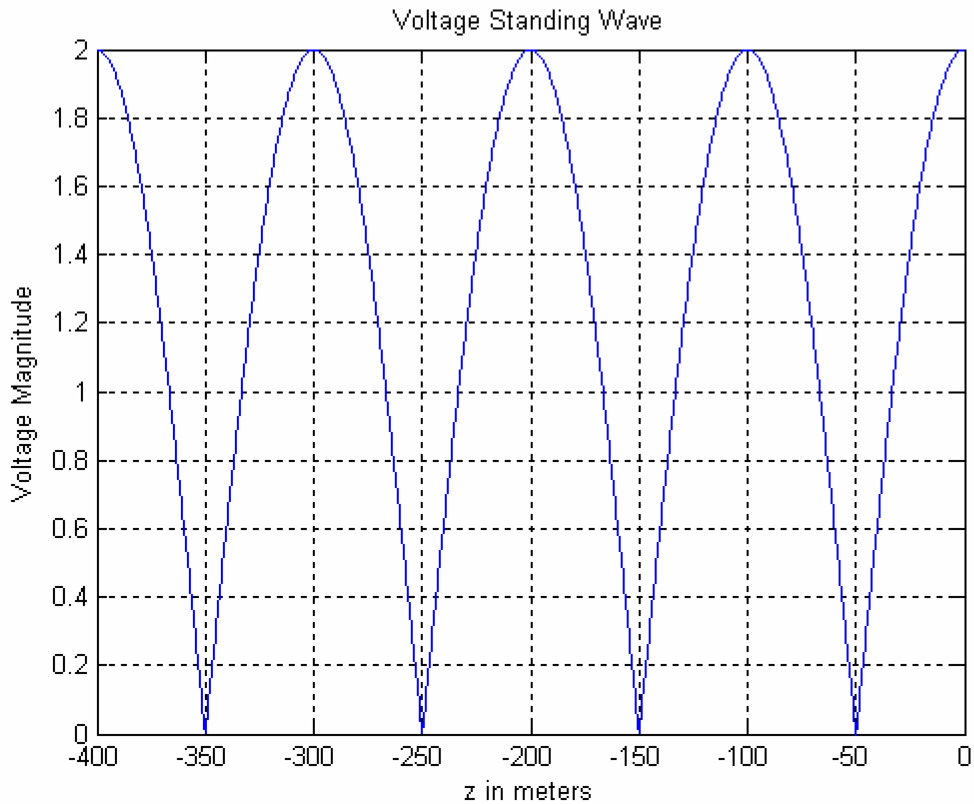


Otherwise, this looks very much like the standing wave pattern for a *50 Ohm* load, which is should since the magnitude of the reflection coefficient is the same. Only the sign is different.

Finally, for an open circuit load, $R_L \to \infty$, the reflection coefficient will be

$$\Gamma_L = \frac{\infty - 50}{\infty + 50} \to 1$$

which has the same magnitude as for the short circuit. Thus, we expect the standing wave pattern to look similar, which it does. The only difference is that the maximum occurs at the load. In general, for resistive loads, the voltage maximum occurs at the load when $R_L > Z_o$ while a minimum occurs at the load for $R_L < Z_o$. Note that the following plot is not generated by setting $R_L \to \infty$, since Matlab has no idea what ∞ is. Rather, $R_L = 10^{100}$ was used. Essentially, any really large number will suffice.



Once again, we can re-write some of the information we know about waves to show that what we observed with these plots is quite general. Start with the expression for the voltage wave

$$v(z) = V_+\left(e^{-j\beta z} + \Gamma_L e^{+j\beta z}\right)$$

Factor out the phase term

$$v(z) = V_+ e^{-j\beta z}\left(1 + \Gamma_L e^{+j2\beta z}\right) = V_+ e^{-j\beta z}\left(1 + \Gamma(z)\right)$$

where we have now defined the generalized reflection coefficient as

$$\Gamma(z) = \Gamma_L e^{+j2\beta z}$$

Since $e^{j0} = e^{-j2\pi} = e^{-j4\pi} = +1$ and $e^{-j\pi} = e^{-j3\pi} = e^{-j5\pi} = -1$, the first two maximas will occur at $2\beta z = 0$ and $2\beta z = -2\pi$ which are separated by $z = \dfrac{\pi}{\beta} = \dfrac{\lambda}{2}$ as we have seen.

(The arguments must be negative since *z* is negative.) The same separation occurs for the minimas. With this more general expression, we can generalize things somewhat by letting the load impedance be complex (i.e. include an inductor or capacitor). When this is the case, the reflection coefficient at the load will also be complex.

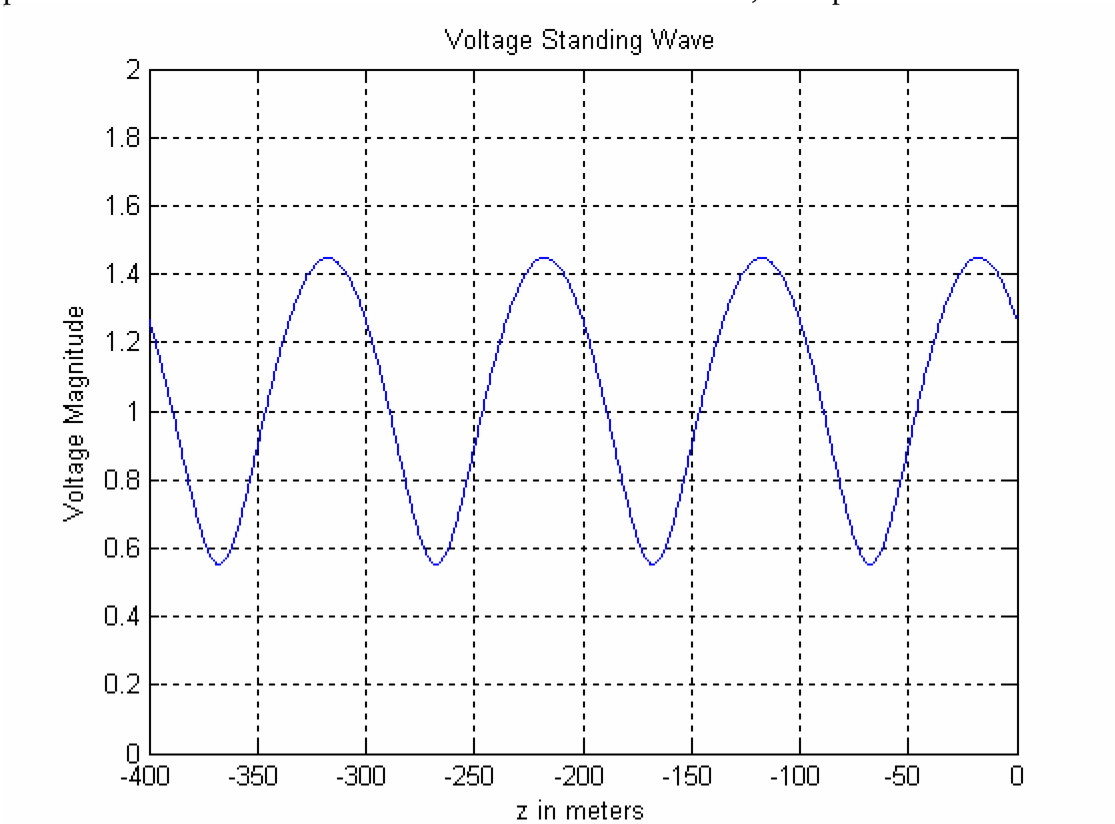
$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \left|\Gamma_L\right| e^{j\theta_\Gamma}$$

The generalized reflection coefficient becomes

$$\Gamma(z) = \Gamma_L e^{+j2\beta z} = \left|\Gamma_L\right| e^{j\theta_\Gamma} e^{+j2\beta z}$$

The first minimum occurs at $\theta_\Gamma + 2\beta z = -\pi$. Since z is negative, the distance to the first minimum is $z_{\min} = \frac{-\pi - \theta_\Gamma}{2\beta} = -\lambda\left(\frac{\pi}{4\pi} + \frac{\theta_\Gamma}{4\pi}\right) = -\frac{\lambda}{4} - \frac{\lambda\theta_\Gamma}{4\pi}$. The first minimum for a resistive load is where we saw it before. The first minimum for a complex load moves further away from the load if the phase is positive and closer if the phase is negative.

