Fields and Waves

Lesson 2.2

ELECTROSTATICS - GAUSS' LAW

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MAXWELL'S FIRST EQUATION

$$\nabla \bullet \vec{D} = \rho$$

Differential Form

$$\oint \vec{D} \bullet d\vec{s} = \int \rho \cdot dv = Q_{encl}$$

Integral Form

Enclosed Charge

- 'dv' integral over volume enclosed by 'ds' integral

$$\vec{D} = \varepsilon_0 \vec{E}$$

constant

For vacuum and air - think of D and E as being the same

D vs E depends on materials

GAUSS' LAW - strategy

Do Problem 1

Use Gauss' Law to find D and E in symmetric problems

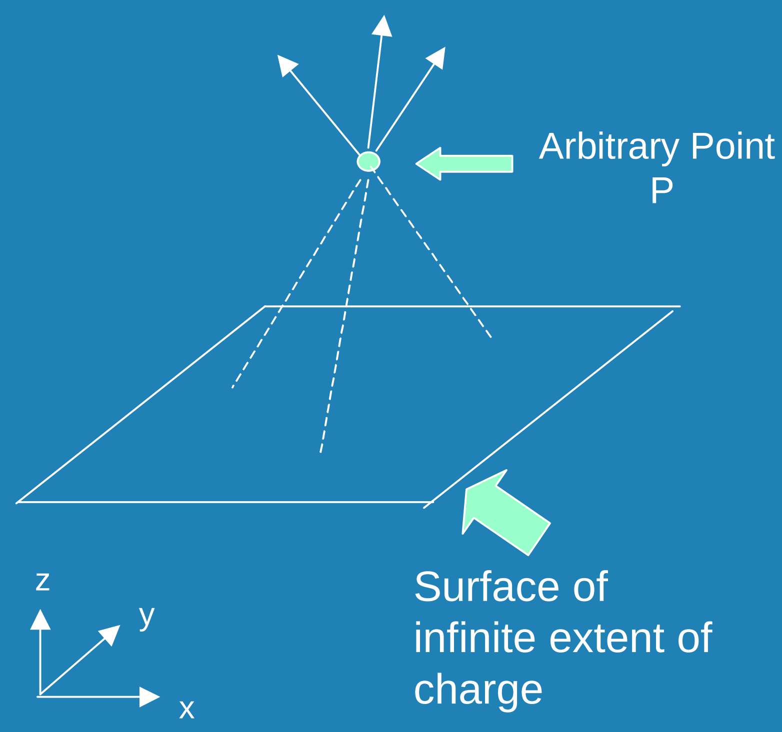
$$\oint \vec{D} \bullet d\vec{s} = \int \rho \cdot dv = Q_{encl}$$

→ *Get D or E out of integral*

Always look at symmetry of the problem - and take advantage of this

GAUSS' LAW - use of symmetry

Example: A sheet of charge



- charges are infinite in extent on say x,y plane

$\vec{E}$, is sum due to all charges

$\vec{E}$, points in $\hat{a}_z$

- all other components cancel
- only a function of z (not x or y)

Can write down:

$$\vec{E} = E_z(z) \cdot \hat{a}_z$$

GAUSS' LAW

Do Problem 2a

Problem 2b → $\int \vec{E} \bullet d\vec{s}$,is constant.
For example a planar sheet of charge, where z is constant

Problem 2c To use GAUSS' LAW, we need to find a surface that encloses the volume

GAUSSIAN SURFACE - takes advantage of symmetry

$\vec{E} = E_r(r) \cdot \hat{a}_r$ - when ρ is only a f(r)

$\vec{E} = E_z(z) \cdot \hat{a}_z$ - when ρ is only a f(z)

GAUSS' LAW

$$\oint \vec{D} \bullet d\vec{s} = \int \rho \cdot dv = Q_{encl}$$

Use Gaussian surface to "pull" this out of integral

Integral now becomes:

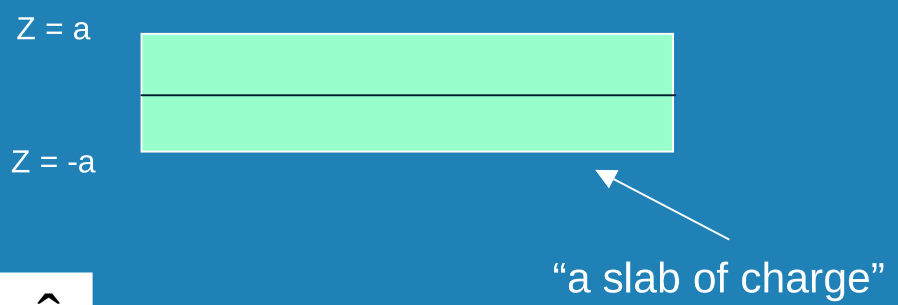
$$\left|\vec{D}\right| \cdot \oint \left|d\vec{s}\right| = \int \rho \cdot dv = Q_{encl}$$

Usually an easy integral for surfaces under consideration

GAUSS' LAW

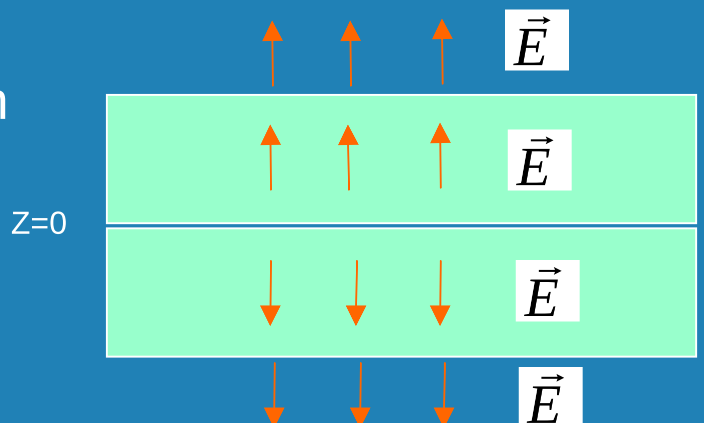
Example of using GAUSS' law to find $\vec{E}$

$$\rho = \begin{cases} \rho_0 & -a < z < a \\ 0 & Z > a \; ; \; z < -a \end{cases}$$



By symmetry: $\vec{E} = E_z(z)\cdot \hat{a}_z$

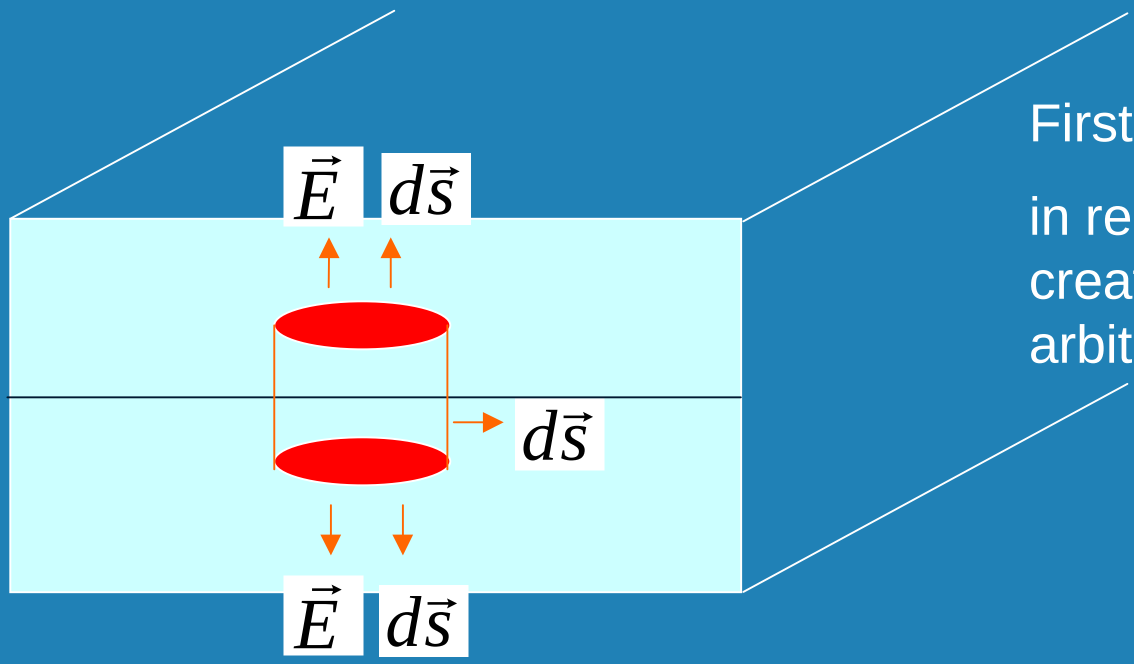
If $\rho_0 > 0$, then



From symmetry

$$\therefore E_z(-z) = -E_z(z)$$

GAUSS’ LAW



First get $\vec{E}$

in region |z| < a and create a surface at arbitrary z

Use Gaussian surface with top at z = z’ and the bottom at -z’

