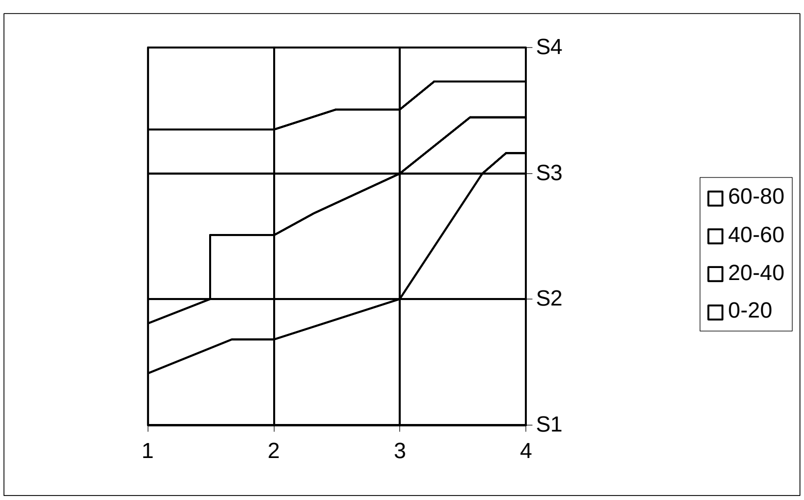
Fields and Waves I
ECSE-2100 Spring 2000

Using a Spreadsheet to Solve Laplace's Equation

Shown below a spreadsheet used to determine the solution to Laplace's equation in a simple 2D structure of the type we address in problem 2 of Lesson 2.6 . Here the top, right, bottom and left boundaries are conductors held at 80, 70, 0 and 30 volts potential, respectively. Such boundary conditions are established by setting the voltage in each cell (which represents a point on the boundary) equal to some fixed number. At the cells in the empty space between the electrodes, the contents are set equal to the average of the four nearest neighboring cells. For example, if the upper left cell is A1, the contents of cell B2 should be the formula =(B1+B3+A2+C2)/4 . Such a formula should be in each cell of this type.

80	80	80	80
30	50	60	70
30	30	40	70
0	0	0	0

Once all of the cells have been filled appropriately, then we must enable iteration to update the values in the cells until they come close to being equal to the average of their neighbors. Go to Tools button and select Options. Click on the Calculation tab. Put a check mark in the box next to the word iteration. Note the default values of the maximum iterations and the maximum change. The iteration will run until it reaches 100 or until the change becomes less than the maximum. The iteration can either be automatic or manual. Manual works better because you can control when it happens. Push key F9 to start the iteration. The potential values shown in the table above result from very few iterations, since only four cells are involved. The equipotentials for this structure are plotted below. The lines are very crude because so few points were used. Also the cells in the corners are not used in the calculation, but show up in the plot. When we use Excel for this purpose, we have to accept that it was designed for something else. Thus, there are certain oddities that appear in the results, particularly when they are plotted. In the plot below, the problem has been flipped over because that is the default view for the 3D plots. You can actually turn the plot right side up by changing the view elevation to -90^o instead of 90^o for this plot.



We can make this calculation much more accurate if we make the spreadsheet larger. We can also see how the process works better, since our small spreadsheet converged too fast. Doing this same problem again for an 11x11 array produces the following spreadsheet before any iterations are done.

80	80	80	80	80	80	80	80	80	80	80
30	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	70
30	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	70
30	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	70
30	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	70
30	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	70
30	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	70
30	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	70
30	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	70
30	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	27.5	70
0	0	0	0	0	0	0	0	0	0	0

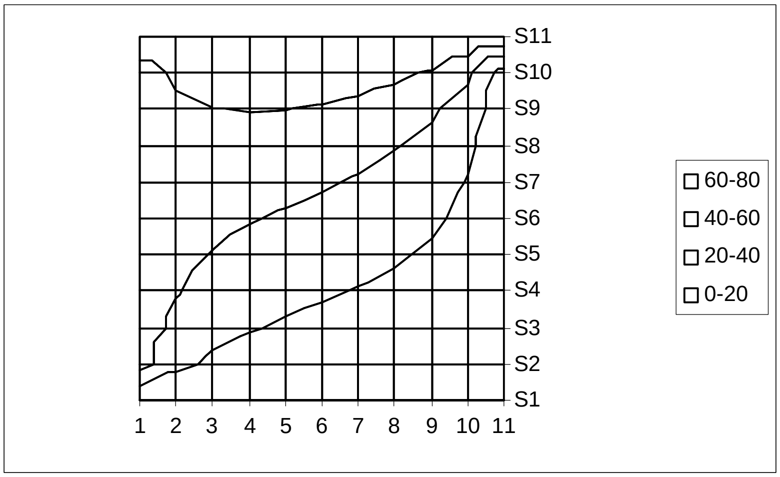
After one pass through all of the cells, it looks like:

80	80	80	80	80	80	80	80	80	80	80
30	41.25	44.063	44.766	44.941	44.985	44.996	44.999	45	55.625	70
30	31.563	32.656	33.105	33.262	33.312	33.327	33.332	33.333	46.614	70
30	29.141	29.199	29.326	29.397	29.427	29.439	29.443	29.444	43.39	70
30	28.535	28.184	28.127	28.131	28.14	28.145	28.147	28.148	42.259	70
30	28.384	27.892	27.755	27.721	27.715	27.715	27.715	27.716	41.869	70
30	28.346	27.809	27.641	27.591	27.576	27.573	27.572	27.572	41.735	70
30	28.336	27.786	27.607	27.549	27.531	27.526	27.525	27.524	41.69	70
30	28.334	27.78	27.597	27.537	27.517	27.511	27.509	27.508	41.675	70
30	21.459	19.185	18.57	18.402	18.355	18.341	18.338	18.336	32.503	70
0	0	0	0	0	0	0	0	0	0	0

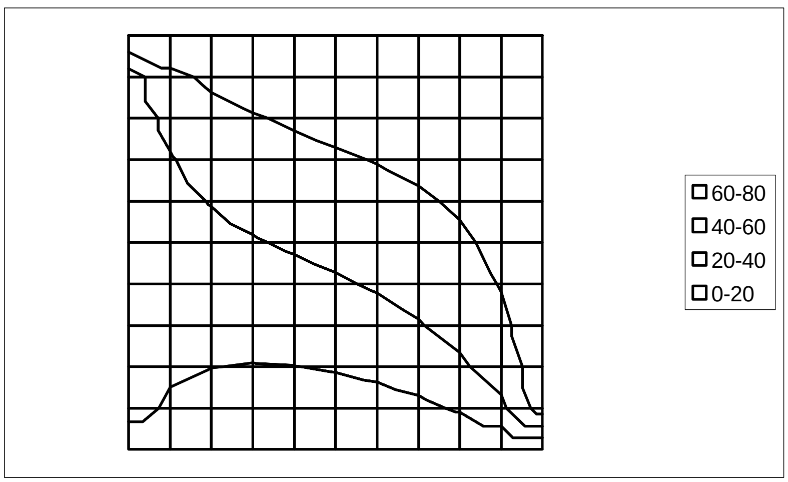
After well over 100 iterations (which takes almost no real time):

80	80	80	80	80	80	80	80	80	80	80
30	54.557	64.042	68.376	70.716	72.194	73.249	74.008	74.346	73.671	70
30	44.186	53.235	58.746	62.292	64.812	66.795	68.438	69.705	70.338	70
30	38.952	45.966	51.081	54.895	57.967	60.681	63.242	65.699	67.975	70
30	35.656	40.597	44.715	48.24	51.481	54.719	58.15	61.874	65.865	70
30	33.074	36.052	38.944	41.866	45	48.562	52.768	57.78	63.611	70
30	30.588	31.592	33.144	35.281	38.091	41.76	46.579	52.868	60.797	70
30	27.687	26.583	26.758	28.025	30.321	33.811	38.919	46.316	56.711	70
30	23.578	20.295	19.28	19.74	21.356	24.243	28.972	36.765	49.73	70
30	16.329	11.738	10.329	10.297	11.121	12.831	15.961	22.042	35.443	70
0	0	0	0	0	0	0	0	0	0	0

The plot now looks somewhat better, but it is still kind of a strange problem.



By changing the view elevation to –90°, we get the correct plot.



Most of the problems we wish to address are more complicated than the one we just considered. Also, even the simple parallel plate capacitor cannot be addressed in the same manner, since it does not have conductors on all sides. For other cases, we need additional boundary conditions.

Dielectric-Dielectric Boundary

For a boundary between two dielectric regions, we will rewrite the condition for the normal component of D in terms of the voltages as

$$D_{n1} = \varepsilon_1 E_{n1} = \varepsilon_1 \frac{\partial V_1}{\partial n} = \varepsilon_2 \frac{\partial V_2}{\partial n} = \varepsilon_2 E_{n2} = D_{n2}$$

where $\frac{\partial V}{\partial n}$ is the derivative of the voltage normal to the boundary. In finite difference form $\frac{\partial V_1}{\partial n} \approx \frac{V_1 - V}{\Delta}$ where V is the voltage at a cell on the boundary, V_1 is the voltage at the nearest cell in region 1 and Δ is the spacing between the cells. The normal boundary condition can thus be written as $e_1(V_1 - V) = e_2(V - V_2)$. Combining terms gives us $V = \frac{e_1V_1 + e_2V_2}{e_1 + e_2}$. For example, at a horizontal boundary between two dielectrics, the cell formula would be based on this expression rather than being set equal to the average of its neighbors.