For another example, let $Z_L = 50 + j50$. Then $\Gamma_L = 0.2 + j0.4$ and the standing wave pattern shows that the first minimum has moved to the left, as expected.



The phase of the reflection coefficient is $\theta_\Gamma = 0.3524\pi$ so that the first minimum will occur at $z_{\min} = -\frac{\lambda}{4} - \frac{\lambda\theta_\Gamma}{4\pi} = -\frac{\lambda}{4} - \frac{\lambda 0.3524\pi}{4\pi} = -\left(0.25 + \frac{0.3524}{4}\right)\lambda = -.338\lambda = 67.6m$

which is where the minimum does indeed occur. The other minimas are separated by $\frac{\lambda}{2}$.

We saw above that the impedance anywhere on the line can be written as

$$Z(z) = \frac{v(z)}{i(z)} = Z_o \frac{V_+ e^{-j\beta z} + V_- e^{+j\beta z}}{V_+ e^{-j\beta z} - V_- e^{+j\beta z}}$$

which can now be written in a more compact form using the generalized reflection coefficient

$$Z(z) = Z_o \frac{V_+ e^{-j\beta z}\left(1 + \Gamma(z)\right)}{V_+ e^{-j\beta z}\left(1 - \Gamma(z)\right)} = Z_o \frac{1 + \Gamma(z)}{1 - \Gamma(z)}$$

To avoid having to separately evaluate the reflection coefficient, this expression can be written more conveniently.

$$Z(z) = Z_o \frac{e^{-j\beta z} + \Gamma_L e^{+j\beta z}}{e^{-j\beta z} - \Gamma_L e^{+j\beta z}} = Z_o \frac{e^{-j\beta z} + \dfrac{Z_L - Z_o}{Z_L + Z_o} e^{+j\beta z}}{e^{-j\beta z} - \dfrac{Z_L - Z_o}{Z_L + Z_o} e^{+j\beta z}}$$

$$Z(z) = Z_o \frac{(Z_L + Z_o)e^{-j\beta z} + (Z_L - Z_o)e^{+j\beta z}}{(Z_L + Z_o)e^{-j\beta z} - (Z_L - Z_o)e^{+j\beta z}}$$

$$Z(z) = Z_o \frac{Z_L\left(e^{-j\beta z} + e^{+j\beta z}\right) + Z_o\left(e^{-j\beta z} - e^{+j\beta z}\right)}{Z_o\left(e^{-j\beta z} + e^{+j\beta z}\right) + Z_L\left(e^{-j\beta z} - e^{+j\beta z}\right)}$$

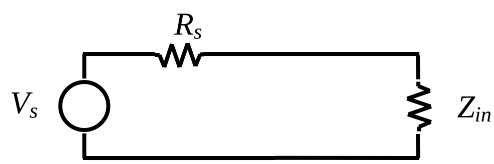
$$Z(z) = Z_o \frac{Z_L 2\cos\beta z + Z_o(-j2\sin\beta z)}{Z_o 2\cos\beta z + Z_L(-j2\sin\beta z)}$$

$$Z(z) = Z_o \frac{Z_L - jZ_o \tan\beta z}{Z_o - jZ_L \tan\beta z}$$

The form of the general reflection coefficient assumes that we have chosen $z = 0$ at the load end for simplicity. With this expression, we can now address the input impedance of the transmission line, which only requires that we evaluate it at $z = -d$.

$$Z_{in} = Z(-d) = Z_o \frac{Z_L + jZ_o \tan\beta d}{Z_o + jZ_L \tan\beta d}$$

This is the impedance seen by the source that drives the line.



One of the more profound differences between transmission lines and low frequency circuits is that a simple combination of resistors and cables can produce almost any impedance. This occurs because of mismatches between the line and the load. When the load is not equal to the line impedance, the voltage wave incident on the load will at least partially reflect, returning to the source end and interfering either constructively or destructively with the source voltage. It is this combination of incident and reflected voltage waves that produces the amazing variety of impedances at the input end of the line. Let us consider several simple cases to see what can happen.

Example 1: Matched Line ($Z_L = R_L = Z_o$) of arbitrary length. Propagation velocity = u.

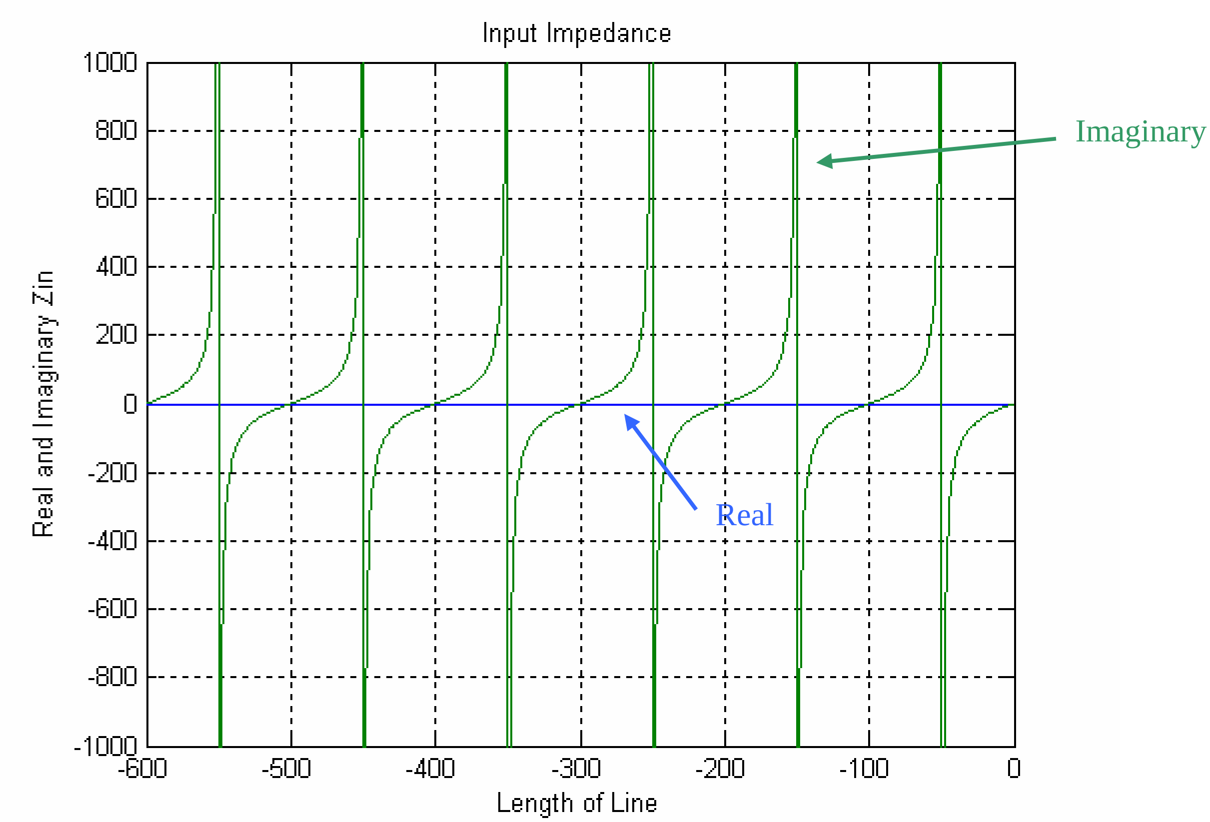
$$Z_{in} = Z_o \frac{Z_0 + jZ_o \tan \beta d}{Z_o + jZ_o \tan \beta d} = Z_o$$

This is the ideal situation where the input impedance is the characteristic impedance no matter how long the line is. Clearly this is the best possible circumstance and the reason why standard function generators are built to drive *50 Ohm* loads since RG58/U cables have $Z_o = 50\Omega$. Maximum power is transferred to the line and no reflections exist as long as all impedances are the same.

