Electromagnetic Waves
PHASOR FORM

TIME HARMONIC FORM OF MAXWELL'S EQUATIONS

$\nabla \times \vec{E}(\vec{r}) = -j\omega\mu\vec{H}(\vec{r})$

$\nabla \times \vec{H}(\vec{r}) = \vec{J}(\vec{r}) + j\omega\varepsilon\vec{E}(\vec{r})$

$\nabla \cdot \vec{E}(\vec{r}) = \rho/\varepsilon$

$\nabla \cdot \vec{B}(\vec{r}) = 0$

$\vec{B} = \mu\vec{H}$

$\vec{D} = \varepsilon\vec{E}$

$\vec{J} = \sigma\vec{E}$

TO CONVERT BACK TO SPACE-TIME FORM

$\vec{E}(x,y,z,t) = \mathrm{Re}\left(\vec{E}(x,y,z)e^{j\omega t}\right)$

POTENTIALS

$\vec{B} = \nabla \times \vec{A}$

$\vec{E} = -\nabla V - j\omega\vec{A}$

CONSTANTS

$\mu_o = 4\pi \times 10^{-7} \approx 1.26 \times 10^{-6}\ H/m \qquad \varepsilon_o \approx \frac{1}{36\pi} \times 10^{-9} \approx 8.854 \times 10^{-12}\ F/m$

BOUNDARY CONDITIONS (Hold for both general time variation and phasor form)

$E_{2t} = E_{1t}$ or $\hat{n} \times \left(\vec{E}_2 - \vec{E}_1\right) = 0$

$H_{1t} - H_{2t} = J_S$ or $\hat{n} \times \left(\vec{H}_2 - \vec{H}_1\right) = \vec{J}_S$

$D_{1n} - D_{2n} = \rho_S$ or $\hat{n} \cdot \left(\vec{D}_1 - \vec{D}_2\right) = \rho_S$

$B_{1n} = B_{2n}$ or $\hat{n} \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0$

PLANE WAVES IN LOSSLESS MEDIA

Arbitrary Direction

$\vec{E} = E_o \exp\left(-j\vec{k} \cdot \vec{r}\right)\hat{a}_E$

$\vec{H} = \frac{E_o}{\eta} \exp\left(-j\vec{k} \cdot \vec{r}\right)\hat{a}_H$

($\hat{a}_E$ and $\hat{a}_H$ are unit vectors)

Incident and Reflected Waves (z-direction)

$E_x(z) = E_{io}\exp(-jkz) + E_{ro}\exp(+jkz)$

$H_y(z) = \frac{E_{io}}{\eta}\exp(-jkz) - \frac{E_{ro}}{\eta}\exp(+jkz)$

MATERIAL PROPERTIES – LOSSLESS MEDIA

Intrinsic Impedance: $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

Wavelength: $\lambda = \frac{2\pi}{k}$

Phase Velocity: $v_{ph} = u = \frac{1}{\sqrt{\mu\varepsilon}}$

Wavenumber: $k = \beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{u}$

PLANE WAVES IN LOSSY MEDIA (z-direction)

$$E_x(z) = E_{io}\exp(-\gamma z) + E_{ro}\exp(+\gamma z) = E_{io}\exp(-\alpha z)\exp(-j\beta z) + E_{ro}\exp(+\alpha z)\exp(+j\beta z)$$

$$H_y(z) = \frac{E_{io}}{\eta_c}\exp(-\gamma z) - \frac{E_{ro}}{\eta_c}\exp(+\gamma z) = \frac{E_{io}}{\eta_c}\exp(-\alpha z)\exp(-j\beta z) - \frac{E_{ro}}{\eta_c}\exp(+\alpha z)\exp(+j\beta z)$$

MATERIAL PROPERTIES – LOSSY MEDIA

Complex Permittivity: $\varepsilon_c = \varepsilon' - j\varepsilon''$ or $\varepsilon_c = \varepsilon - j\,\sigma/\omega$

Wavelength: $\lambda = \dfrac{2\pi}{\beta}$

Propagation Constant: $\gamma = j\omega\sqrt{\mu\varepsilon_c} = \alpha + j\beta$

Skin Depth: $\delta = \dfrac{1}{\alpha}$

Good Conductor: $\sigma/\omega\varepsilon >> 1$

$\alpha = \beta \approx \sqrt{\pi f \sigma \mu}$ $\qquad \eta \approx (1+j)\sqrt{\dfrac{\pi f \mu}{\sigma}}$

Low-Loss Dielectric

$\alpha \approx \dfrac{\sigma}{2}\sqrt{\dfrac{\mu}{\varepsilon}} \approx \dfrac{\omega\varepsilon''}{2}\sqrt{\dfrac{\mu}{\varepsilon'}}$ $\qquad \beta \approx \omega\sqrt{\mu\varepsilon'}\left(1 + \dfrac{1}{8}\left(\dfrac{\varepsilon''}{\varepsilon'}\right)^2\right)$ $\qquad \eta \approx \sqrt{\dfrac{\mu}{\varepsilon'}}\left(1 + j\dfrac{\varepsilon''}{2\varepsilon'}\right)$

AVERAGE POWER FLUX

$\vec{P}_{av} = \vec{S}_{av} = \dfrac{1}{2}\mathrm{Re}\left(\vec{E} \times \vec{H}^*\right)$

ENERGY DENSITY

$w_e = \dfrac{1}{2}\varepsilon\vec{E}\cdot\vec{E}$ and $w_m = \dfrac{1}{2}\varepsilon\vec{H}\cdot\vec{H}$

NORMAL INCIDENCE ON MULTIPLE BOUNDARIES

Wave Impedance:

$$Z(z) = \frac{TotalE_x(z)}{TotalH_y(z)} = \eta\frac{e^{-j\beta_1 z} + \Gamma e^{+j\beta_1 z}}{e^{-j\beta_1 z} - \Gamma e^{+j\beta_1 z}}$$

Input Impedance:

$$Z_{IN}(z) = \eta_2\frac{\eta_3 + j\eta_2 \tan\beta_2 d}{\eta_2 + j\eta_3 \tan\beta_2 d}$$

OBLIQUE INCIDENCE

Snell's Law: $\theta_i = \theta_r$

$k_1 \sin\theta_i = k_2 \sin\theta_t$ or $n_1 \sin\theta_i = n_2 \sin\theta_t$

Index of Refraction: $n = \sqrt{\varepsilon_r}$

Perpendicular Polarization:

$$\Gamma_\perp = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$\tau_\perp = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$\tau_\perp = 1 + \Gamma_\perp$$

Parallel Polarization:

$$\Gamma_\parallel = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$\tau_\parallel = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$\tau_\parallel = \left(1 + \Gamma_\parallel\right)\frac{\cos\theta_i}{\cos\theta_t}$$

Brewster Angle: $\sin\theta_{B\parallel} = \dfrac{1}{\sqrt{1 + \varepsilon_1/\varepsilon_2}}$

Critical Angle: $\sin\theta_c = \sqrt{\varepsilon_2/\varepsilon_1}$