Fields and Waves

Lesson 1.3

VECTOR CALCULUS - Line Integrals,Curl & Gradient

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DIFFERENTIAL LENGTHS

Representation of differential length **dl** in coordinate systems:

rectangular

$$d\vec{l} = dx \bullet \hat{a}_x + dy \bullet \hat{a}_y + dz \bullet \hat{a}_z$$

cylindrical

$$d\vec{l} = dr \bullet \hat{a}_r + r d\phi \bullet \hat{a}_\phi + dz \bullet \hat{a}_z$$

spherical

$$d\vec{l} = dr \bullet \hat{a}_r + r d\theta \bullet \hat{a}_\theta + r \sin\theta d\phi \bullet \hat{a}_\phi$$

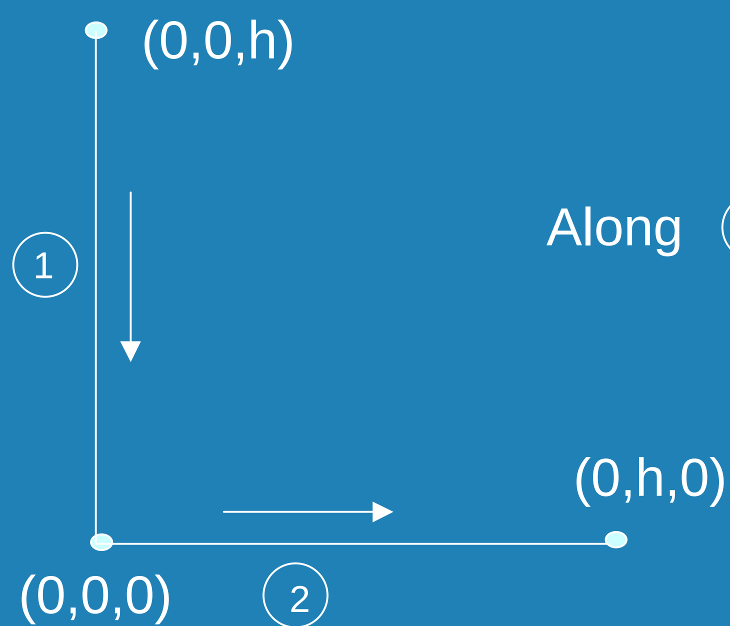
LINE INTEGRALS

EXAMPLE: *GRAVITY*

Define Work or Energy Change:

$$W = \int \vec{F} \bullet d\vec{l}$$

$$= \int_1 \vec{F} \bullet d\vec{l} + \int_2 \vec{F} \bullet d\vec{l}$$



Along ①

$$d\vec{l} = dz \bullet \hat{a}_z$$ (dx & dy = 0)

$$\vec{F} \bullet d\vec{l} = -mg\, dz$$

$$\int \vec{F} \bullet d\vec{l} = -\int_h^0 mg\, dz = mgh$$

LINE INTEGRALS

Along ② $d\vec{l} = dy \bullet \hat{a}_y$ (dx & dy = 0)

$\vec{F} \bullet d\vec{l} = -0 \bullet dy$ → Vectors are perpendicular to each other

$$\int \vec{F} \bullet d\vec{l} = -\int_0^h 0 \bullet dy = 0$$

Final Integration: $W = \int \vec{F} \bullet d\vec{l}$ → $W = mgh$

LINE INTEGRALS

Note: For the first integral, DON'T use $d\vec{l} = -dz \bullet \hat{a}_z$

Negative Sign comes in through integration limits

For PROBLEM 1a - use Cylindrical Coordinates:

Differential Line Element

$$d\vec{l} = dr \bullet \hat{a}_r + r d\phi \bullet \hat{a}_\phi + dz \bullet \hat{a}_z$$

NOTATION: $\oint$ Implies a CLOSED LOOP Integral

LINE INTEGRALS - ROTATION or CURL

$$\oint \vec{A} \bullet d\vec{l}$$

Measures Rotation or Curl

For example in Fluid Flow: $$\oint \vec{v} \bullet d\vec{l} \neq 0$$

Means ROTATION or "EDDY CURRENTS"

$$\oint \vec{v} \bullet d\vec{l}$$

Integral is performed over a large scale (global)

However, $\nabla \times \vec{v}$, the "CURL of v" is a local or a POINT measurement of the same property

ROTATION or CURL

NOTATION: $\nabla \times \vec{v}$ is NOT a CROSS-PRODUCT

Result of this operation is a VECTOR

Note: We will not go through the derivation of the connection between

$$\oint \vec{A} \bullet d\vec{l} \Longleftrightarrow \nabla \times \vec{A}$$

More important that you understand how to apply formulations correctly

To calculate CURL, use formulas in the TEXT

ROTATION or CURL

Example: SPHERICAL COORDINATES

$$\vec{A} = \frac{c}{r^2} \bullet \sin\theta \bullet \hat{a}_\theta$$

$$A_\theta$$

$$\vec{A} = A_\theta \bullet \hat{a}_\theta$$

We quickly note that: $A_r = A_\phi = 0$

ROTATION or CURL

$\nabla \times \vec{A}$, has 6 terms - we need to evaluate each separately

Start with last two terms that have ϕ - dependence:

$$\hat{a}_\phi \cdot \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right] = \hat{a}_\phi \cdot \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{c \cdot \sin\theta}{r}\right)$$

0

$$= \hat{a}_\phi \cdot \frac{c\sin\theta}{r^2} \cdot \left(-\frac{1}{r^2}\right)$$

$$= \frac{-c\sin\theta}{r^3} \cdot \hat{a}_\phi$$

ROTATION or CURL

4 other terms include: two A_ϕ terms and two A_r terms ⇒ 0

Example:

$$\hat{a}_r \cdot \frac{1}{r\sin\theta}\left(-\frac{\partial A_\theta}{\partial \phi}\right) = 0$$

, because A_θ has NO PHI DEPENDENCE

THUS,

$$\nabla \times \vec{A} = -\frac{c\sin\theta}{r^3} \cdot \hat{a}_\phi$$

Do Problem 1b and Problem 2

GRADIENT

GRADIENT measures CHANGE in a SCALAR FIELD

- the result is a VECTOR pointing in the direction of increase

For a Cartesian system:

$$\nabla f = \frac{\partial f}{\partial x} \cdot \hat{a}_x + \frac{\partial f}{\partial y} \cdot \hat{a}_y + \frac{\partial f}{\partial z} \cdot \hat{a}_z$$

Do Problems 3 and 4

You will find that $\nabla \times \nabla f = 0$ ALWAYS

IF $\vec{F} = \nabla f$ *, then* $\nabla \times \vec{F} = 0$ *and* $\oint \vec{F} \bullet d\vec{l} = 0$