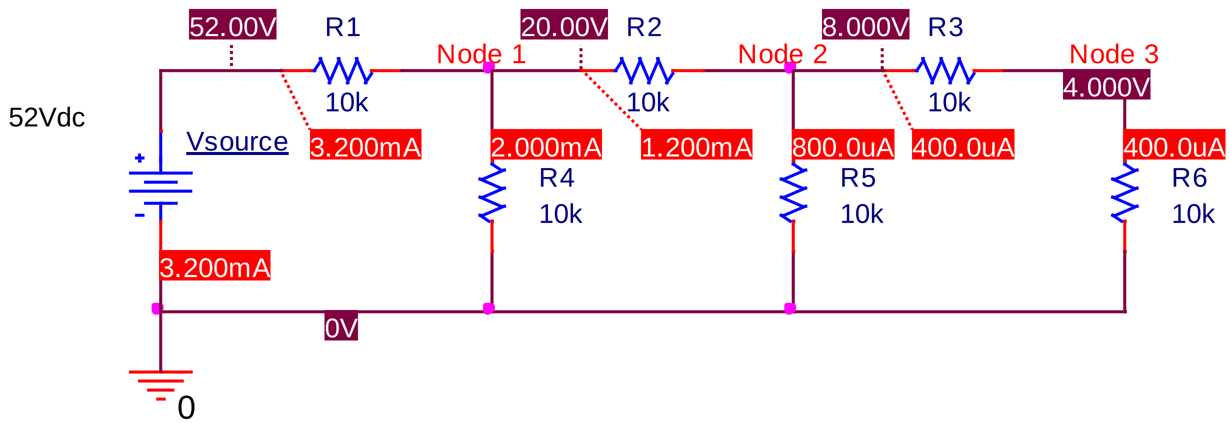
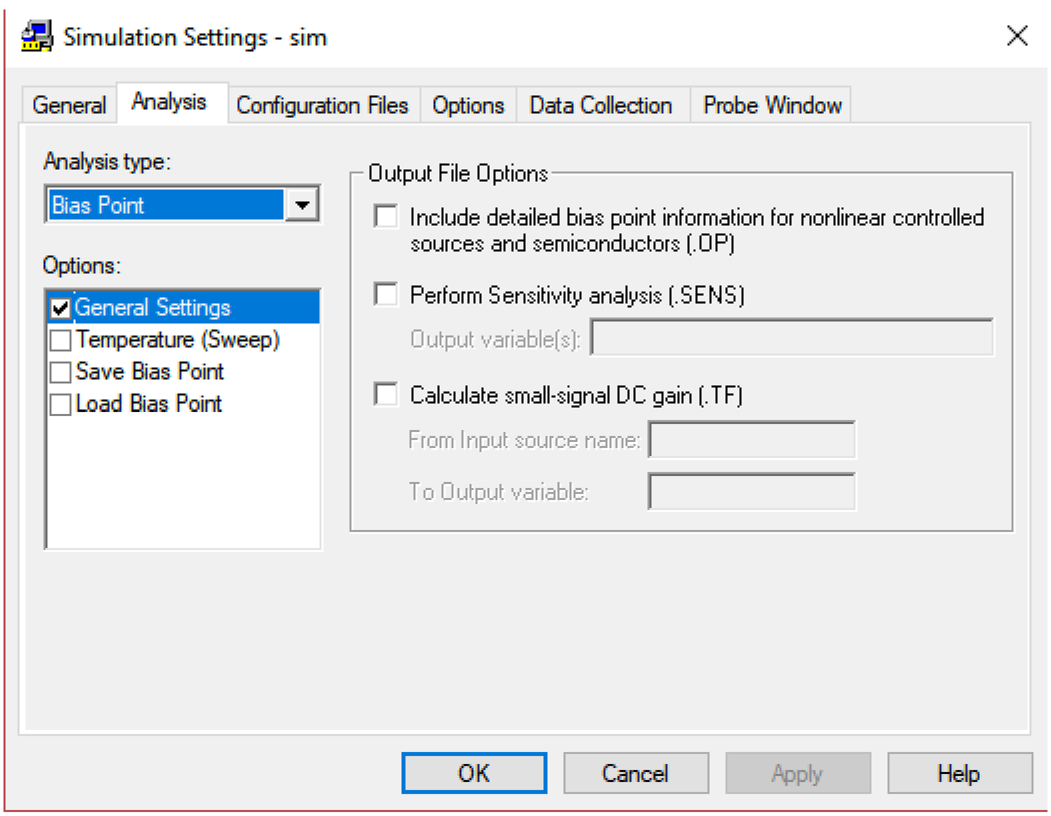
A simple six resistor ladder circuit to solve many ways:

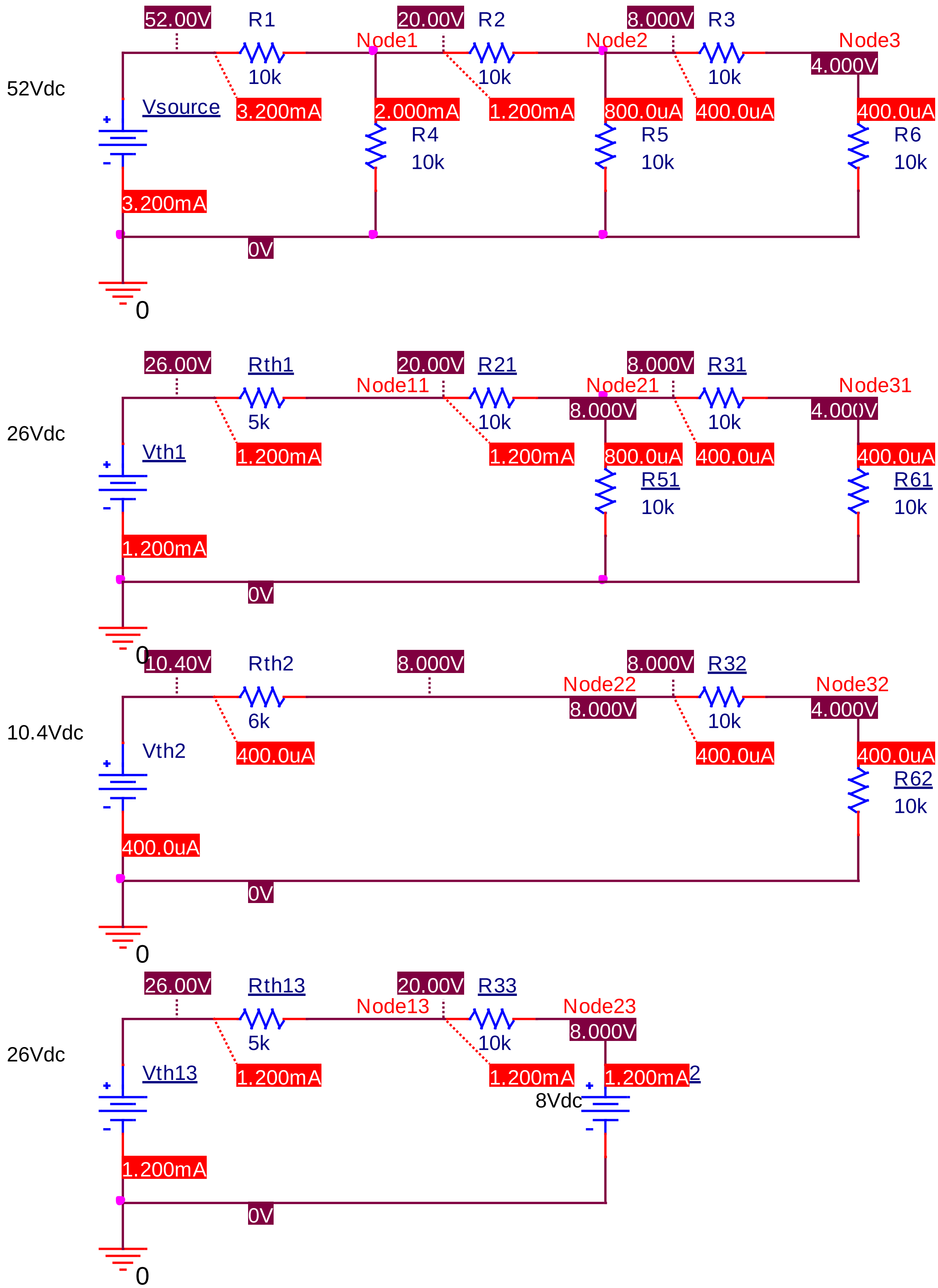
Concepts: Adding resistors in series and parallel; Ohm's Law; voltage dividers with two or three resistors; Thevenin equivalent sources; loop (mesh) equations; node equations; node voltages; loop currents; vectors; matrices; PSpice; solving matrix equations; Gaussian Elimination; matrix inverse; KVL; KCL; Voltage dividers with non-zero voltages at each end; Linear Algebra …

Method 1: Use PSpice to find the node voltages and currents through each resistor.





Analysis was done to find the DC bias voltages and currents.



Various reconfigurations showing that the Thevenin sources work.

Method 2: Solve by combining resistors – in this method, we first combine resistors R_2, R_3, R_4, R_5, R_6 to form a voltage divider with R_1. Begin by combining R_3 and R_6 in series by summing their values and then find the parallel combination of this sum with R_5. Then find the sum of the combination with R_2 and then its parallel combination with R_4. Note that, in this method, we simplify the circuit from right to left.