Note: Gaussian Surface is NOT a material boundary

GAUSS’ LAW

$$\varepsilon_0 \oint \vec{E} \bullet d\vec{s} = \int \rho \cdot dv$$

=0, since $\vec{E} \perp d\,\vec{s}$

Evaluate LHS:

$$\oint \vec{E} \bullet d\vec{s} = \int_{TOP} \vec{E} \bullet d\vec{s} + \int_{BOTTOM} \vec{E} \bullet d\vec{s} + \int_{SIDE} \vec{E} \bullet d\vec{s}$$

These two integrals are equal

$$\oint \vec{E} \bullet d\vec{s} = 2 \int_{TOP} \vec{E} \bullet d\vec{s}$$

GAUSS' LAW

$$2 \cdot \int_{TOP} \vec{E} \bullet d\vec{s} = 2 \cdot E_z \int ds = 2 \cdot E_z \cdot \pi \cdot r^2$$

Key Step: Take E out of the Integral

Computation of enclosed charge

$$\int \rho \cdot dv = \int_{-z'}^{z'} \rho_0 \cdot \pi \cdot r^2 \cdot dz = 2 \cdot \rho_0 \cdot z' \cdot \pi \cdot r^2$$

GAUSS' LAW

$$\varepsilon_0 \oint \vec{E} \bullet d\vec{s} = \int \rho \cdot dv$$

$$\varepsilon_0 \cdot 2 \cdot E_z \cdot \pi \cdot r^2 = 2 \cdot \rho_0 \cdot z' \pi \cdot r^2$$

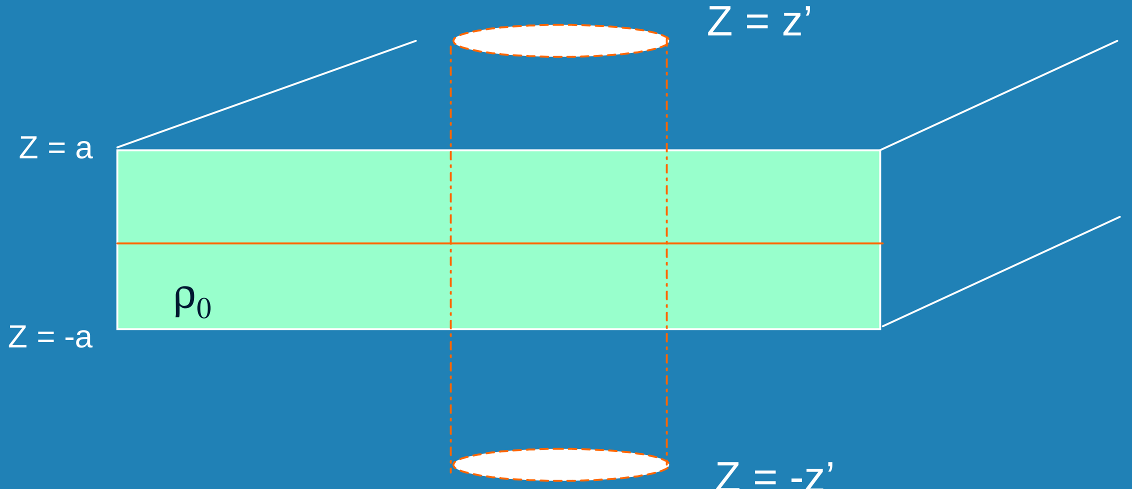
$$\Rightarrow \quad E_z = \frac{\rho_0}{\varepsilon_0} \cdot z$$ (drop the prime)

Do Problem 3a

GAUSS’ LAW

Back to rectangular, slab geometry example…..