Example 2: Short Circuit Load ($Z_L = R_L = 0$), line length *d*, other conditions the same.

$$Z_{in} = Z_o \frac{0 + jZ_o \tan \beta d}{Z_o + j0 \tan \beta d} = jZ_o \tan \beta d$$

The input impedance of a short circuited line is completely imaginary and can have any value from $-\infty$ to $+\infty$, depending on the length of the line. Using Matlab, we can plot Z_{in} for line lengths up to three wavelengths.

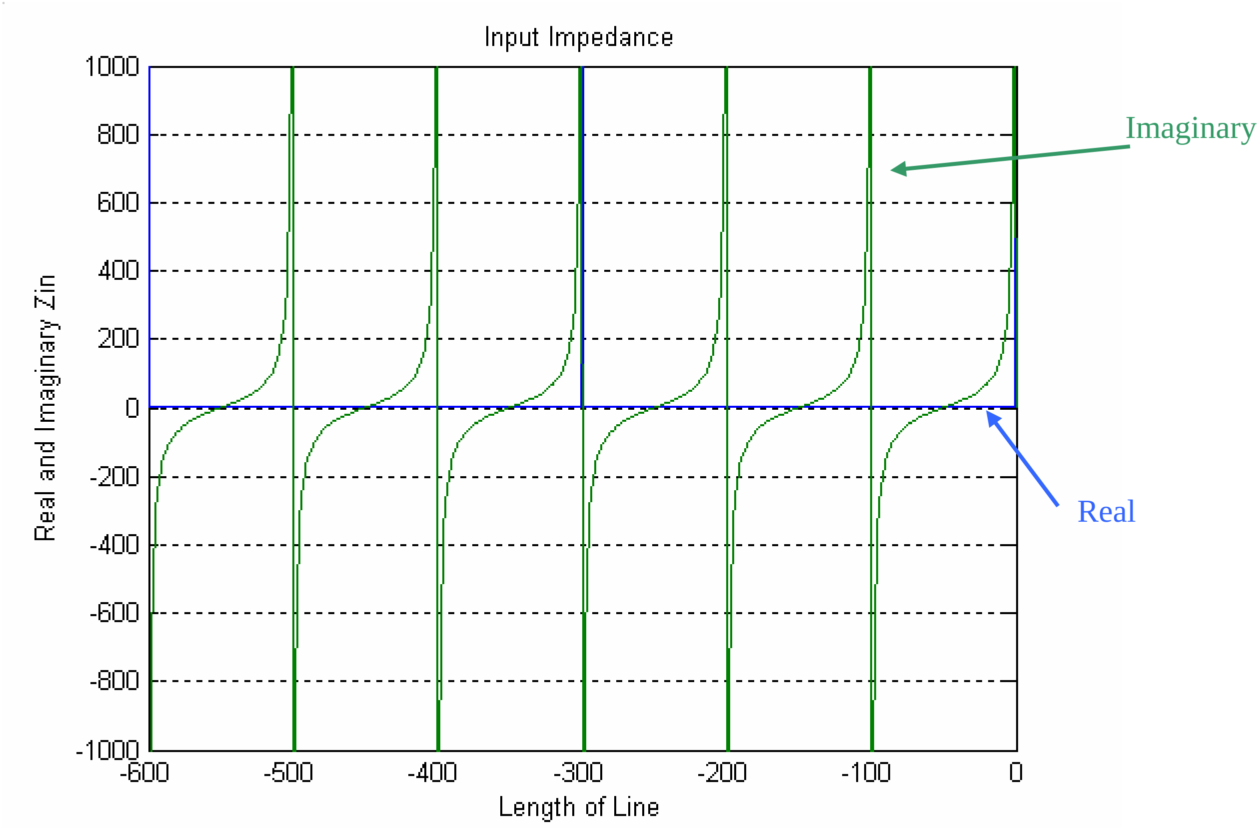


Note that the impedance can indeed achieve any imaginary value and that it repeats every half wavelength, just like the standing wave pattern.

Example 2: Open Circuit Line ($Z_L = R_L \to \infty$), line length *d*, other conditions the same.

$$Z_{in} = Z_o \frac{\infty + jZ_o \tan \beta d}{Z_o + j\infty \tan \beta d} = \frac{1}{jZ_o \tan \beta d} = -jZ_o \cot \beta d$$

Again, plotting this for line lengths up to three wavelengths



Note that Z_{in} for this case looks exactly like that of the short circuit except that it is shifted to the right by a quarter wavelength.

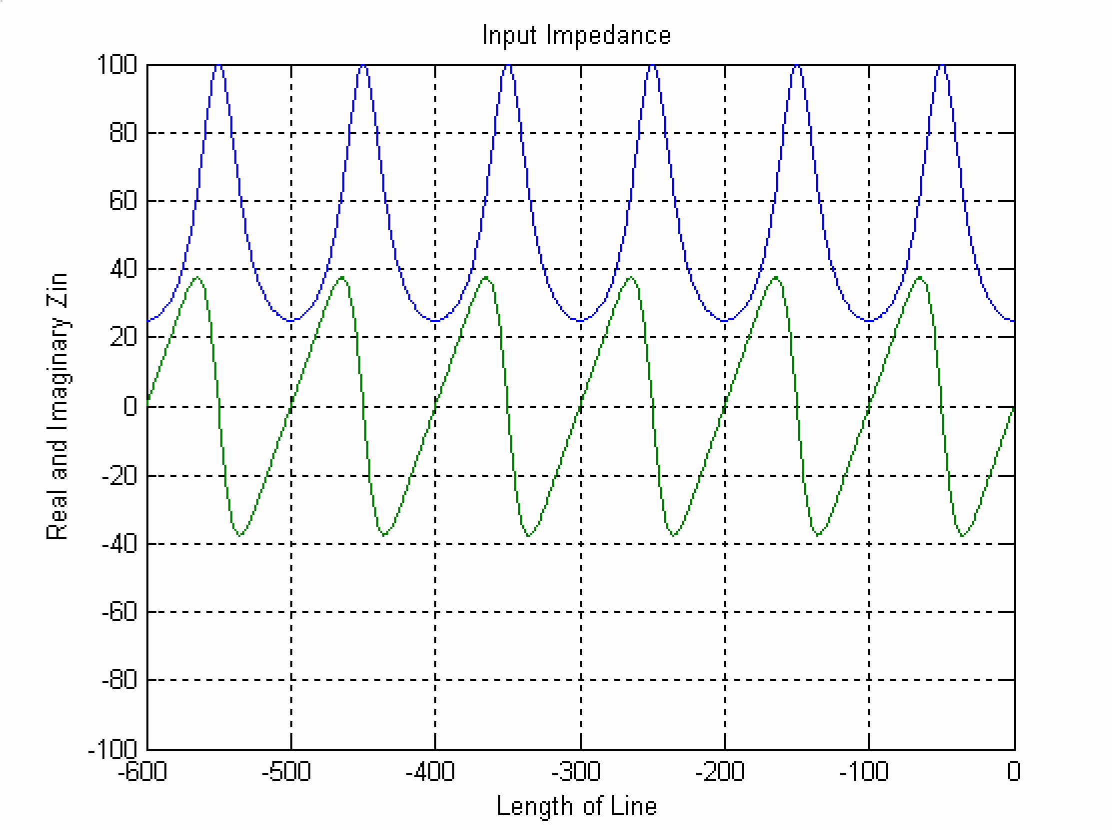
Example 3: Mismatched Load Smaller than Z_o ($Z_L = R_L = 25\Omega$), line length *d*, other conditions the same.