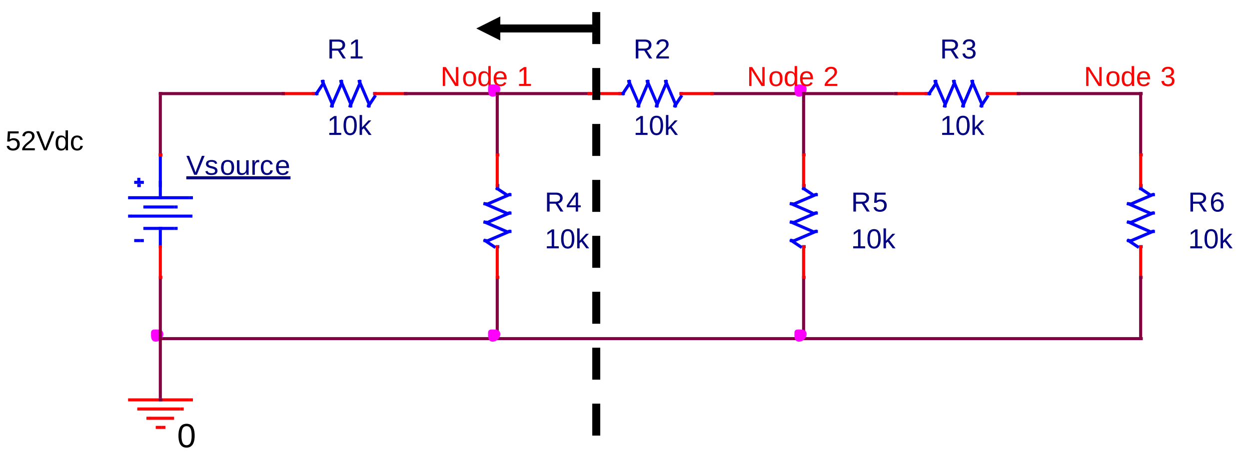
- $(R_6 + R_3) \| R_5 = \frac{20(10)}{30} k = \frac{20}{3} k$
- $(R_2 + (R_6 + R_3) \| R_5) \| R_4 = \frac{10\left(10 + \frac{20}{3}\right)}{20 + \frac{20}{3}} = \frac{500}{80} k = \frac{25}{4} k$

The voltage divider relationship then gives us $v_1 = \frac{\frac{25}{4}}{10 + \frac{25}{4}} 52 = \frac{25}{65} 52 = 20V$

Apply a second voltage divider relationship to obtain $v_2 = \frac{\frac{20}{3}}{10 + \frac{20}{3}} v_1 = \frac{20}{50} 20 = 8V$

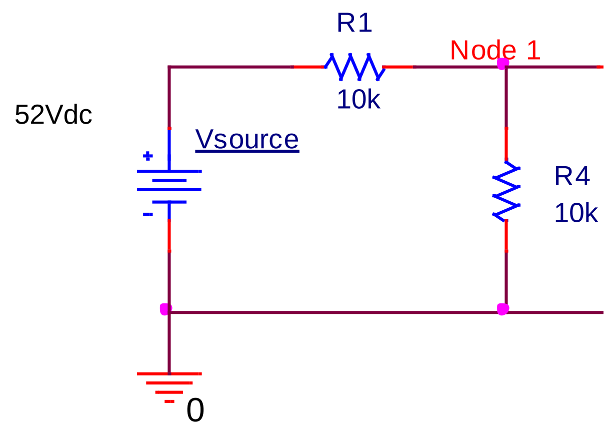
Apply a third voltage divider relationship to obtain $v_1 = \frac{1}{2} v_2 = \frac{1}{2} 8 = 4V$