Line of Symmetry

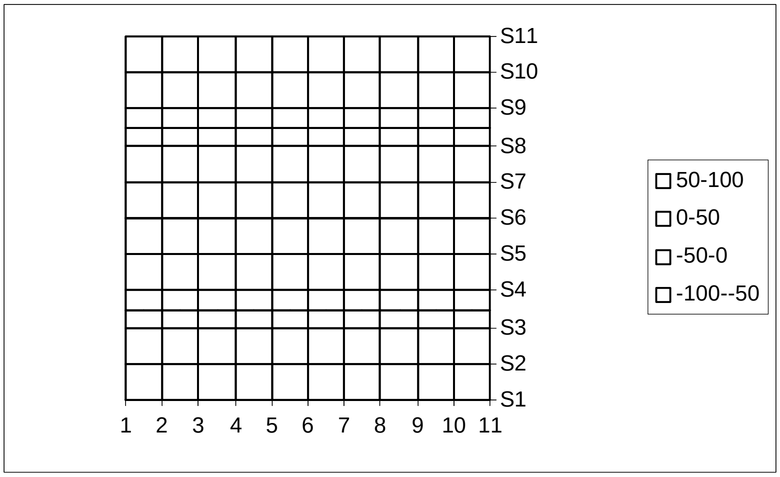
For a parallel plate capacitor or for a symmetric structure that contains an exactly vertical or horizontal electric field line, we know that the equipotentials must be perpendicular to the boundary of the problem. That means the boundary cells should be set equal to their immediate neighbor in a direction perpendicular to the boundary. Shown below is a parallel plate capacitor configuration, with 100 volts on the top plate and -100 volts on the bottom plate. There have been no iterations of this spreadsheet.

100	100	100	100	100	100	100	100	100	100	100
0	25	25	25	25	25	25	25	25	25	0
0	25	25	25	25	25	25	25	25	25	0
0	25	25	25	25	25	25	25	25	25	0
0	25	25	25	25	25	25	25	25	25	0
0	25	25	25	25	25	25	25	25	25	0
0	25	25	25	25	25	25	25	25	25	0
0	25	25	25	25	25	25	25	25	25	0
0	25	25	25	25	25	25	25	25	25	0
0	25	25	25	25	25	25	25	25	25	0
-100	-100	-100	-100	-100	-100	-100	-100	-100	-100	-100

The cells showing 25 volts all have the formula setting them equal to the average of their neighbors. The cells showing 0 are set equal to their immediate neighbor, since we know that the equipotentials must be horizontal. For example, cell A2 is set equal to cell B2 and cell K2 is set equal to J2. These cells show a value of 0 because they were defined before the interior cells. After many iterations, the spreadsheet converges to the solution we expect.

100	100	100	100	100	100	100	100	100	100	100
80	80	80	80	80	80	80	80	80	80	80
60	60	60	60	60	60	60	60	60	60	60
40	40	40	40	40	40	40	40	40	40	40
20	20	20	20	20	20	20	20	20	20	20
3E-13	3E-13	3E-13	3E-13	3E-13	3E-13	3E-13	3E-13	3E-13	3E-13	3E-13
-20	-20	-20	-20	-20	-20	-20	-20	-20	-20	-20
-40	-40	-40	-40	-40	-40	-40	-40	-40	-40	-40
-60	-60	-60	-60	-60	-60	-60	-60	-60	-60	-60
-80	-80	-80	-80	-80	-80	-80	-80	-80	-80	-80
-100	-100	-100	-100	-100	-100	-100	-100	-100	-100	-100