$$Z_{in} = 50\frac{25 + j50\tan\beta d}{50 + j25\tan\beta d}$$

For this case, Z_{in} will have both a real and imaginary part at most line lengths. There are a couple of conditions that produce totally real input impedances. First, for $d = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, 2\lambda, \ldots$ $Z_{in} = 25 = Z_L$. Thus, for line length equal to any integer multiple of a half wavelength, $Z_{in} = Z_L$. We can see that this is a general result by setting $d = \frac{n\lambda}{2}$

$$Z_{in} = Z_o\frac{Z_L + jZ_o\tan\left(\beta\frac{n\lambda}{2}\right)}{Z_o + jZ_L\tan\left(\beta\frac{n\lambda}{2}\right)} = Z_o\frac{Z_L + jZ_o\tan\left(\frac{2\pi}{\lambda}\frac{n\lambda}{2}\right)}{Z_o + jZ_L\tan\left(\frac{2\pi}{\lambda}\frac{n\lambda}{2}\right)}$$

$$Z_{in} = Z_o\frac{Z_L + jZ_o\tan(n\pi)}{Z_o + jZ_L\tan(n\pi)} = Z_o\frac{Z_L + jZ_o 0}{Z_o + jZ_L 0} = Z_o\frac{Z_L}{Z_o} = Z_L$$



Second, for $d = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots\ Z_{in} = 100$. To show the general form of this result, set $d = \frac{n\lambda}{4}$, for n odd.

$$Z_{in} = Z_o \frac{Z_L + jZ_o \tan\left(\beta \frac{n\lambda}{4}\right)}{Z_o + jZ_L \tan\left(\beta \frac{n\lambda}{4}\right)} = Z_o \frac{Z_L + jZ_o \tan\left(\frac{2\pi}{\lambda}\frac{n\lambda}{4}\right)}{Z_o + jZ_L \tan\left(\frac{2\pi}{\lambda}\frac{n\lambda}{4}\right)}$$

$$Z_{in} = Z_o \frac{Z_L + jZ_o \tan\left(\frac{n\pi}{2}\right)}{Z_o + jZ_L \tan\left(\frac{n\pi}{2}\right)} = Z_o \frac{Z_L + jZ_o \infty}{Z_o + jZ_L \infty} = Z_o \frac{Z_o}{Z_L} = \frac{Z_o^{\ 2}}{Z_L}$$

For the specific case considered here, $Z_{in} = \frac{Z_o^{\ 2}}{Z_L} = \frac{50^2}{25} = 100$, as expected.

Since it is possible to obtain real Z_{in} for these two sets of line lengths, designers can achieve optimal power transfer conditions even when transmission lines and loads do not match. We will return to this issue later.

Power Delivered to the Load Impedance

Our primary goal is to deliver the maximum possible power to the load. Recall from circuits, that Phasor notation provides us with a simple way to determine average power.

$$P_{ave} = \mathrm{Re}\left(\frac{VI^*}{2}\right) = \mathrm{Re}\left(\frac{VV^*}{2Z^*}\right) = \mathrm{Re}\left(\frac{V^2}{2Z^*}\right) = \mathrm{Re}\left(\frac{II^*Z}{2}\right) = \mathrm{Re}\left(\frac{I^2Z}{2}\right)$$

The power delivered to the transmission line is, thus

$$P_{ave} = \mathrm{Re}\left(\frac{V_{in}I_{in}^*}{2}\right) = \mathrm{Re}\left(\frac{V_{in}^2}{2Z_{in}^*}\right)$$

Once we know Z_{in}, we can easily determine the power delivered to the line. Thus far, we have only considered lossless lines, so the power delivered to the line will equal the power delivered to the load (no power is dissipated in the line). Therefore

$$P_{ave} = \mathrm{Re}\left(\frac{V_{in}^2}{2Z_{in}^*}\right) = \mathrm{Re}\left(\frac{V_L^2}{2Z_L^*}\right)$$

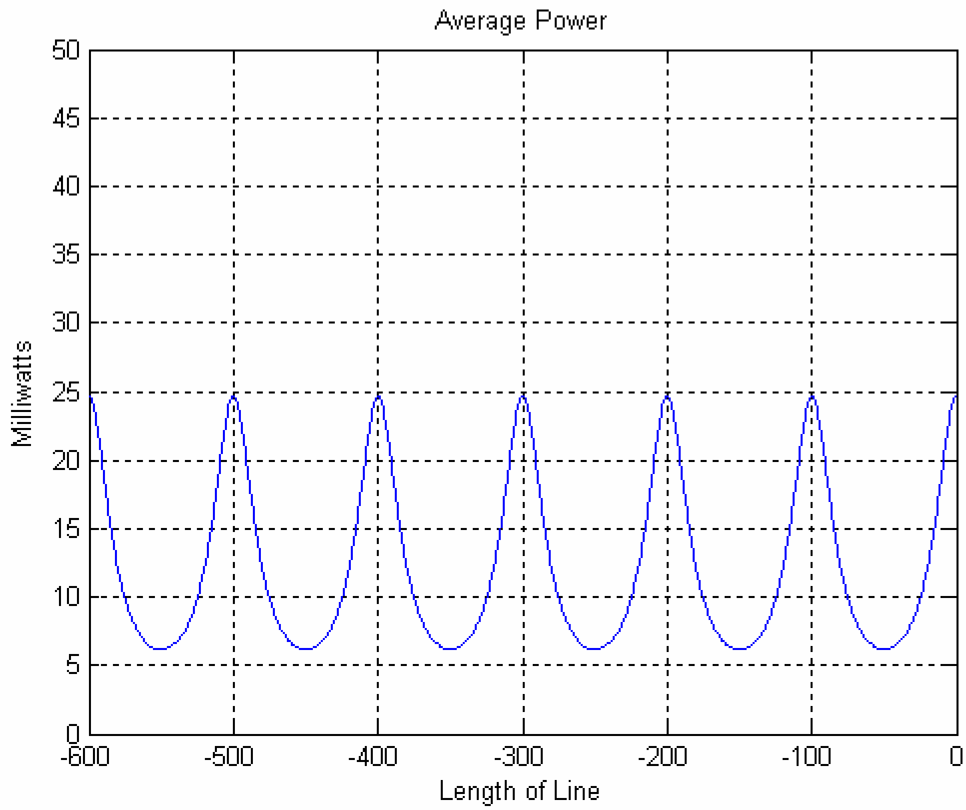
For lossy lines, things will not be so simple.

For the three examples we have considered above, the average power is

Examples 1 & 2: Z_{in} is imaginary in both cases. $P_{ave} = \mathrm{Re}\left(\frac{V_{in}^2}{2Z_{in}^*}\right) = \mathrm{Re}\left(\frac{V_{in}^2}{-j2|Z_{in}|}\right) = 0$

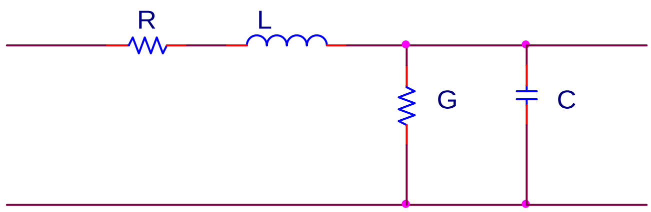
since V_{in}^2 is real. Neither an open nor short circuit can dissipate any power, so this is the expected answer.

Example 3: To determine the input voltage, we also need to specify the source impedance and voltage. Assume we have set up for matched conditions to the line, if not the load. That is $V_s = 2V$, $R_s = Z_o = 50\Omega$. Note that the maximum power is transferred when the input impedance is equal to the load impedance. Power is transferred no matter what the length of cable is, but the best conditions occur for particular lengths. The power delivered also depends on the frequency, since the length of the line in wavelengths varies with frequency. This topic is left for the reader to investigate and will addressed in a design project.



Lossy Transmission Lines

Another major advantage of using Phasors to analyze transmission lines is that we can simply extend the results obtained for lossless lines. Return to the circuit representation of a small piece of a transmission line.