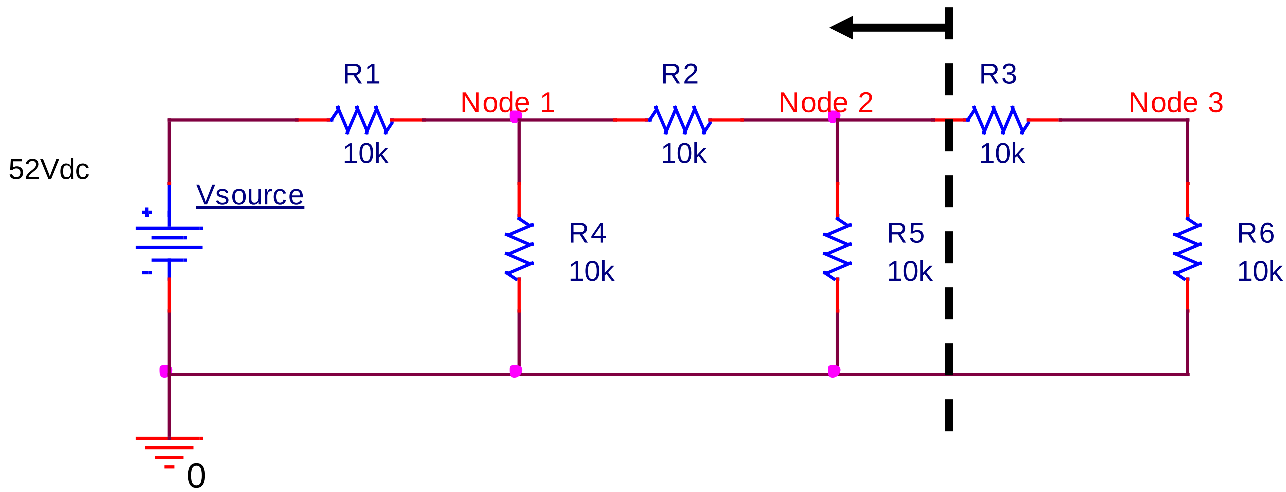
Method 3: It is also possible to simplify the circuit from left to right using Thevenin equivalent sources. Begin by finding the Thevenin equivalent source to the left of the dashed line in the figure below, which can be used to find the voltage at Node 1.

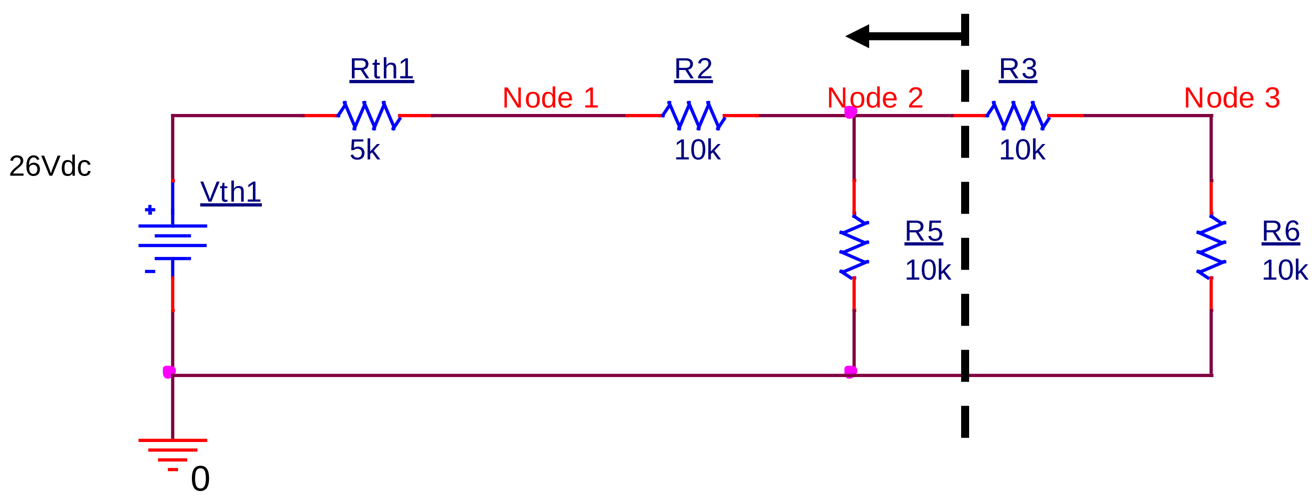


The voltage divider expression gives us the open circuit voltage $v_{oc} = v_{TH} = \frac{1}{2} 52 = 26V$.

The Thevenin resistance is the parallel combination of R_1 and R_4 or $R_{TH} = 5k$. Next find the Thevenin source to the left of the second dashed line shown in the circuit below, which can be used to find the voltage at Node 2. Begin by using the Node 1 Thevenin source, to simplify the circuit.