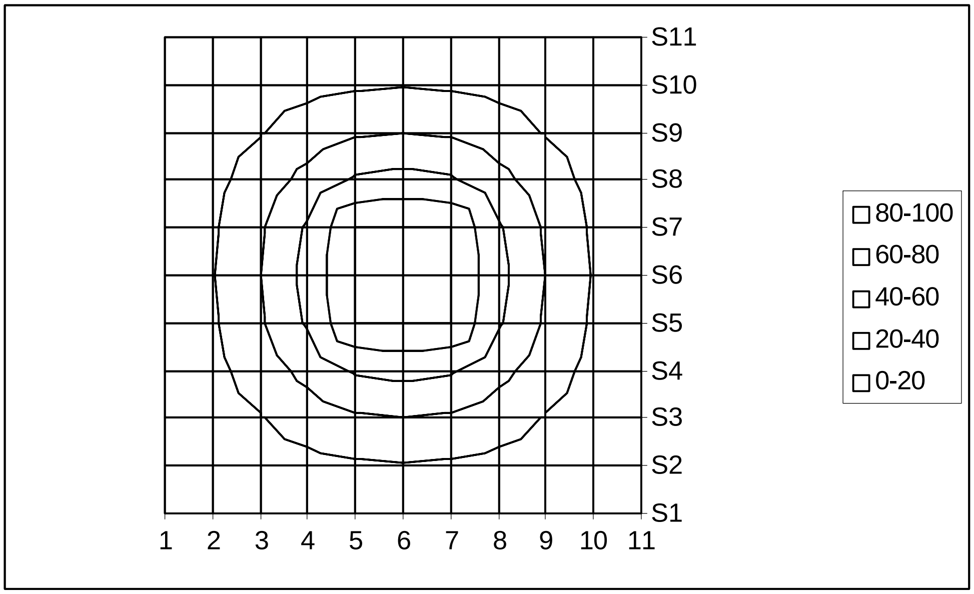
Note that the middle equipotential never converges exactly to zero, but it will get arbitrarily close if we let the iterations continue forever. The equipotential plot also looks very good in this case. The lines are so straight that the look like the grid lines.



Finally, we need to address how we use this information to figure out capacitance. Let us look at a problem similar to the coaxial structure of Lesson 2.6 – a square cross-section coaxial cable with 100 volts on the center conductor and a grounded outer conductor. The solved spreadsheet (11x11) looks like:

0	0	0	0	0	0	0	0	0	0	0
0	4.736	9.472	13.95	17.4	18.62	17.4	13.95	9.472	4.736	0
0	9.472	19.2	28.93	37.03	39.67	37.03	28.93	19.2	9.472	0
0	13.95	28.93	45.53	62.14	65.99	62.14	45.53	28.93	13.95	0
0	17.4	37.03	62.14	100	100	100	62.14	37.03	17.4	0
0	18.62	39.67	65.99	100	100	100	65.99	39.67	18.62	0
0	17.4	37.03	62.14	100	100	100	62.14	37.03	17.4	0
0	13.95	28.93	45.53	62.14	65.99	62.14	45.53	28.93	13.95	0
0	9.472	19.2	28.93	37.03	39.67	37.03	28.93	19.2	9.472	0
0	4.736	9.472	13.95	17.4	18.62	17.4	13.95	9.472	4.736	0
0	0	0	0	0	0	0	0	0	0	0

Since this is up-down symmetric, the plot of the equipotentials looks fine.



To determine the capacitance, we need to know the total charge on one of the conductors. We will choose to look at the outer conductor for this purpose, since the potentials near larger objects are more accurately determined. To find the charge we need the surface charge density. To get the charge density we need the electric flux density D and to get D we need the electric field E. At any point on the outer conductor, we can approximate E by subtracting the voltage at the neighboring cell from the voltage at the boundary and dividing by the separation distance of the cells.

$$\rho_s = D_n = \varepsilon E_n = \varepsilon \frac{V_{neighbor} - V_{boundary}}{\Delta}$$

where for each boundary cell, the neighboring cell is the closest cell in the direction normal to the boundary. For example, for cell D1, the neighboring cell is D2. Δ is the distance between cells. We could find the surface charge density at each point and multiply it by the small area represented by the boundary of each cell and then add them up to find the total charge Q. However, there is an easier way. If we find the average surface charge density for the outer boundary, then we need only multiply this number by the total area of the conductor to obtain the total charge. The good news is that the average function is standard in spreadsheets. If we take the average of the voltages neighboring the outer conductor, we get 12.193. Since the boundary voltages are all zero, this number is the average difference voltage. Now divide this number by the cell separation (let that be 1mm so that the cable is 10mmx10mm). Then we have $\rho_s = 12193\varepsilon$. The area per unit length for this cable is .04m^2. Thus the capacitance is

$$C = \frac{Q}{V} = \frac{12193\varepsilon(0.04)}{100} \approx 5\varepsilon$$

For a cylindrical coaxial cable with similar dimensions, $C = \dfrac{2\pi\varepsilon}{\ln\left(\dfrac{5}{1}\right)} \approx 4\varepsilon$, so the answer is plausible.