The series resistance comes from losses in the transmission line conductors while the parallel conductance comes from losses in the transmission line insulator. For lossless lines, the series impedance per unit length was given by $j\omega l$ and the parallel admittance was $j\omega c$. For lossy lines, we only need to replace these terms by $r + j\omega l$ and $g + j\omega c$, respectively. Thus, we should be able to take all of the results obtained so far, replace $j\omega l$ by $r + j\omega l$ and $j\omega c$ by $g + j\omega c$ and we will have a complete characterization of lossy lines. To do this properly, we have to generalize some of our results first. Listed below are the basic building blocks for transmission line analysis for both lossless and lossy transmission lines. The terms in the left column are the results obtained thus far while the terms in the right column are the corresponding lossy expressions.

Expression	Lossless Line	Lossy Line
Series Impedance	$j\omega l$	$r + j\omega l$
Parallel Admittance	$j\omega c$	$g + j\omega c$
Z_o	$\sqrt{\dfrac{j\omega l}{j\omega c}} = \sqrt{\dfrac{l}{c}}$	$\sqrt{\dfrac{r + j\omega l}{g + j\omega c}}$
Propagation Constant	$j\beta = \sqrt{j\omega l j\omega c} = j\omega\sqrt{lc}$	$\gamma = \alpha + j\beta = \sqrt{(r + j\omega l)(g + j\omega c)}$
Z_{in}	$Z_o \dfrac{Z_L + jZ_o \tan\beta d}{Z_o + jZ_L \tan\beta d}$	$Z_o \dfrac{Z_L + Z_o \tanh\gamma d}{Z_o + Z_L \tanh\gamma d}$

Note that the terms in the lossy column reduce to those in the lossless column when r and g are both zero. Given these expressions, we can write the solutions for voltage and current, etc. for lossy lines. The basic solution becomes

$$v(z) = V_+ e^{-\gamma z} + V_- e^{+\gamma z}$$
$$i(z) = I_+ e^{-\gamma z} + I_- e^{+\gamma z}$$

which, when re-written in the several useful ways we found for lossless lines, become

$$v(z) = V_+ e^{-\gamma z} + V_- e^{+\gamma z}$$
$$i(z) = \frac{V_+}{Z_o} e^{-\gamma z} - \frac{V_-}{Z_o} e^{+\gamma z}$$

and

$$v(z) = V_+\left(e^{-\gamma z} + \Gamma_L e^{+\gamma z}\right)$$
$$i(z) = \frac{V_+}{Z_o}\left(e^{-\gamma z} - \Gamma_L e^{+\gamma z}\right)$$

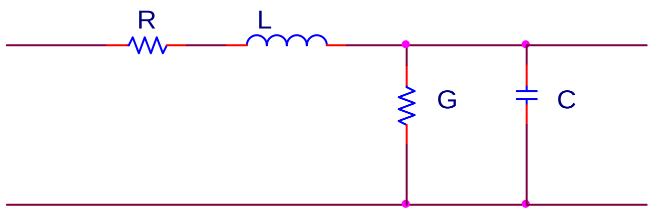
It is left to the reader to check each expression to confirm that they reduce correctly for lossless lines. We will consider what these expressions look like in real space and time. Consider first just the positive traveling wave $V_+ e^{-\gamma z} = V_+ e^{-\alpha z - j\beta z}$. Multiplying by $e^{j\omega t}$ and taking the real part, one obtains

$$\mathrm{Re}\left(V_+ e^{-\alpha z - j\beta z} e^{j\omega t}\right) = \mathrm{Re}\left(|V_+| e^{j\theta_+} e^{-\alpha z - j\beta z} e^{j\omega t}\right) = |V_+| e^{-\alpha z} \cos(\omega t - \beta z + \theta_+)$$

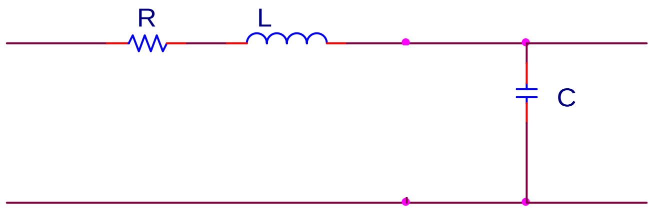
which looks almost exactly like the expressions we saw for lossless lines except for the exponential decay term. This makes intuitive sense because the wave should get smaller as it propagates in a lossy line. Thus, the power input at the source end will not equal the power delivered to the load end, since some of the wave energy is dissipated in the line. If one looks at practical transmission lines (e.g. the RG58/U commonly used in a lab or cable TV lines like RG59/U), it is seen that loss is significant.

Low Loss Transmission Lines

Practical transmission lines are not lossless. However, they are generally considered to be low loss. That is, the lines generally behave as if they are lossless, except that there is a small amount of attenuation. To see more precisely what is meant by low loss, we need only return to the basic circuit diagram for a small element of line.



First, we note that the insulators used for transmission lines are very, very good so that g is always very much less than $j\omega c$. Thus, we simplify our diagram by removing G.



For a low loss line, then, we will only consider the effects of the wire resistance as given by r. Low loss conditions occur when $r << j\omega l$. This does not seem like a very precise definition, but we will see below that there are several functional conditions under which we can consider this to be satisfied. That is, we will be able to use some simplifying approximations when this is the case.

Recall that we can fully describe the voltage and current waves if we know $\gamma = \alpha + j\beta$ and Z_o. For the latter

$$Z_o = \sqrt{\frac{r + j\omega l}{g + j\omega c}} \approx \sqrt{\frac{r + j\omega l}{j\omega c}} \approx \sqrt{\frac{j\omega l}{j\omega c}}\sqrt{1 + \frac{r}{j\omega l}} \approx \sqrt{\frac{l}{c}}\left(1 - j\frac{r}{2\omega l}\right)$$

where we have used the approximation that $\sqrt{1+x} \approx 1 + \frac{x}{2}$ for $x << 1$. (The Binomial Theorem) Usually, for Z_o the small correction term is not very important since it is only used to find the magnitude of currents, etc. On the other hand, the correction for $\gamma = \alpha + j\beta$ will always be significant since, for some length, any α will eventually absorb all wave energy.

$$\gamma = \alpha + j\beta = \sqrt{(r + j\omega l)(g + j\omega c)} \approx \sqrt{(r + j\omega l)(j\omega c)} \approx \sqrt{(j\omega l)(j\omega c)}\sqrt{1 + \frac{r}{j\omega l}}$$

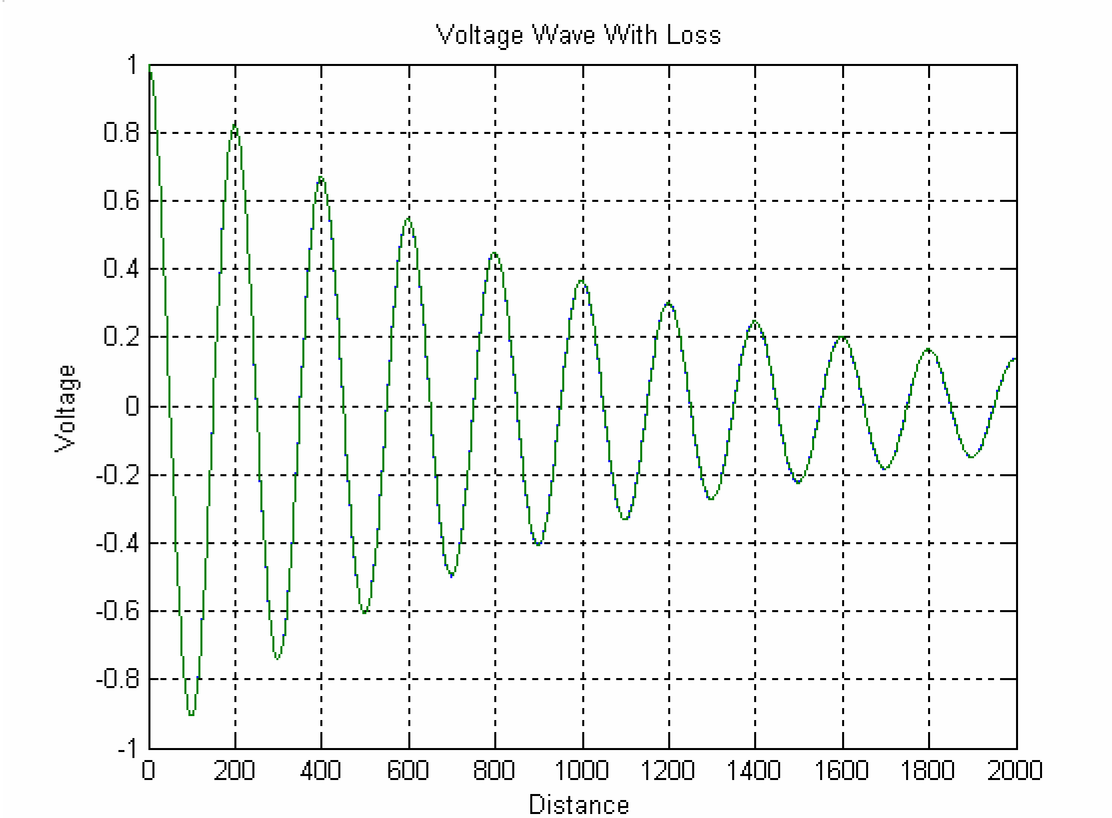
$$\gamma = \alpha + j\beta \approx j\omega\sqrt{lc}\left(1 - j\frac{r}{2\omega l}\right)$$