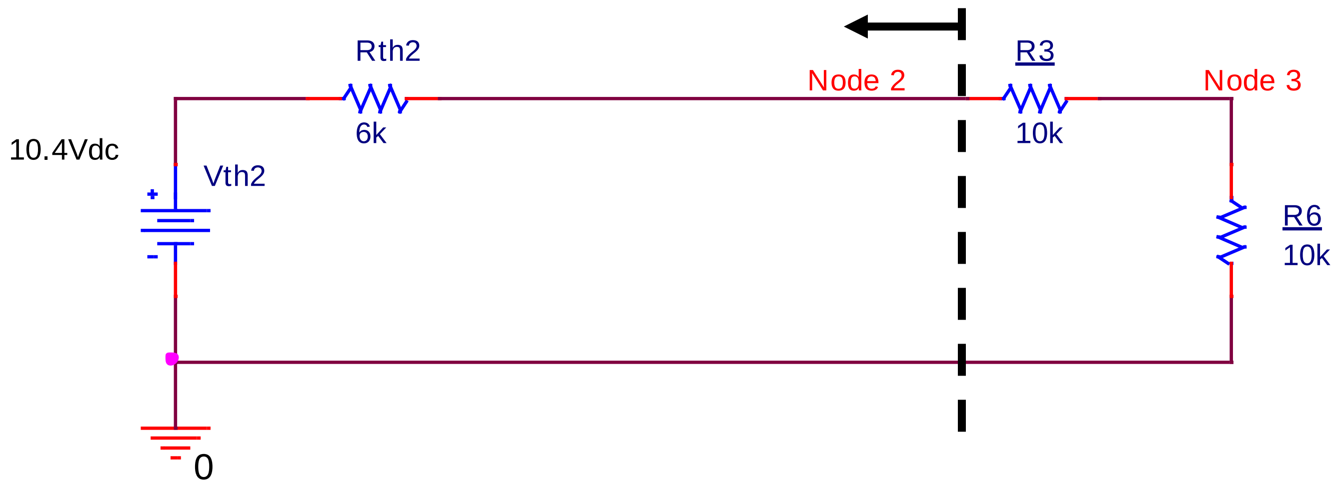




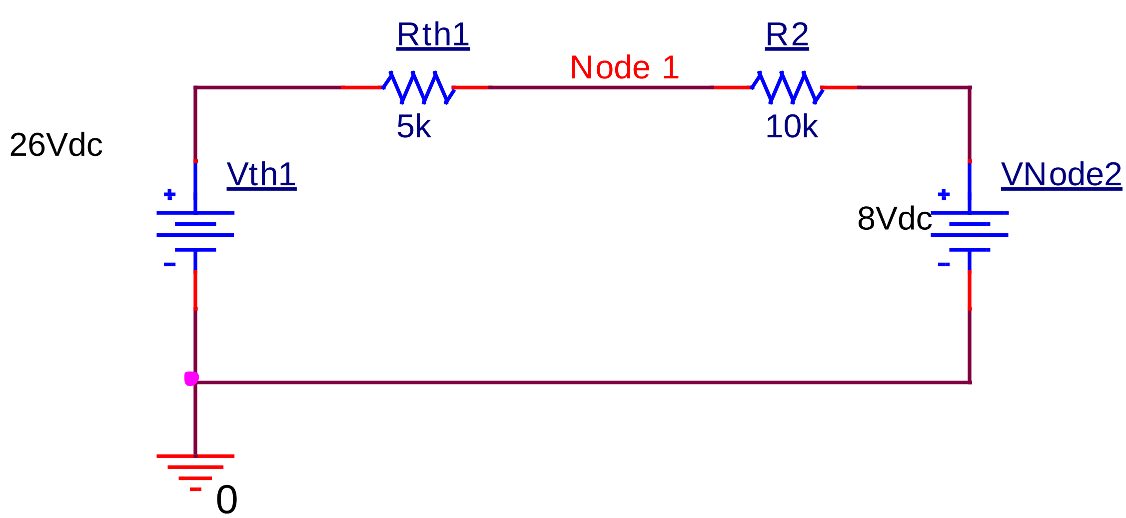


The voltage divider expression gives us the open circuit voltage

$v_{oc} = v_{TH} = \frac{R_5}{R_{TH1} + R_2 + R_5} 26 = \frac{10}{5+10+10} 26 = 10.4V$. The Thevenin resistance is the parallel combination of R_5 and (R_{TH1} +R_2) or $R_{TH} = \frac{150}{25}k = 6k$. Since the resulting simplified circuit is now just a three resistor divider, the voltage at Nodes 2 and 3 and be easily found from the voltage divider expression: $v_3 = \frac{10}{10+10+6}10.4 = 4V$ and $v_3 = \frac{10+10}{10+10+6}10.4 = 8V$.



If the only voltage we are interested in is at Node 3, the problem is complete. However, if the voltages at all three nodes are of interest, the value of the voltage at Node 2 can be used to find the voltage at Node 1, using the following circuit. This requires the use of the voltage divider expression for the case where the voltage at neither end of the divider is zero. For this case, the voltage at node 1 will be $v_1 = 8 + (26-8)\frac{10}{10+5} = 8 + 18\frac{2}{3} = 20V$. Note that this voltage can also be determined by taking the superposition of the contributions from both voltage sources. That is, $v_1 = 8\frac{5}{10+5} + 26\frac{10}{10+5} = \frac{8}{3} + \frac{52}{3} = 20V$.