Need to find $\vec{E}$, for |z| > a



GAUSS' LAW

As before,

$$\oint \vec{E} \bullet d\vec{s} = 2 \cdot \int_{TOP} \vec{E} \bullet d\vec{s} = 2 \cdot E_z \cdot \pi \cdot r^2$$

Computation of enclosed charge

$$\int \rho \cdot dv = \int_{-a}^{a} \rho_0 \cdot \pi \cdot r^2 \cdot dz = 2 \cdot a \cdot \pi \cdot r^2 \cdot \rho_0$$

Note that the z-integration is from -a to a ; there is NO CHARGE outside |z|>a

GAUSS' LAW

Once again,

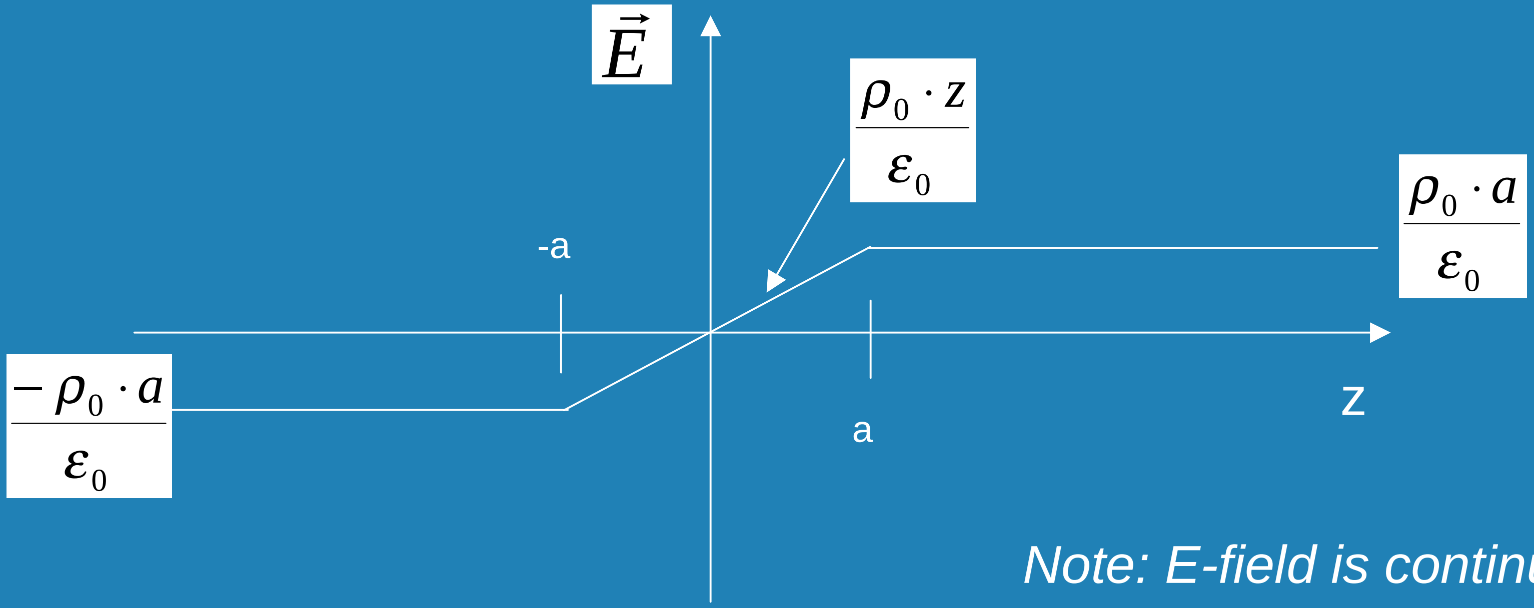
$$\varepsilon_0 \oint \vec{E} \bullet d\vec{s} = \int \rho \cdot dv$$

For the region outside |z|>a

$$\varepsilon_0 \cdot 2 \cdot E_z \cdot \pi \cdot r^2 = 2 \cdot a \cdot \pi \cdot r^2 \cdot \rho_0$$

$$\Rightarrow \quad \vec{E} = \frac{\rho_0 \cdot a}{\varepsilon_0} \cdot \hat{a}_z$$

GAUSS' LAW



Note: E-field is continuous

Plot of E-field as a function of z for planar example