or

$$j\beta \approx j\omega\sqrt{lc}$$

$$\alpha \approx \omega\sqrt{lc}\left(\frac{r}{2\omega l}\right) = \frac{r}{2\sqrt{\frac{l}{c}}} = \frac{r}{2Z_o}$$

An example or two will demonstrate that this approximation works quite well for a wide variety of conditions. Assume the following: $f = 1MHz$, $l = .25\mu H/m$, $c = 100\,pF/m$. For $r = 0.1\Omega/m$, the wave is seen to attenuate markedly in *2000 meters.*

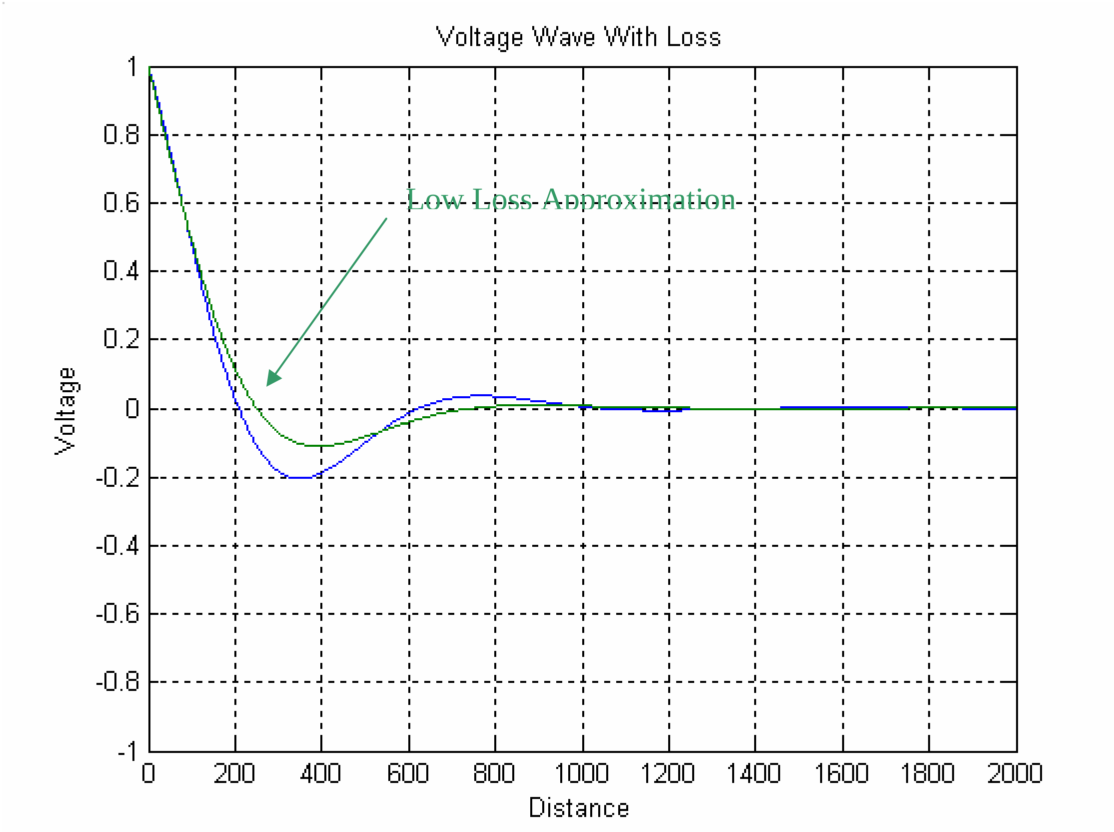


Both the exact and approximate voltages are plotted for this case. Note that they overlap exceedingly well. If we lower the frequency by a factor of 5 and increase the resistance per unit length by the same amount, we begin to see differences between the two expressions. The low loss approximation is the smaller of the two since it over estimates the attenuation. However, even here the approximation is not too bad. Checking to see the difference between the impedance terms for these cases:

$$\text{Case 1: } r = 0.1 << j\omega l = j\left(2\pi x10^{6}\right)\left(0.25x10^{-6}\right) = j\frac{\pi}{2}$$

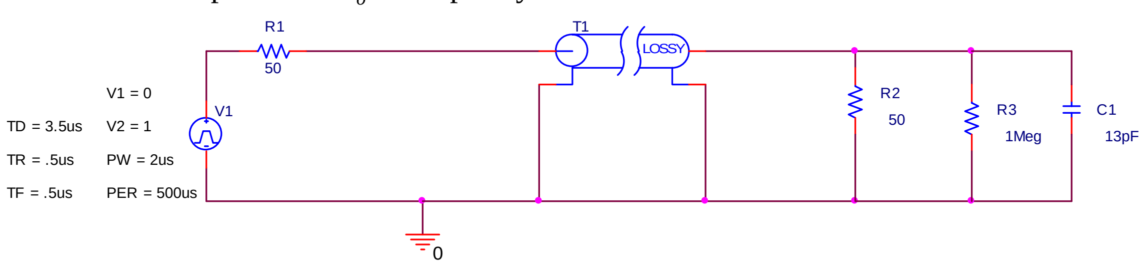
$$\text{Case 2: } r = .5 << j\omega l = j\left(2\pi 0.2x10^{6}\right)\left(.25x10^{-6}\right) = j\frac{\pi}{10}$$

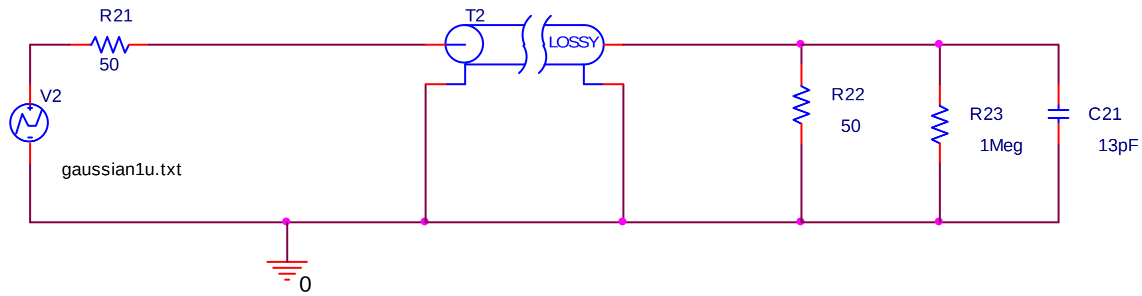
so the second case is clearly not satisfied. The voltage plotted as a function of distance shows the differences between the exact and approximate expressions. The wavelength is about correct, but the attenuation is quite a bit too large. However, if one only wishes to know that the voltage is a lot smaller, then the approximation can be used to get a ballpark estimate even when it is not particularly accurate.