Method 4: Loop (Mesh) Equations (from KVL)

$$52-(10k)i_1-(10k)i_1+(10k)i_2=0$$

$$-(10k)i_2-(10k)i_2-(10k)i_2+(10k)i_1+(10k)i_3=0$$

$$-(10k)i_2+(10k)i_3+(10k)i_3+(10k)i_3=0$$

Simplifying

$$20i_1-10i_2=52$$

$$10i_1-30i_2+10i_3=0$$

$$-10i_2+30i_3=0$$

Rewriting the 3rd equation: $i_3=\left(\frac{1}{3}\right)i_2$

Rewriting the 2nd equation using the 3rd equation: $i_2=\left(\frac{3}{8}\right)i_1$

Rewriting the 1st equation using the 2nd equation: $i_1=3.2mA$

Using the other two simplified equations: $i_2=\left(\frac{3}{8}\right)i_1$=1.2mA and $i_3=\left(\frac{1}{3}\right)i_2=0.4mA$

Solving for the three node voltages: $v_a=52-3.2(10)=20V$; $v_b=(1.2-0.4)(10)=8V$; $v_c=0.4(10)=4V$

Method 5: Solving Using Matlab (It is assumed that the reader knows how to write a Matlab script (aka m-file). If this is not the case, review Matlab basics before going through this problem.

```
% Simple Ladder Circuit
% K. A. Connor 21 Oct 17

% Solve System of Equations
syms i1 i2 i3
eqn1 = 20*i1 - 10*i2 == 52;
eqn2 = 10*i1 + -30*i2 + 10*i3 == 0;
eqn3 = -10*i2 + 30*i3 == 0;

% Convert Equations to Matrix
[A,B] = equationsToMatrix([eqn1, eqn2, eqn3], [i1, i2, i3])

% Use linsolve to solve AI = B for the vector of unknowns I.
I = linsolve(A,B)
```

```
% Determine Voltages
Vc=I(3)*10
Vb=(I(2)-I(3))*10
Va=52-I(1)*10
```

When the Matlab script is run, the following will be see in the command window.

```
>> LadderCktSolve

 A =

 [20, -10, 0]

[10, -30, 10]

[0, -10, 30]

 B =

  52

  0

  0

 I =

  16/5

  6/5

  2/5

 Vc =

 4

 Vb =

 8

 Va =

 20
```

Method 6: Solving Matrix Equations by Hand (Gaussian Elimination)

It is possible to solve matrix equations without the use of Matlab or other matrix tools, but usually only for small matrices (e.g. 2x2, 3x3 or somewhat larger if one is ambitious.) Here the loop equations are 3x3, so it is reasonable to try. The most common approach to solving matrix equations is some form of Gaussian Elimination. This is pretty much what was done to find the solution to the three simultaneous equations, but it is more systematic and thus provides a more reliable approach. The loop equations are in the form of Ax=b where A is a matrix and x and b are vectors for the unknowns and constants, respectively.

The matrix equation for the three loop currents is written in general form as:

$$\begin{bmatrix} 20 & -10 & 0 \\ 10 & -30 & 10 \\ 0 & -10 & 30 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 52 \\ 0 \\ 0 \end{bmatrix}$$

For Gaussian Elimination, this is rewritten as an augmented matrix.

$$\left[\begin{array}{ccc|c} 20 & -10 & 0 & 52 \\ 10 & -30 & 10 & 0 \\ 0 & -10 & 30 & 0 \end{array}\right]$$

First, we manipulate the rows to get the matrix in row echelon form, where the first terms in each row are diagonal terms (e.g. other terms in lower left are zeros) and the diagonal term is unity.

To begin, replace line 2 with $L_2 - \frac{1}{2}L_1 \rightarrow L_2$

$$\left[\begin{array}{ccc|c} 20 & -10 & 0 & 52 \\ 0 & -25 & 10 & -26 \\ 0 & -10 & 30 & 0 \end{array}\right]$$

Next, $L_2 - \frac{1}{3}L_3 \rightarrow L_2$

$$\left[\begin{array}{ccc|c} 20 & -10 & 0 & 52 \\ 0 & -25+\frac{10}{3} & 0 & -26 \\ 0 & -10 & 30 & 0 \end{array}\right] \text{ or } \left[\begin{array}{ccc|c} 20 & -10 & 0 & 52 \\ 0 & \frac{65}{3} & 0 & 26 \\ 0 & -10 & 30 & 0 \end{array}\right] \text{ or } \left[\begin{array}{ccc|c} 20 & -10 & 0 & 52 \\ 0 & 1 & 0 & 1.2 \\ 0 & -10 & 30 & 0 \end{array}\right]$$