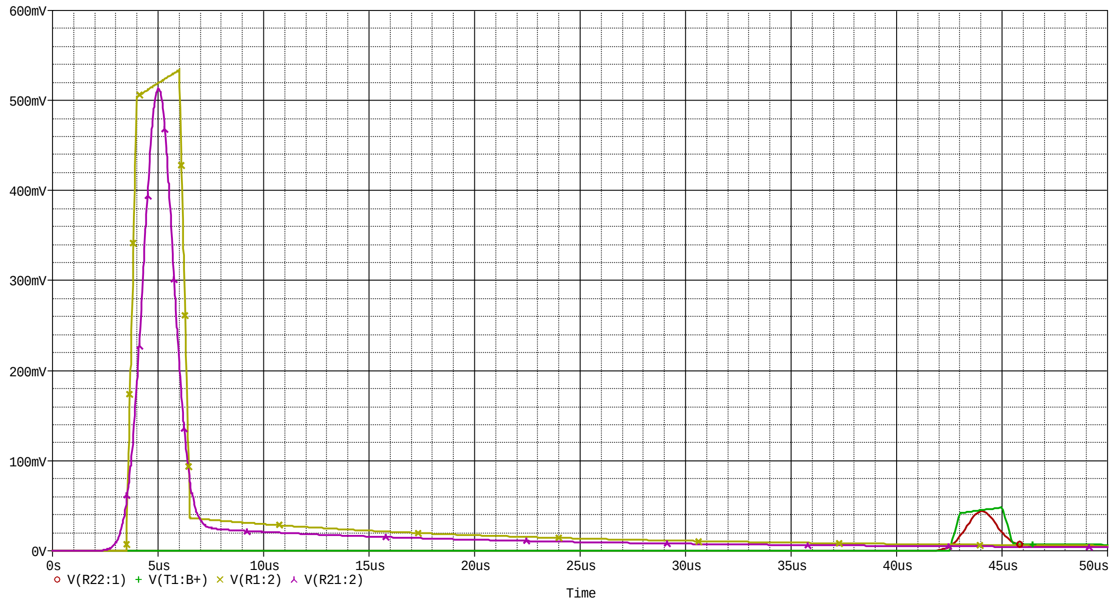


Distortionless Lines

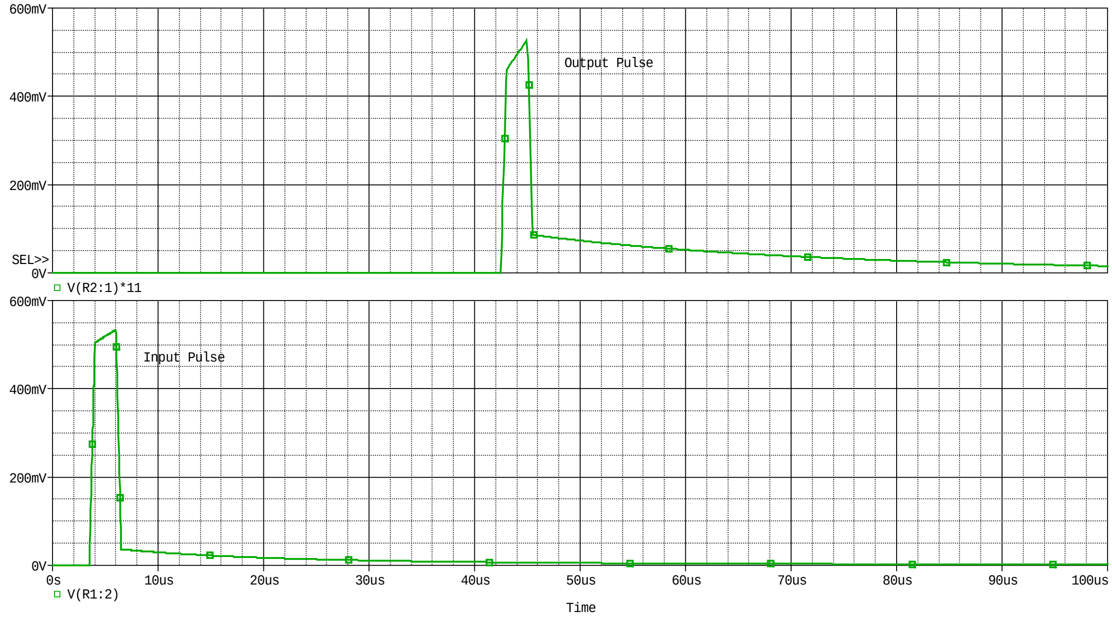
Note that, in general, the propagation constant $\gamma = \alpha + j\beta = \sqrt{(r + j\omega l)(g + j\omega c)}$ will vary with frequency. Thus, when a pulse is launched on a transmission line, the different frequencies that make up the pulse will propagate differently and the pulse will be distorted. A simple PSpice experiment shows that such distortion does indeed occur. For a transmission line with $r = 0.025\,\Omega/m$, $l = 0.195\,\mu H/m$, $c = 78\,pF/m$, $g = 0\,S/m$ and length $d = 10{,}000m$, square and Gaussian pulses appear somewhat distorted at both the input and output ends of the line. The distortion at the input is caused by the non-ideal characteristic impedance. Z_o is not purely resistive when there is some loss on the line.



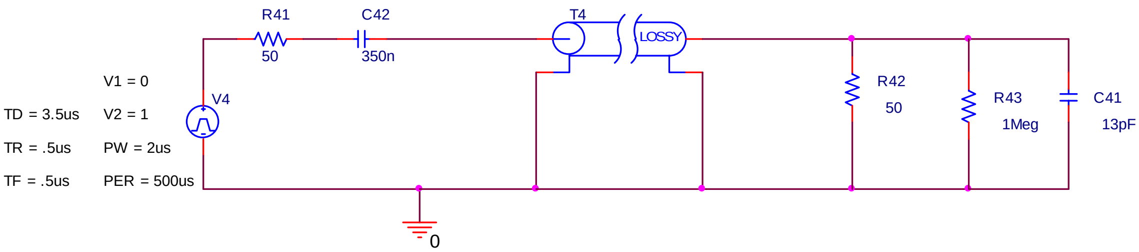


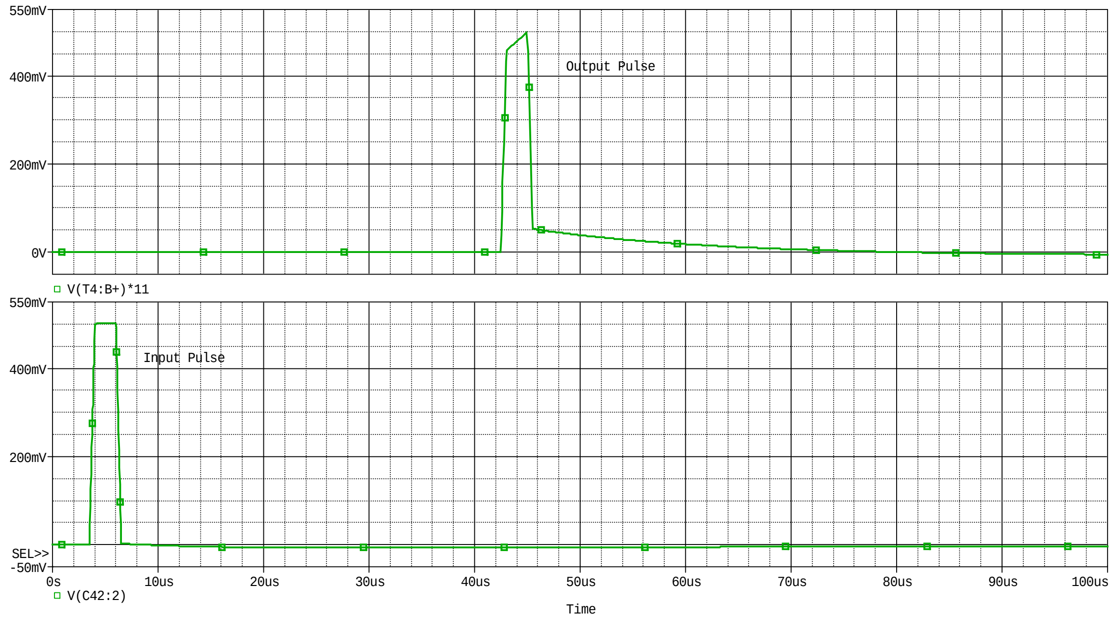


The distortion of the pulse at the input can be somewhat separated from that produced by propagation by looking at the two square pulses. The output pulse has been scaled up to display with the same scale as the input pulse. Note that it is further distorted.