Next, $L_1 + 10L_2 \rightarrow L_1$

$$\left[\begin{array}{ccc|c} 20 & 0 & 0 & 64 \\ 0 & 1 & 0 & 1.2 \\ 0 & -10 & 30 & 0 \end{array}\right] \text{ or } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3.2 \\ 0 & 1 & 0 & 1.2 \\ 0 & -10 & 30 & 0 \end{array}\right]$$

Finally, $L_3 + 10L_2 \rightarrow L_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3.2 \\ 0 & 1 & 0 & 1.2 \\ 0 & 0 & 30 & 12 \end{array}\right] \text{ or } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3.2 \\ 0 & 1 & 0 & 1.2 \\ 0 & 0 & 1 & 0.4 \end{array}\right]$$

The three currents can then be found by inspection from the right hand column (in mA).

Method 7: Node Equations (from KCL)

$$\frac{v_1 - 52}{10k} + \frac{v_1}{10k} + \frac{v_1 - v_2}{10k} = 0$$

$$\frac{v_2 - v_1}{10k} + \frac{v_2}{10k} + \frac{v_2 - v_3}{10k} = 0$$

$$\frac{v_3}{10k} + \frac{v_3 - v_2}{10k} = 0$$

Simplifying

$$3v_1 - v_2 = 52$$

$$-v_1 + 3v_2 - v_3 = 0$$

$$-v_2 + 2v_3 = 0$$

Rewriting 3rd equation: $v_3 = 0.5v_2$ and rewriting 1st equation: $v_1 = \frac{v_2 + 52}{3}$

Equation 2 becomes: $-\frac{v_2 + 52}{3} + 3v_2 - 0.5v_2 = 0$ or $v_2 = 8V$

and the other two voltages are $v_3 = 4V$ and $v_1 = 20V$

Method 8: Solving Using Matlab

```
% Solve System of Equations from Node Voltages
syms v1 v2 v3
eqn4 = 3*v1-v2 == 52;
eqn5 = -v1+3*v2-v3 == 0;
eqn6 = -v2 + 2*v3 == 0;

% Convert Equations to Matrix
[C,D] = equationsToMatrix([eqn4, eqn5, eqn6], [v1, v2, v3])

% Use linsolve to solve AV = B for the vector of unknowns V.
V = linsolve(C,D)
```

When the script is run, we see the following in the command window.

C =

[3, -1, 0]

[-1, 3, -1]

[0, -1, 2]

D =

52

0

0

V =

20

8

4

Which agrees with the loop equations.

Method 9: Solving by Hand Using Gaussian Elimination

The matrix equation for the three node voltages is written in general form as:

$$\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 52 \\ 0 \\ 0 \end{bmatrix}$$

For Gaussian Elimination, this is rewritten as an augmented matrix.

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & 52 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

The method is the same as for the loop equations.

To begin, replace line 2 with $L_2 + \frac{1}{3}L_1 \rightarrow L_2$

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & 52 \\ 0 & \frac{8}{3} & -1 & \frac{52}{3} \\ 0 & -1 & 2 & 0 \end{array}\right] \text{ or } \left[\begin{array}{ccc|c} 3 & -1 & 0 & 52 \\ 0 & 8 & -3 & 52 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

Next, $L_2 - \frac{3}{2}L_3 \rightarrow L_2$

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & 52 \\ 0 & 6.5 & 0 & 52 \\ 0 & -1 & 2 & 0 \end{array}\right] \text{ or } \left[\begin{array}{ccc|c} 3 & -1 & 0 & 52 \\ 0 & 1 & 0 & 8 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

Next, $L_3 + L_2 \rightarrow L_3$

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & 52 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 2 & 8 \end{array}\right] \text{ or } \left[\begin{array}{ccc|c} 3 & -1 & 0 & 52 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Finally, $L_1 + L_2 \rightarrow L_1$

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 60 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 4 \end{array}\right] \text{ or } \left[\begin{array}{ccc|c} 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

The three voltages can then be found by inspection from the right hand column.

Method 10: Finding the solution using the inverse of the matrix. Again, both the loop and node equations are in the form of Ax=b where A is a matrix and x and b are vectors for the unknowns and constants, respectively. The inverse of a matrix A^{-1} is defined by $(A^{-1})(A)=1$, where 1 is the unit matrix, which, for a 3x3, looks like $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. If the inverse is known, then multiply Ax=b by A^{-1} to obtain the values for x. That is $A^{-1}Ax = x = A^{-1}b$. Thus, for either of the matrix expressions that describe this circuit, the inverse can provide the currents and/or voltages. For the loop equations, $\begin{bmatrix} 20 & -10 & 0 \\ 10 & -30 & 10 \\ 0 & -10 & 30 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 52 \\ 0 \\ 0 \end{bmatrix}$, the matrix is $\begin{bmatrix} 20 & -10 & 0 \\ 10 & -30 & 10 \\ 0 & -10 & 30 \end{bmatrix}$ and its inverse is

$\begin{bmatrix} 4/65 & -3/130 & 1/130 \\ 3/130 & -3/65 & 1/65 \\ 1/130 & -1/65 & 1/26 \end{bmatrix}$. The inverse was obtained using the Matlab function inv(..).

For the node equations, $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 52 \\ 0 \\ 0 \end{bmatrix}$, the matrix is $\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ and its inverse is $\begin{bmatrix} 5/13 & 2/13 & 1/13 \\ 2/13 & 6/13 & 3/13 \\ 1/13 & 3/13 & 8/13 \end{bmatrix}$. The inverse was obtained using the Matlab function inv(..). Finding the inverse manually requires some effort. There is a reasonably clear discussion of how this is done at https://www.mathsisfun.com/algebra/matrix-inverse-minors-cofactors-adjugate.html. Following the recipe given there, we must do the following:

- Step 1: calculating the Matrix of Minors,
- Step 2: then turn that into the Matrix of Cofactors,
- Step 3: then the Adjugate, and
- Step 4: multiply that by 1/Determinant.

To see how this works, apply each step to the node equation matrix and then compare with the result from Matlab. The definitions of each type of matrix can be found online.

Matrix of minors: $\begin{bmatrix} 5 & -2 & 1 \\ -2 & 6 & -3 \\ 1 & -3 & 8 \end{bmatrix}$

Matrix of cofactors: $\begin{bmatrix} 5 & 2 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 8 \end{bmatrix}$

Adjugate Matrix: $\begin{bmatrix} 5 & 2 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 8 \end{bmatrix}$

Divide each term by the determinant of the original matrix (which = 13) $\begin{bmatrix} 5/13 & 2/13 & 1/13 \\ 2/13 & 6/13 & 3/13 \\ 1/13 & 3/13 & 8/13 \end{bmatrix}$

Which agrees with the inverse from Matlab. This process is a good example of what is possible with Linear Algebra. Linear systems are complex, but exceptionally well-behaved, so there are nearly always systematic approaches to solving problems. The rules of algebra also make it possible to demonstrate the existence of solution approaches when they are in no way intuitively obvious.

Method 11: Using Excel to Solve the Matrix Equations

It is also possible to use Excel to solve the 3x3 matrix equations. This approach is a little less elegant but does provide some insights into how matrix arithmetic works. The following is a screen capture from Excel showing both the loop and node equations. Note that matrix expressions all begin with m so we are using MINVERSE and MMULT and that is it. Note also that, because these are arrays, it is necessary to use CTRL-SHIFT-ENTER rather than ENTER when working with arrays. To see how these operations work, check out any of the online references on solving simultaneous equations with Excel. A particularly good video: https://www.youtube.com/watch?v=gSNa3fQX0WQ

	A	B	C	D	E	F	G	H	I	J	K	L
1	Solving The Simple Ladder Circuit by Finding The Inverse of a Matrix											
2												
3												
4	First: Solving for Currents from AI=V											
5		Matrix A				V		I using mmult(inverse of A,V) = A^{-1} V				
6		20	-10	0		52		3.2	mA			
7		10	-30	10		0		1.2	mA			
8		0	-10	30		0		0.4	mA			
9												
10		Inverse A using minverse(A)			aka A^{-1}			Checking the Inverse by Multiplying it Times the Matrix				
11		0.061538	-0.02308	0.007692				1	-5.55112E-17	2.78E-17		
12		0.023077	-0.04615	0.015385				0	1	5.55E-17		
13		0.007692	-0.01538	0.038462				0	0	1		
14												
15	Second: Solving for Voltages from CV=I											
16		Matrix C				I		V using mmult(inverse of C,I) = C^{-1} I				
17		3	-1	0		52		20	V			
18		-1	3	-1		0		8	V			
19		0	-1	2		0		4	V			
20												
21		Inverse of C using minverse(C)			aka C^{-1}			Checking the Inverse by Multiplying it Times the Matrix				
22		0.384615	0.153846	0.076923				1	5.55112E-17	0		
23		0.153846	0.461538	0.230769				0	1	0		
24		0.076923	0.230769	0.615385				0	0	1		
25												

Two comments on these results: 1) The inverse does not exactly give just 1's on the diagonal because of the limits of numerical solutions and 2) Matrices are just arrays of numbers and Excel forces us to highlight specific cells to identify matrices and vectors. The latter can help us think in the more general algebra of matrices rather than just individual numbers (aka scalars).