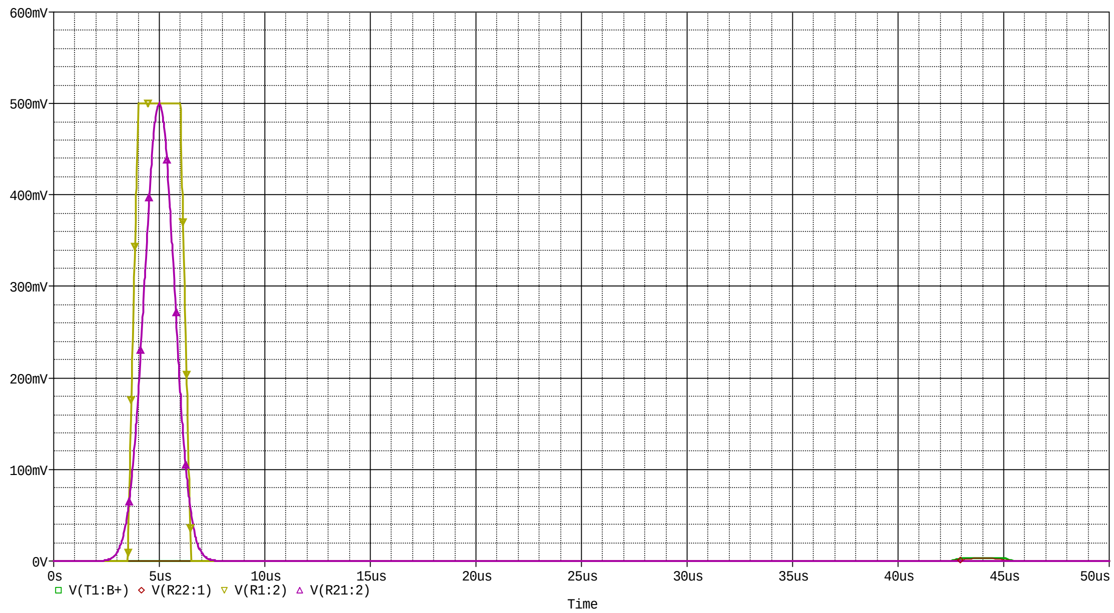


One can also do a modified simulation with a capacitance added to the voltage source to compensate for the distortion at the input end due to the characteristic impedance. In this case the input pulse now looks reasonably square but the output pulse still has the characteristic distortion observed above. This is not perfect compensation, but it does show the effect.

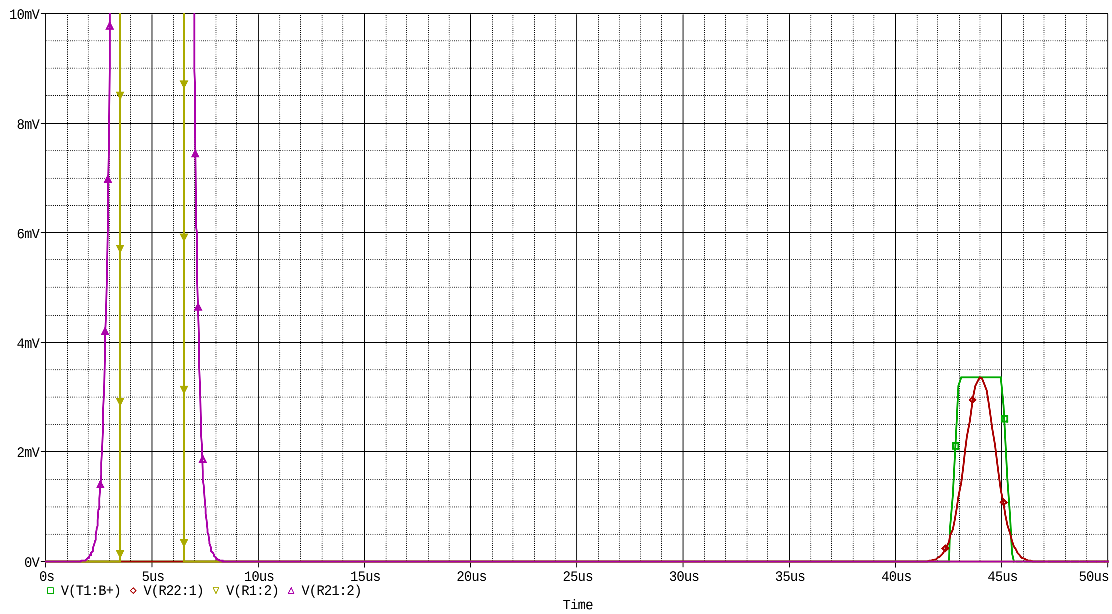




A really remarkable result can be obtained by adding some loss to the line. To see this, note that $\gamma = \alpha + j\beta = \sqrt{(r + j\omega l)(g + j\omega c)}$ significantly simplifies if we can select $\frac{r}{l} = \frac{g}{c}$. Then $\gamma = \alpha + j\beta = \sqrt{(r + j\omega l)(\frac{rc}{l} + j\omega c)} = \sqrt{\frac{c}{l}}(r + j\omega l)$. We see then that $\alpha = r\sqrt{\frac{c}{l}}$ and $\beta = \omega\sqrt{lc}$. Since our line has no conductance g per unit length, we can set it equal to $g = \frac{rc}{l}$ rather than changing any of the other 3 parameters. When we do this, we find $g = \frac{rc}{l} = 10\frac{\mu S}{m}$. This results in more attenuation, since the loss is greater. The input pulses are now, however, undistorted.



and the output pulses are also undistorted, if quite a bit smaller.



This has a huge practical advantage for long distance communication, since amplifiers can be used to restore the original signal. If the signal is distorted by the line, no amount of amplification will reproduce the signal. The condition $\frac{r}{l} = \frac{g}{c}$ also simplifies the characteristic impedance $Z_o = \sqrt{\frac{r + j\omega l}{g + j\omega c}} = \sqrt{\frac{r + j\omega l}{\frac{rc}{l} + j\omega c}} = \sqrt{\frac{l}{c}}\sqrt{\frac{\frac{r}{l} + j\omega}{\frac{r}{l} + j\omega}} = \sqrt{\frac{l}{c}}$. Thus, if we can achieve these conditions, then signals will propagate undistorted and amplifiers can be used to give them a useful level when necessary.

Some interesting links discussing distortionless lines:

http://www.hep.princeton.edu/~mcdonald/examples/distortionless.pdf

http://www.du.edu/~jcalvert/tech/cable.htm

In particular, the latter reference discusses how to add lumped resistances across lines to achieve a reasonable approximation to a distortionless line. This technique was used in the early days of telephony so that customers in New York could talk to Chicago. This is maybe the very best example of why a solid, math-based education can produce some non-intuitive results in engineering. To add resistance and make the signal better is hard to accept without some serious theoretical basis.

Possibly the following sections should be appendices.

Propagation of pulses on lossy transmission lines

To be written -- this should include the experimental work alluded to at the end of unit XII. Thus, some experimental work must be done first.

Combining Transmission Lines and Loads

To be written – what happens when one wishes to combine transmission lines or add arbitrary loads to lines? This is background for the channel blocker project.

Matching Transmission Lines and Loads

To be written – how to match sources, loads and transmission lines. Some notes on stubs exist.

The Smith Chart

To be written – how to use a Smith Chart and why they are useful. Some short slides exist.